TORUS SUM FORMULA OF SIMPLE INVARIANTS
FOR 4-MANIFOLDS

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1. Introduction.

The topology of moduli spaces of anti-self-dual (ASD) connections is closely related with differentiable structures on 4-manifolds. In his celebrated paper [6], Donaldson has defined the polynomial invariants to distinguish differentiable structures on a 4-manifold. Though a significant result on the vanishing has been obtained at the same time, these invariants remain to be very difficult to determine completely. In fact, many examples have been calculated using an identification of irreducible ASD connections and stable vector bundles by Donaldson ([6], [7], [9], [12], [18]). But there are another direct approaches in the case that the dimension of ASD moduli is zero and that the invariant is just a number of the points in the moduli. For example Gompf and Mrowka have defined an invariant for 4-manifolds with torus end, using 0 or 1 dimensional ASD moduli, and proved that the invariant of the glued manifolds with solid torus can be emerged as the number of ASD connections which can be extended to over the solid torus. From a topological argument on K3 surface with elliptic fibration, they calculated the above numbers for fake K3 surfaces obtained by performing logarithmic transformations on embedded 2-tori. After that, Kromheimer has observed that the ASD moduli of Kummer surface comes down to the flat moduli as all (−2) curves tends to infinity, so the invariant could be computed algebraically [13]. The invariant obtained by 0-dimensional ASD moduli is said to be a simple invariant.

In this paper we give a torus sum formula of simple invariants for 4-manifolds. Our idea and formula are simple. Suppose that we have two simply connected closed 4-manifolds which contain a 2-torus with the trivial normal bundle. We assume that the complements are simply connected and the second Stiefel-Whitney class to define the $SO(3)$ bundle does not vanish on the 2-torus. Then any ASD connection converges to some ASD connections as the 2-torus tends to infinity. On the other hand, any ASD connection over the new 4-manifold obtained by torus sum also converges to some ASD connections as the bi-collar of the intermediate 3-torus is stretched to infinity. Hence we prove that the simple invariant of the new 4-manifold is the product of that of the

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given 4-manifolds. This formula is a variant of a relation of Donaldson invariants to Floer homology in Atiyah's exposition [1].

We can apply this formula to compute the simple invariant for regular elliptic surfaces. The rational elliptic surface has the least Euler number among them. The K3 surface is obtained by gluing two rational elliptic surfaces as fiber sum [17]. Gluing more rational elliptic surfaces yields all the other regular elliptic surfaces without multiple fibers. On the other hand, the simple invariant of the K3 surface has been known to be 1 for all second Stiefel-Whitney classes ([5], [13]). Hence the value is 1 for second Stiefel-Whitney classes whose restriction to the fiber are non-zero. In particular, if the geometric genus is even, then all the value is 1.

We remark that these calculations improve two known facts: the first, Sato and the author have shown that the value is non-zero for a second Stiefel-Whitney class, by using stable vector bundles [11]. The second, Ue has shown that the value of the above is independent of the choice of second Stiefel-Whitney class with the additional condition, by analyzing the action of the diffeomorphism groups on second cohomology groups [21].

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2. Review of simple invariant and main result

We recall the simple invariant γ defined by Donaldson ([6], [7, Chapter 9]) let X be an oriented closed smooth 4-manifold with the following properties;

(A1) \( \pi_1(X) = 1 \),

(A2) \( b_+(X) \geq 3 \) and \( b_+(X) \) is odd,

where \( b_+ \) denotes the dimension of the maximal positive subspace for the intersection on \( H^+(X) \). Then there is a well defined lift of mod 2 cup square called the Pontrjagin square \( H^b(X ; \mathbb{Z}_2) \rightarrow H^b(X ; \mathbb{Z}_4) \). Composing this map with evaluation on the fundamental class defines a quadratic map \( H^b(X ; \mathbb{Z}_4) \rightarrow \mathbb{Z}_4 \). Let \( P \rightarrow X \) be the principal \( SO(3) \) bundle with \( w_2(P) = \eta \) and \( p_1(P) = l \). They satisfy \( \eta^2 \equiv l \pmod{4} \). A theorem of Dold and Whitney [4] tells us that \( SO(3) \) bundles over a compact 4-manifold are determined completely by \( \eta \) and \( l \). Conversely, given \( \eta \) and \( l \) with \( \eta^2 \equiv l \pmod{4} \), we can easily construct the corresponding \( SO(3) \) bundle over \( X \). We give \( X \) a Riemannian metric. Then the affine space \( \mathcal{A}_P \) of \( L^2_4 \) connections has a natural Banach structure from the \( L^2_4 \) norm. The \( L^2_4 \) gauge group \( \mathcal{G}_P \) is a Banach Lie group. The Lie algebra of \( \mathcal{G}_P \) is \( L^2_4(\text{Ad } P) \). Let \( \mathcal{A}_P^\text{reg} \subset \mathcal{A}_P \) denote the subset of irreducible connections. Then the quotient \( \mathcal{M}_X = \mathcal{A}_P^\text{reg} / \mathcal{G}_P \) is a \( C^\infty \)-Banach manifold such that the projection \( \pi : \mathcal{A}_P \rightarrow \mathcal{M}_X \) defines a principal \( \mathcal{G}_P \)-bundle ([8], [7, 4.2]). The tangent space to \( [A] \in \mathcal{M}_X \) is isomorphic to \( \{ a \in L^2_4(\text{Ad } P) \mid d_A^* a = 0 \} \). We say an element \( [A] \) in \( \mathcal{M}_X \) to be regular if the operator \( d_A^* \) is surjective. Let \( \mathcal{M}_X(l, \eta, g) \) denote the moduli space of \( g \)-ASD connections on \( P \). Then the formal dimension is equal to
We choose $l_X$ so that $-2l_X - 3(1 + b_+(X)) = 0$. Let $C_X^r (r \geq 3)$ denote the space of $C^r$-metrics on $X$. By ([8], [7, 4.3]), there is a Baire set $C_X^r$ in $C_X^r$ such that for all $g \in C_X^r$, $\mathcal{M}_X(l_X, \eta, g)$ is a finite set consisting of irreducible regular connections. We fix $g \in C_X^r$. Then we can define a sign at any $[A] \in \mathcal{M}_X(l_X, \eta, g)$ using a line bundle over $\mathcal{B}_P^p$. For $A \in \mathcal{A}_P^p$, we consider the deformation complex

$$\delta_A = d \star d_A : L^2_\alpha(\Omega_k^1(Ad P)) \longrightarrow L^2_\alpha((\Omega^1_k \oplus \Omega^p_1)(Ad P)).$$

We choose a linear map $S : R^N \rightarrow L^2_\alpha(\Omega^1_k(Ad P))$ so that $\delta_A \oplus S$ is surjective. Then we define the determinant line of $\delta_A \oplus S$ by

$$A_{P,A} = (A^{\text{def}} \text{Ker}(\delta_A \oplus S)) \otimes (A^N R^N)^*.$$ 

This line has an intrinsic sense by the exact sequence

$$0 \longrightarrow \text{Ker} \delta_A \longrightarrow \text{Ker}(\delta_A \oplus S) \longrightarrow \mathcal{P} \longrightarrow \text{Coker} \delta_A \longrightarrow 0.$$ 

Since the surjectivity holds in a neighborhood of $A$, these lines are patched together to get a locally trivial line bundle over $\mathcal{A}_P^p$. It descends to the determinant line bundle $A_{P} \rightarrow \mathcal{B}_P^p$ by the free action of $\mathcal{B}_P$. The bundle $A_{P} \rightarrow \mathcal{B}_P^p$ is topologically trivial ([5], [7, 5.4]). Since $\delta_A$ is an isomorphism for any $[A] \in \mathcal{M}_X(l_X, \eta, g)$, we can define a section on $A_{P}$ at $[A]$ by

$$(\pi^A_1(e_1) \wedge \cdots \wedge \pi^A_N(e_N)) \otimes (e_1 \wedge \cdots \wedge e_N)^*,$$

where $e_1, \ldots, e_N$ form a basis of $R^N$. This defines an orientation $o([A])$ of the line bundle $A_{P} \rightarrow \mathcal{B}_P^p$. For a later use, we remark that this section is defined on a connected region $U([A])$ consisting of irreducible regular connections about $[A]$.

On the other hand, for an orientation $\Omega$ of $H^*(X)$ and an integral lift $c$ of $\eta$, there is another orientation $\sigma(\Omega)$ determined at a connection obtained by attaching some standard instantons over $S'$ with reducible connection determined by $c$ ([5], [7, 7.1.6]). Then the sign $\epsilon([A])$ at $[A] \in \mathcal{M}_X(l_X, \eta, g)$ is given by $\sigma([A]) = \epsilon([A]) \sigma(\Omega)$ and the simple invariant is defined by

$$\gamma_x(\eta) = \sum_{[A] \in \mathcal{M}_X(l_X, \eta, g)} \epsilon([A]).$$

This function $\gamma_x$ is independent of the choice of the element $g$ in $C_X^r$ and satisfies the following ([6], [7]): If $\phi : X \rightarrow X'$ is an orientation preserving diffeomorphism, then $\gamma_x(\phi^*(\eta')) = \epsilon(\phi)\gamma_x(\eta')$ where $\epsilon(\phi)$ is $-1$ if either $\phi^*$ is an orientation reversing map from $H^*(X')$ onto $H^*(X)$ or $(c - \phi^*(c'))/2 \equiv 1 \pmod 2$ but not both and is $1$ otherwise. (Here $c'$ denotes the integral lift of $\eta'$ and $c$ is the integral lift of $\phi^*(\eta')$ used in orientaring their respective moduli spaces.) Hence the absolute value $|\gamma_x|$ can be thought of a function on

$$C_X = \{ \eta \in H^*(X; Z_2) | \eta \neq 0, \eta^2 \equiv l_X \pmod 4 \}.$$
We return to our main theorem. Let $K$ be a compact oriented smooth 4-manifold with boundary $Z = T^4$, satisfying the following:

(B1) $\tau_1(K) = 1$,
(B2) $b_*(K) \geq 2$ and $b_*(K)$ is even.

For two such manifolds $K_1, K_2$, and an orientation reversing diffeomorphism $\phi: \partial K_1 \to \partial K_2$, the oriented 4-manifold $X = K_1 \cup_\phi K_2$ always satisfies (A1), (A2).

Let $\sigma: Z \to X$, $\sigma_i: K_i \to X$ $(i = 1, 2)$ be the inclusion. For each $\eta \in H_2(X, Z)$ with $\sigma^*(\eta) \neq 0$, we can define an orientation reversing diffeomorphism $\phi_i: \partial K_i \to Z$ such that $\phi_i^*(\eta)$ can be extended to a class $\eta_i \in H^2(K_i^*; Z)$. The oriented closed 4-manifold $K^*_i = K_i \cup_\phi W$ also satisfies (A1), (A2). Here $W$ is the solid torus $T^2 \times D^2$.

**Theorem 2.1.** If $\eta_i^* s$atisfies $(\eta_i^*)^2 \equiv l_{\pi_i^*} (\text{mod } 4)$ for each $i = 1, 2$, then 
$$|\gamma_x(\eta)| = |\gamma_{K_i^*}(\eta_i)| = |\gamma_{K_i^*}(\eta_i)|.$$

**Theorem 2.2.** If $\eta_i^*$ does not satisfy $(\eta_i^*)^2 \equiv l_{\pi_i^*} (\text{mod } 2)$ for some $i = 1, 2$, then $\gamma_x(\eta) = 0$.

Remarks. (1) Let $P. D.$ be the mod 2 Poincaré dual. Then $\eta_i^* + P. D. [T^2 \times 0]$ satisfies $(\eta_i^* + P. D. [T^2 \times 0])^2 \equiv (\eta_i^*)^2 + 2 (\text{mod } 4)$ and by the exact sequence

$$0 \to H^2(W, Z) \to H^2(K_i^*; Z) \to H^2(K_i; Z),$$

it is only another choice for $\eta_i^*$. So the conditions of Theorem 2.1 and 2.2 are complementary to each other.

(2) In our application, if $K^*_i$ and $\eta_i^* (i = 1, 2)$ are chosen, then we will write $X = K^*_i \cup K^*_i$ and $\eta = \eta \cap \eta^i$.

3. Setting up gauge theory

We first argue the ASD moduli over a 4-manifold with torus end. According to Taubes [20], we study a gauge theory on a convenient subspace of connections to apply a known analysis and to contain all ASD connections by some gauge. The uniqueness of flat connections over the torus enable us to apply his argument directly. For $\eta = 2, 3$, we denote by $\mathcal{X}(T^n)$ the set of the conjugacy classes of representations from $\pi_1(T^n)$ to $SO(3)$. The topology of $\mathcal{X}(T^n)$ has been discussed in [10, Proposition V. 2.1]. Given a representations $\rho$, we form the associated flat $\mathbb{R}^3$ bundle $\xi_\rho$. Then we define a map

$$\omega_\rho: \mathcal{X}(T^n) \to H^2(T^n, Z),$$

by $\omega_\rho(\rho) = \omega_\rho(\xi_\rho)$. This is surjective. We denote by $\mathcal{X}_a(T^n)$ the preimage of $a \in H^2(T^n, Z)$. Then the decomposition

$$\mathcal{X}(T^n) = \bigcup_{a \in H^2(T^n, Z)} \mathcal{X}_a(T^n).$$
decomposes \( \chi(T^n) \) into connected components. \( \chi(T^n) \) is homeomorphic to \( T^n/\pm 1 \) and the other 7 components are isolated points whose stabilizers are isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Any representation \( p \) in the isolated points is regular, that is \( H^1(T^n; \mathbb{Z}) = 0 \) \( (0 \leq i \leq 2) \).

For the unique non-zero element \( \alpha \in H^2(W; \mathbb{Z}_2) \), we let \( Q, \Gamma \) and \( \mathcal{R} \) be the corresponding flat bundle, flat connection and stabilizer respectively. We write \( Q = \bar{Q} \mid_z \) and \( \Gamma = \bar{\Gamma} \mid_z \). Then the stabilizer of \( \Gamma \) is just \( \mathcal{R} \).

Let \( K \) denote a compact oriented smooth 4-manifold with boundary \( Z \), satisfying (B1) and (B2). Let \( \sigma : Z \to K \) be the inclusion. Define \( Y = K \cup Z \times [0, \infty) \). We choose \( \eta \in H^1(K; \mathbb{Z}_2) \) with \( \tau^*(\eta) \neq 0 \). Let \( P_0 \) be the \( \text{SO}(3) \) bundle over \( K \) with \( w_3(P_0) = \eta \) and \( p_1(P_0) = 0 \in H^1(K; \mathbb{Z}) = 0 \). For a bundle isomorphism \( \zeta : \sigma^*(P_0) \to Q \), define \( P = P_0 \cup_\pi \Gamma Q \), where \( \pi : Z \times [0, \infty) \to Z \) is the projection. We give \( Y \) a metric which is the product metric \( h + dt^2 \) on \( Z \times [0, \infty) \), where \( h \) is a metric on \( Z \). Let \( \tau : Y \to \mathbb{R} \) be a smooth function such that \( \tau(x, t) = t \) on \( Z \times [1, \infty) \) and \( \tau = 0 \) on \( K \). We fix a smooth connection \( A_\delta \) on \( P \) such that \( A_\delta \) is equal to \( \pi^* \Gamma \) over \( Z \times [0, \infty) \). We write \( (-1/4\pi^2) \int_Y \text{Tr}(F_{A_\delta} \wedge F_{A_\delta}) = \ell (\in \mathbb{Z}) \).

For \( \rho \geq 1 \), \( k \geq 0 \) and \( \delta > 0 \), we define the weighted norm \( L^p_{k, \delta}(Y) \) by

\[
\| s \|_{L^p_{k, \delta}(Y)} = \left( \int_Y e^{\delta \sum |A_{\delta} s|^k} \right)^{1/p}.
\]

Then we consider the following class of connections and gauge group on \( P \)

\[
\mathcal{A}_\rho = \{ A_\delta + a \mid a \in L^1_{k, \text{loc}}(\Omega^1(\text{Ad} P)), \| a \|_{L^1_{k, \text{loc}}(\text{Ad} P)} < \infty \},
\]

\[
\mathcal{G}_\rho = \{ u \in L^1_{k, \text{loc}}(\text{Aut} P) \mid \| \nabla_{A_\delta} u \|_{L^1_{k, \text{loc}}(\text{Ad} P)} < \infty \}.
\]

By the argument in [20, Section 7], we see that there is a well defined map \( r : \mathcal{G}_\rho \to \mathcal{R} \) given by

\[
r(u) = \lim_{t \to \infty} u_t,
\]

where the limit is in \( C^0 \)-convergence. We note that the automorphisms in \( \mathcal{G}_\rho \) are continuous by the Sobolev embedding theorem \( L^1_{k, \text{loc}}(\mathbb{R}^4) \to C^0 \).

**Lemma 3.1.** The automorphism \( a \in \mathcal{R} \) over \( \partial K \) can be extended to all over \( K \) continuously if and only if \( a = 1 \).

**Proof.** The primary difference \( b(1 \mid_{\partial K}, a) \) [19, §36] between the identity \( 1 \in C^0(\text{Aut} P_0) \) and \( a \neq 1 \) is a non-zero element in \( H^1(\partial K; \mathbb{Z}_2) \). If \( a \) can be extended to \( K \), then by the extension theorem [19, 37.11], there is an element \( v \) in \( H^1(K; \mathbb{Z}_2) \) such that \( \sigma^*(v) = b(1 \mid_{\partial K}, a) \). But this contradicts to \( H^1(K; \mathbb{Z}_2) = 0 \).

**Corollary 3.2.** The image of the map \( r \) is \( \{ 1 \} \) and there are no reducible connections in \( \mathcal{G}_\rho \).
A gauge theory on 4-manifolds with cylindrical end has been studied by Taubes [20, Section 7]. Using his argument, we can prove the following two lemmas (see also [16]). We topologize $\mathcal{A}_P$, $\mathcal{G}_P$ by the norm $\| \cdot \|_{L^2_{\delta}(\delta Y)}$, $\| \nabla_{\mathcal{A}_P} \|_{L^2_{\delta}}$ respectively. Then

**LEMMA 3.3.** $\mathcal{G}_P$ is a Banach Lie group. The Lie algebra of $\mathcal{G}_P$ is

$$\mathfrak{g}_P = \{ \sigma \in L^1_{\text{loc}}(\text{Ad } P) \mid \| \nabla_{\mathcal{A}_P} \|_{L^2_{\delta}(\delta Y)} < \infty \}.$$  

$\mathcal{G}_P$ acts smoothly on $\mathcal{A}_P$. The set $\{ u \in L^1_{\text{loc}}(\text{Aut } P) \mid u \mathcal{A}_P = \mathcal{A}_P \}$ is $\mathcal{G}_P$.

**LEMMA 3.4.** The quotient $\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}_P$ is a $C^\infty$-Banach manifold such that the projection $\mathcal{A}_P \rightarrow \mathcal{B}_P$ defines a principal $\mathcal{G}_P$-bundle. The tangent space to $[A] \in \mathcal{B}_P$ is isomorphic to

$$\{ a \in L^2_{\delta}(\text{Aut } P) \mid \| a \|_{L^2_{\delta}(\delta Y)} < \infty, \ e^{-\delta\pi\delta} a = 0 \}.$$  

**LEMMA 3.5.** For bundle isomorphisms $c, c' : \sigma_\delta(P_0) \rightarrow Q$, we consider $SO(3)$ bundles $P = P_0 \cup \pi^* Q$, $P' = P_0 \cup \pi^* Q$, and $C^m$ ($m \in N$)-connection $A_0, A'_0$ on $P, P'$ such that $A_0, A'_0$ are equal to $\pi^* \Gamma$ over $Z \times [0, \infty)$ respectively. Then the following two conditions are equivalent.

1. The automorphism $(c')^{-1} c$ can be extended to all over of $K$ continuously for some $r \in \mathcal{R}$ with $b(1, (c')^{-1} c) = 0$.

2. $(-1/4\pi^3) \int_Y \text{Tr}(F_{A_0} \wedge F_{A_0}) = (-1/4\pi^3) \int_Y \text{Tr}(F_{A'_0} \wedge F_{A'_0})$.

**Proof.** We consider $SO(3)$ bundles $P^* = P_0 \cup \bar{\mathcal{Q}}, (P')^* = P_0 \cup \bar{\mathcal{Q}}$ over $K \cup W$ and $C^k$-connections $A^*, (A')^*$ extended by $I^*$ respectively. If the condition (1) holds, then the connections $A_0$ and $A'_0$ have the same integral by Chern-Weil formula. Conversely if the condition (2) holds, then obviously $\bar{\pi}_*(P^*) = \bar{\pi}_*((P')^*)$ and, moreover, $w_0(P^*) = w_0((P')^*)$ for we can write $w_0(P^*) = w_0((P')^*) + PD$. $[T^2 \times 0]$, which induces $\bar{\pi}_*(P^*) = \bar{\pi}_*((P')^*) + 2 \text{ mod } 4$, a contradiction. By Dold-Whitney theorem, there exist bundle isomorphisms $f : P_0 \rightarrow P_0$ and $h : \mathcal{Q} \rightarrow \mathcal{Q}$ with $\bar{\epsilon} f = h t$. We prove that $A$ is homotopic to some $r \in \mathcal{R}$. It suffices to prove that $A |_{\mathcal{T}_2}$ is homotopic to some $r \in \mathcal{R}$. We remove an open 2-disk $D^2$ in $T^2$. By the homotopy classification theorem [19, 37.12], the assignment of the primary difference $b(1, \cdot)$ in $H^1(T^2 \setminus D^2)$ is isomorphic to each homotopy class in $[T^2 \setminus D^2; \text{Aut } Q]$ sets up a 1-1 correspondence between each sets. So $A |_{\mathcal{T}_2}$ is homotopic to some $r \in \mathcal{R}$ by a homotopy $H_t \ (0 \leq t \leq 1)$. We extend $H_t$ to all over to $T^2$, using the collar of $\partial D^2$. Since $[(D^2, \partial D^2), (SO(3), \text{id})] = \pi_2(SO(3)) = 1$, $H_t |_{D^2}$ is homotopic to $r$ relative to $\partial D^2$, and so $(c')^{-1} c$ is homotopic to $r$. We see that $b(1, (c')^{-1} c)$ is zero by the argument in Lemma 3.1. D

**COROLLARY 3.6.** $\mathcal{B}_P$ depends only on $\eta$ and $/$. 
We return to the moduli of ASD connections. By [15, Theorem 1.1], there exists \( \delta > 0 \) such that for \( 0 < \delta < \delta \), \( p \geq 2 \) and \( k \geq 0 \), if \( A \) is an ASD connection on \( P \), then the AHS complex
\[
0 \rightarrow \Omega^p(\text{Ad}P) \xrightarrow{d_A} \Omega^p(\text{Ad}P) \xrightarrow{d^+_A} \Omega^p(\text{Ad}P) \rightarrow 0
\]
defines a Fredholm complex
\[
0 \rightarrow L^p_{k+1, \delta}(Y) \xrightarrow{d_A} L^p_{k+1, \delta}(Y) \xrightarrow{d^+_A} L^p_k(\text{Ad}P) \rightarrow 0.
\]
We denote by \( H^*_A(0 \leq i \leq 2) \) the cohomology of the above complex. Then its index is given by \([10, VI. 3]\)
\[-\dim H^*_A - \dim H_*^A - \dim H^*_A = -2l - 3(2 + b_*(Y)).\]

Let \( \mathcal{C}_Y^r (r \geq 3) \) be the space of all conformal classes of \( C^r \)-metrics on \( Y \) which are fixed metric \( h + dt^2 \) on \( Z \times [0, \infty) \). If we fix a metric \([g_0] \in \mathcal{C}_Y^r\), then \( \mathcal{C}_Y^r \) is identified with
\[
\{ m : A^+ \rightarrow A^- \ C^r \text{-bundle map}, \sup |m| < 1, m|_{Y \times K^0} = 0 \},
\]
where \( A^* \) is \( \pm \) self-dual space with respect to \( g_0 \) and \( K^0 \) is the interior of \( K \) ([7, 1.1.5]). We choose \( l_Y \) with \(-2l_Y - 3(2 + b_*(Y)) = 0 \).

**Proposition 3.7.** There is a Baire set \( \mathcal{C}_Y^r \subset \mathcal{C}_Y^r \) such that for all \( l_Y \leq l < 0 \) and \( g \in \mathcal{C}_Y^r \), the ASD moduli
\[
\mathcal{M}_Y(l, \eta, g) = \{ [A] \in \mathcal{B}_P | F_A = -*F_A, \frac{-1}{4\pi^2} \int_Y \text{Tr}(F_A \wedge F_A) = l \}
\]
is a finite set consisting of irreducible regular connections. Any element in \( \mathcal{M}_Y(l, \eta, g) \) has a smooth representative. Its dimension is equal to \(-2l - 3(2 + b_*(Y))\). In particular if \( l_Y < l < 0 \), then \( \mathcal{M}_Y(l, \eta, g) \) is empty. Here the regularity at \([A] \in \mathcal{M}_Y(l, \eta, g) \) means that \( H^*_A = 0 \).

**Proof.** The regularity follows from the same argument as in [7, 4.3] (see also [14, 5 (iii)]). By the argument in [7, 4.2.3], we see that the representative in Coulomb gauge relative to some nearby smooth gauge is also smooth. We will prove the compactness in Appendix 2. \( \square \)

**Remark.** The proposition above has been stated in [10, Theorem V.3.3]. But we do not know how they orient the ASD moduli \( \mathcal{M}_Y(l, \eta, g) \) in spite of the non-existence of reducible connections in \( \mathcal{B}_P \).

Let \( \eta \in \mathcal{C}_X \), \( \eta_+ \in \mathcal{C}_K^* \) and \( \eta_- \in \mathcal{C}_K^* \) satisfy the assumption of Theorem 2.1. Let \( P^* \) be the \( SO(3) \) bundle over \( X \) with \( w_2(P^*) = \eta \) and \( p_i(P^*) = l_{K^*} \), and let \( P^* \) be the \( SO(3) \) bundle over \( K^* \) with \( w_2(P^*) = \eta \) and \( p_i(P^*) = l_{K^*} \). We write \( P^* = P_{01} \cup_{\varepsilon_1} Q \) for some bundle isomorphism \( \varepsilon_1 : P_{01} \rightarrow Q \) and write \( P^* = P_{01} \cup_{\varepsilon_1} P_{00} \).
for some bundle isomorphism $\iota_z: P_{8z} \to Q$. We put $P_8^z = P_{8z} \cup \iota_z Q$. We fix a smooth connection $A_t$ on $P_t^z$ which is $f$ on $Q$ $(t=1, 2)$, and a smooth connection $A^*_8$ on $P_8$ which is $A^*_t$ on $P_t$. Then

$$p_1(P^z_8) = -\frac{1}{4\pi^2} \int_{K^t_s} \text{Tr} (F_{A_t} \wedge F_{A^*_t}) = -\frac{1}{4\pi^2} \int_{K^t_s} \text{Tr} (F_A \wedge F_A) - \frac{1}{4\pi^2} \int_{K^t_s} \text{Tr} (F_{A^*} \wedge F_{A^*_t})$$

$$= l_x - l_{K^s_1} = l_{K^s_2},$$

and $w_a(P^z_8) = \eta^z_1$. If not, then it must be $w_a(P^z_t) = \eta^z_t + \text{P.D. } [T^s \times 0]$, which induces $l_{K_{s1}} = l_{K^s_2} + 2$ (mod 4). This is a contradiction. We define $Y_1 = K_1 \cup Z \times [0, \infty)$ and $P_i = P_{8i} \cup \iota_i \pi_t^* Q$ for each $i=1, 2$. We fix a connection $A_{8t}$ on $P_t$ by

$$A_{8t} = \begin{cases} A^*_t & \text{over } K^t, \\ \pi^*_t \Gamma & \text{over } Z \times [0, \infty). \end{cases}$$

Using $A_{8t}$ and $0 < \delta_0 < \delta$, we define the space $\mathcal{A}_{P_t}$, the gauge group $\mathcal{G}_{P_t}$, and the quotient $\mathcal{B}_{P_t}$.

We move on the gauge theory on 4-closed manifolds. We choose a lift $\tilde{\phi}: Q \to Q$ of $\phi: \partial K_1 \to \partial K_2$, so that $\tilde{\phi} \Gamma = \Gamma$. For $n \in \mathbb{N}$, let $X_n$ be the oriented closed 4-manifold defined by

$$X_n = K_1 \cup Z \times [0, n + 1] \cup Z \times [0, n + 1] \cup Z \times [0, n + 1] \cup Z \times [n - 1, n + 1],$$

where $\phi_n$ is the diffeomorphism on $Z \times [n-1, n+1]$ defined by using $\phi$ and the reflection on $[n-1, n+1]$. Let $P_n$ be the $SO(3)$ bundle over $X_n$ defined by

$$P_n = P_{81} \cup Q \times [0, n + 1] \cup Q \times [0, n + 1] \cup P_{82},$$

where $\tilde{\phi}_n$ is the automorphism on $Q \times [n-1, n+1]$ defined by using $\phi$ and the reflection on $[n-1, n+1]$. We put

$$\mathcal{A}_{P_n} = \begin{cases} A_{81} & \text{over } K_1 \cup Z \times [0, n + 1], \\ A_{82} & \text{over } K_2 \cup Z \times [0, n + 1]. \end{cases}$$

Let $\mathcal{C}_{X_n}(r \geq 3)$ be the space of all conformal classes of $C^r$-metrics on $X_n$ which are the fixed metric $h + dt^2$ on $Z \times [0, n + 1] \cup Z \times [n - 1, n + 1]$. If we fix a metric $[g_6]$ in $\mathcal{C}_{X_n}$, then $\mathcal{C}_{X_n}$ is identified with

$$\{ m: A^* \to A^-; C^r \text{-bundle map, } \sup |m| < 1, m|_{X_n \setminus K^s_1 \setminus K^s_2} = 0 \},$$

where $A^*$ is ± self-dual space with respect to $g_6$. Then we have the following theorem, whose proof is also the same as in ([7, 4.3], [14, 5 (iii)]).

**Proposition 3.8.** There is a Baire set $\mathcal{C}'_{X_n} \subset \mathcal{C}_{X_n}$ such that for all $g \in \mathcal{C}'_{X_n}$, the ASD moduli $\mathcal{M}_{X_n}(l_{X_n}, \eta, g)$ is a finite set consisting of irreducible regular connections.
For each \(n \in \mathbb{N}\), we fix a metric \(g_n\) in \(C'(K)\) such that \(g_n\) is independent of \(n\) on \(K\). The metric \(g_n\) lies in \(C'(Z)\) for \(t = 1, 2\). We can find these metrics because the intersection of countable Baire sets is again a Baire set. For \(n \in \mathbb{N}\) we fix a function \(\tau_n : X_n \to \mathbb{R}\) by

\[
\tau_n = \begin{cases} 
\tau_1 & \text{on } \tau_1^t([0, n - \varepsilon]), \\
\tau_2 & \text{on } \tau_2^t([0, n - \varepsilon]),
\end{cases}
\]

to a small \(0 < \varepsilon < 1\). We use the weighted norm \(L^p_{\delta}(X_n)\) defined by

\[
\|s\|_{L^p_{\delta}(X_n)} = \left( \int_{X_n} e^{\tau_n(x)} \sum_{i=0}^{k} |\nabla_{X_n} s|^p \right)^{1/p}.
\]

We note that if \(s\) is supported in \(\tau_1^t([0, n + 1])\), then \(\|s\|_{L^p_{\delta}(X_n)} \leq \|s\|_{L^p_{\delta}(Y_t)}\) and if the support of \(s\) is contained in \(\tau_2^t([0, n - 1])\), then \(\|s\|_{L^p_{\delta}(X_n)} = \|s\|_{L^p_{\delta}(Y_t)}\). We fix \(p > 4, q > 8, 0 < \delta \in (0, \delta)\), and \(\lambda \in (0, \lambda)\) by

\[
\frac{1}{q} + \frac{1}{q} = \frac{1}{p}, \quad \frac{1}{2} < \frac{\delta}{\lambda} < \frac{\delta}{2},
\]

where \(\lambda\) is the first eigenvalue of \(\Delta F\) on \(\text{Ker} d^*_q \subset \Omega^1(\text{Ad} Q)\). The second inequality implies that \(L^p_{\delta}(Y_t) \subset L^p_{\delta}(Y_t)\) [15, Lemma 7.2].

4. Decay estimate.

In [22], Uhlenbech has proved that the curvature of ASD connections controls the uniform norm of the connection matrices in Coulomb gauge. Taubes has extended the result to ASD connections on the trivial bundle over 4-manifolds ([20], see also [3, Appendix A]). We show that his argument is applicable to ASD connections on flat bundles with an unique flat connection in the same way.

**Lemma 4.1.** Let \(U\) be an oriented open noncompact Riemannian 4-manifold. Let \(U' \subset U\) be an interior domain with compact closure \(U\). We let \(Q : U \to U'\) be a flat SO(3) bundle and \(\Gamma_0\) be a canonical flat connection. Suppose that \(Q_0|_{U'} \to U'\) cannot admit any other flat connection topologically. Then there are constants \(\varepsilon > 0\) and \(\zeta_m, m > 0\) which depend on \(U'\) and \(m \in \mathbb{N}\) with the following significance:

Let \(A\) be an ASD connection on \(P\) with \(\|F_A\|_{U'} < \varepsilon\). Then there exists \(h \in C^\infty(\text{Aut} Q_0|_{U'})\) such that
Proof. We fix a locally finite open cover of $U$ by geodesic balls $\{B_t\}_{t \in \mathbb{N}}$ such that the small balls $\{B_t\}_{t \in \mathbb{N}}$ with $1/2$ radius cover $U$ and the metric on $B_t$ is close to the Euclidean metric. Then by ([22], [7, Proposition (2.3.7), Theorem (2.3.8)]), there exists $\{h_t \in C^\infty(\text{Iso}(Q_0|_{B_t} B_t \times SO(3)))\}_{t \in \mathbb{N}}$ such that $a_t = h_t A$ obeys $d^* a_t = 0$ and

$$\sup_{B_t} \left\{ \sum_{i=0}^m |\nabla^{(i)} a_t|^2 \right\} \leq \xi_m \int_{B_t} |F_A|^2.$$  

(4.1)

Here $B_t$ is the ball of radius $3/4$ (radius $B_t$) and $\nabla$ is the covariant derivative defined by the product structure. We denote by $\{\{B_t \cap B_{t'}\}_{t,t'}\}$ the connected components of $B_t \cap B_{t'}$. To obtain a desired gauge, we modify $h_t$, inductively. First we put $h_t = h_1$. Suppose that for $j < t$ we defined $h_j \in C^\infty(\text{Iso}(Q_0|_{B_j} B_j \times SO(3)))$ such that $a_j = h_j A$ obeys

$$\sup_{B_j} \left\{ \sum_{i=0}^m |\nabla^{(i)} a_j|^2 \right\} \leq \xi_j \int_{B_j} |F_A|^2,$$  

(4.2)

for some modified constant $\xi_j > 0$. On $B_t \cap B_j$, $h_{tj} = h_t(h_j)^{-1} \in C^\infty(\text{SO}(3))$ obeys

$$d h_{tj} = h_{tj} a_j' - a_j h_{tj}'.$$

By bootstrapping, we see that $dh_{tj}$ has the same $C^m$-bound as (4.1). If we choose $\varepsilon > 0$ so small, we can write $h_{tj} = \exp(\xi_{tjk}) z_{tjk}$ for some constant $z_{tjk} \in \text{SO}(3)$ on $B_t \cap B_j$. We define $h_t \in C^\infty(\text{Iso}(Q_0|_{B_t} B_t \times SO(3)))$ by

$$h_t' = \begin{cases} \exp(-\phi_{tjk} \xi_{tjk}) h_t, & \text{on } B_t \cap B_j, \\ h_t, & \text{on } B \cap B_j', \end{cases}$$

where $\phi_{tjk}$ is a cut-off function equal to 1 on $(B_t \cap B_j)_k$ and is supported in $(B_t \cap B_j)_k$. Then $a_t' = h_t' A$ obeys the estimate (4.2) for a modified constant $\xi_j > 0$. In the construction above, we also obtained that $h_t(h_j)^{-1} = z_{tjk}$ on $(B_t \cap B_j)_k$. The data $\{B_t \cap B_t', z_{tjk}\}$ define a flat bundle $Q_t' \rightarrow U'$ and a flat connection $\Gamma_t'$ on $Q_t'$. The data $\{B_t \cap U', h_t'|_{\overline{U} \cap B_t}\}$ define an isomorphism $h^*: Q_t' \rightarrow Q_t|_{\overline{U}}$. By the assumption, there is an isomorphism $h^*: Q_t'|_{\overline{U}} \rightarrow Q_t'$. By the assumption, there is an isomorphism $h^*: Q_t' \rightarrow Q_t|_{\overline{U}}$ such that $h^* \Gamma_t' = \Gamma_t'$. (4.2) guarantees that $h - h^* h$ obeys the desired estimate. \square

**Proposition 4.2.** There are constants $\varepsilon > 0$ and $\xi = \xi_m > 0$, independent of $8 \leq T \leq \infty$, with the following significance. Let $A$ be an ASD connection on $Q \times (0, T)$ with $\int_{\mathbb{R} \times (0, t)} F_A < \varepsilon$, then there exists $h \in C^\infty(\text{Aut}(Q \times (1, T-1)))$ such that for $8 \leq t \leq T-8$. 


Proof. The proof is essentially the same as that of [20, Lemma 10.5]. We apply Lemma 4.1 with \( U = Z \times (j - 1, j + 8) \) and \( U' = Z \times [j, j + 7] \) for each \( j \in 4N \). Then there exists \( h_j \in C^\infty(\text{Aut}(Q \times [j, j + 7])) \) such that
\[
\sup_{Z \times [j - 1, j + 8]} \left\{ \sum_{l=0}^{\infty} |\nabla_{\xi l}^\Gamma(h_j A - \pi^* \Gamma)|^2 \right\} \leq \zeta_j \sup_{Z \times [j - 1, j + 8]} |F_A|^2.
\]
(4.3)
On \( Z \times [j + 4, j + 7] \), \( h_{j, j+4} = h_j h_{j+4}^{-1} \in C^\infty(\text{Aut}(Q \times [j + 4, j + 7])) \) obeys
\[
d_{\pi^* \Gamma} h_{j, j+4} = h_{j, j+4} (h_{j+4, A} - \pi^* \Gamma) - (h_{j, A} - \pi^* \Gamma) h_{j, j+4}.
\]
By bootstrapping, we see that \( h_{j, j+4} \) has the same estimate as (4.3). If we choose \( \epsilon > 0 \) so small, the argument in Lemma 4.1 can be repeated with the data \( \{h_{j, j+4}\} \) to produce \( h'_j \in C^\infty(\text{Aut}(Q \times [j, j + 7])) \) such that \( h'(h'_{j+4})^{-1} = z_{j, j+4} \in \mathbb{R} \) on \( Z \times [j + 5, j + 6] \) and for some \( \zeta' > 0, \)
\[
\sup_{Z \times [j - 1, j + 8]} \left\{ \sum_{l=0}^{\infty} |\nabla_{\xi l}^\Gamma(h'_j A - \pi^* \Gamma)|^2 \right\} \leq \zeta' \sup_{Z \times [j - 1, j + 8]} |F_A|^2.
\]
Now we change \( h'_j \) to
\[
\tilde{h}_j = z_{0, 4} \cdots z_{j-4, j} h'_j.
\]
Then \( \tilde{h}_j \) obeys the same bound as above and \( \tilde{h}_j (\tilde{h}_j)^{-1} = 1 \). We define \( h = \tilde{h}_j \) on \( Z \times [j + 1, j + 6] \). Then \( A \) satisfies the desired estimate. □

We prove the decay estimate of the curvature of ASD connections over the cylinder \( Z \times [0, \infty) \) by the parallel discussion in [7, 7.3]. The following three lemmas can be proved by replacing the trivial connection in the argument of [7, 2.3.4, 2.3.5, 2.3.6, 2.3.7, 2.3.8] by the flat connection \( \Gamma \). We prove the first only.

**LEMMA 4.3.** There are constants \( N, \eta > 0 \) such that if \( B \) is a connection on \( Q \) in Coulomb gauge relative to \( \Gamma \) (i.e. \( d^\Gamma(B - \Gamma) = 0 \)) and satisfies \( \|B - \Gamma\|_{L^4} \leq \eta \), then \( \|B - \Gamma\|_{L^2} \leq \|B - \Gamma\|_{L^4} + \|\nabla R(B - \Gamma)\|_{L^4} \leq N \|F_B\|_{L^4}. \)

Proof. Since \( H^1(Z, \text{Ad } Q) = 0 \), the basic elliptic estimate for the operator \( d^\Gamma + d^R \) on 1-forms gives a bound
\[
\|B - \Gamma\|_{L^2} \leq c_1 \|d^R(B - \Gamma)\|_{L^2}.
\]
Using the Sobolev multiplication theorem, we get
\[
\|B - \Gamma\|_{L^2} \leq c_1 \|F_B\|_{L^2} + c_1 c_2 \|B - \Gamma\|_{L^4} \|B - \Gamma\|_{L^4}.
\]
If \( \|B - \Gamma\|_{L^4} < 1/(2c_1c_2) \), then we can rearrange it as
\[
\|B - \Gamma\|_{L^4} \leq 1 - c_1c_2 \|B - \Gamma\|_{L^4} \leq c_1 \|F_B\|_{L^4},
\]
to get \( \|B - \Gamma\|_{L^4} \leq 2c_1 \|F_B\|_{L^4} \). □

For a connection \( B \) on \( Q \) and \( t \geq 1 \), put:
\[
Q_t(B) = \|F_B\|_{L^4} + \frac{1}{t} \sum_{j=0}^{1} \|\nabla^j B\|_{L^4}.
\]

**Lemma 4.4.** There is a constant \( \eta > 0 \) such that if the connection \( B \) of Lemma 4.3 has \( \|B - \Gamma\|_{L^4} < \eta' \), then for each \( t \geq 1 \), a bound,
\[
\|B - \Gamma\|_{L^4} = \frac{1}{t} \sum_{j=0}^{1} \|\nabla^j B\|_{L^4} \leq f_t(Q_t(B))
\]
holds for a universal continuous function \( f_t \), independent of \( B \), with \( f_t(0) = 0 \).

In the lemma below, by a one-parameter family we mean that they are smooth in the \( z \) variable, and all partial derivatives are continuous in both variables.

**Proposition 4.5.** There is a constant \( \varepsilon > 0 \) such that if \( B_t \) \( (0 \leq t \leq 1) \) is a one-parameter family of connections on \( Q \) with \( \|F_{B_t}\|_{L^4} < \varepsilon \) for all \( t \) and with \( B_1 = \Gamma \), then for each \( t \) there exists a one-parameter family of gauge transformations \( u_t \) such that \( u_t \) satisfies
\[
d^*_t(B_t - \Gamma) = 0,
\]
\[
\|B_t - \Gamma\|_{L^4} \leq 2N \|F_{B_t}\|_{L^4},
\]
where \( N \) is the constant in Lemma 4.3.

**Proposition 4.6.** There are constants \( \varepsilon > 0 \) and \( \zeta > 0 \) independent of \( T \) such that if \( A \) is an ASD connection on \( Q \times (-T, T) \) with \( \|F_A\|_{L^4} \leq \varepsilon \), then for all \( (x, t) \in Z \times [-T + 8, T - 8] \),
\[
|F_A|_{(x, t)} \leq \zeta e^{-\frac{3}{2} |t|} \left( \int_{Z \times (-T, T)} |F_A|^2 \right)^{1/2}.
\]

**Proof.** We apply Proposition 4.2 over \( Z \times (-T, T) \) to obtain a gauge transformation \( h \in C^\infty(\text{Aut}(Q \times (-T + 1, T - 1))) \) such that for \(-T + 8 \leq t \leq T - 8\),
\[
\sup_{Z \times (-T + 8, T - 8)} \left\{ \sum_{i=0}^{1} \|\nabla^i h r(hA - \pi^* \Gamma)\|_{L^2} \right\} \leq \zeta \left\{ \int_{Z \times (-T + 8, T - 8)} |F_A|^2 \right\}^{1/2}.
\]
We henceforth omit \( h \) for simplicity. We write \( A_t \) for the restriction of \( A \) to
We take the path
\[ A_{t,s} = \Gamma + s(A_t - \Gamma) \quad (0 \leq s \leq 1) \]
from \( \Gamma \) to \( A_t \). By (4.4) we can assume that \( \|F_{A_t, \delta}\|_{L^2} < \delta \) for small \( \xi > 0 \). Then we apply Proposition 4.5 to get a gauge transformation \( l_t \) on \( Q \) which is homotopic to the identity and satisfies
\[ \|l_tA_t - \Gamma\|_{L^2} < 2N\|F_{A_t}\|_{L^2}. \tag{4.5} \]
We use the 'Chern-Simons' invariant relative to \( \Gamma' \) defined by
\[ T_z(\Gamma, A_t) = \int \left( \frac{1}{2}Tr((\partial_t A_t - \Gamma)^2) + \frac{2}{3}(A_t - \Gamma) \cdot (A_t - \Gamma) \cdot (A_t - \Gamma) \right). \]
A direct calculation shows that
\[ T_z(\Gamma, l_t A_t) = T_z(\Gamma, A_t) + \frac{1}{3} \deg l_t, \tag{4.6} \]
Here the last term
\[ \deg l_t = \int_{\mathbb{Z}} \text{Tr}(d_r l_t^{-1} \wedge d_r l_t^{-1} \wedge d_r l_t^{-1}) \]
is a homotopy invariant by Stoke's theorem. So we have \( \deg l_t = 0 \). For \(-T + 8 \leq t \leq T - 8\), we write
\[ \nu(t) = \int_{Z \times \{-u, t\}} |F_A|^2. \]
Then we can easily verify that
\[ \frac{d\nu}{dt} = 2\left( \|F_{A_t}\|_{L^2} + \|F_{A_t - \delta}\|_{L^2} \right) \]
and
\[ \nu(t) = T_z(A_t, \Gamma) - T_z(A_t - \delta, \Gamma) \]
We need a simple lemma; any \( a \in \Omega^2(\text{Ad} Q) \) satisfies
\[ \int_{\mathbb{Z}} \text{Tr}(d_r a \wedge a) \leq \frac{1}{\lambda} \int_{\mathbb{R}} |d_r a|^2. \tag{4.7} \]
We prove it quickly. If \( a \) is replaced by \( a + d_r f \) for some \( f \in \Omega^0(\text{Ad} Q) \), each of the integrals is unchanged, so we may assume that \( a \) satisfies \( d_r^* a = 0 \). Then
\[ \int_{\mathbb{Z}} \text{Tr}(d_r a \wedge a) \leq \|a\|_{L^2} \|d_r a\|_{L^2} \]
and \( \int_{\mathbb{Z}} d_r a \|^2 = \langle a, \Delta_r a \rangle \). So \( \|d_r a\|_{L^2} \geq \lambda\|a\|_{L^2} \). Since \( H^1(Z; \text{Ad} Q) = 0 \), we see
that \( \int_{\mathbb{S}^2} \text{Tr} (d_T a \wedge a) \leq \lambda^{-1} \int_{\mathbb{S}^2} |d_T a|^2. \)

Using (4.5), (4.6), (4.7) and the Sobolev embedding theorem \( L^4 \to L^q \), we get
\[
|T_{\mathbb{S}^2}(T', A_t)| \leq \frac{1}{\lambda} \| F_{A_t} \|_{L^2} + c \| A_t - T' \|_{L^1}^2
\]
where the constant \( c \) depends only on \( Q \). So we have
\[
\frac{d\nu}{dt} \geq 2\lambda \nu - c \left( \frac{d\nu}{dt} \right)^{2/3},
\] (4.8)

We also know that \( \nu \) and \( d\nu/dt \) are small by (4.4). We use an elementary inequality; if \( y + Cy^{3/2} \geq 2\lambda x \) for some \( C \), and \( x \) and \( y \) are small, then \( y \geq 2\lambda x - C' x^{3/2} \) for another constant \( C' \). Hence (4.8) gives
\[
\frac{d\nu}{dt} \geq 2\lambda \nu - c\nu^{3/2}.
\]

We choose \( \varepsilon > 0 \) so small that \( \varepsilon^{1/3} \leq (1/2)\lambda \). Then the inequality above gives \( d\nu/dt \geq (2\lambda - \delta)\nu \), which we can integrate to get an exponential bound
\[
\nu(t) \leq e^{(2\lambda - \delta) (t - T)} \nu(T).
\]

Feeding back this into the differential inequality, we get
\[
\frac{d\nu}{dt} \geq 2\lambda \nu - c\delta e^{(2\lambda - \delta) (t - T)/3} \nu
\]
\[
\geq 2\lambda \nu - \frac{c\delta e^{(2\lambda - \delta) (t - T)/3} d\nu}{2\lambda - \delta} dt.
\]

It follows that
\[
\log \nu(T) - \log \nu(t) \geq 2\lambda \int_t^T \frac{d\tau}{1 + c\delta e^{(2\lambda - \delta) (t - \tau)/3}}
\]
\[
\geq 2\lambda \int_t^T (1 - c\delta e^{(2\lambda - \delta) (t - \tau)/3}) d\tau
\]
\[
\geq 2\lambda (T - t) - \frac{4c\lambda}{2\lambda - \delta}.
\]

Taking exponentials, we have the bound
\[
\nu(t) \leq K \nu(T) e^{4\lambda (t - T)}
\]
with \( K = \exp(4c\lambda/(2\lambda - \delta)). \) Finally we use the following for any element \( f \) in \( \mathcal{Q}_0(\text{Ad}(Q \times (-1, 1))) \) with \( d_A^* f = d_A f = 0 \), we have an elliptic estimate:
Since $A$ obeys the uniform $C^1$-estimate by (4.4), we can take $c$ which is independent of $A$. Applying this to $f=F_A$ on $Z \times (-T+8, T-8)$, we get
\[ |F_A|_{\epsilon, \tau} \leq c \nu(t)^{1/2} \leq c e^{\frac{1}{2} (t+T) \nu(T)^{1/2}} \]
and the proof is completed.

We apply the above proposition over $Z \times (n_0, 2t-n_0)$ to obtain that

**COROLLARY 4.7.** // $A$ is an ASD connection on $\pi^*Q$ with $J_{\chi}$ for some $n_0 \in \mathbb{N}$, then for all $(x, t) \in Z \times [n_0+1, \infty)$,
\[ |F_A|_{\epsilon, \tau} \leq c e^{\frac{1}{2} (t+T) \nu(T)^{1/2}}. \]

5. **Gluing ASD connections.**

According to [7, 7.1], we shall construct a gluing map from $\mathcal{M}_{Y_1}(l_{x_1}, \sigma_{x_1}^*(\eta), g_1)$ \times $\mathcal{M}_{Y_2}(l_{x_2}, \sigma_{x_2}^*(\eta), g_2)$ to $\mathcal{M}_{X_0}(l_{x_0}, \eta, g_0)$. To obtain the gluing map globally, we need a technical lemma below. In this section, we use $c$ for a constant independent of $n$ and use $c_\tau$ for a constant with respect to $F_\tau$.

**LEMMA 5.1.** There are constants $\varepsilon > 0$, $t_0 = t_0, m > 0$ and $\rho = \rho_m > 0$ with the following significance: Let $A$ be an ASD connection on $\pi^*Q$. Suppose that $n_0 \in \mathbb{N}$ exists such that $\int_{Z \times [n_0+1, \infty)} |F_A|^2 < \varepsilon$. Then there exists $h \in C^\infty(\text{Aut}(Q \times [n_0+1, \infty)))$ such that for $t \geq n_0 + t_0$,
\[ \sup_{Z \times [t-1, t+1]} \left\{ \sum_{\ell=0}^{m} |\nabla_{\pi^*}^\ell(h(A - \pi^* \Gamma_\ell)|^2 \right\} \leq \rho e^{-\frac{1}{2} (t-n_0)}, \]
\[ i_{\partial \pi}(h(A - \pi^* \Gamma_\ell)) = 0. \]
Such a $h$ is unique up to $\mathcal{R}$. Moreover, if $A$ satisfies
\[ \int_{Z \times [n_0+1, \infty)} \sum_{\ell=0}^{m} |\nabla_{\pi^*}^\ell(A - \pi^* \Gamma_\ell)|^2 < \infty, \]
then we can choose $h$ so that $h|_{t=n_0}$ is homotopic to the identity, and such a $h$ is unique.

**Proof.** By Proposition 4.2 and Corollary 4.7, there exists $h \in C^\infty(\text{Aut}(Q \times [n_0+1, \infty)))$ such that for $t \geq n_0 + 8$,
Solve the ordinary differential equation for \( \tilde{h} \in C^\infty(\text{Aut}(Q \times [\mathbb{Z}^n \pm 1, \infty])) \):

\[
\tilde{h}(hA) - \pi^* \Gamma = - \frac{\partial}{\partial t} \tilde{h}^{-1} + \tilde{h} \tilde{h}(hA - \pi^* \Gamma) \tilde{h}^{-1} = 0
\]

with initial condition \( \lim_{t \to \infty} \tilde{h} = 1 \). From the differential equation we have

\[
-\frac{\partial}{\partial t} (d_t \tilde{h}) = d_t \tilde{h} \tilde{h}(hA - \pi^* \Gamma) + \tilde{h} d_t (i_{\text{Ad}}(hA - \pi^* \Gamma))
\]

This gives us

\[
-\frac{\partial}{\partial t} |d_t \tilde{h}|^2 \leq c e^{-\lambda\overline{t}(t-n)} (|d_t \tilde{h}|^2 + |d_t \tilde{h}|),
\]

which integrates over \([t-1, \infty)\) to get an inequality

\[
\|d_t \tilde{h}\|_{C^0([t-1, \infty))} \leq c e^{-\lambda\overline{t}(t-n)} (\|d_t \tilde{h}\|_{C^0([t-1, \infty))} + \|d_t \tilde{h}\|_{C^0([t-1, \infty))}).
\]

For \( t > n + t_0 \) with \( \epsilon e^{\lambda\overline{t}<1/2} \), we can rearrange this to get

\[
\|d_t \tilde{h}\|_{C^0([t-1, \infty))} \leq \|d_t \tilde{h}\|_{C^0([t-1, \infty))} \leq 2 c e^{-\lambda\overline{t}(t-n)}.
\]

Now we can bootstrap the equation

\[
\tilde{h}(hA) - \pi^* \Gamma = - d_t \tilde{h}^{-1} - \tilde{h} \tilde{h}(hA - \pi^* \Gamma) \tilde{h}^{-1} + \tilde{h}(hA - \pi^* \Gamma) \tilde{h}^{-1}.
\]

Then we see that \( h' = \tilde{h} \) satisfies the desired properties.

If \( h' \) also satisfies the properties of Lemma 5.1, then \( h'h^{-1} \) is independent of \( t \) and satisfies

\[
d_t (h'h^{-1}) = h'h^{-1} i_{\text{Ad}}(hA - \pi^* \Gamma) - i_{\text{Ad}}(hA - \pi^* \Gamma) h'h^{-1}.
\]

This implies that \( h'h^{-1} \) converges to a flat gauge as \( t \to \infty \). So \( h'h^{-1} \) must be an element in \( \mathcal{R} \).

If \( A \) satisfies the additional condition, then by the equation

\[
d_{\ast \ast} r h = h(A - \pi^* \Gamma) - (hA - \pi^* \Gamma) h
\]

and the Sobolev embedding theorem \( L^2 \to C^s \), we get

\[
\|d_{\ast \ast} r h\|_{C^s([t-1, \infty])} \longrightarrow 0 \ (t \to \infty).
\]

Hence \( A \) factors as \( h = r(h) \exp \xi \) with \( r(h) \in \mathcal{R} \) and \( \xi \in C^\infty(\text{Ad}(Q \times [\mathbb{Z}^n \pm 1, \infty])) \) for some \( \mathbb{N} \in \mathcal{N} \). So \( h' = r(h)^{-1} h \) has the desired properties and it is unique, since the primary difference \( b(1, r) \) is non-zero for any non-identity element \( r \in \mathcal{R} \).

The above unique gauge is said to be exponential gauge. We combine Lemma 5.1 with Lemma 3.1 to deduce
COROLLARY 5.2. If $B$ is an ASD connection in $\mathcal{A}_{P_t}$ then there are $n_0 \in \mathbb{N}$ and $u_i \in \mathcal{G}_P$, such that $A_i = u_i B$, satisfies the following conditions.

(i) $A$ is smooth over $\tau^{-1}([n_0 + 1, \infty))$

(ii) For $t \geq n_0 + l_0$,

\[
\sup_{Z \times [t-1, t+1] \cup [\tau^{-1}([n_0 + l_0, \infty))]} \left( \sum_{\ell=0}^{m} |\nabla^{(\ell)}(A_i - \pi^* \gamma)|^2 \right)^{1/2} \leq e^{-2\lambda(t-n_0)},
\]

\[
i_{\partial Z}(A_i - \pi^* \gamma) = 0.
\]

// $A'$ also satisfies the above condition, then $A_i = A'_i$ over $\tau^{-1}([n_0 + l_0, \infty))$

Let $\beta_t, \gamma_t$, and $\mu_t$ be smooth cut off functions satisfying

\[
\beta_t(x, t) = \begin{cases} 
1 & \text{on } \tau^{-1}([0, n+1]), \\
(n+N+1-t)/N & \text{on } \tau^{-1}([n+1, n+N+1-\varepsilon]), \\
0 & \text{on } \tau^{-1}([n+N+1, \infty)),
\end{cases}
\]

\[
\gamma_t(x, t) = \begin{cases} 
1 & \text{on } \tau^{-1}([0, n-1]), \\
0 & \text{on } \tau^{-1}([n+1, \infty)),
\end{cases}
\]

\[
\mu_t(x, t) = \begin{cases} 
1 & \text{on } \tau^{-1}([0, n-N-2]), \\
0 & \text{on } \tau^{-1}([n-N-1, \infty)).
\end{cases}
\]

$\gamma_t(x, t) + \gamma_t(n-x, t) = 1$ if $(x, t) \in \tau^{-1}([n-1, n+1])$, for a small $0 < \varepsilon < 1$. Then a direct calculation shows that

\[
\|\nabla \beta_t\|_{L^2(\partial \mathcal{V} \cap \mathcal{F})} \leq KN^{-(q-1)/q},
\]

where $K$ is independent of $a$ and $N$. For $A_i \in \mathcal{A}_{P_t}$, we define a connection $A'_i$ on $P_t$ by $A'_i = \mu_t(A_i - \pi^* \gamma) + (1 - \mu_t) \pi^* \gamma$ and a connection $A'$ on $P_n$ by

\[
J_n(A_1, A_2) = A' = \begin{cases} 
A'_i & \text{over } \tau^{-1}([0, n+1]), \\
A_i & \text{over } \tau^{-1}([0, n+1]).
\end{cases}
\]

LEMMA 5.3. If $A_i$ is an ASD connection in $\mathcal{A}_{P_t}$ satisfying the conditions (i)

(ii), then there is a constant $c > 0$ such that

(1) $\|A_i - A_i\|_{L^2(\partial \mathcal{V} \cap \mathcal{F})} < ce^{-2(\lambda-N)}$

(2) $\|A_i - A_i\|_{L^2(\partial \mathcal{V} \cap \mathcal{F})} + \|\nabla \pi^* \gamma_i\|_{L^2(\partial \mathcal{V} \cap \mathcal{F})} < ce^{-(\lambda-N)}$

(3) $\|F_i - F_i\|_{L^2(\partial \mathcal{V} \cap \mathcal{F})} < ce^{-(\lambda-N)}$

Proof. This is obvious. □

We are in the position to argue the right inverse to $d_{\mathcal{A}}$. For an ASD connection $A_i$ in $\mathcal{A}_{P_t}$, we take the Laplacian
The condition $H_{A_t}=0$ implies that $\Delta_{A_t}$ is invertible [15, Lemma 7.3]. Then there is the right inverse $P_t$ to the operator $d_{A_t}^{-1}$:

$$P_t=e^{-\iota \Delta_{A_t}}: L^p_{\omega}(Y_t) \longrightarrow L^p_{\omega}(Y_t),$$

which satisfies $\|P_t\xi\|_{L^p_{\omega}(Y_t)} \leq c_t\|\xi\|_{L^2_{\omega}(Y_t)}$ for some constant $c_t$. Composing with the Sobolev embedding

$$L^p_{\omega}(Y_t) \longrightarrow L^{q\ell p}_0(Y_t),$$

[15, Lemma 7.2], we have $\|P_t\xi\|_{L^{q\ell p}_0(Y_t)} \leq c_t\|\xi\|_{L^2_{\omega}(Y_t)}$. We also need Hölder's inequality, for 1-forms $a, f$,

$$\|(a \wedge \wedge b)\|_{L^{q\ell p}_0} \leq \sqrt{2} \|a\|_{L^{q\ell p}_0} \|b\|_{L^{q\ell p}_0} \leq \sqrt{2} \|a\|_{L^{q\ell p}_0} \|b\|_{L^{q\ell p}_0},$$

over $X_n$ or $Y_t$. Now let $Q_t: L^p_{\omega}(X_n) \rightarrow L^p_{\omega}(X_n)$ be the operator defined by

$$Q_t\xi = \beta_i P_t(\gamma_i \xi).$$

Then we obtain a bound $\|Q_t\xi\|_{L^p_{\omega}(X_n)} \leq c_t e^{\delta t}\|\xi\|_{L^p_{\omega}(X_n)}$.

**Lemma 5.4.** There is a constant $\epsilon_1 = \epsilon_1(N, n) \rightarrow 0$ as $n \rightarrow \infty$ and $N \rightarrow \infty$ in order such that

$$\|d_{A_t}^*Q_t\xi\|_{L^p_{\omega}(X_n)} \leq \epsilon_1(N, n)\|\xi\|_{L^p_{\omega}(X_n)}$$

**Proof.** If we write $A_t = A_t + a_t$, then

$$d_{A_t}^*Q_t\xi = d_{A_t}^*\xi(\beta_i P_t(\gamma_i \xi))$$

$$= \beta_i(d_{A_t}^*P_t(\gamma_i \xi)) + (\nabla \beta_i)P_t(\gamma_i \xi) + [\beta_i a_t, \gamma_i \xi]$$

$$= \gamma_i \xi(\nabla \beta_i)P_t(\gamma_i \xi) + [\beta_i a_t, P_t(\gamma_i \xi)].$$

By Lemma 5.3, we get

$$\|d_{A_t}^*P_t(\gamma_i \xi)\|_{L^p_{\omega}(X_n)} \leq \|d_{A_t}^*\xi(\beta_i P_t(\gamma_i \xi))\|_{L^p_{\omega}(X_n)}$$

$$\leq \sqrt{2} \|\beta_i P_t(\gamma_i \xi)\|_{L^2_{\omega}(X_n)} \leq c_t\|\xi\|_{L^2_{\omega}(X_n)},$$

$$\|\beta_i a_t, P_t(\gamma_i \xi)\|_{L^p_{\omega}(X_n)} \leq \|\beta_i a_t, P_t(\gamma_i \xi)\|_{L^2_{\omega}(X_n)}$$

$$\leq \sqrt{2} \|\beta_i a_t\|_{L^2(X_t)} \|P_t(\gamma_i \xi)\|_{L^2_{\omega}(X_n)}$$

$$\leq c_t e^{-\lambda (n-N)+\delta t}\|\xi\|_{L^p_{\omega}(X_n)}.$$
We put
\[ Q = Q_1 + Q_2 : L^p_0(X_n) \longrightarrow L^p_0(X_n). \]
The operator \( R = d_A^* Q - 1 \) obeys a bound
\[ \| R(\xi) \|_{L^p_0(x_n)} \leq (\varepsilon_1(N, n) + \varepsilon_2(N, n)) \| \xi \|_{L^p_0(x_n)}. \]
We choose \( N_0 \) and \( n_0 = n_0(N_0) \) so that \( \varepsilon_1(N_0, n) \leq 1/3 \) for all \( n \geq n_0 \). Then the operator norm of \( R \) is at most 2/3. So \( 1 + R \) is invertible and the norm of the inverse is at most 3. Thus the right inverse \( P = Q(1 + R)^{-1} \) to \( d_A^* \) satisfies
\[ \| P\xi \|_{L^p_0(x_n)} \leq \| Q_1(1 + R)^{-1}\xi \|_{L^p_0(x_n)} + \| Q_2(1 + R)^{-1}\xi \|_{L^p_0(x_n)} \]
\[ \leq C_1 e^{3\delta} \| (1 + R)^{-1}\xi \|_{L^p_0(x_n)} + C_2 e^{3\delta} \| (1 + R)^{-1}\xi \|_{L^p_0(x_n)} \]
\[ \leq C \| \xi \|_{L^p_0(x_n)} \] (5.3)
with \( C = 3e^{3\delta}(C_1 + C_2) \). Combining with the Sobolev embedding, we get \( \| P\xi \|_{W^{k,p}(X_n)} \leq C \| \xi \|_{L^p_0(x_n)} \). We seek a solution \( A' + a \) to the ASD equations in the form \( a = P(\xi) \). If we write \( q(\xi) = (P\xi) + a \), the ASD equation becomes
\[ \xi + q(\xi) = -F_A^+, \] (5.4)
By Hölder’s inequality,
\[ \| q(\xi) - q(\xi) \|_{L^p_0(x_n)} \leq \sqrt{2} C e^{3\delta} \| \xi \|_{L^p_0(x_n)} \| \xi \|_{L^p_0(x_n)} + \| \xi \|_{L^p_0(x_n)} \].

**Lemma 5.5.** ([7, Lemma (7.2.23)]) Let \( B : B \rightarrow B \) be a smooth map on a Banach space and \( \| S\xi - S\eta \| \leq k \{ \| \xi \| + \| \eta \| \} \| \xi - \eta \| \) for some \( k > 0 \) and all \( \xi, \eta \in B \). Then for each \( \eta \in B \) with \( \| \eta \| \leq 1/(10k) \) there is a unique \( \xi \) with \( \| \xi \| \leq 1/(5k) \) such that
\[ \xi + S(\xi) = \eta. \]
We apply Lemma 5.5 to the above equation with \( S = q, \eta = -F_A^+, \) and \( k \geq \sqrt{2} e^{3\delta} \). Then

**Proposition 5.6.** Let \( A_1 \) be an ASD connection in \( \mathcal{A}_{P_1} \) satisfying the conditions (i), (ii). Then there exists \( N_0 \) and \( n_0 = n_0(N_0) \) such that for all \( n \geq n_0 \), we can find an ASD connection \( I_n (A_1 A_2) = A' + a \) on \( P_n \) with \( \| a \|_{L^p_0(x_n)} \leq C e^{3\delta} \| \xi \|_{L^p_0(x_n)} \) for some \( \xi \in \mathcal{A}_{P_1} \). If \( u_i A_i \) also satisfies the conditions (i), (ii) for some \( u_i \in \mathcal{A}_{P_i} \), then
\[ I_n(u_i A_i, u_a A_2) = (u_i \cup u_a) I_n(A_i, A_2). \]
Moreover we can choose \( N_0 \) and \( n_0 = n_0(N_0) \) so that for all \( n \geq n_0, A_i = A' + t a, (0 \leq t \leq 1) \) is regular, that is \( \{ A' \} \in U(I_n(A A_2)). \)

**Proof.** By bootstrapping (5.4), we see that the solutions we solved are in
C∞ if so are A_i. The second part is obvious from the gauge equivalence of the above construction. We prove the last part. By (5.3), we have
\[ \| (d_{A_t}^+ - d_{A_t}^-) \xi \|_{L^p_0(\mathcal{M})} = \| [a, \xi] \|_{L^p_0(\mathcal{M})} \leq c e^{-(\lambda - \delta)p(n - n_0)} \| \xi \|_{L^p_0(\mathcal{M})}. \]
If we choose n_0 so large that \( c e^{-(\lambda - \delta)p(n_0 - n_0)} \leq 1/2 \), then for \( n \geq n_0 \), the operator norm of \((d_{A_t}^+ - d_{A_t}^-)P\) is at most 1/2 and \( d_{A_t}^+ P = 1 + (d_{A_t}^+ - d_{A_t}^-)P \) is invertible. So \( A_t \) is regular. \( \Box \)

Now we obtain the gluing map
\[ I_n : M_{\mathcal{M}}(l_{Y_1}, \rho^*(\eta), g_1) \times M_{\mathcal{M}}(l_{Y_2}, \sigma^*(\eta), g_2) \rightarrow M_{\mathcal{M}}(l_{x_1}, \eta, g_n), \]
for large \( N \geq N_0 \) and \( n \geq n_0(N_0) \). Here \( N_0, n_0(N_0) \) are the maximal values of all \([A_i]\) in \( M_{\mathcal{M}}(l_{Y_1}, \sigma^*(\eta), g_1) \). We may suppose that \( N_0 = 0 \).
We prove that \( I_n \) is injective for large \( n \). Let \( A_1, B_1 \) be as above. Suppose that \( I_n(A_1, A_2) = A' + a \) is gauge equivalent to \( I_n(B_1, B_2) = B' + b \) by some gauge \( u_n \in \mathcal{G}_{x_n} \). We expand the equation \( u_n(A' + a) = B' + b \) over \( \tau^{-1}([n - 3, n - 2]) \). Then
\[ a_{\pi \Gamma} u_n = u_n(A_1 - \pi \Gamma) + u_n a - (B_1 - \pi \Gamma) u_n - b u_n. \]
By Lemma 5.3,
\[ \| d_{\pi \Gamma} u_n \|_{C^1([n - 3, n - 2])} \leq c \| d_{\pi \Gamma} u_n \|_{\tau^{-1}([n - 3, n - 2])} \| \tau^{-1}([n - 3, n - 2]) \|_{L^p_0(\mathcal{M})} \leq c e^{-(\lambda - \delta)p(n - 2)}. \]
Hereafter \( \| I \|_{L^p_0(\mathcal{M})} \) is the integral over the restriction. This implies that for large \( n \), \( u_n \) factors as \( u_n = r(u_n) \exp \delta_i h_i \) over \( \tau^{-1}([n - 3, n - 2]) \), where \( r(u_n) \in \mathcal{G} \) satisfies
\[ \| r(u_n) \|_{C^1([n - 3, n - 2])} \leq c \| d_{\pi \Gamma} u_n \|_{C^1([n - 3, n - 2])} \| \tau^{-1}([n - 3, n - 2]) \|_{L^p_0(\mathcal{M})} \leq c e^{-(\lambda - \delta)p(n - 2)}. \]
Lemma 3.1 asserts that \( r(u_n) \) is the identity. Since the exponential map on \( \mathfrak{s}_0(3) \) is a local diffeomorphism, we get
\[ \| h_i \|_{C^1([n - 3, n - 2])} \leq \| d_{\pi \Gamma} r(u_n) \|_{C^1([n - 3, n - 2])} \| \tau^{-1}([n - 3, n - 2]) \|_{L^p_0(\mathcal{M})} \leq c e^{-(\lambda - \delta)p(n - 2)}. \]
We put
\[ u_{i\pi} = \begin{cases} u_n & \text{over } \tau^{-1}([0, n - 3]), \\ \exp(\delta_i h_i) & \text{over } \tau^{-1}([n - 3, \infty)), \end{cases} \]
where \( \delta_i \) is a cut off function such that \( \delta_i = 1 \) on \( \tau^{-1}([0, n - 3]) \) and \( \delta_i = 0 \) on \( \tau^{-1}([n - 2, \infty)) \). By estimating all the term in the right hand side of the equation
\[ u_{i\pi} A_i - B_i = \begin{cases} -u_n a u_n^{-1} + b & \text{over } \tau^{-1}([0, n - 3]), \\ -d_{\pi \Gamma} r(\exp(\delta_i h_i)) u_n^{-1} \\ + u_n(A_i - \pi \Gamma) - (B_i - \pi \Gamma) & \text{over } \tau^{-1}([n - 3, n - 2]), \\ (A_i - \pi \Gamma) - (B_i - \pi \Gamma) & \text{over } \tau^{-1}([n - 2, \infty)), \end{cases} \]
we see that \( u_n A \) converges to \( B \) in \( L^p_\infty(Y) \) as \( n \to \infty \). On the other hand, \( \mathcal{M}_Y(\mathcal{F}_I(\eta), g_\tau) \) consists of isolated points with respect to the metric (c.f. [7, Lemma (4.2.4)])

\[
d_p([A], [B]) = \inf_{u \in \mathcal{B}_p} \| A - uB \|_{L^p_\infty(Y)}.\]

This implies that \([A_n] = [B_n]\).

We will prove that \( I_n \) is surjective for larger \( n \). For a moment, we work with the space of \( L^p_\infty \) connections \( \mathcal{A}_{p_n} \), \( L^p_\infty \) gauge group \( \mathcal{G}_{p_n} \) and the quotient \( \mathcal{B}_{p_n} = \mathcal{A}_{p_n} / \mathcal{G}_{p_n} \). Let \( d_{p_n} \) be the metric in \( \mathcal{B}_{p_n} \) given by [7, Lemma (4.2.4)]

\[
d_{p_n}([A], [B]) = \inf_{u \in \mathcal{B}_{p_n}} \| A - uB \|_{L^q_{\delta/p}(X_n)}.\]

We define

\[
J_n : \mathcal{M}_Y(l_{\tau}, \sigma^I(\eta), g_\tau) \times \mathcal{M}_Y(l_{\tau}, \sigma^J(\eta), g_\tau) \to \mathcal{B}_{p_n}
\]

by \( J_n([A], [A_2]) = [J_n(A, A_2)] = [A'] \). For \( \nu > 0 \), let \( U(\nu) \subseteq \mathcal{B}_{p_n} \) be the open set

\[
U(\nu) = \{ [A] \in \mathcal{B}_{p_n} | d_{p_n}([A], \text{Im} J_n) < \nu, \| F^\lambda_L \|_{L^q_{\delta/p}(X_n)} < \nu^{1/k} \}.
\]

The solutions we have constructed lie in \( U(\nu) \), if \( n \geq n_0 = n_0(\nu) \) with \( c e^{-\nu (1 - 0^{2/p})} < \nu \). Conversely.

**Proposition 5.7.** There is a constant \( \nu_0 > 0 \) such that for \( 0 < \nu < \nu_0 \), some \( n_0 = n_0(\nu) \in \mathbb{N} \) satisfies the following: If \( n > n_0 \), then \([A] \in U(\nu) \) can be represented by a connection \( A' + P\xi \) with \( \xi \in L^q_{\delta/p}(X_n) \) and \( \| \xi \|_{L^q_{\delta/p}(X_n)} < 1/(5k) \) (\( k = \sqrt{2\alpha} \)).

**Proof.** Let \( B \) be an element of \( U(\nu) \). Then there is a connection \([A'] \in \text{Im} J_n \) with

\[
\| A' - B \|_{L^q_{\delta/p}(X_n)} < \nu.
\]

We write \( B = A' + b \) and consider the path

\[
B_t = A' + tb \ (0 \leq t \leq 1),
\]

then \( \| A' - B_t \|_{L^q_{\delta/p}(x_n)} < \nu \) and

\[
F^t_{B_t} = (1-t)F^t_{A'} + tF^t_b + (t^2-t)(b \wedge b)^*.
\]

So

\[
\| F^t_{B_t} \|_{L^q_{\delta/p}(x_n)} \leq (1-t)\| F^t_{A'} \|_{L^q_{\delta/p}(x_n)} + t\| F^t_b \|_{L^q_{\delta/p}(x_n)} + \sqrt{2}\| (t-t^2)\| b \|_{L^q_{\delta/p}(x_n)}<
\]

\[
\leq (1-t)c e^{-((d-0)^2/p)n} + t\nu^{1/k} + \sqrt{2}(t-t^2)\nu^k.
\]

Thus we can find \( \nu_0 > 0 \) with the following: For \( 0 < \nu < \nu_0 \), there is a constant \( n_0 = n_0(\nu) \in \mathbb{N} \) such that if \( n > n_0 \), then \([B_1] \ (0 \leq t \leq 1) \) is contained in \( U(\nu) \). We
define \( S \subset [0, 1] \) to be the set of times for which there are \( u_t \in \mathcal{C}_P^0 \) and \([A']\) \in \text{Im} \, J_n\) such that

\[
u_t B_t = A' + P \xi
\]

with \( \|\xi\|_{L_b^p(x_n)} < 1/(5k) \). We will prove that \( S \) is closed and open. Suppose that \( t \) is in \( S \). We may take \( u_t = 1 \). Then the representation \( B_t = A' + P \xi \) gives

\[
F_t = F_{A'} + \xi + (P \xi \wedge P \xi)^{\ast}.
\]

This gives a bound

\[
\|\xi\|_{L_b^p(x_n)} \leq \|F_{B_t}\|_{L_b^p(x_n)} + \|F_{A'}\|_{L_b^p(x_n)} + \sqrt{2} \|P \xi\|_{L^2_{\delta p}(x_n)} + \sqrt{2} c^2 \|\xi\|_{L_b^p(x_n)}.
\]

Arranging this, we get

\[
\|\xi\|_{L_b^p(x_n)} \leq \frac{5}{4} \left( \sqrt{3} + c e^{-\langle \delta_{p}/p \rangle} \right).
\]

This implies that for larger \( n_0 \in \mathcal{N} \) and smaller \( v_0 > 0 \), \( \|\xi\|_{L_b^p(x_n)} \leq 1/(10k) \). That is, \( \|\xi\|_{L_b^p(x_n)} < 1/(5k) \) implies \( \|\xi\|_{L_b^p(x_n)} \leq 1/(10k) \), so this open condition is also closed. We will prove that \( S \) is closed. Suppose that \( \{t_i\} \) is in \( S \) with \( t_i \to t \). Then we have connections \( A_{t_i} = A' + P \xi_i \), with \( \|\xi_i\|_{L_b^p(x_n)} \), and \( u_{t_i} \in \mathcal{C}_P^0 \) with \( u_i B_{t_i} = A_{t_i} \). By the uniform convergence bound on the \( \xi_i \) above, we may suppose that, taking a subsequence, the \( \xi_i \) converge to a limit \( \xi \) weakly in \( L_b^p(X_n) \), with \( \|\xi\|_{L_b^p(x_n)} < 1/(5k) \). Then the connections \( A_{t_i} \) converge weakly in \( L_b^p(x_n) \) and the equation

\[
d_{B_t} u_t = u_t (B_{t_i} - B_t) - (A_{t_i} - B_t) u_t
\]

implies that, after taking a subsequence, \( u_t \) converges to a limit \( u \) weakly in \( L_b^p(x_n) \). The gauge relation is preserved under weak limit, so

\[
u B_t = A' + P \xi
\]

and \( t \) is in \( S \). We will prove that \( S \) is open. Suppose that \( t \) is in \( S \). Then \( B_t = A' + P \xi \) with \( \|\xi\|_{L_b^p(x_n)} < 1/(10k) \). We define a map

\[
M: \Omega_{b,n}^+(\text{Ad} \, P) \times \Omega_{b,n}^+(\text{Ad} \, P) \to \Omega_{b,n}^+(\text{Ad} \, P)
\]

by

\[
M(\mathcal{X}, \eta) = (\exp (\mathcal{X})(A' + P (\xi + \eta)) - B_t.
\]

Let \( B_t \) be the completion of \( \Omega_{b,n}^+(\text{Ad} \, P) \times \Omega_{b,n}^+(\text{Ad} \, P) \) in the norm:

\[
\|\mathcal{X}, \eta\|_{B_t} = \|d_{B_t} \mathcal{X}\|_{L_b^p(x_n)} + \|\eta\|_{L_b^p(x_n)}.
\]

Since \( A' \) is irreducible by the unique continuation theorem [7, Lemma (4.3.2)], we have an elliptic estimate
for some constant $c_0 > 0$ (c.f. [7, (7.2.30)]). So $B_1$ is a norm. Let $B_2$ be the completion of $\Omega^k_n(\text{Ad } P_n)$ in the norm:

$$\|\alpha\|_{B_2} = \|\alpha\|_{L^{q_0/p}(x_n)} + \|d^*_\alpha\|_{L^{q_0}(x_n)}.$$ 

Then $M$ can be extended to a map from $B_1$ to $B_2$ and the derivative at $(0, 0)$ is given by

$$T(\alpha, \eta) = DM(0, 0)(\alpha, \eta) = d_A \alpha + P \eta.$$

By definition, $T$ is a bounded map from $B_1$ to $B_2$.

**Lemma 5.8.** There is a larger $n_0 \in \mathbb{N}$ such that if $n > n_0$ then

$$\|\alpha\|_{B(A, \eta)} \leq 4 \|T(\alpha, \eta)\|_{B_2}.$$ 

**Proof.** Let $\alpha = d_A \alpha + P \eta$, so that $\mathcal{FJ} = [F^\alpha] + \eta$. Since

$$d_A \phi \alpha = d_A \alpha + \gamma_1 ((A_1 - \pi^* \Gamma), \lambda)$$

on $\tau^{-1}(]-2, n-1])$, we have an elliptic estimate

$$\|\alpha\|_{C^0(\tau^{-1}(]-n+2, n-1]))} \leq c \|d_A \alpha\|_{C^0(\tau^{-1}(]-n+2, n-1]))} \|\phi\|_{L^8(x_n)} + c e^{-\lambda n} \|\alpha\|_{C^0(\tau^{-1}(]-n+2, n-1]))}.$$

For $n > n_0$ with $ce^{-\lambda n_0} < 1/2$, we can rearrange this to get

$$\|\alpha\|_{C^0(\tau^{-1}(]-n+2, n-1]))} \leq 2c \|d_A \alpha\|_{C^0(\tau^{-1}(]-n+2, n-1]))} \|\phi\|_{L^8(x_n)}.$$

Noting that the support of $[F^\alpha, \lambda]$ is contained in $\tau^{-1}(]-2, n-1])$, we see that

$$\|\eta\|_{L^8(x_n)} \leq \|\alpha\|_{B_2} + \|F^\alpha, \lambda\|_{L^8(x_n)}$$

and

$$\|\alpha\|_{B_2} + \|F^\alpha, \lambda\|_{L^8(x_n)} \|\alpha\|_{C^0(\tau^{-1}(]-n+2, n-1]))} \leq c \|d_A \alpha\|_{C^0(\tau^{-1}(]-n+2, n-1]))} \|\phi\|_{L^{q_0/p}(x_n)} + \alpha - P \eta \|L^{q_0/p}(x_n)\|$$

Thus, for $n > n_0$ with $ce^{-(1-\delta/p)n_0} < 1/2$, we obtain

$$\|\eta\|_{L^8(x_n)} \leq 3 \|\alpha\|_{B_2}.$$

This gives us then

$$\|\phi\|_{B_2} \leq 4 \|\alpha\|_{B_2}.$$
On the other hand, the operator $P$ is a pseudo-differential operator and the symbol is homotopic to that of $(d_{A'})^*(1+\Delta_{A'})^{-1}$. It follows that $d_{A'}+P$ is Fredholm and its index equals that of $d_{A'}^*+(d_{A'})^*$.

So

$$\text{index}(T)=2l_{x_n}+3(1+b_1(X_n))=0.$$  

Thus $T$ is an isomorphism from $B_1$ to $B_2$ with operator norm $\|T^{-1}\| \leq 4$. This implies that $M$ is invertible near $(0,0)$ and $5$ is open. It follows that $S=[0,1]$ and the proof is completed.  

**COROLLARY 5.9.** For $0<\nu<\nu_0$ and $n>n_0(\nu)$, the intersection $U(\nu)\cap \mathcal{M}_0(l_{x_n},\eta, g_n)$ is equal to the image of $I_n$.

Suppose that $I_n$ is not surjective for any large $n$. Then there are a subsequence $\{n\}$ (now we relabeled) with $n\to\infty$ and a sequence of $g_n$-ASD connections $\{A_n\}$ not coming from the map $I_n$. Uhlenbeck's compactness principle ([22], [7, 4.4]) and the preservation of $w_4(P)$ under weak limit imply that, after taking a subsequence, the following data exists:

1. A bundle $P'\to Y_1\cup Y_2$ with $w_4(P'|_{Y_1})=\sigma^?(\eta)$, $w_4(P'|_{Y_2})=\sigma^?(\eta)$,
2. ASD connections $A_1$ on $P'|_{Y_1}$ and $A_2$ on $P'|_{Y_2}$ with $(-1/4\pi)^2\int_{Y_1} \text{Tr}(F_{A_1}\wedge F_{A_2})=l_1$ and $(-1/4\pi)^2\int_{Y_2} \text{Tr}(F_{A_2}\wedge F_{A_2})=l_2$,
3. A collection of points $\{x_1,\ldots, x_a\} \subseteq Y_1$, $\{x_{a+1},\ldots, x_{a+b}\} \subseteq Y_2$,
4. $C^\infty$-gauge transformations $\{k_n\}$ over $X_n\setminus \{x_1,\ldots, x_{a+b}\}$,
5. $\text{Hilb}/\text{Law}$ converges to $A_1, A_2$ in $C^\infty$ on compact subsets of $Y_1\setminus \{x_1,\ldots, x_a\}$.
6. ASD connections $A_1$ on $P'|_{Y_1}$ and $A_2$ on $P'|_{Y_2}$ with $(-1/4\pi)^2\int_{Y_1} \text{Tr}(F_{A_1}\wedge F_{A_2})=l_1$ and $(-1/4\pi)^2\int_{Y_2} \text{Tr}(F_{A_2}\wedge F_{A_2})=l_2$.

Since $\sigma^?(\eta)\neq 0$ and $\sigma^?(\eta)\neq 0$, there are no flat connections on $P'|_{Y_1}, P'|_{Y_2}$, which implies that $l_1<0$ and $l_2<0$. Lemma 5.1 and Lemma 3.5 supply $h_t\in C^\infty(\text{Aut}(P))$ such that $[h_t(A_n)]$ lies in $\mathcal{M}_{\nu}(l_t, \sigma^?(\eta), g_n)$. By Proposition 3.7, we obtain $l_1=l_2$ and $a=b=0$.

**LEMMA 5.10.** There are constants $\epsilon>0$, $t_0=t_{0,n}>0$ and $\rho=\rho_m>0$, independent of $T$, with the following significance Let $A$ be an ASD connection on $\mathbb{Q} \times (-T,T)$ with $\{z\times (-T,T) \mid |F_A|^2<\epsilon\}$. Then there exists $h_t \in C^\infty(\text{Aut}(\mathbb{Q} \times (-T+1, T-1)))$ such that for $-T+t_0 \leq t \leq T-t_0$,

$$\left\{ \sum_{n=0} |\nabla_{\pi}^2 h_{-\tau} A - \pi^* \Gamma|^2 \right\} \leq \rho e^{-z^2(1-t_0)}$$

$$\text{Hom}(h_{-\tau} A - \pi^* \Gamma) = 0.$$

**Proof.** The proof is very similar to that of Lemma 5.1. We apply Proposition 4.2 over $\mathbb{Q} \times (-T, T)$ to obtain a gauge transformation $h \in C^\infty(\text{Aut}(\mathbb{Q} \times [-T+1, T-1]))$ such that for $-T+8 \leq t \leq T-8$,
Solve the ordinary differential equation for $\dot{h} \in C^\infty(\text{Aut}(Q \times [-T+1, T-1]))$:

$$i\partial/h_0(hA-\pi*\Gamma)=0$$

with initial condition $\dot{h}|_{t=0}=1$. Then by (5.1) we have

$$\frac{\partial}{\partial t} |d^r \dot{h}|^2 \leq Ce^{-2t\lambda} + \|d^r \dot{h}\|_{C^0(Q \times [0, t+1])}.$$ 

If $t \geq 0$, then we integrate over $[0, t+1]$ to get an inequality

$$\|d^r \dot{h}\|_{C^0(Q \times [0, t+1])} \leq Ce^{-t\lambda}.$$ 

For $t \geq 0$ with $Ce^{-t\lambda} \leq 1/2$ and $0 \leq t \leq T-t_0-1$, we obtain

$$\|d^r \dot{h}\|_{C^0(Q \times [0, t+1])} \leq 2e^{-t\lambda}.$$ 

In the case $t \leq 0$, we can also prove the above by the parallel discussion. So by bootstrapping (5.2), we see that $\dot{h} = h\dot{h}$ satisfies the desired properties for some $p > 0$. D

We choose $n_0 \in \mathbb{N}$ so that

$$\int_{\tau^{-1}(Q \times \infty)} |F_{A_1}|^2 < \frac{1}{2} \varepsilon.$$ 

Since

$$\lim_{n \to \infty} \int_{\tau^{-1}(Q \times \infty)} |F_{A_n}|^2 = \int_{\tau^{-1}(Q \times \infty)} |F_{A_1}|^2 + \int_{\tau^{-1}(Q \times \infty)} |F_{A_2}|^2,$$

we have

$$\int_{\tau^{-1}(Q \times \infty)} |F_{A_n}|^2 < \varepsilon \quad (n > n_0)$$

for some $n_1 > n_0$. By Lemma 5.10, we can find $h_n \in C^\infty(\text{Aut}(Q \times [n_0+1, n+1]) \cup \bar{\gamma}_n Q \times [n_0+1, n+1])$ such that for $n_0 + t_0 \leq t \leq n$,

$$\sup_{Z \times [t-1, t+1]} \left\{ \sum_{i=0}^{m} |Y_i(h_{A_n} - \pi*\Gamma)|^2 \right\} \leq C e^{-2t\lambda}.$$

So the connection $(h_n A_n)' = \gamma_t(h_n A_n - \pi*\Gamma) + (1-\gamma_t)\pi*\Gamma$ over $\tau^{-1}_1([n_0+1, \infty))$ also satisfies the same condition as above for $n_0 + t_0 \leq t \leq n-1$. Now we can apply Ascoli-Arzela's theorem with diagonal argument to deduce that, after taking a subsequence, $(h_n A_n)'$ converges to an ASD connection $A'$ in $C^{m-1}$ over compact sets in $\tau^{-1}_1([n_0+t_0, \infty))$. So $A'$ satisfies
for $t \geq n_0 + t_0$. By Lebesgue convergence theorem we have

$$\lim_{n \to \infty} \| (\tilde{A}_n - A_0) \|_{L^2_{\partial K \cap \partial t}} = 0.$$  \hspace{1cm} (5.6)

We bootstrap the equation

$$d_{A_t}(h_n k^{-1}) = \overline{\partial}^* (h_n \bar{A}_0) - (h_n \bar{A}_0 - A_t) h_n k^{-1},$$

to deduce that there is a subsequence $\{h_n k^{-1}\}$ (now we relabeled) such that $h_n k^{-1}$ converges to some $u_t$ in $C^\infty$ on $\tau_n^1 ((n_0 + t_0, n_0 + t_0 + 1))$. By Theorem 5.1, if we replace $\{h_n\}$ by $\{r t h_n\}$ for some $r_t \in \mathbb{R}$, we can suppose that $u_t$ can be extended to $u_t^*$ over $\tau_n^1 ((0, n_0 + t_0 + 1))$ and $b(1, f_t) = 0$. Then for large $n \in \mathbb{N}$, $b(1, r_t h_n k^{-1}) = b(1, \tilde{A}_n) = 0$. So we have $b(1, r_t \tau_n^1) = b(1, r_t h_n k^{-1}(r_t h_n k^{-1})^{-1}) = b(1, r_t h_n k^{-1}) + b(1, r_t h_n k^{-1}) = 0$. Since $(r_t \tau_n^1) = \Gamma$ we see that $r_t = r$. Now we can apply the argument of [7, Lemma (4.4.5)] (see also Appendix 1) to patch gauge transformations $k_n$ and $\bar{A}_t$ over $\tau_n^1 ((n_0 + t_0, n_0 + t_0 + 1))$. Then taking a subsequence, we can find $C^m$ gauge transformations $\{u_n\}$ on $X$ such that $u_n = h_n$ on $\tau_n^1 ([n_0 + t_0 + 1, n])$ and $u_n \bar{A}_0$ converges to an ASD connection $\bar{A}_t$ on $P$ in $C^m$ over compact subsets of $Y$, which is in the exponential gauge. If we choose $m \geq 5$, then $\bar{A}_t$ is gauge equivalent to a $C^\infty$-connection and the uniqueness in Lemma 5.1 implies that $\bar{A}_t$ is smooth over $Z \times [n_0 + t_0, \infty)$ and satisfies the condition (ii) for $t \geq n_0 + t_0$. So we obtain

$$\| u_n A_t - f_n (\bar{A}_t A_0) \|_{L^2_{\partial K \cap \partial t}(X_n)} \to 0 \mbox{ (n \to \infty)}.$$
where $A^\pm$ is ± self-dual space with respect to $g_{10}$. Then we have the following theorem, whose proof is also the same as in ([7, 4.3], [14, 5 (iii)]).

**PROPOSITION 5.11.** There is a Baire set $C^\kappa_1 \subset C^\kappa_2$ such that for all $g \in C^\kappa_2$, the ASD moduli $\mathcal{M}^\kappa_1(l^{\kappa_1}, \eta^\kappa_1, g)$ is a finite set consisting of irreducible regular connections.

For each $n \in \mathcal{N}$, we fix a metric $g_{1n}$ in $C^\kappa_2$ such that $g_{1n}$ is independent of $n$ on $K_1$ and the metric

$$g_1 = \begin{cases} g_{1n} & \text{on } K_1, \\ h + dr^2 & \text{on } Z \times [0, \infty) \end{cases}$$

lies in $C^\gamma_1$ for $i=1, 2$.

We extend $Q$ and $f$ naturally over $W \cup Z \times [0, \infty)$, which we also denote by $F$ and $Q$ respectively. We replace $Y_2$ by $W \cup Z \times [0, \infty)$ and set a function $\tau_2 : Y_2 \to \mathbb{R}$ as before. Then we can define the gluing map

$$I_{1n} : \mathcal{M}^\kappa_1(l_1, \sigma^\kappa_1(\eta), g_1) \times \{\tilde{T}\} \to \mathcal{M}^\kappa_2\kappa_1(l^{\kappa_1}, \eta^\kappa_1, g_{1n})$$

for large $n \in \mathcal{N}$ just as before. We can prove that $I_{1n}$ is bijective in the same way as that of $I$. But in the proof, we need to correct it at some points. First, after Corollary 5.9, we treat a sequence of connections $\{\Lambda_{1n}\}$. Then we obtain $l_1 = l_{1n}, l_2 = 0$ and $a = h = 0$. So $u_{1n}^\kappa k_n A_{1n}$ converges to $T$ in $C^\infty$ on compact subsets of $W \cup Z \times [0, \infty)$. Second, after (5.6), on $\tau_2^{-1}((n_0 + t_0, n_0 + t_0 + 1))$, if we replace $\{h_n\}$ by $\{r_i h_n\}$ for some $r_i \in \mathbb{R}$, we can suppose that $r_i h_n k_n^{-1}$ converges to $u_1$ over $\tau_2^{-1}((n_0 + t_0, n_0 + t_0 + 1))$. We bootstrap the equation

$$d_{\ast\Gamma}(r_i h_n(u_{1n}^\kappa k_n)^{-1}) = r_i h_n(u_{1n}^\kappa k_n)^{-1}(u_{1n}^\kappa k_n A_n - \pi^\kappa \Gamma) - r_i(h_n A_n - \pi^\kappa \Gamma) h_n(u_{1n}^\kappa k_n)^{-1}$$

to obtain that, if we choose $n_0$ so large, then there is a subsequence $\{r_i h_n(u_{1n}^\kappa k_n)^{-1}\}$ (now we relabeled) such that $r_i h_n(u_{1n}^\kappa k_n)^{-1}$ converges to some $u_{1n}^\kappa k_n$ in $C^{m-1}$ on $\tau_2^{-1}((n_0 + t_0, n_0 + t_0 + 1))$ and, since the right hand side is so small, $u_{1n}^\kappa k_n$ can be extended to $u_{2n}^\kappa k_n$ over $\tau_2^{-1}((0, n_0 + t_0 + 1))$. Now we can apply the argument of [7, Lemma 4.4.5] to patch gauge transformations $k_n$ (resp. $u_{1n}^\kappa k_n$) and $r_i h_n$ over $\tau_2^{-1}((n_0 + t_0, n_0 + t_0 + 1))$ (resp. $\tau_2^{-1}((n_0 + t_0, n_0 + t_0 + 1))$). Then as was shown there, we see that $I_{1n}$ is surjective for large $n \in \mathcal{N}$. (This bijection has been stated in [10, Theorem V. 3.4].) Of course, the same holds for $Y_2$ and $I_{2n}$ defined as before. Composing three bijections, we have a bijection

$$K_n : \mathcal{M}^\kappa_1(l^{\kappa_1}, \eta^\kappa_1, g_{1n}) \times \mathcal{M}^\kappa_2\kappa_1(l^{\kappa_1}, \eta^\kappa_1, g_{2n}) \to \mathcal{M}^\kappa_2(l^{\kappa_2}, \eta, g_{2n})$$

for large $n \in \mathcal{N}$. Hence Theorem 2.1 follows from the lemma below.

**LEMMA 5.12.** If $\{B_{10}\}, \{B_{11}\} \in \mathcal{M}^\kappa_1(l^{\kappa_1}, \eta^\kappa_1, g_{1n})$ satisfies $o([B_{1i}]) = e_i o([B_{10}])$ ($i=1, 2$), then $o(K_n([B_{11}]), [B_{11}]) = e_i e_0 o(K_n([B_{10}]), [B_{10}])$. 

Proof. We write \([B_{i0}]=[I_{i0}(A_{i0},\hat{\Gamma})], [B_{i1}]=[I_{i1}(A_{i1},\hat{\Gamma})]\) for some \([A_{i0}], [A_{i1}]\in \mathcal{M}_{X_i}(l_{X_i}, \varphi_i(\eta), g_i)\). We choose any path \(A_{it}(0 \leq t \leq 1)\) in \(\mathcal{A}_{P_i}\) from \(A_{i0}\) to \(A_{i1}\). Define a connection \(A'_{it}\) on \(P_{st}\) by

\[
A'_{it} = \begin{cases} 
\mu_t(A_{it} - \pi^*\Gamma) + (1 - \mu_t)\pi^*\Gamma & \text{over } \mathcal{T}_n(\{0, n+1\}), \\
\hat{\Gamma} & \text{over } W \cup Z \times [0, n+1],
\end{cases}
\]

and a connection \(A'_i\) on \(P_n\) by \(A'_i = I_{n}(A_{it}, A_{it})\). Since \([A_{i0}]\) (resp. \([A_{i1}]\)) lies in \(U([A_{i0}])\) (resp. \(U([A_{i1}])\)) for large \(n \in \mathbb{N}\), we have a nowhere zero section \(s_i\) over \([A_{i0}]\) for \(0 \leq t \leq 1\) such that \(s_i([A_{i0}]) = \epsilon_i([B_{i0}]), \quad s_i([A_{i1}]) = \epsilon([B_{i1}]).\)

Since \(\ker \delta^*_A\) is supported in \(\mathcal{T}_n([0, n-1])\), we can choose a linear map \(S_i: R \to L((\mathcal{O}_{X_{i0}} \oplus \mathcal{O}_{X_{i1}}) \text{Ad } P_{i0})\) so that for all \(0 \leq t \leq 1\), \(\delta_A + S_i\) is surjective and the image of \(S_i\) is supported in \(\mathcal{T}_n([0, n-1])\). So \(\ker (\delta_A + S_i)([0, n-1])\) is supported in \(\mathcal{T}_n(\{0, n-1\}) \times R \times (i=1, 2)\) and \(\ker (\delta_A + S_i)([0, n-1])\) is supported in \(\mathcal{T}_n(\{0, n-1\}) \times R \times (i=1, 2)\). Then we have a natural isomorphism

\[
\theta_i: \ker (\delta_A + S_i) \to \ker (\delta_A + S_i)([0, n-1]) \\
(t_1, t_2) \mapsto t_1 + t_2,
\]

by which we obtain a nowhere zero section \((A_{\max} \theta_i)(s_1 \otimes s_2)\) on \(P_{i0}\) over \([A_{i0}]\) (0 \(\leq t \leq 1\)). Since \(\ker \delta_A^i\) is supported in \(\mathcal{T}_n([0, n-1])\), \(\delta_A + S_i\) is surjective. Now we get

\[
(A_{\max} \theta_i)(s_1 \otimes s_2)([A_{i0}]) = (A_{\max} \theta_i)(s_1([A_{i0}]) \otimes s_2([A_{i0}]))
\]

\[
= \epsilon_1 \cdot \epsilon_2 (A_{\max} \theta_i) \cdot o([B_{i0}]) \otimes o([B_{i0}]) = \epsilon_1 \cdot \epsilon_2 o([I_n(A_{i0}, A_{i0})])
\]

\[
= \epsilon_1 \cdot \epsilon_2 o([B_{i0}], [B_{i0}]),
\]

\[
(A_{\max} \theta_i)(s_1 \otimes s_2)([A_{i1}]) = o([B_{i1}], [B_{i1}]).
\]

This implies the desired equality. \(\square\)

Proof of Theorem 2.2. Suppose that \(\gamma(\eta) = 0\). Then for each \(n\) there is an ASD connection \([A_{i0}]\) in \(\mathcal{M}_{X_i}(l_{X_i}, \varphi_i(\eta), g_i)\). (Here, in the construction of \(P_n\), \(\epsilon_1\) and \(\epsilon_2\) are chosen so that \(w_1(P_n) = \eta, \varphi_i(P_n) = I_{X_i}\) are only satisfied.) Then after taking a subsequence, we have the data from (1) to (6) in the above. As was shown there, we see that \(l_1 = l_{Y_1}\) and \(l_2 = l_{Y_2}\). The resulting connection \(A_i\) on \(P_i\) can be thought of an element in \(\mathcal{M}_{X_i}(l_{X_i}, \varphi_i(\eta), g_i)\). The following lemma contradicts to the assumption. \(\square\)

**Lemma 5.13.** For any connection \(A\) on \(P\) which is isomorphic to \(\pi^*\Gamma\) on \(Z \times [n, \infty)\) for some \(n \in \mathbb{N}\),
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\[-1 \over 4\pi^2 \int_X \text{Tr} (F_A \wedge F_A) \equiv (\eta^*)^2 \pmod 2.\]

Proof. Define \( \tilde{P} = P \mid_{\tau-1(S, \eta^*)} \cup \tilde{Q}. \) \( \sigma^*(w_2(\tilde{P})) = \sigma^*(\eta^*) \) implies that \( w_2(\tilde{P}) = \eta^* + P . D. [T^g \times 0]. \) In any case, we obtain

\[-1 \over 4\pi^2 \int_X \text{Tr} (F_A \wedge F_A) \equiv (\eta^*)^2 \equiv (\mod 2). \]

Remark. The vanishing of \( \gamma_X(\eta) \) can be observed for more general elements \( \eta \in C_X. \) In a forthcoming paper, we will treat it.

6. Explicit calculations on elliptic surfaces.

In this section we calculate values of the simple invariant for the regular elliptic surfaces without multiple fibers. Let \( \pi: S_k \to \mathbb{C}P^1 \) be a regular minimal elliptic surface with \( p_g = k - l \) and without multiple fibers. Then \( S_k \) satisfies

\[ \pi_1(S_k) = 1, \quad b_2(S_k) = 2k - l \quad \text{and} \quad l_2(S_k) = -3k. \]

It admits a differentiable section \( \Sigma \), \( \mathbb{C}P^1 \to S_k \), which has the self-intersection number \( (\Sigma)^2 = -k. \) We take a general fiber in \( S_k. \) Then we can interpret this surface \( S_k \) as a fiber sum up to fiber preserving diffeomorphism as follows [17]: Given \( S_k \) and \( S_k \cup S_k \), identify the tubular neighborhood of a general fiber in each with \( T^2 \times D^2 \) so that the fibrations correspond to projection onto \( D^2 \). Remove the interior of tubular neighborhoods from \( S_k \) and \( S_k \cup S_k \), and glue the two remaining manifolds together by an orientation reversing and fiber preserving diffeomorphism on the boundaries. Then we get an oriented manifold \( S_k \cup S_k \), a fibration \( \pi: S_k \cup S_k \to \mathbb{C}P^1 \), and a section \( \Sigma \cup \Sigma : \mathbb{C}P^1 \to S_k \). We note that \( S_k \) contains the Kummer surface, which is one of the K3 surface.

We use a well known result by Donaldson ([6], [7], [13]).

PROPOSITION 6.1. (Donaldson) If we fix the orientation of \( H^*(S_k) \) determined by the complex structure of \( S_k \), then \( \gamma_{S_k}(\eta) = 1 \) for any \( \eta \in C_{S_k}. \)

LEMMA 6.2. \( |\gamma_{S_k}(P . D. (\Sigma s) + [f])| = 1. \)

Proof. Because \( S_k = S_k \cup S_k = S_k \cup S_k \times S_k \) and \( \Sigma s = \Sigma s + \Sigma s = \Sigma s + \Sigma s + \Sigma s : \mathbb{C}P^1 \to S_k \), we apply Theorem 2.1 three times to deduce that

\[ |\gamma_{S_k}(P . D. (\Sigma s))| = |\gamma_{S_k}(P . D. (\Sigma s) + 2[f])| \]

\[ = |\gamma_{S_k}(P . D. (\Sigma s))|^2 = 1. \]

So \( |\gamma_{S_k}(P . D. (\Sigma s) + [f])| = 1. \)

COROLLARY 6.3. For integer \( k \geq 2, \)

\[ |\gamma_{S_k}(P . D. (\Sigma s))| = 1 \quad \text{if} \ k \ \text{is even}, \]
Remark. In a recent paper [11], we found that for integer $k \geq 2$,

| $\gamma_{S_k}(P \cdot D(\Sigma_s) + [f])$ | if $k$ is even, 

| $\gamma_{S_k}(P \cdot D(\Sigma_s))$ | if $k$ is odd,

using the moduli of stable vector bundles.

For $k$ odd, we can determine the image of $|\gamma_{S_k}|$.

**Lemma 6.4.** $|\gamma_{S_k}(\eta)| = 1$ for any $\eta \in C_{S_k}$.

**Proof.** Since the characteristic element is $w_2(S_3) = P \cdot D(f) \pmod{2}$, $\langle \eta, [f] \rangle \equiv 1 \pmod{2}$. We construct $\eta \cdot \eta \in C_{S_{S_k}}$ by identifying a tubular neighborhood of a general fiber in two copies of $S_3$. If we identify $S_3$ with $S_3 \times S_3$, then $\eta \in C_{S_k}$ factors as $\eta = \eta_2 \cdot \eta_1$ for some $\eta_2 \in H^3(S_3; \mathbb{Z}_2)$ and $\eta_1 \in H^3(S_3; \mathbb{Z}_2)$ with $\eta \equiv (\eta_2)^2 + (\eta_1)^4 \pmod{4}$. If $(\eta_2)^2 \equiv 0 \pmod{4}$, then $(\eta_2 + P \cdot D(f))^2 \equiv (\eta_2)^2 + 2 = 2 \equiv 2 \pmod{4}$. So we may assume that $\eta_2 \in C_{S_k}, \eta_1 \in C_{S_k}$. In the same way as above we have an element $\eta_2 \cdot \eta_1 \in C_{S_{S_k}}$. Now by Theorem 2.1 and Proposition 6.1, we get

| $\gamma_{S_k}(\eta \cdot \eta_1)$ | $\gamma_{S_k}(\eta_1)$ | $\gamma_{S_k}(\eta_2)$ | $\gamma_{S_k}(\eta_2) = 1$. 

D

We can apply the above argument on $S_{S_k} = S_{S_k} \cdot S_{S_k}$ to deduce that

**Corollary 6.5.** // if $k$ is odd and $k \geq 3$, then $|\gamma_{S_k}(\eta)| = 1$.

**Corollary 6.6.** // if $k$ is even, then $|\gamma_{S_k}(\eta)| = 1$ for $\eta \in C_{S_k} \cdot \eta \cdot [\eta, [f]] \equiv 1 \pmod{2}$.

Remark. In fact, Ue has obtained that the value $|\gamma_{S_k}(\eta)|$ is independent of $\eta \in C_{S_k}$ with $\langle \eta, [f] \rangle \equiv 1 \pmod{2}$, by analyzing the action of diffeomorphism group of $S_k$ on $C_{S_k}$ [21].

**Appendix 1**

Lemma (4.4.5) in [7] omits a necessary hypothesis on extension of gauge group. We write a precise statement, but omit its proof, since it is same as that of Lemma (4.4.5).

**Lemma (4.4.5).** We fix $m \in \mathbb{N}$. Suppose that $A_n$ is a sequence of unitary $C^m$-connections on a $SO(3)$ bundle $P$ over a base manifold $\Omega$ (possibly non-compact), and let $\bar{\Omega} \subseteq \bar{\Omega}$ be an interior domain. Suppose that there are gauge transformations $u_n \in C^{m+1}(\text{Aut } P)$ and $\tilde{u}_n \in C^{m+1}(\text{Aut } P)$ such that $u_n A_n$ converges in $C^m$ over $\Omega$ and $\tilde{u}_n A_n$ converges in $C^m$ over $\bar{\Omega}$. Then we may assume that; taking a subsequence $\{n'\}$, the $u_n' u_n'$ converge in $C^m$ over $\bar{\Omega}$ to a limit $\tilde{u}$.
// u extends over Ω, then for any compact set K ⊂ ⨁ we can find gauge transformations \( w_n \in C^m(\text{Aut} P) \) such that \( w_n \rightarrow \bar{u}_n \) in a neighborhood of K and the connections \( w_n A_n \) converge in \( C^{m-1} \) over Ω.

In Section 5, we apply the above lemma with \( \Omega = \tau^{-1}([0, n_0 + t_0 + 1]), \bar{\Omega} = \tau^{-1}([n_0 + t_0, n_0 + t_0 + (3/4)]) \) and \( K = \tau^{-1}([n_0 + t_0 + (1/4), n_0 + t_0 + (1/2)]) \).

**Appendix 2.**

We prove the compactness in Proposition 3.7. For \( l_Y < l < 0 \), it is vacuous. Let \( \{[A_\lambda]\} \) be a sequence in \( \mathcal{H}_Y(l_Y, \eta, g) \). Then after taking a subsequence, the following data exists:

1. A bundle \( P' \rightarrow Y \) with \( w_\eta(P') = \eta \),
2. An ASD connection \( A \) on \( P' \) with \( -(1/4\pi^2) \int_Y \text{Tr}(F_A \wedge F_A) = l \),
3. A collection of points \( \{x_1, \ldots, x_a\} \in Y \),
4. \( C^m \)-gauge transformations \( \{k_n\} \) over \( Y \setminus \{x_1, \ldots, x_a\} \),
5. \( k_n A_n \) converges to \( A \) in \( C^m \) on compact subsets of \( Y \setminus \{x_1, \ldots, x_a\} \),
6. \( 4a - l \leq -l_Y \).

Since \( \eta \neq 0 \), there are no flat connections on \( P' \), and \( l < 0 \). By Lemma 5.1 and Lemma 3.5, there is \( h \in C^m(\text{Aut} P) \) such that \([hA]\) lies in \( \mathcal{H}_Y(l, \eta, g) \). So we have \( l = l_Y, a = 0 \). We choose \( \varepsilon > 0 \) so that

\[
\int_{\tau^{-1}([t_0, \infty))} |F_A|^2 < \varepsilon,
\]

where we choose \( \varepsilon > 0 \) by Lemma 5.1. Since

\[
\lim_{n \to \infty} \int_{\tau^{-1}([t_0, \infty))} |F_{A_n}|^2 = \int_{\tau^{-1}([t_0, \infty))} |F_A|^2
\]

we have

\[
\int_{\tau^{-1}([t_0, \infty))} |F_{A_n}|^2 < \varepsilon \quad (n > n_0)
\]

for some \( n_0 \in \mathbb{N} \). Then there exists \( h_n \in C^m(\text{Aut} (Q \times [t_0, \infty))) \) such that

\[
\sup_{t \geq t_0} \left\{ \sum_{t=m}^{m-1} |\nabla_{(t')} (h_n A_n - \pi^* F')|^2 \right\} \leq \rho e^{-2t(t-t_0)}
\]

for \( t \geq t_0 \). Now we apply Ascoli-Arzela's theorem with diagonal argument to deduce that \( h_n A_n \) converges to an ASD connection \( A' \) in \( C^{m-1} \) over compact sets in \( \tau^{-1}([t_0, \infty)) \). So

\[
\sup_{t \geq t_0} \left\{ \sum_{t=m}^{m-1} |\nabla_{(t')} (A' - \pi^* F')|^2 \right\} \leq \rho e^{-2t(t-t_0)}
\]

for \( t \geq t_0 \). By bootstrapping the equation
we see that there is a subsequence \( \{h_n k_n^{-1}\} \) (now we relabeled) such that \( h_n k_n^{-1} \) converges to some \( u \) in \( C^{m-1} \) on \( \tau^{-1}([t_0, t_0+1]) \). By Lemma 3.5, if we replace \( \{h_n\} \) by \( \{r h_n\} \) for some \( r \in \mathbb{R} \), we can suppose that \( u \) can be extended to \( u^* \) over \( \tau^{-1}([0, t_0+1]) \). So we can apply the argument of [7, Lemma (4.4.5)] (see also Appendix 1) to patch gauge transformations \( k_n \) and \( r h_n \) over \( \tau^{-1}([t_0, t_0+1]) \). Then taking a subsequence, we can find \( C^{m-1} \)-gauge transformations \( \{u_n\} \) on \( Y \) such that \( u_n h_n \) on \( \tau^{-1}([t_0+1, \infty)) \) and \( u_n A_n \) converges to an ASD connection \( A^* \) on \( P \) in \( C^{m-1} \) on compact subsets of \( Y \). If we choose \( m \geq 5 \), then

\[
\|u_n A_n - A^*\|_{L^2(Y)} \to 0 \quad (n \to \infty),
\]

and \([u_n A_n], [A^*] \in \mathcal{M}_Y(l, \eta, g)\). Now by Lemma 3.3, \( \{u_n\} \) is in \( G_P \). □

REFERENCES


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