ON MINIMAL SURFACES WITH THE RICCI CONDITION IN SPACE FORMS

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0. Introduction

A 2-dimensional Riemannian metric $ds^2$ is said to satisfy the Ricci condition with respect to $c$ if its Gaussian curvature $K$ satisfies $K < c$ and the new metric $d\hat{s}^2 = \sqrt{c - K}ds^2$ is flat.

Let $X_N(c)$ denote the $N$-dimensional simply connected space form of constant curvature $c$, and in particular, let $R^N = X_N(0)$. The induced metric $ds^2$ on a minimal surface in $X^3(c)$ satisfies the Ricci condition with respect to $c$ except at points where the Gaussian curvature $= c$. Conversely, assume that a Riemannian metric $ds^2$ on a 2-dimensional simply connected manifold $M$ satisfies the Ricci condition with respect to $c$. Then there exists a smooth $2\pi$-periodic family of isometric minimal immersions $f_\theta : (M, ds^2) \to X^3(c)$; $\theta \in \mathbb{R}$, which is called the associated family. Moreover, up to congruences, the maps $f_\theta$; $0 \leq \theta < \pi$ represent all local isometric minimal immersions of $(M, ds^2)$ into $X^3(c)$ (see [5]). So, the Ricci condition with respect to $c$ is an intrinsic characterization of minimal surfaces in $X^3(c)$.

Here we consider the following problem, which may be seen as a kind of rigidity problem.

PROBLEM. Classify those minimal surfaces in $X_N(c)$ whose induced metrics satisfy the Ricci condition with respect to $c$, or equivalently, classify those minimal surfaces in $X_N(c)$ which are locally isometric to minimal surfaces in $X^3(c)$.

A submanifold in $X_N(c)$ is said to lie fully in $X_N(c)$ if it does not lie in a totally geodesic submanifold of $X_N(c)$. Let $S(N, c)$ denote the set of all Riemannian structures of minimal surfaces lying fully in $X_N(c)$. Then the problem is to determine the intersection of $S(3,c)$ and $S(N, c)$.

1. Examples

In this section, we give examples of minimal surfaces in $X_N(c)$ which do not lie in a totally geodesic $X^3(c)$ and whose induced metrics satisfy the Ricci condition with respect to $c$. The following three types of examples are known.

Example 1 ([6]). Let $f_\theta : (M, ds^2) \to \mathbb{R}^3$; $\theta \in \mathbb{R}$ be the associated family of isometric minimal immersions of a 2-dimensional Riemannian manifold $(M, ds^2)$ into
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Then we can construct an isometric minimal immersion $f : (M, ds^2) \rightarrow \mathbb{R}^6$ by setting

$$f = f_\theta \cos \varphi \oplus f_{\theta + \pi/2} \sin \varphi,$$

where the symbol $\oplus$ denotes the direct sum with respect to an orthogonal decomposition $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$. The metric induced by $f$ is $ds^2$, which satisfies the Ricci condition with respect to $0$ except at points where the Gaussian curvature $= 0$. Furthermore, in general, $f(M)$ lies fully in $\mathbb{R}^6$ if $\varphi \not\equiv 0 \ (\text{mod} \pi/2)$.

Example 2 ([6]). Let $c > 0$. Let $f_\theta : (M, ds^2) \rightarrow X^3(c) (\subset \mathbb{R}^4)$; $\theta \in \mathbb{R}$ be the associated family of isometric minimal immersions of a 2-dimensional Riemannian manifold $(M, ds^2)$ into $X^3(c)$. Then we can construct an isometric minimal immersion $f : (M, ds^2) \rightarrow X^{4m+3}(c) (\subset \mathbb{R}^{4m+4})$ by setting

$$f = a_0 f_{\theta_0} \oplus \cdots \oplus a_m f_{\theta_m},$$

where $0 \leq \theta_0 < \theta_1 < \cdots < \theta_m < \pi$, each $f_{\theta_i}$ is viewed as an $\mathbb{R}^4$-valued function with $|f_{\theta_i}| = 1/\sqrt{c}$, and the symbol $\oplus$ denotes the direct sum with respect to an orthogonal decomposition $\mathbb{R}^{4m+4} = \mathbb{R}^4 \oplus \cdots \oplus \mathbb{R}^4$. The metric induced by $f$ is $ds^2$, which satisfies the Ricci condition with respect to $c$ except at points where the Gaussian curvature $= c$. Furthermore, in general, $f(M)$ lies fully in $X^{4m+3}(c)$.

Example 3 ([11] and [4]). Every 2-dimensional flat metric automatically satisfies the Ricci condition with respect to $c > 0$, and there are flat minimal surfaces lying fully in $X^{2n+1}(c)$ where $c > 0$.

2. Known results

In the Euclidean case where $c = 0$, Lawson solved the problem completely as follows.

Theorem 1 ([6] and [7, Chapter IV]). Let $f : M \rightarrow \mathbb{R}^N$ be a minimal immersion of a 2-dimensional manifold $M$ into $\mathbb{R}^N$. Suppose that the induced metric $ds^2$ satisfies the Ricci condition with respect to $0$ except at isolated points where the Gaussian curvature $= 0$. Then either (i) $f(M)$ lies in a totally geodesic $\mathbb{R}^3$, or (ii) $f(M)$ lies fully in a totally geodesic $\mathbb{R}^5$ and $f$ is of the form of (1) in Example 1 for $\varphi \not\equiv 0 \ (\text{mod} \pi/2)$.

Remark 1. Theorem 1 says that $S(3,0)$ and $S(N,0)$ are disjoint if $N = 4$, $N = 5$ or $N \geq 7$. Theorem 1 says also that $S(3,0)$ is included in $S(6,0)$ through Example 1.

Concerning the spherical case where $c > 0$, Lawson posed the following conjecture.

Conjecture ([6]). Let $f : M \rightarrow X^N(c)$ be a minimal immersion of a 2-dimensional manifold $M$ into $X^N(c)$ where $c > 0$. Suppose that the induced metric $ds^2$ satisfies the Ricci condition with respect to $c$ except at isolated points where the Gaussian curvature $= c$. Then $f$ must be of the form of (2) in Example 2.

As a matter of fact, there are easy counter-examples to this conjecture (cf. Example 3). So one should consider the conjecture for non-flat minimal surfaces. In [8], with some
global assumptions, Naka (= Miyaoka) obtained partial positive answers to this question.

3. Our results

First we solve the problem in the case where \( N = 4 \).

**THEOREM 2 ([10]).** Let \( f : M \rightarrow X^4(c) \) be a minimal immersion of a 2-dimensional manifold \( M \) into \( X^4(c) \). Suppose that the induced metric \( ds^2 \) satisfies the Ricci condition with respect to \( c \) except at isolated points where the Gaussian curvature = \( c \). Then \( f(M) \) lies in a totally geodesic \( X^3(c) \).

**Remark 2.**

(i) Theorem 2 says that \( S(3,c) \) and \( S(4,c) \) are disjoint.

(ii) When \( c = 0 \), Theorem 2 is included in [6].

(iii) In the case where \( c > 0 \), Theorem 2 is not true if we replace \( X^4(c) \) by \( X^5(c) \) (cf. Example 3).

(iv) In [10], with an additional assumption, we give a result also in higher codimensional cases.

In [3] Johnson studied a class of minimal surfaces in \( X^N(c) \), which are called exceptional minimal surfaces and are related to the theory of harmonic sequences (cf. [2] and [11]). Next we discuss exceptional minimal surfaces in \( X^N(c) \) whose induced metrics satisfy the Ricci condition with respect to \( c \).

**THEOREM 3 ([9]).** Let \( f : M \rightarrow X^N(c) \) be an exceptional minimal immersion of a 2-dimensional manifold \( M \) into \( X^N(c) \) where \( c > 0 \). Suppose that the induced metric \( ds^2 \) satisfies the Ricci condition with respect to \( c \) except at isolated points where the Gaussian curvature = \( c \). Then either (i) \( f(M) \) lies fully in a totally geodesics \( X^{4m+1}(c) \) and \( ds^2 \) is flat, or (ii) \( f(M) \) lies fully in a totally geodesic \( X^{4m+3}(c) \).

**THEOREM 4 ([9]).** Let \( f : M \rightarrow X^N(c) \) be an exceptional minimal immersion of a 2-dimensional manifold \( M \) into \( X^N(c) \) where \( c < 0 \). Suppose that the induced metric \( ds^2 \) satisfies the Ricci condition with respect to \( c \) except at isolated points where the Gaussian curvature = \( c \). Then \( f(M) \) lies in a totally geodesic \( X^3(c) \).

**Remark 3.**

(i) There are flat exceptional minimal surfaces lying fully in \( X^{2n+1}(c) \), where \( c > 0 \) (see [9]).

(ii) There are non-flat exceptional minimal surfaces lying fully in \( X^{4m+3}(c) \) whose induced metrics satisfy the Ricci condition with respect to \( c \), where \( c > 0 \) (see [9]).

**Reference**


