SURFACES WITH PARALLEL MEAN CURVATURE VECTOR IN $P^n(C)$

TAKASHI OGATA

1. Introduction

The set of surfaces with parallel mean curvature vector in Riemannian manifold, which includes all minimal surfaces in the manifold, has been studied by many geometers. Especially, Chen [1] and Yau [7] studied them in the case that the ambient space is an $n$-dimensional real space form $\mathcal{M}^n(c)$ of constant sectional curvature $c$. They proved that if $x: M \rightarrow \mathcal{M}^n(c)$ is an isometric immersion with parallel mean curvature vector of a two-dimensional Riemannian manifold $M$ into $\mathcal{M}^n(c)$, then $x(M)$ is one of the following surfaces: (1) a minimal surface in $\mathcal{M}^n(c)$, (2) a minimal surface of a small hypersphere of $\mathcal{M}^n(c)$, and (3) a surface with constant mean curvature in a 3-sphere of $\mathcal{M}^n(c)$. This shows that the study of surfaces in $\mathcal{M}^n(c)$ with parallel mean curvature vector is reduced to that of minimal surfaces except the case (3).

On the other hand, concerning the surfaces with parallel mean curvature vector in a complex space form, we know several minimal surfaces in the $n$-dimensional complex projective space $P^n(C)$ with the Fubini-Study metric of constant holomorphic sectional curvature $4\rho$. Moreover, many results characterizing them have been obtained (cf. [2], [3], [4], [5], [6]). However, when we concern with non-minimal surfaces in $P^n(C)$ with parallel mean curvature vector, not many such examples are known so far, even for $n=2$.

In Sections 1 and 2 of the previous paper [5], we developed a local theory of surfaces in $P^n(C)$ by using the Kaehler function. By applying it, in this paper we shall study non-minimal immersions $x: M \rightarrow P^n(C)$ with parallel mean curvature vector. In fact, in Section 2 we obtain basic formulas for such surfaces in a 2-dimensional Kaehler manifold of constant holomorphic sectional curvature $4\rho$. Then, in Sections 3 and 4, we show a method of the local construction of such immersions. Finally, in Section 5 we determine isometric immersions with parallel mean curvature vector field of a Riemannian 2-manifold with constant Gaussian curvature into $P^n(C)$. Theorem 5.2 generalizes a theorem by Ludden, Okumura and Yano [4].

---

*Mathematical Subject Classification* (1991): 53A10, 53C42, 53C55

Received July 20, 1994; revised July 25, 1995.

397
The author would like to express particular thanks to Professor Kenmotsu for his advice and encouragement during the development of this work.

2. The fundamental theorem of surfaces in a Kaehler manifold

Let $X$ be a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature $4\rho$. We consider some basic properties of immersed surfaces in $X$. Let $\{\omega_a\}$ be a local field of unitary coframes on $X$, so that the Kaehler metric is represented by $\sum \omega_a \bar{\omega}_a$. Here and in what follows, we will agree on the following range of indices: $1 \leq \alpha, \beta, \gamma \leq 2$. We denote by $\omega_{a\bar{\beta}}$ the unitary connection forms with respect to $\{\omega_a\}$. The structure equations of $X$ are given by

$$
\begin{align*}
\text{d}\omega_a &= \sum \omega_{a\beta} \wedge \omega_{\beta}, \quad \omega_{a\beta} + \bar{\omega}_{\beta a} = 0, \\
\text{d}\omega_{a\bar{\beta}} &= \sum \omega_{a\gamma} \wedge \omega_{\gamma \bar{\beta}} + \Omega_{a\bar{\beta}}, \\
\Omega_{a\bar{\beta}} &= -\rho (\omega_a \wedge \bar{\omega}_{\beta} + \delta_{a\beta} \sum \omega_t \wedge \bar{\omega}_t).
\end{align*}
$$

Let $(M, ds^2)$ be an oriented connected 2-dimensional Riemannian manifold. The tangent bundles of $M$ and $X$ are denoted by $TM$ and $TX$, respectively. Let $x: M \rightarrow X$ be an isometric immersion of $M$ into $X$. By means of the differential $dx$ we may consider $TM$ as a subbundle of the induced bundle $x^*TX$ over $M$, so that we get the orthogonal decomposition $x^*TX = TM \oplus NM$, where $NM$ denotes the normal bundle of $x$.

Let $\{e_1, e_2\}$ be an oriented orthonormal local frame on $M$. Let $\langle , \rangle$ denote the Riemannian metric of $X$ induced by the Kaehler metric $\sum \omega_a \bar{\omega}_a$ and $J$ the complex structure of $X$. The Kaehler function $\cos(\alpha)$ on $M$ is defined by

$$
\cos(\alpha) = \langle J e_1, e_2 \rangle,
$$

which is independent of the choice of oriented orthonormal frames on $M$. The immersion is said to be holomorphic if $\cos(\alpha) = 1$ on $M$, anti-holomorphic if $\cos(\alpha) = -1$ on $M$, and totally real if $\cos(\alpha) = 0$ on $M$.

Recall that in Sections 1 and 2 of [5], it was assumed that $x$ is neither holomorphic nor anti-holomorphic at a neighbourhood of any point of $M$. In this paper, we also assume the same conditions on $x$, and use the same formulas obtained in Sections 1 and 2 of [5]. Let $H$ be the mean curvature vector field of $x$, which is defined by

$$
H = \frac{1}{2} \sum h_{\alpha \bar{\alpha}} e_\alpha,
$$

where $h_{\alpha \bar{\alpha}}$'s are the components of the second fundamental form of $x$ (cf. Section 1 [5]), and $e_1$ and $e_2$ are the adapted frames along $x$. The immersion $x$ is called minimal if $H = 0$ on $M$. Let $D^\perp$ denote the connection of the normal bundle $NM$. If

$$
D^\perp H = 0
$$
on $M$, then $H$ is called the parallel mean curvature vector field.

We assume that $H \neq 0$, $D^H = 0$ on $M$, and the Kaehler function is $\cos(\alpha)$. We can construct a unique system of global orthonormal vector fields $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ along $M$ such that $\tilde{e}_1$ and $\tilde{e}_2$ are tangent to $M$ by the following: First we put $
abla = -H/\|H\|$, then the normal vector field $\tilde{e}_4$ of $N\!M$ is uniquely determined by choosing it to be compatible with the fixed orientations of $M$ and $X$. The system of vectors $\{\tilde{e}_5, \tilde{e}_4, \tilde{e}_3, \tilde{e}_2\}$ is linearly independent, because $x$ is neither holomorphic nor anti-holomorphic. We have the identity

$$\cos(\alpha) = \langle J\tilde{e}_4, \tilde{e}_3 \rangle$$

which is easily proved by using the fact that $\cos(\alpha)$ is independent of the choice of the oriented orthonormal frame on $M$. By using the Schmidt orthonormalization, we get a new frame $\{\tilde{e}_1, \tilde{e}_2\}$ on $M$, which is explicitly represented as follows

$$\tilde{e}_1 = \text{cot}(\alpha)\tilde{e}_3 - \text{cosec}(\alpha)J\tilde{e}_4,$$

$$\tilde{e}_2 = \text{cosec}(\alpha)J\tilde{e}_3 + \text{cot}(\alpha)\tilde{e}_4.$$ 

It is easy to see that $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is an adapted frame on $M$ in $X$, that is, $\tilde{e}_1$ and $\tilde{e}_2$ are sections on $T\!M$ and $\tilde{e}_3$ and $\tilde{e}_4$ are sections on $N\!M$. Moreover, we define vector fields $e_1$ and $e_2$ as follows:

$$e_1 = \frac{1}{2} \sec(\alpha) (\tilde{e}_1 - J\tilde{e}_2),$$

$$e_2 = \frac{1}{2} \text{cosec}(\alpha) (\tilde{e}_1 + J\tilde{e}_2),$$

and put

$$e_3 = Je_1$$

and

$$e_4 = Je_2.$$ 

Then $\{e_1, e_2, e_3, e_4\}$ is a $J$-canonical frame along $x$ (cf. Section 1 [5]). We extend $\{\tilde{e}_A\}$ and $\{e_4\}$ to a neighbourhood of $M$ in $X$, where $A, B$ and $C$ run from 1 through 4.

Let $\{\theta_A\}$ and $\{\theta_A\}$ be the dual coframes of $\{e_A\}$ and $\{e_4\}$ respectively. Let $\bar{\theta}_{AB}$ and $\theta_{AB}$ be the Riemannian connection forms with respect to the canonical 1-forms $\{\theta_A\}$ and $\{\theta_A\}$, respectively and put

$$\omega_a = \theta_{za-z\alpha},$$

$$\omega_a^b = \theta_{za-z\alpha-1-z\beta-1} + i \theta_{za-z\alpha-1-z\beta-1},$$

where $i = \sqrt{-1}$. Then we have the following relations (cf. [5]):

$$\bar{\theta}_{zz} = -i \left( \sin(\frac{\alpha}{2}) \omega_{11} - \cos(\frac{\alpha}{2}) \omega_{22} \right),$$

$$\bar{\theta}_{zz} = -i \left( \sin(\frac{\alpha}{2}) \omega_{11} + \cos(\frac{\alpha}{2}) \omega_{22} \right).$$

(2.2)
We denote the restriction of \( \{ \bar{\theta}_a \} \) to \( M \) by the same letters and put
\[
\phi = \bar{\theta}_1 + i\bar{\theta}_2.
\]
By the assumptions, \( \bar{\theta}_4 \) is a parallel vector field along \( M \), hence so is \( \bar{\theta}_4 \). This implies
\[
(2.3) \quad \bar{\theta}_4 = 0.
\]
Then it is proved that there exist a positive number \( b \), complex-valued smooth functions \( a \) and \( c \) defined locally on \( M \), which satisfy the followings (cf. (2.1) and (2.2) in \([5]\)):
\[
\begin{align*}
\bar{\theta}_{12} &= i \cot(\alpha) \{(a - b)^2 - (\bar{a} - b)^2\}, \\
da + d\bar{a} &= (a + b) \phi + (\bar{a} + b) \bar{\phi}, \\
(d(c + 3ic\bar{\theta}_{12}) \wedge \bar{\theta} = \cot(\alpha)(b - a)c\phi \wedge \bar{\phi}, \\
H &= -2b\bar{\theta}_3.
\end{align*}
\]
The third and fourth formulas of (2.4) are the Codazzi equations of \( x \).

Denoting by \( K \) the Gaussian curvature of \( M \), the Gauss equation is written as
\[
(2.5) \quad K = 6\rho \cos^2(\alpha) - 4(|a|^2 - b^2).
\]
Let \( K_N \) be the normal curvature of \( x \) defined by
\[
d\bar{\theta}_4 = -K_N \bar{\theta}_1 \wedge \bar{\theta}_2.
\]
By taking the exterior derivative of the second formula of (2.2) and using the formula (2.1) in \([5]\), we have
\[
K_N = (3 \cos^2(\alpha) - 1)\rho + 2(|c|^2 - |a|^2).
\]
Since now the normal curvature vanishes, we get
\[
(2.6) \quad |c|^2 = |a|^2 - \frac{\rho}{2}(3 \cos^2(\alpha) - 1).
\]
Combining formulas (2.5) and (2.6), we get
\[
(2.7) \quad K = (1 + 3 \cos^2(\alpha))\rho - 2(|a|^2 - 2b^2 + |c|^2).
\]
For a neighbourhood \( U \) of a point of \( M \), there exists an isothermal coordinate
\[ z = u + iv \]
such that \( ds^2 = \lambda^2 |dz|^2 \), where \( \lambda \) is a positive function defined on \( U \), and we have
\[ \phi = \lambda dz. \]

Then the set of the first three formulas of (2.4) is rewritten as the following system of differential equations:
\[ \begin{aligned}
\frac{\partial \lambda}{\partial z} &= -\lambda \cot(\alpha)(a-b), \\
\frac{\partial \alpha}{\partial z} &= \lambda(a+b), \\
\frac{\partial a}{\partial z} &= \lambda \left\{ 2 \cot(\alpha)(\bar{a} - b)a + \frac{3}{4} \rho \sin(2\alpha) \right\}.
\end{aligned} \]

(2.8)

By using (2.8), we have that
\[ \frac{\partial^2 \lambda}{\partial z \partial \bar{z}} = \frac{\partial^2 \lambda}{\partial \bar{z} \partial z} \quad \text{if and only if} \quad \bar{a} = a. \]

Therefore \( a \) is a real-valued function defined locally on \( M \). This implies that \( \lambda, \alpha \) and \( a \) are functions of single variable, and (2.8) is seen to be a system of ordinary differential equations. Consequently, if \( M \) is a non-minimal surface with parallel mean curvature in \( X \), then there exists a positive number \( b \) and real-valued smooth functions of single variable \( \lambda, \alpha \) and \( a \) which are defined locally on \( M \) and satisfy the system of ordinary differential equations (cf. 3.1).

\textbf{Remark.} The fourth formula of (2.4) is equivalent to the equation
\[ \frac{\partial (\lambda^4 c)}{\partial z} = 0. \]

(2.10)

In the next section, we shall consider a converse problem to the result obtained above, that is, a local existence problem for non-minimal surface in \( X \) with parallel mean curvature vector. To this end, we need the fundamental theorem of surfaces theory in \( X \). When \( X \) is a real space form, the fundamental theorem of submanifolds is well-known. On the other hand, for a surface in a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature \( 4\rho \), the following fundamental theorem is proved by Eschenburg et al. [2], which is in essential use in this paper:

\textbf{Theorem 2.1 ([2])}. Let \((M, ds^2)\) be a connected, simply connected 2-dimensional Riemannian manifold. Given complex-valued 1-forms \( \omega_1, \omega_2, \omega_1, \omega_2 \), and \( \omega_{12} \) defined on \( M \) satisfying the structure equations (2.1) and...
there exist an isometric immersion \( x: M \rightarrow X \) and a unitary frame \( \{ E_1, E_2 \} \) along \( x \) such that \( \{ \omega_1, \omega_2 \} \) is the unitary coframe of \( \{ E_1, E_2 \} \) and \( \omega_{11}, \omega_{22} \) and \( \omega_{12} \) are the unitary connection forms with respect to \( \{ \omega_1, \omega_2 \} \).

3. Local construction of surfaces in \( \mathbb{P}^2(\mathbb{C}) \)

It was B.Y. Chen who constructed surfaces with constant mean curvature in a 3-dimensional real space form (cf. [1], p. 121). In Theorem 3.11 of [2], Eschenburg et al. proved a local existence theorem for minimal surface in \( \mathbb{P}^2(\mathbb{C}) \). In this section, we consider a method of the local construction of a non-minimal surface with parallel mean curvature vector in a complex 2-dimensional Kaehler manifold.

**Theorem 3.1.** Let \( b \) and \( \rho \) be real numbers \((b>0)\), and \( \lambda, \alpha \) and \( a \) be real-valued smooth functions of single variable \( u \) defined on an interval \( I \), which satisfy the following system of ordinary differential equations:

\[
\begin{align*}
\frac{d\lambda}{du} &= -2\lambda^2 \cot(\alpha)(a-b), \\
\frac{d\alpha}{du} &= 2\lambda(a+b), \\
\frac{da}{du} &= 2\lambda \left( 2 \cot(\alpha)(a-b)a + \frac{3}{4} \rho \sin(2\alpha) \right).
\end{align*}
\]

Let \( M \) be an open domain of \((u, v)\)-plane contained in \( I \times (-1, 1) \). Define

\[
ds^2 = \lambda^2 (du^2 + dv^2)
\]

on \( M \). Suppose that for any constants \( k_1 \) and \( k_2 \), \( \lambda \), \( \alpha \) and \( a \) satisfy

\[
\log \left( \lambda^2 \left( a^2 - \frac{\rho}{2} (3 \cos^2(\alpha) - 1) \right) \right) = k_1 u + k_2.
\]

Then we can construct an isometric immersion \( x: M \rightarrow X \) of \( M \) into a complex 2-dimensional Kaehler manifold \( X \) which satisfies the following:

1. \( x \) has a non-zero parallel mean curvature vector field whose length is \( 2b \),
2. the Kaehler function of \( x \) is \( \cos(\alpha) \),
3. the second fundamental form of \( x \) is explicitly written in terms of \( a, b, \lambda \) and \( \alpha \).

**Proof.** Let \((r, s, t)\) be the standard coordinate of \( \mathbb{R}^3 \) and \( D \) a domain in \( \mathbb{R}^3 \) such that \( r>0 \) and \( 0<s<\pi \). We define a \( \mathbb{R}^3 \)-valued function \( f(r, s, t) \) on \( D \) by
SURFACES IN \(P^3(C)\)

\[
f(r, s, t) = \begin{pmatrix}
-r^2 \cot(s)(t-b) \\
r(t+b) \\
r{2 \cot(s)(t-b)t+3\rho \sin(2s)/4}
\end{pmatrix}.
\]

\(f(r, s, t)\) has continuous partial derivatives on \(D\), so that it satisfies Lipschitz condition on \(D\). Hence, a solution of the system (3.1) exists and is unique under preassigned initial conditions.

Let \((\lambda, \alpha, a)\) be a solution of (3.1) which satisfy (3.2) and we put 

\[
z = u + iv \quad \text{and} \quad \phi = \lambda dz.
\]

We define a complex-valued function \(c\) on \(M\) by

\[
(3.3) \quad c = \frac{\beta}{\lambda^2} \exp\left(\frac{k_1}{2}(u - iv)\right)
\]

where \(\beta\) is a complex constant. Then it is proved that \(\lambda^2c\) is anti-holomorphic, which is equivalent to (2.10) and \(|c|^2\) satisfies (2.6). We define \(\omega_1\), \(\omega_2\), \(\omega_{11}\), \(\omega_{12}\) and \(\omega_{13}\) on \(M\) as follows:

\[
\omega_1 = \cos\left(\frac{\alpha}{2}\right) \phi,
\]

\[
\omega_2 = \sin\left(\frac{\alpha}{2}\right) \phi,
\]

\[
(3.4) \quad \omega_{11} = \frac{1}{2} \cot\left(\frac{\alpha}{2}\right)\{(a - b)\phi - (b - a)\phi\},
\]

\[
\omega_{22} = \frac{1}{2} \tan\left(\frac{\alpha}{2}\right)\{(a - b)\phi - (b - a)\phi\},
\]

\[
\omega_{13} = -\omega_{11} = b\phi + c\phi.
\]

Note that these satisfy (2.1) because of (3.1). Therefore, by Theorem 2.1, we have an isometric immersion \(x : M \to X\) which has a non-zero, parallel mean curvature vector field and \(\cos(\alpha)\) the Kaehler function. The second fundamental form of \(x\) is explicitly written in terms of \(a, b, \lambda\) and \(\alpha\) by (2.2) of [5].

q. e. d.

4. Associated family of isometric immersion

It is well known that there exists a one-parameter family of isometric surfaces in \(\mathbb{R}^3\) with the same constant mean curvature. The following theorem shows that an analogous property holds in the case that the amiant space is a 2-dimensional complex space form \(X\) and that the mean curvature vector field \(H\) of an immersed surface is parallel. Note that Eschenburg et al. [2] have proved that there exists a one-parameter family of isometric minimal immersions of a simply connected surface into \(P^3(C)\) with the same normal curvature and
Kaehler function. Theorem 4.1 is an extension of Theorem B in [2] stated above.

**Theorem 4.1.** Let $(M, ds^2)$ be a simply connected oriented 2-dimensional Riemannian manifold, $x: M \to X$ an isometric immersion with non-zero parallel mean curvature vector field $H$ and $\cos(\alpha)$ the Kaehler function. Assume that the immersion $x$ is neither holomorphic nor anti-holomorphic. Then there exists a one-parameter family of isometric immersions $x_t: M \to X$, $t \in (-\pi, \pi)$, which satisfies the following properties:

1. $x_0 = x$,
2. $x_t$ is isometric to $x$ for each $t$,
3. $\|H_t\| = \|H\| \neq 0$, where $H_t$ denotes the mean curvature vector field of $x_t$,
4. $D^t H_t = 0$, where $D^t$ is the normal connection of $x_t$,
5. $\cos(\alpha_t) = \cos(\alpha)$,
6. $x_t$ is not congruent to each other.

**Proof.** By the assumptions, we can use results in Section 2. The first formula in (2.4) implies

$$a \phi \wedge \bar{\phi} = i \tan(\alpha) \tilde{\phi} \wedge b \phi \wedge \bar{\phi}.$$  

This shows that the real valued function $a$ is uniquely determined by the Riemannian metric $ds^2$, the mean curvature vector $H$ and $\cos(\alpha)$. By (2.6), $|c|^2$ is also uniquely determined by $ds^2$, $H$ and $\cos(\alpha)$. We put

$$c = |c| e^{\tau r}, \quad 0 \leq \tau < 2\pi$$

where $\tau$ is a real-valued function on $M$. Then (3.3) shows that $\tau$ is uniquely determined by $ds^2$, $H$ and $\cos(\alpha)$, up to additive constants. Hence, if we put

$$c_t = c e^{\tau t} \quad \text{for some } t \in (-\pi, \pi),$$

then $c_t$ also satisfies the fourth formula in (2.4). We put

$$\omega_{12} = -\bar{\omega}_{11} = b \phi + c_t \bar{\phi},$$

and the other connection forms are defined similarly as in (3.4). Then $\omega_1$, $\omega_2$, $\omega_{11}$, $\omega_{12}$ and $\omega_{13}$ satisfy (2.1) for each $t$. Hence, by Theorem 2.1, for each $t$ we have an isometric immersion $x_t: M \to X$ for which the adapted frame

$$\{\tilde{\phi}(t), \tilde{\phi}(t), \tilde{\phi}_3(t), \tilde{\phi}_4(t)\}$$

along $x_t$ satisfies

$$H_t = -2b \tilde{\phi}_3(t), \quad D^t H_t = 0,$$

and $\cos(\alpha)$ is the Kaehler function of $x_t$ for each $t$. q.e.d.

**Corollary 4.2.** Let $x_i: M \to X$ ($i=1, 2$) be an isometric immersion with non-zero, parallel mean curvature vector field $H$, and the Kaehler function $\cos(\alpha_i)$. 


Assume that \( x_1 \) are neither holomorphic nor anti-holomorphic, and that \( x_1 \) is isometric to \( x_2 \). Then \( x_1 \) is congruent to \( x_2 \) if and only if
\[
\cos(\alpha_1) = \cos(\alpha_2), \quad \|H_1\| = \|H_2\| \quad \text{and} \quad c_1 = c_2.
\]

5. Complete flat surface with parallel mean curvature vector

In this section we apply the results obtained in this paper for the case that \((M, ds^2)\) is a Riemannian manifold of constant Gaussian curvature. As a result, we determine all isometric immersions of the \((M, ds^2)\) into \( P^3(C) \) with parallel mean curvature vector field. We put \( \rho = 1 \) for simplicity.

Let \( M^3[K] \) denote an oriented connected 2-dimensional Riemannian manifold of constant Gaussian curvature \( K \) and \( x : M^3[K] \rightarrow P^3(C) \) be an isometric immersion whose mean curvature vector field \( H \) is parallel but non-vanishing. Differentiating (2.5) and using \( \overline{\alpha} = \alpha \), we have
\[
2\alpha \frac{da}{du} + 3 \cos(\alpha) \sin(\alpha) \frac{da}{du} = 0.
\]
Since the system (3.1) is valid for the immersion \( x \), the formulas (5.1) and (3.1) give
\[
\cos(\alpha) = 0
\]
or
\[
3 \sin^2(\alpha) = \frac{-4a^2(a-b)}{2a+b}.
\]
It follows from these formulas and the Gauss equation (2.5) that \( a \) is constant, \( \alpha = \pi/2 \) and hence \( K = 0 \). Note that we have \( k_1 = 0 \) in (3.2). In consequence, we obtain the following.

**Proposition 5.1.** Let \( M^3[K] \) be an oriented 2-dimensional Riemannian manifold of constant Gaussian curvature \( K \) and \( x : M^3[K] \rightarrow P^3(C) \) an isometric immersion such that the mean curvature vector field is parallel and not zero. Then \( x \) is totally real and \( K = 0 \).

Now we are going to determine isometric immersions with parallel mean curvature vector field of a complete flat surface into \( P^3(C) \). Let \( R^2 \) be the Euclidean 2-plane with the standard flat metric \( ds^2 = du^2 + dv^2 \). We put
\[
\phi = dz \quad \text{and} \quad z = u + iv.
\]
Let \( x : R^2 \rightarrow P^3(C) \) be an isometric immersion with non-zero parallel mean curvature vector field. It follows from Proposition 5.1 that the \( x \) must be totally real and \( \alpha = \pi/2 \). By (2.4), we have \( a = -b \). By (2.6) and (3.2), \( c \) is a complex constant with...
On account of Theorem 4.1, we may assume that $c$ is real. Therefore we have

$$|c|^2 = b^2 + \frac{1}{2}.$$  

We can solve the system (5.2) in the same way as in Kenmotsu [3, p.p. 679-681]: Let $\lambda_i$, $i=0, 1, 2$, be the eigenvalues of the matrix $A$ defined by

$$(5.3)$$

$$A = \begin{pmatrix}
0 & 1 & 0 \\
0 & b & c \\
\frac{1}{\sqrt{2}} & -b & b \\
\end{pmatrix}.$$  

It is easy to see that, if necessary renumbering $\lambda_i$, $\lambda_0$ is a non-zero real number which is not rational, $\lambda_i$ is a complex number which is not real and $\lambda_2$ is the complex conjugate of $\lambda_1$. Put

$$G = \{(\exp(\lambda_i z - \lambda_0 \bar{z}) \delta_{ij}) | z = u + iv, (u, v) \in \mathbb{R}^2 \}.$$  

Then $x(\mathbb{R}^3)$ is an orbit of the abelian Lie subgroup $G$ of the unitary group $U(3)$. We remark that $G$ is homeomorphic to the cylinder $S^1 \times \mathbb{R}^1$.

Summarizing our results of this section, we obtain the following.

THEOREM 5.2. Let $x: \mathbb{R}^3 \to P^2(\mathbb{R})$ be an isometric immersion with non-zero parallel mean curvature vector field $H$. Then $x(\mathbb{R}^3)$ is an orbit of the abelian Lie subgroup $G$ of $U(3)$ and $G$ is algebraically determined by the constant $b$, where $2b$ is the length of $H$.

It should be remarked that when $x$ is minimal and totally real, this theorem was proved in [4].

REFERENCES

SURFACES IN $P^3(C)$


DEPARTMENT OF MATHEMATICS
YAMAGATA UNIVERSITY
YAMAGATA 990
JAPAN