A MORSE INDEX THEOREM FOR GEODESICS ON A GLUED RIEMANNIAN SPACE

MASAKAZU TAKIGUCHI

Abstract

A glued Riemannian space is obtained from Riemannian manifolds $M_1$ and $M_2$ by identifying their isometric submanifolds $B_1$ and $B_2$. A curve on a glued Riemannian space which is a geodesic on each Riemannian manifold and satisfies certain passage law on the identified submanifold $B := B_1 \cong B_2$ is called a $B$-geodesic. Considering the variational problem with respect to arclength $L$ of piecewise smooth curves through $B$, a critical point of $L$ is a $B$-geodesic. A $B$-Jacobi field is a Jacobi field on each Riemannian manifold and satisfies certain passage condition on $B$. In this paper, we extend the Morse index theorem for geodesics in Riemannian manifolds to the case of a glued Riemannian space.

0. Introduction

In Riemannian manifolds, various results have been given on geodesics by many authors. Recently, N. Innami studied a geodesic reflecting at a boundary point of a Riemannian manifold with boundary in [5]. Let $M$ be a Riemannian manifold with boundary $B$ which is a union of smooth hypersurfaces. A curve on $M$ is said to be a reflecting geodesic if it is a geodesic except at reflecting points and satisfies the reflection law. He dealt with the index form, conjugate points and so on, as in the case of a usual geodesic. Moreover, in [6], he generalized these to the case of a glued Riemannian manifold which is a space obtained from Riemannian manifolds with boundary by identifying their isometric boundary hypersurfaces. Some collapsing Riemannian manifolds are considered to be a kind of glued Riemannian manifolds. In [10] the author gave the definition of a glued Riemannian space which is obtained from Riemannian manifolds by identifying their isometric submanifolds $B_1$ and $B_2$ and is a generalization of a glued Riemannian manifold. A curve on a glued Riemannian space which is a geodesic on each Riemannian manifold and satisfies certain passage law on the identified submanifold $B := B_1 \cong B_2$ was called a $B$-geodesic. Considering the variational problem with respect to arclength $L$ of piecewise smooth curves through $B$, a critical point of $L$ is a $B$-geodesic. Also, the definitions of the index form of $B$-geodesics, $B$-Jacobi fields and $B$-conjugate
points were given. A $B$-Jacobi field is a Jacobi field on each Riemannian manifold and satisfies certain passage condition on $B$. The purpose of this paper is to generalize the Morse index theorem for geodesics to the case of a glued Riemannian space. In Section 1, we review fundamental definitions, and results ([10]) on a glued Riemannian space. In Section 2, we give a precise statement of a Morse index theorem for $B$-geodesics, which relates the number of $B$-conjugate points on a $B$-geodesic $\gamma$, counted with their multiplicities, to the index of $\gamma$, and prove this theorem. Moreover, we make a comparison of the indices of $B$-geodesics in different glued Riemannian spaces, in Section 3.

The author would like to express his sincere gratitude to Professor N. Abe for suggesting this problem and his helpful advice.

1. Preliminaries

Let $N_\mu$ and $M_\lambda$ be manifolds (possibly with boundary) for $\mu = 1, \ldots, k$ and $\lambda = 1, \ldots, l$. We allow the case where $\dim N_\mu \neq \dim N_v$ and $\dim M_\kappa \neq \dim M_\lambda$ for $\mu \neq v$ and $\kappa \neq \lambda$. A map $\varphi : N \to \overline{M}$ from the topological direct sum $N := N_1 \amalg \cdots \amalg N_k$ to $\overline{M} := M_1 \amalg \cdots \amalg M_l$ is smooth if $\varphi|N_\mu$ is smooth. A tangent bundle $T\overline{M}$ of $\overline{M}$ is the direct sum $T\overline{M} = TM_1 \amalg \cdots \amalg TM_l$, where $TM_\lambda$ denotes the tangent bundle of $M_\lambda$. We note that a tangent bundle $T\overline{M}$ on $\overline{M}$ is not constant rank vector bundle on $\overline{M}$. We put $T_p\overline{M} := T_pM_\lambda$ for $p \in M_\lambda$. We define a map $\pi_{\overline{M}} : T\overline{M} \to \overline{M}$ by

$$\pi_{\overline{M}}(v_p) := p \quad \text{for } v_p \in T_pM_\lambda.$$ 

A vector field $\overline{V}$ on $\overline{M}$ is a map $\overline{V} : \overline{M} \to T\overline{M}$ such that $\pi_{\overline{M}} \circ \overline{V} = \text{id}_{\overline{M}}$, where $\text{id}_{\overline{M}}$ is the identity map on $\overline{M}$. If $\overline{V}|M_\lambda : M_\lambda \to TM_\lambda$ is smooth vector field on each $M_\lambda$, then $\overline{V}$ is smooth. Let $I_{1,\mu}$ be a closed interval in $R$ which is a manifold with boundary, for $\mu = 1, \ldots, k$. A map $\varphi : I_1 \amalg \cdots \amalg I_k \to \overline{M}$ is called a curve on $\overline{M}$ if $\varphi$ is smooth.

Let $M_\lambda$ be a manifold (possibly with boundary) with a submanifold $B_\lambda$ for $\lambda = 1, 2$ and $\psi$ a diffeomorphism from $B_1$ to $B_2$. A glued space $M = M_1 \cup_\psi M_2$ is defined as follows: $M$ is the quotient topological space obtained from the topological direct sum $\overline{M} = M_1 \amalg M_2$ of $M_1$ and $M_2$ by identifying $p \in B_1$ with $\psi(p) \in B_2$. We allow the case where $B_1 = B_2 = \emptyset$, $M_1 = \emptyset$ or $M_2 = \emptyset$, where $\psi$ is the empty map. Let $\pi : \overline{M} \to M$ be the natural projection which is defined by $\pi(p) = [p]$, where $[p]$ is the equivalence class of $p$. Let $N_{\lambda}$ be a manifold with a submanifold $C_\lambda$ ($\lambda = 1, 2$), $\tau : C_1 \to C_2$ a diffeomorphism and $N = N_1 \cup_\tau N_2$ a glued space. A glued smooth map $\varphi : \overline{N} \to \overline{M}$ on $\overline{N}$ derived from a smooth map $\bar{\varphi} : \overline{N} \to \overline{M}$ or, simply, a smooth map on $N$ is defined by $\varphi = \pi \circ \bar{\varphi}$. We note that a glued smooth map on $\overline{N}$ is considered as a map on $N$ which, possibly, take two values at $[p]$ ($p \in C_2$). A glued smooth map $\varphi$ is continuous if $\varphi(p) = \varphi(\tau(p))$ holds for any $p \in C_1$.

A glued tangent bundle $TM$ of $M$ is the glued space $TM_1 \cup_\psi TM_2$, where $\psi_* : TB_1 \to TB_2$ is the differential map of $\psi$. Let $\hat{\pi} : T\overline{M} \to TM$ be the natural projection which is defined by $\hat{\pi}(v) = [v]$, where $[v]$ is the equivalence class of $v$. 

For $p \in \overline{M}$, we set $T_p \overline{M} := \{ \hat{\pi}(T_p \overline{M}) = [v] \in T\overline{M} | v \in T_p \overline{M} \}$. We define a map $\pi_M : T\overline{M} \to M$ by

$$\pi_M([v]) := [p] \quad \text{for } v \in T_p \overline{M}.$$  

We note that $\pi \circ \pi_M = \pi_M \circ \hat{\pi}$ holds. A glued vector field $V : \overline{M} \to T\overline{M}$ derived from a vector field $\overline{V}$ on $\overline{M}$ or, simply, a vector field on $M$ is defined by $V = \hat{\pi} \circ \overline{V}$. A glued vector field $V$ is called a smooth glued vector field provided $V$ is glued smooth. If a glued vector field $V$ on $\overline{M}$ is continuous, then we can regard it as a cross section of $T\overline{M}$ over $M$; that is $\pi_M \circ V = \text{id}_M$. Similarly, we can define a glued vector field (or vector field) along a curve $\overline{\alpha} : I := I_1 \bigcup I_2 \to \overline{M}$.

Let $T_p^* \overline{M}$ be the dual vector space of $T_p \overline{M}$. We put $T^* \overline{M} = T^* M_1 \bigcup T^* M_2$, where $T^* M_j$ is the cotangent bundle of $M_j$. For $\tilde{\partial}_p \in T_p^* \overline{M}$, $\tilde{\omega}_q \in T_q^* \overline{M}$, we define an equivalence relation $\sim$ as follows: $\tilde{\partial}_p \sim \tilde{\omega}_q$ if and only if $\tilde{\partial}_p|_{T_p B_1} = \psi^*(\tilde{\omega}_q)$ $(p \in B_1, q = \psi(p))$ or $\tilde{\omega}_q|_{T_q B_1} = \psi^*(\tilde{\partial}_p)$ $(q \in B_1, p = \psi(q))$, where $\psi^*$ is the dual map of $\psi$. The quotient space obtained from $T^* \overline{M}$ by this equivalence relation is denoted by $T^* \overline{M}$. Let $\hat{\pi} : T^* \overline{M} \to T^* M$ be the natural projection, that is, $\hat{\pi}(\tilde{\partial}) := \overline{\partial}$, where $\overline{\partial}$ is the equivalence class of $\tilde{\partial}$. For $p \in \overline{M}$, we set $T^*_p \overline{M} := \hat{\pi}(T^*_p \overline{M})$ and define a map $[\overline{\partial}] : T^*_p \overline{M} \to \mathbb{R}$ by $[\overline{\partial}](v) := \overline{\partial}(v)$ for $\overline{\partial} \in T^*_p \overline{M}$ and $v \in T_p \overline{M}$. Then we can regard $T^*_p \overline{M}$ as the dual of $T_p \overline{M}$. We put $T^{r,s}(\overline{M}) := T^{r,s}(M_1) \bigcup T^{r,s}(M_2)$, where $T^{r,s}(M_j)$ is the $(r,s)$-tensor bundle of $M_j$. A $(r,s)$-tensor field on $\overline{M}$ is a cross section of $T^{r,s}(\overline{M})$. The definition of the smoothness of a tensor field on $\overline{M}$ is similar to that of a vector field on $\overline{M}$. Similarly, we can define the equivalence relation on $T^{r,s}(\overline{M})$ induced from those on $T\overline{M}$ and $T^* \overline{M}$, and denote the quotient space by $T^{r,s}(\overline{M})$. Let $\hat{\pi} : T^{r,s}(\overline{M}) \to T^{r,s}(M)$ be the natural projection. A glued tensor field $T$ derived from a tensor field $\overline{T}$ on $\overline{M}$ is (glued) smooth if $T$ is smooth.

**Definition 1.1.** Let $(M_\lambda, g_\lambda)$ be a Riemannian manifold with a Riemannian submanifold $B_\lambda$ for $\lambda = 1, 2$ and $\psi$ an isometry from $B_1$ to $B_2$. Let $\overline{g}$ be the metric on $\overline{M}$ which is defined to be $\overline{g}_p = (g_\lambda)_p$ for $p \in M_j$. A glued Riemannian space $(M, g) = (M_1, g_1) \cup \psi(M_2, g_2)$ is a pair of a glued space $M = M_1 \cup \psi M_2$ and a glued metric $g$ on $\overline{M}$ derived from $\overline{g}$ which is a glued tensor field derived from the $(0,2)$-tensor field $\overline{g}$.

We note that, for any glued smooth vector fields $V$ and $W$ on $\overline{M}$ derived from smooth vector fields $\overline{V}$ and $\overline{W}$ on $\overline{M}$, respectively, a map $g(V, W) : \overline{M} \to \mathbb{R}$ defined by

$$g(V, W)(p) := \overline{g}(\overline{V}_p, \overline{W}_p)$$  

is glued smooth on $\overline{M}$ derived from a smooth map $\overline{g}(\overline{V}, \overline{W}) : \overline{M} \to \mathbb{R}$.

From now on, identifying $B_1$ with $B_2$ by $\psi$, we put $B := B_1 \cong B_2$ and $T_p B := T_p B_1 \cong T_p B_2$ for $p \in B$ and omit the symbol $[\cdot]$ of the equivalence
class. In particular, \([M_2] := \pi(M_2)\) will be denoted by \(M_2\). We call a map \(\alpha: [a, t_0] \bigoplus [t_0, b] \to M\) a glued curve derived from a curve \(\bar{\alpha}: [a, t_0] \bigoplus [t_0, b] \to \overline{M}\) or, simply, a curve on \(M\) if \(\alpha: [a, t_0] \bigoplus [t_0, b] \to M\) is a continuous glued smooth map derived from \(\bar{\alpha}\). Let \(\alpha: [a, t_0] \bigoplus [t_0, b] \to M\) be a glued curve derived from a curve \(\bar{\alpha}: [a, t_0] \bigoplus [t_0, b] \to \overline{M}\). The (glued) velocity vector field of \(\alpha\) is \(\alpha' := \pi \circ \bar{\alpha}'\). We put \(\alpha'(t_0 - 0) := \pi \circ \bar{\alpha}'_1(t_0)\) and \(\alpha'(t_0 + 0) := \pi \circ \bar{\alpha}'_2(t_0)\), where \(\bar{\alpha}_1 := \bar{\alpha}| [a, t_0] : [a, t_0] \to \overline{M}\) and \(\bar{\alpha}_2 := \bar{\alpha}| [t_0, b] : [t_0, b] \to \overline{M}\). We note that a glued velocity vector field is considered as a glued vector field along \(\bar{\alpha}\) and not generally continuous. We call \(\alpha: [a, b] \to M\) a piecewise smooth curve on \(M\) provided there is a partition \(a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b\) of \([a, b]\) such that \(\alpha| [a_{i-1}, a_{i+1}] : [a_{i-1}, a_i] \bigoplus [a_i, a_{i+1}] \to M\) is a glued curve. We call \(a_j\) \((j = 1, \ldots, k)\) the break.

A function \(\lambda: [a, t_0] \bigoplus [t_0, b] \to \{1, 2\}\) is defined by

\[
\lambda(t) := \begin{cases} 
1 & \text{on } [a, t_0] \\
2 & \text{on } [t_0, b].
\end{cases}
\]

For simplicity, we put \(\lambda := \lambda(t)\).

If \(M\) is a glued Riemannian space such that \((M, g) = (M_1, g_1) \cup_{\psi} (M_2, g_2)\), then, for \(t_0 \in (a, b)\), let \(\Omega_{t_0}(M_1, M_2; B) =: \Omega_{t_0}\) be the set of all piecewise smooth curves \(\alpha: [a, b] \to M\) such that \(\alpha(t_0) \in B\), \(\alpha([a, t_0]) \subset M_1\) and \(\alpha([t_0, b]) \subset M_2\). Moreover, if \(p\) and \(q\) are points of \(M_1\) and \(M_2\), respectively. Then let \(\Omega_{t_0}(p, q) \subset \Omega_{t_0}\) be the set of all piecewise smooth curves \(\alpha \in \Omega_{t_0}\) such that \(\alpha(a) = p\) and \(\alpha(b) = q\). The projection from \(T_p M_2\) to \(T_p B\) is denoted by \(\tan\). Let \(D^\lambda\) be Levi-Civita connection of Riemannian manifold \(M_2\) for \(\lambda = 1, 2\). A curve \(\gamma \in \Omega_{t_0}\) is a \textit{B-geodesic} if \(\gamma\) satisfies the following conditions:

\[
D^\lambda_{\gamma'} \gamma' = 0 \quad \text{on } M_2, 
\]

that is, \(\gamma| [a, t_0]\) and \(\gamma| [t_0, b]\) are geodesics on \(M_1\) and \(M_2\), respectively,

\[
\tan \gamma'(t_0 - 0) = \tan \gamma'(t_0 + 0), \tag{1.2}
\]

\[
g_1(\gamma'(t_0 - 0), \gamma'(t_0 - 0)) = g_2(\gamma'(t_0 + 0), \gamma'(t_0 + 0)). \tag{1.3}
\]

We assume that geodesics and \(B\)-geodesics are parametrized by arclength.

Let \(q \in B\), \(u \in T_q M_1\) and \(v \in T_q M_2\) with \(\|u\|_1 = \|v\|_2\), \(\tan u = \tan v\) and \(v \notin T_q B\). We define a linear map \(Q_{u, v}: T_q B \oplus \text{Span}\{\text{nor}_1 u\} \to T_q B \oplus \text{Span}\{\text{nor}_2 v\}\) as

\[
Q_{u, v}(w) = \left\{ w - \frac{g_1(w, \text{nor}_1 u)}{g_1(u, \text{nor}_1 u)} \text{nor}_1 u + \frac{g_1(w, \text{nor}_1 u)}{g_1(u, \text{nor}_1 u)} \text{nor}_2 v \right\}
\]

for any \(w \in T_q B \oplus \text{Span}\{\text{nor}_1 u\}\), where \(\text{nor}_\lambda: T_q M_\lambda \to T_q B^\perp\) is the projection. The following hold:

\[
Q_{u, v}(x) = x \quad \text{for any } x \in T_q B.
\]

\[
Q_{u, v}(\text{nor}_1 u) = \text{nor}_2 v.
\]

\[
g_2(Q_{u, v}(w), x) = g_1(w, x)
\]
for any \( x \in T_q B \) and \( w \in T_q B \oplus \text{Span}\{\text{nor}_1 u\} \).
\[
g_2(Q_{u,v}(w), Q_{u,v}(w)) = g_1(w,w)
\]
for any \( w \in T_q B \oplus \text{Span}\{\text{nor}_1 u\} \). Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)} B \). Then we have
\[
Q_{\gamma'(t_0-0), \gamma'(t_0+0)}(\gamma'(t_0)) = \gamma'(t_0 + 0).
\]

**Remark.** Let \( q \in B, u \in T_q M_1 \) and \( v \in T_q M_2 \) with \( \|u\|_1 = \|v\|_2 \), \( \tan u = \tan v \) and \( v \notin T_q B \). If we define a linear map \( Q_{v,u} : T_q B \oplus \text{Span}\{\text{nor}_2 v\} \to T_q B \oplus \text{Span}\{\text{nor}_1 u\} \) as
\[
Q_{v,u}(z) = \left\{ \frac{z - g_2(z, \text{nor}_2 v)}{g_2(v, \text{nor}_2 v)} \text{nor}_2 v \right\} + \frac{g_2(z, \text{nor}_2 v)}{g_2(v, \text{nor}_2 v)} \text{nor}_1 u
\]
for any \( z \in T_q B \oplus \text{Span}\{\text{nor}_2 v\} \). The following hold:
\[
Q_{v,u} \circ Q_{v,u} = \text{id}, \quad Q_{v,u} \circ Q_{u,v} = \text{id},
\]
\[
g_2(Q_{v,u}(w), z) = g_1(w, Q_{v,u}(z))
\]
for \( w \in T_q B \oplus \text{Span}\{\text{nor}_1 u\} \) and \( z \in T_q B \oplus \text{Span}\{\text{nor}_2 v\} \).

If \( \gamma \in \Omega_{t_0}(p,q) \) is a \( B \)-geodesic with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)} B \), the set \( T_{\gamma} \Omega_{t_0} \) consists of all vector fields \( Y \) along \( \gamma \) which satisfy the following condition:
\[
Q_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0 - 0)) = Y(t_0 + 0). \tag{1.4}
\]
A subspace \( T_{\gamma} \Omega_{t_0}(p,q) \) in \( T_{\gamma} \Omega_{t_0} \) is defined by
\[
T_{\gamma} \Omega_{t_0}(p,q) := \{ \, Y \in T_{\gamma} \Omega_{t_0} \mid Y(a) = 0, Y(b) = 0 \}. \]

For \( \lambda = 1, 2 \), let \( R^\lambda \) be the Riemannian curvature tensor of a Riemannian manifold \( M_\lambda \), defined as
\[
R^\lambda(X, Y) W := D^\lambda_Y D^\lambda_X W - D^\lambda_X D^\lambda_Y W - D^\lambda_{[X,Y]} W,
\]
for any vector field \( X, Y \) and \( W \) on \( M_\lambda \), and \( S^\lambda_Z \) the shape operator of \( B \subset M_\lambda \) defined as
\[
S^\lambda_Z(V) := \ominus \tan D^\lambda_Y Z,
\]
for any vector field \( V \) tangent to \( B \) and \( Z \) normal to \( B \). Especially, if \( B = \{p\} \), we have that \( S^1_Z = 0 \) for \( Z \in T_p M_2 \). A vector field \( Y \) along a piecewise smooth curve \( \alpha \in \Omega_{t_0} \) is a tangent to \( \alpha \) if \( Y = f \alpha' \) for some function \( f \) on \([a,b]\) and perpendicular to \( \alpha \) if \( g_\lambda(Y, \alpha') = 0 \). If \( \|\alpha'\|_\lambda \neq 0 \), then each tangent space \( T_{\alpha(t)} M_\lambda \) has a direct sum decomposition \( \text{Span}\{\alpha'(t)\} \perp \{\alpha'(t)\} \perp \). Hence each vector field \( Y \) along \( \alpha \) has a unique expression \( Y = Y^T + Y^\perp \), where \( Y^T \) is tangent to \( \alpha \) and \( Y^\perp \) is perpendicular to \( \alpha \), that is,
\[
Y^\perp = Y - \frac{g_\lambda(Y, \alpha')}{g_\lambda(\alpha', \alpha')} \alpha'.
\]
If \( \alpha \) is a \( B \)-geodesic, then \((Y^T)') = (Y')^T\) and \((Y^\perp)') = (Y^\perp)'\).
Let \( q \in B \) and \( v \in T_qM_{\lambda} (\lambda = 1, 2) \) is not tangent to \( B \). A linear operator \( P^v_{\lambda} : T_qB \oplus \text{Span}\{\text{nor}_\lambda v\} \rightarrow T_qB \) is defined by

\[
P^v_{\lambda}(w) := w - \frac{g_\lambda(w, \text{nor}_\lambda v)}{g_\lambda(v, \text{nor}_\lambda v)} \text{nor}_\lambda v
\]

for any \( w \in T_qB \oplus \text{Span}\{\text{nor}_\lambda v\} \) (\( \subset T_qM_{\lambda} \)). We note that \( P^v_{\lambda} \) is surjective and \( P^v_{\lambda}(v) = 0 \).

Let \( q \in B, u \in T_qM_1 \) and \( v \in T_qM_2 \) with \( \|u\| = \|v\|_2 \), \( \tan u = \tan v \) and \( v \notin T_qB \). We define a symmetric linear map \( A_{u,v} : T_qB \oplus \text{Span}\{\text{nor}_2 v\} \rightarrow T_qB \oplus \text{Span}\{\text{nor}_2 v\} \) as

\[
A_{u,v}(w) = (S^1_{\text{nor}_1 u} - S^2_{\text{nor}_2 v})(P^v_{\lambda}(w)) - \frac{g_2((S^1_{\text{nor}_1 u} - S^2_{\text{nor}_2 v})(P^v_{\lambda}(w)), v)}{g_2(v, \text{nor}_2 v)} \text{nor}_2 v
\]

for any \( w \in T_qB \oplus \text{Span}\{\text{nor}_2 v\} \). We call this map \( A_{u,v} \) a passage endomorphism. The following hold:

\[
A_{u,v}(w) \perp v \quad \text{and} \quad A_{u,v}(v) = 0.
\]

The index form \( I_\gamma : T_\gamma\Omega_0 \times T_\gamma\Omega_0 \rightarrow \mathbb{R} \) of a \( B \)-geodesic \( \gamma \in \Omega_0 \) with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B \) is the symmetric bilinear form defined as

\[
I_\gamma(Y, W) = \int_{t_0}^{t_1} \{g_1(Y_{-t'}, W_{-t'}) - g_1(R^1(Y, \gamma')\gamma', W)\} dt
\]

\[
+ \int_{t_0}^{t_1} \{g_2(Y_{-t'}, W_{-t'}) - g_2(R^2(Y, \gamma')\gamma', W)\} dt
\]

\[
+ g_2(A_{\gamma'(t_0 - 0), \gamma'(t_0 + 0)}(Y(t_0), W(t_0 + 0))
\]

for all \( Y, W \in T_\gamma\Omega_0 \). It follows that

\[
I_\gamma(Y, W) = I_\gamma(Y_{\perp}, W_{\perp}) \quad \text{for all} \quad Y, W \in T_\gamma\Omega_0.
\]

Thus there is no loss of information in restricting the index form \( I_\gamma \) to

\[
T_{\gamma_{\perp}}\Omega_0 := \{Y \in T_\gamma\Omega_0 \mid Y \perp \gamma'\}.
\]

We write \( I_{\gamma_{\perp}} \) for this restriction. For \( \gamma \in \Omega_0(p, q) \), we put

\[
T_{\gamma_{\perp}}\Omega_0(p, q) := \{Y \in T_\gamma\Omega_0(p, q) \mid Y \perp \gamma'\}
\]

and write \( I_{\gamma_{\perp}} \) for the restriction of the index form \( I_\gamma \) to this.

Let \( \text{pr}_1 : T_{\gamma(t_0)}M_1 \rightarrow T_{\gamma(t_0)}B \oplus \text{Span}\{\text{nor}_1 \gamma'(t_0 - 0)\} \) and \( \text{pr}_2 : T_{\gamma(t_0)}M_2 \rightarrow T_{\gamma(t_0)}B \oplus \text{Span}\{\text{nor}_2 \gamma'(t_0 + 0)\} \) be orthogonal projections. For proofs of Lemmas without the proof in this section we refer the reader to [10]. The following holds:

**Lemma 1.2.** Let \( \gamma \in \Omega_0(p, q) \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B \). If \( Y \) and \( W \in T_\gamma\Omega_0(p, q) \) have breaks \( a_1 < \cdots < t_0 = a_j < \cdots < a_k \), then we have that
\[ I_\gamma(Y, W) = - \left\{ \int_a^b g_1(Y^{\perp''} + R^1(Y, \gamma')\gamma', W^\perp) \, dt + \int_{t_0}^b g_2(Y^{\perp''} + R^2(Y, \gamma')\gamma', W^\perp) \, dt \right\} + g_2(A_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0 + 0)), W(t_0 + 0)) + g_1(\text{pr}_1(Y^{\perp'}(t_0 - 0)), W^\perp(t_0 - 0)) - g_2(\text{pr}_2(Y^{\perp'}(t_0 + 0)), W^\perp(t_0 + 0)) + \sum_{i=1}^{j-1} g_1(Y^{\perp'}(a_i - 0) - Y^{\perp'}(a_i + 0), W^\perp(a_i)) + \sum_{i=j+1}^k g_2(Y^{\perp'}(a_i - 0) - Y^{\perp'}(a_i + 0), W^\perp(a_i)) + g_2(Y^{\perp'}(b), W^\perp(b)) - g_1(Y^{\perp'}(a), W^\perp(a)). \]

Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic. If it holds \( a \leq t_1 < t_2 \leq t_0 \), we set \( T_{\gamma|[t_1, t_2]} = \{ Y \mid \text{vector fields along } \gamma \mid [t_1, t_2] \} \). Then we define the map \( \tilde{I}_{\gamma|[t_1, t_2]} : T_{\gamma|[t_1, t_2]} \times T_{\gamma|[t_1, t_2]} \rightarrow \mathbb{R} \) by

\[ \tilde{I}_{\gamma|[t_1, t_2]}(Y, W) = \int_{t_1}^{t_2} \{ g_1(Y^{\perp'}, W^\perp') - g_1(R^1(Y, \gamma')\gamma', W) \} \, dt, \]

for all \( Y, W \in T_{\gamma|[t_1, t_2]} \Omega \). If it holds \( t_0 < t_1 < t_2 \leq b \), we set \( T_{\gamma|[t_1, t_2]} = \{ Y \mid \text{vector fields along } \gamma \mid [t_1, t_2] \} \). Then we define the map \( \tilde{I}_{\gamma|[t_1, t_2]} : T_{\gamma|[t_1, t_2]} \Omega \times T_{\gamma|[t_1, t_2]} \Omega \rightarrow \mathbb{R} \) by

\[ \tilde{I}_{\gamma|[t_1, t_2]}(Y, W) = \int_{t_1}^{t_2} \{ g_2(Y^{\perp'}, W^\perp') - g_2(R^2(Y, \gamma')\gamma', W) \} \, dt, \]

for all \( Y, W \in T_{\gamma|[t_1, t_2]} \Omega \).

Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B \). If \( Y \in T_{\gamma} \Omega_{t_0} \) satisfies

\[ Y'' + R^\lambda(Y, \gamma')\gamma'' = 0 \quad \text{on } M_\lambda \quad (\lambda = 1, 2), \]

(1.5)

\[ -A_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0 + 0)) = Q_{\gamma'(t_0-0), \gamma'(t_0+0)}(\text{pr}_1(Y'(t_0 - 0))) - \text{pr}_2(Y'(t_0 + 0)), \]

(1.6)

and

\[ g_1(Y'(t_0 - 0), \gamma'(t_0 - 0)) = g_2(Y'(t_0 + 0), \gamma'(t_0 + 0)), \]

(1.7)

then \( Y \) is called a \( B \)-Jacobi field along \( \gamma \). Let \( \mathcal{J}_\gamma \) be the set of all \( B \)-Jacobi fields along \( \gamma \). A \( B \)-Jacobi field \( Y \) along \( \gamma \) is perpendicular if \( Y \) is perpendicular to \( \gamma \). Let \( \mathcal{J}_\gamma^\perp \) be the set of all perpendicular \( B \)-Jacobi fields along \( \gamma \). Let \( \mathcal{J}_\gamma^0 \) be the set of all \( B \)-Jacobi field \( Y \in \mathcal{J}_\gamma \) such that \( Y(a) = 0 \).

If \( Y \) is a \( B \)-Jacobi field along \( \gamma \), then we have that

\[ I_\gamma(Y, Y) = g_2(Y^{\perp'}(b), Y^\perp(b)) - g_1(Y^{\perp'}(a), Y^\perp(a)). \]

(1.8)
Lemma 1.3. Let \( \gamma \in \Omega_{t_0}(p, q) \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B \). Then \( Y \in T_{t_0}^\perp \Omega_{t_0}(p, q) \) is an element of the nullspace of \( J_{t_0}^0 \) if and only if \( Y \) is a \( B \)-Jacobi field along \( \gamma \).

Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B \). We say that \( \gamma(t_2) \) \((t_2 \in (a, b))\) is a \( B \)-conjugate point to \( \gamma(t_1) \) \((t_1 \in [a, b], t_1 < t_2)\) along \( \gamma \) if there exists a \( B \)-Jacobi field \( Y \) along \( \gamma \) such that \( Y(t_1) = 0, \ Y(t_2) = 0 \) and \( Y|_{[t_1, t_2]} \) is nontrivial.

\( B \)-conjugate points in \( M_1 \) are always usual ones but the converse is not true in general. We give an example which shows this:

Example 1. Let \( M = M_1 \cup_{id} M_2 \) be a glued Riemannian space which consists of the following \( M_\lambda \) and \( B \) a submanifold of \( M_\lambda \) (\( \lambda = 1, 2 \)):

\( M_1 = S^2(1) = \{(x, y, z)|x^2 + y^2 + z^2 = 1\}, \ M_2 = \mathbb{E}^3, \ B = \{(0, -1, 0)\}, \)

and \( g_1 \) is a Riemannian metric induced from the natural Euclidean metric of \( \mathbb{E}^3 \) and \( g_2 \) is the natural Euclidean metric of \( \mathbb{E}^3 \). We defined a \( B \)-geodesic \( \gamma: [-\pi/2, +\infty) \to M \) by

\[
\gamma(t) = \begin{cases} 
(0, \cos t, \sin t) & \text{on } [-\pi/2, \pi] \\
(0, -t + \pi - 1, 0) & \text{on } [\pi, +\infty) 
\end{cases}.
\]

Then, \( T_{t_0}^\perp \Omega_{t_0} \) is the set of all vector fields \( Y \) along \( \gamma \) such that \( Y|_{[a, t_0]} \) and \( Y|_{[t_0, b]} \) are piecewise smooth vector fields on \( M_1 \) and \( M_2 \), respectively, and, \( Y(t_0 - 0) = d\gamma'(t_0 - 0) \) and \( Y(t_0 + 0) = d\gamma'(t_0 + 0) \) for some \( d \in \mathbb{R} \). Hence, \( \gamma(\pi/2) \) is a conjugate point to \( \gamma(-\pi/2) \) but not a \( B \)-conjugate point.

We define the function \( \rho_K : [a, b] \to \mathbb{R} \) and \( f_K : [a, b] \to \mathbb{R} \) by

\[
\rho_K(t) = \begin{cases} 
t & \text{if } K = 0 \\
\frac{1}{\sqrt{K}} \tan \sqrt{K}t & \text{if } K > 0 \\
\frac{1}{\sqrt{-K}} \tanh \sqrt{-K}t & \text{if } K < 0
\end{cases}
\]

and

\[
f_K(t) = \begin{cases} 
t & \text{if } K = 0 \\
\frac{1}{\sqrt{K}} \sin \sqrt{K}t & \text{if } K > 0 \\
\frac{1}{\sqrt{-K}} \sinh \sqrt{-K}t & \text{if } K < 0
\end{cases},
\]

respectively.

Lemma 1.4. Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B \). Then there are \( \tilde{a} \) and \( \tilde{b} \) \((a \leq \tilde{a} < t_0 < \tilde{b} \leq b)\) such that \( \gamma(t) \) is not a conjugate point to \( \gamma(\tilde{a}) \) for any \( t \in (\tilde{a}, t_0] \) and \( \gamma(t) \) is not a \( B \)-conjugate point to \( \gamma(\tilde{a}) \) for any \( t \in (t_0, \tilde{b}] \).

To show this lemma it is necessary to use the following proposition:
Proposition ([11]). Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \neq T_{\gamma(t_0)} B \). Let \( K_1 \) be any real number such that \( f_{K_1}(t-a) > 0 \) for any \( t \in (a, t_0] \). Let \( \delta \) be any real number. We assume that \( K_2 := K_1 \) if \( \delta = 0 \) and \( K_2 \) is any real number if \( \delta \neq 0 \). Let \( b_1(> t_0) \) be the smallest value which satisfies
\[
\delta = \frac{-1}{\rho_{K_1}(t_0 - a)} + \frac{-1}{\rho_{K_2}(t - t_0)},
\]
and \( b_2(> t_0) \) the smallest value which satisfies \( f_{K_2}(t - t_0) = 0 \), where \( b_i := \infty \) (\( i = 1, 2 \)) if there are no such \( b_i \). Moreover, we put \( \tilde{b} := \min \{b, b_1, b_2\} \). Assume that \( \dim B > 0 \),
\[
(\text{the maximal eigenvalue of } R_i^\gamma) \leq K_i \quad \text{for any } t \in [a, b]
\]
and
\[
(\text{the minimal eigenvalue of } A) \geq \delta.
\]
Then there are no conjugate points along \( \gamma|\{a, t_0\} \) and no \( B \)-conjugate points along \( \gamma|\{a, \tilde{b}\} \) to \( \gamma(a) \).

Proof of Lemma 1.4. In case where \( \dim B = 0 \), the assertion is trivial. We assume that \( \dim B > 0 \). Choose a real number \( K \) and \( \delta \) such that
\[
(\text{the maximal eigenvalue of } R_i^\gamma) \leq K \quad \text{for any } t \in [a, b]
\]
and
\[
(\text{the minimal eigenvalue of } A) \geq \delta.
\]
Moreover, choose \( \tilde{a} \) (\( a < \tilde{a} < t_0 \)) such that
\[
f_K(t - \tilde{a}) > 0 \quad \text{for any } t \in (\tilde{a}, t_0].
\]
Let \( b_1(> t_0) \) be the smallest value which satisfies
\[
\delta = \frac{-1}{\rho_K(t_0 - \tilde{a})} + \frac{-1}{\rho_K(t - t_0)},
\]
and \( b_2(> t_0) \) the smallest value which satisfies \( f_K(t - t_0) = 0 \), where \( b_i := \infty \) (\( i = 1, 2 \)) if there are no such \( b_i \). Moreover, we put \( b_0 := \min \{b, b_1, b_2\} \). Then, by taking \( \tilde{b} \) as \( t_0 < \tilde{b} < b_0 \) the assertion holds from the above proposition. \( \square \)

Lemma 1.5. Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \neq T_{\gamma(t_0)} B \). We assume that \( \gamma(t_0) \) and \( \gamma(b) \) are not \( B \)-conjugate points to \( \gamma(a) \). Then, for any \( v_1 \in T_{\gamma(a)} M_1 \) and \( v_2 \in T_{\gamma(b)} M_2 \), there is a unique \( Y \in \mathcal{J}_\gamma \) with \( Y(a) = v_1 \) and \( Y(b) = v_2 \).

Lemma 1.6. Let \( \gamma \in \Omega_{t_0} \) be a \( B \)-geodesic with \( \gamma'(t_0 + 0) \neq T_{\gamma(t_0)} B \). If \( \gamma(t) \) is not a conjugate point to \( \gamma(a) \) for any \( t \in (a, t_0] \) and \( \gamma(t) \) is not a \( B \)-conjugate point
to $\gamma(a)$ for any $t \in (t_0, b]$, then, for any $Y \in T_{\gamma(t_0)}\Omega$ with $Y(a) = 0$, there exist a unique $B$-Jacobi field $J \in J^0_\gamma$ such that $J(b) = Y(b)$ and

$$I_\gamma(J, J) \leq I_\gamma(Y, Y).$$

In particular, the equality holds if and only if $J^\perp = Y^\perp$.

**Lemma 1.7.** Let $\gamma \in \Omega_{t_0}$ be a $B$-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. If $\gamma(t)$ is not a conjugate point to $\gamma(a)$ for any $t \in (a, t_0]$ and $\gamma(t)$ is not a $B$-conjugate point to $\gamma(a)$ for any $t \in (t_0, b]$, then, for any $Y \in T_{\gamma(t_0)}\Omega$, there exist a unique $B$-Jacobi field $J \in J^0_\gamma$ such that $J(a) = Y(a)$, $J(b) = Y(b)$ and

$$I_\gamma(J, J) \leq I_\gamma(Y, Y).$$

In particular, the equality holds if and only if $J^\perp = Y^\perp$.

**Proof.** By Lemma 1.6, we obtain that

$$0 \leq I_\gamma(J - Y, J - Y) = I_\gamma(J, J) - 2I_\gamma(J, Y) + I_\gamma(Y, Y). \quad (1.9)$$

Moreover, from (1.8), we get

$$I_\gamma(J, Y) = g_2(J^\perp(b), Y^\perp(b)) - g_1(J^\perp(a), Y^\perp(a))$$

$$= g_2(J^\perp(b), J^\perp(b)) - g_1(J^\perp(a), J^\perp(a)) = I_\gamma(J, J).$$

It follows that $I_\gamma(J, J) \leq I_\gamma(Y, Y)$, and the equality of (1.9) holds if and only if $J^\perp - Y^\perp = (J - Y)^\perp = 0$. \qed

## 2. Index theorem

Let $\gamma \in \Omega_{t_0}$ be a $B$-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. Given a $B$-conjugate point $\gamma(c)$, $a < c < b$, to $\gamma(a)$, its multiplicity (or order) $\tilde{\mu}$ is defined to be the dimension of the space of all $B$-Jacobi fields along $\gamma$ which vanish at $a$ and $c$. We note that if $\gamma(c)$ is not $B$-conjugate point to $\gamma(a)$, the multiplicity of $\gamma(c)$ is zero. Moreover, we note that, for $B$-conjugate point $\gamma(c)$ ($a < c < t_0$) to $\gamma(a)$, (the multiplicity of $\gamma(c)$) $\leq$ (the multiplicity of $\gamma(c)$ as a conjugate point), since $B$-conjugate points in $M_1$ are always usual ones but the converse is not true. We assume that $\gamma(t_0)$ is not conjugate point to $\gamma(a)$, then $\tilde{\mu} \leq m_2 - 1$ since $\dim J^0_{\gamma} = m_2 - 1$ where $J^0_{\gamma} := J^0_{\gamma} \cap J^\perp_{\gamma}$ and $m_2 = \dim M_2$ (see [10]).

In general, given a symmetric bilinear form $I$ on a vector space $V$, the index $i(I)$, the augmented index $a(I)$ and the nullity $n(I)$ of $I$ are defined by

- $i(I)$ := the maximum dimension of those subspaces of $V$ on which $I$ is negative definite;
- $a(I)$ := the maximum dimension of those subspaces of $V$ on which $I$ is negative semi-definite;
- $n(I)$ := $\dim \{ v \in V | I(v, w) = 0 \text{ for all } w \in V \}$. 

Lemma 2.1 ([7]). If $I$ is a symmetric bilinear form on a finite-dimensional vector space $V$, then $a(I) = i(I) + n(I)$.

For a $B$-geodesic $g \in \Omega_{t_0}(p, q)$ with $t'(t_0 + 0) \neq T_{\gamma(t_0)}B$, we put

$$L := \{ Y \in T^\perp_{\gamma} \Omega_{t_0}(p, q) \mid I^\perp_{\gamma}(Y, W) = 0 \text{ for all } W \in T^\perp_{\gamma} \Omega_{t_0}(p, q) \}.$$  

We consider the index, the augmented index and the nullity of the index form $I^0_{\gamma}$ restricted $I$ to $T^\perp_{\gamma} \Omega_{t_0}(p, q)$. The purpose of this section is to give a proof of the index theorem:

Theorem 2.2 (Index theorem). Let $g \in \Omega_{t_0}(p, q)$ be a $B$-geodesic such that $t'(t_0 + 0) \notin T_{\gamma(t_0)}B$ and $g(t_0)$ is not conjugate point to $g(a)$. Then there are only finitely many points $g(t_n)$ ($a < t_1 < \cdots < t_m < t_0$) which are conjugate to $g(a)$ along $\gamma$ and finitely many points $g(t_{m+1}), \ldots, g(t_l)$ ($t_0 < t_{m+1} < \cdots < t_l < b$) other than $g(b)$ which are $B$-conjugate to $g(a)$ along $\gamma$. Let $\mu_i$ be the multiplicity of $g(t_i)$ ($i = 1, \ldots, m$) as a conjugate point to $g(a)$ and $\tilde{\mu}_i$ ($i = 1, \ldots, l$) the multiplicity of $g(t_i)$. Then it holds that

$$i(I^0_{\gamma}) = \mu_1 + \cdots + \mu_m + \tilde{\mu}_{m+1} + \cdots + \tilde{\mu}_l \geq \tilde{\mu}_1 + \cdots + \tilde{\mu}_l.$$  

We give an example where $\mu_1 + \cdots + \mu_m + \tilde{\mu}_{m+1} + \cdots + \tilde{\mu}_l \neq \tilde{\mu}_1 + \cdots + \tilde{\mu}_l$ holds.

Example 2. In example 1, $\gamma(\pi/2)$ is a conjugate point to $\gamma(-\pi/2)$ but not a $B$-conjugate point. Let $\mu_1$ be the multiplicity of $\gamma(\pi/2)$ as a conjugate point to $\gamma(-\pi/2)$ and $\tilde{\mu}_1$ the multiplicity of $\gamma(\pi/2)$. Then it holds that

$$i(I^0_{\gamma}) = \mu_1 = 1 > \tilde{\mu}_1 = 0.$$  

Theorem 2.3. Let $g \in \Omega_{t_0}(p, q)$ be a $B$-geodesic with $t'(t_0 + 0) \neq T_{\gamma(t_0)}B$. Then

1. $n(I^0_{\gamma}) = 0$ if $g(b)$ is not $B$-conjugate point to $g(a)$,
2. $n(I^0_{\gamma}) = \text{the multiplicity of } g(b) \text{ if } g(b) \text{ is } B\text{-conjugate point to } g(a)$.

Proof. By Lemma 1.3, we have

$$n(I^0_{\gamma}) = \dim L = \dim \{ Y \in T^\perp_{\gamma} \Omega_{t_0}(p, q) \mid Y \in \mathcal{F}_g \}.$$  

This proves (1) and (2). ⊠

Theorem 2.4. Let $g \in \Omega_{t_0}(p, q)$ be a $B$-geodesic such that $t'(t_0 + 0) \neq T_{\gamma(t_0)}B$ and $g(t_0)$ is not conjugate point to $g(a)$. Then

$$a(I^0_{\gamma}) = i(I^0_{\gamma}) + n(I^0_{\gamma}).$$  

Proof. We will construct a finite-dimensional subspace $L_1$ of $T^\perp_{\gamma} \Omega_{t_0}(p, q)$ such that $i(I^0_{\gamma}) = i(I_{\gamma}|L_1)$, $a(I^0_{\gamma}) = a(I_{\gamma}|L_1)$ and $n(I^0_{\gamma}) = n(I_{\gamma}|L_1)$. By
Lemma 1.4, we can take a subdivision \( a = a_0 < a_1 < \cdots < a_j = t_0 < a_{j+1} < \cdots < a_k < a_{k+1} = b \) of the interval \([a, b]\) such that \( \gamma(t) \) is not a conjugate point to \( \gamma(a_i) \) for any \( t \in (a_i, a_{i+1}) \) \( (i = 0, 1, \ldots, j - 2, j + 1, \ldots, k) \), \( \gamma(t) \) is not a conjugate point to \( \gamma(a_{j-1}) \) for any \( t \in (a_{j-1}, t_0) \) and \( \gamma(t) \) is not a \( B \)-conjugate point to \( \gamma(a_{j-1}) \) for any \( t \in (t_0, a_{j+1}) \). We set

\[
L_1 := L(a_0, \ldots, a_{k+1}) = \{ Y \in T^\perp_{\gamma(t_0)}(p, q) \mid Y \text{ is a Jacobi field along } \gamma \mid [a_i, a_{i+1}] \text{ for } i = 0, \ldots, j - 2, j + 1, \ldots, k \text{ and a } B\text{-Jacobi field along } \gamma \mid [a_{j-1}, a_{j+1}] \}. 
\]

Let \( N(a_i) \) be the normal space to \( \gamma \) at \( \gamma(a_i) \), that is,

\[
N(a_i) = \{ \gamma'(a_i) \}^\perp := \{ v \in T_{\gamma(t_0)}M_k \mid g_L(v, \gamma'(a_i)) = 0 \},
\]

and define a linear map

\[
\mathcal{N} : L_1 \to N := N(a_1) \times \cdots \times N(a_{j-1}) \times N(a_{j+1}) \times \cdots \times N(a_k)
\]

by

\[
\mathcal{N}(Y) := \langle Y(a_1), \ldots, Y(a_{j-1}), Y(a_{j+1}), \ldots, Y(a_k) \rangle.
\]

Lemma 2.5. (1) \( \mathcal{N} \) is a linear isomorphism of \( L_1 \) onto \( N \); (2) Define a map \( \rho : T^\perp_{\gamma(t_0)}(p, q) \to L_1 \) by setting

\[
\rho(Y) := \mathcal{N}^{-1}(Y(a_1), \ldots, Y(a_{j-1}), Y(a_{j+1}), \ldots, Y(a_k))
\]

for \( Y \in T^\perp_{\gamma(t_0)}(p, q) \). Then

\[
I_\gamma(Y, Y) \geq I_\gamma(\rho(Y), \rho(Y)) \text{ for } Y \in T_{\gamma(t_0)}(p, q),
\]

and the equality holds if and only if \( Y \in L_1 \).

(3) \( i(I_\gamma^{\perp}) = i(I_{\gamma\mid L_1}) \), \( a(I_\gamma^{\perp}) = a(I_{\gamma\mid L_1}) \) and \( n(I_\gamma^{\perp}) = n(I_{\gamma\mid L_1}) \).

Lemma 2.1 and Lemma 2.5(3) imply Theorem 2.4. \( \square \)

Proof of Lemma 2.5. (1) Suppose \( Y \in L_1 \) and \( \mathcal{N}(Y) = 0 \) so that \( Y(a_i) = 0 \) for \( i = 1, \ldots, j - 1, j + 1, \ldots, k \). By our choice of \( a_i \), \( Y = 0 \), proving that \( \mathcal{N} \) is injective. To show that \( \mathcal{N} \) is surjective, it suffices to prove that, given vectors \( v_i \) at \( \gamma(a_i) \) and \( v_{i+1} \) at \( \gamma(a_{i+1}) \), there is a Jacobi field \( Y \) along \( \gamma \mid [a_i, a_{i+1}] \) which extends \( v_i \) and \( v_{i+1} \) for \( i = 1, \ldots, j - 2, j + 1, \ldots, k - 1 \), and given vectors \( v_{j-1} \) at \( \gamma(a_{j-1}) \) and \( v_{j+1} \) at \( \gamma(a_{j+1}) \), there is a \( B \)-Jacobi field \( Y \) along \( \gamma \mid [a_{j-1}, a_{j+1}] \) which extends \( v_{j-1} \) and \( v_{j+1} \). Since \( \gamma(a_{i+1}) \) is not conjugate to \( \gamma(a_i) \), \( Y \mapsto (v_i, v_{i+1}) \) defines a linear isomorphism of the space of Jacobi fields along \( \gamma \mid [a_i, a_{i+1}] \) into the direct sum of the tangent spaces at \( \gamma(a_i) \) and \( \gamma(a_{i+1}) \) for \( i = 1, \ldots, j - 2, j + 1, \ldots, k - 1 \). Moreover since \( \gamma(t_0) \) and \( \gamma(a_{i+1}) \) are not conjugate points to \( \gamma(a_{j-1}) \), \( Y \mapsto (v_{j-1}, v_{j+1}) \) defines a linear isomorphism of the space of \( B \)-Jacobi fields along \( \gamma \mid [a_{j-1}, a_{j+1}] \) into the direct sum of the tangent spaces at \( \gamma(a_{j-1}) \) and \( \gamma(a_{j+1}) \). Since they are linear isomorphisms of a vector space into a vector space...
of the same dimension (cf. Lemma 1.5), it must be surjective. This completes the proof of (1).

(2) With the notations in the Section 1, we have

\[ I^\perp_\gamma(Y, Y) = \sum_{i=0}^{j-2} \tilde{I}_\gamma[a_i, a_{i+1}](Y, Y) + I_\gamma[a_{j-1}, a_{j+1}](Y, Y) + \sum_{i=j+1}^{k} \tilde{I}_\gamma[a_i, a_{i+1}](Y, Y) \]

and

\[ I^\perp_\gamma(\rho(Y), \rho(Y)) = \sum_{i=0}^{j-2} \tilde{I}_\gamma[a_i, a_{i+1}](\rho(Y), \rho(Y)) + I_\gamma[a_{j-1}, a_{j+1}](\rho(Y), \rho(Y)) \]

+ \sum_{i=j+1}^{k} \tilde{I}_\gamma[a_i, a_{i+1}](\rho(Y), \rho(Y)).

By Proposition 3.1 in [7], we have

\[ \tilde{I}_\gamma[a_i, a_{i+1}](Y, Y) \geq \tilde{I}_\gamma[a_i, a_{i+1}](\rho(Y), \rho(Y)) \]

for \( i = 0, \ldots, j-2, j+1, \ldots, k \) and the equality holds if and only if \( Y \) is a Jacobi field along \( \gamma|a_i, a_{i+1} \). By Lemma 1.7, we have

\[ I_\gamma[a_{j-1}, a_{j+1}](Y, Y) \geq I_\gamma[a_{j-1}, a_{j+1}](\rho(Y), \rho(Y)) \]

and the equality holds if and only if \( Y \) is a \( B \)-Jacobi field along \( \gamma|a_{j-1}, a_{j+1} \).

(3) If \( U \) is a subspace of \( T^\perp_\gamma \Omega_{t_0}(p, q) \) on which \( I^0_{\gamma, \perp} \) is negative semi-definite, then \( I^0_{\gamma, \perp} \) is negative semi-definite on \( \rho(U) \) by (2). Moreover, \( \rho|U : U \to \rho(U) \) (\( \subset L_1 \)) is a linear isomorphism. In fact, if \( Y \in U \) and \( \rho(Y) = 0 \), then (2) implies

\[ 0 \geq I_\gamma(Y, Y) \geq I_\gamma(\rho(Y), \rho(Y)) = 0, \]

and hence \( I_\gamma(Y, Y) = I_\gamma(\rho(Y), \rho(Y)) \). Again by (2), we have \( Y = \rho(Y) = 0 \). Thus \( \rho|U \) is injective. It is clear that \( \rho|U \) is surjective and linear. Moreover we have \( a(I^0_{\gamma, \perp}) \leq a(I_\gamma|L_1) \). The reverse inequality is obvious. The proof for the index \( i(I^0_{\gamma, \perp}) \) is similar. Finally, to prove \( n(I^0_{\gamma, \perp}) = n(I_\gamma|L_1) \), let \( Y \) be an element of \( L_1 \) such that \( I^\perp_\gamma(Y, W) = 0 \) for all \( W \in L_1 \). Since \( Y \) is a Jacobi field along \( \gamma|a_i, a_{i+1} \) for \( i = 0, \ldots, j-2, j+1, \ldots, k \) and a \( B \)-Jacobi field along \( \gamma|a_{j-1}, a_{j+1} \), we have that

\[ I_\gamma(Y, W) = \sum_{i=1}^{j-1} g_1(Y'(a_i - 0) - Y'(a_i + 0), W(a_i)) \]

+ \sum_{i=j+1}^{k} g_2(Y'(a_i - 0) - Y'(a_i + 0), W(a_i))

from Lemma 1.2. In the same way as we prove Lemma 1.3, we conclude that \( Y'(a_i - 0) = Y'(a_i + 0) \) for \( i = 1, \ldots, j-1, j+1, \ldots, k \) so that \( Y \) is a \( B \)-Jacobi field along \( \gamma \). This means that \( n(I^0_{\gamma, \perp}) \geq n(I_\gamma|L_1) \). The reverse inequality is obvious. \( \square \)
Proof of Theorem 2.2. Since \( \dim L_1 < \infty \), (3) of Lemma 2.5 implies that both \( a(I^0_{\gamma, -}) \) and \( i(I^0_{\gamma, -}) \) are finite. The finiteness of \( B \)-conjugate points follows from the next lemma.

**Lemma 2.6.** For any finite number of conjugate points \( \gamma(t_1), \ldots, \gamma(t_m) \) (\( a < t_1 < \cdots < t_m < t_0 \)) to \( \gamma(a) \) along \( \gamma \) with multiplicity \( \mu_1, \ldots, \mu_m \) as conjugate points and \( B \)-conjugate points \( \gamma(t_{m+1}), \ldots, \gamma(t_l) \) (\( t_0 < t_{m+1} < \cdots < t_l < b \)) to \( \gamma(a) \) along \( \gamma \) with multiplicity \( \tilde{\mu}_{m+1}, \ldots, \tilde{\mu}_l \), we have

\[
a(I^0_{\gamma, -}) \geq \mu_1 + \cdots + \mu_m + \tilde{\mu}_{m+1} + \cdots + \tilde{\mu}_l.
\]

**Proof.** For simplicity, we put \( \mu_i := \tilde{\mu}_i \) (\( i = m + 1, \ldots, l \)). For each \( i \), let \( \tilde{Y}^i_1, \ldots, \tilde{Y}^i_{\tilde{\mu}_i} \) be a basis for the Jacobi fields along \( \gamma \mid [a, t_0] \) or the \( B \)-Jacobi fields along \( \gamma \) which vanish at \( t = a \) and \( t = t_i \). We put, \( j = 1, \ldots, \mu_i \),

\[
Y^i_m := \begin{cases} 
\tilde{Y}^i_m & \text{on } [a, t_i] \\
0 & \text{on } [t_i, b].
\end{cases}
\]

It suffices to prove that \( \mu_1 + \cdots + \mu_l \) vector fields \( Y^1_1, \ldots, Y^l_{\mu_l}, i = 1, \ldots, l \), along \( \gamma \) are linearly independent and that \( I^i_j \) is negative semi-definite on the space spanned by them. Suppose

\[
\sum_{i=1}^l Y^i = 0,
\]

where

\[
Y^i = c^i_1 Y^1_1 + \cdots + c^i_{\mu_i} Y^i_{\mu_i}.
\]

Since \( Y^1, \ldots, Y^{l-1} \) vanish on \( \gamma \mid [t_{l-1}, b] \), \( Y^l \) must vanish along \( \gamma \mid [t_{l-1}, t_l] \). Being a \( B \)-Jacobi field or a Jacobi field along \( \gamma \mid [a, t_l] \), \( Y^l \) must vanish identically along \( \gamma \), since \( \gamma(t_0) \) is not a conjugate point to \( \gamma(a) \). Thus, \( c^l_1 = \cdots = c^l_{\mu_l} = 0 \). Continuing this argument, we obtain \( c^{l-1}_1 = \cdots = c^{l-1}_{\mu_{l-1}} = 0 \), and so on. To prove that \( I^i_j \) is negative semi-definite on the space spanned by \( Y^1_1, \ldots, Y^l_{\mu_l}, i = 1, \ldots, l \), let

\[
Y = Y^1 + \cdots + Y^l,
\]

where each \( Y^i \) is a linear combination of \( Y^1_1, \ldots, Y^l_{\mu_l} \) as above. Then

\[
I^i_j(Y, Y) = \sum_{i=1}^l I^i_j(Y^i, Y^i) + 2 \sum_{1 \leq s < i \leq l} I^i_j(Y^i, Y^s).
\]

For each pair \( (i, s) \) with \( s \leq i \), we shall show that \( I^i_j(Y^i, Y^s) = 0 \). Let \( \tilde{\gamma} = \gamma \mid [a, t_i] \). Since \( Y^i \) and \( Y^s \) vanish beyond \( t = t_i \), we have \( I^i_j(Y^i, Y^s) = I^i_{\tilde{\gamma}}(Y^i, Y^s) \). As \( Y^i \) is a \( B \)-Jacobi field or a Jacobi field along \( \tilde{\gamma} \), \( I^i_{\tilde{\gamma}}(Y^i, Y^s) = 0 \) by Lemma 1.3. Thus, \( I^i_j(Y, Y) = 0 \), proving our assertion. \( \Box \)
Let \( \gamma_r \) denote the restriction of \( \gamma \) to the interval \([a, b_r]\), where \( b_r = rb + (1 - r)a \) for \( 0 < r \leq 1 \). Thus \( \gamma_r : [a, b_r] \to M \) is a \( B \)-geodesic from \( \gamma(a) \) to \( \gamma(b_r) \) if \( (t_0 - a)/(b - a) < r \leq 1 \) and a geodesic in \( M_1 \) if \( 0 < r \leq (t_0 - a)/(b - a) \). Let \( I_r \) denote the index form associated with this \( B \)-geodesic or geodesic. Thus \( i(I_1) \) is the index which we are actually trying to compute. First note that:

**Assertion (1).** \( i(I_r) = 0 \) for small values of \( r \). (cf. [8])

**Assertion (2).** \( i(I_r) \) is a monotone function of \( r \).

In fact, if \( r < r' \) then there exists a \( i(I_r) \) dimensional space \( \mathcal{V} \) of vector fields along \( \gamma_r \) which vanish at \( a \) and \( b_r \) such that the index form \( I_r \) is negative definite on this vector space. Each vector field in \( \mathcal{V} \) extends to a vector field along \( \gamma_r \) which vanishes identically between \( b_r \) to \( b_{r'} \). Thus we obtain a \( i(I_r) \) dimensional vector space of vector fields along \( \gamma_{r'} \) on which \( I_{r'} \) is negative definite. Hence \( i(I_r) \leq i(I_{r'}) \). \( \square \)

Now let us examine the discontinuity of the function \( i(I_r) \). First note that \( i(I_r) \) is continuous from the left:

**Assertion (3).** For all sufficiently small \( \varepsilon > 0 \) we have \( i(I_{r-\varepsilon}) = i(I_r) \).

**Proof.** According to (3) of Lemma 2.5 the number \( i(I_1) \) can be interpreted as the index of a quadratic form on a finite dimensional vector space \( L_1 = L(a_0, \ldots, a_{k+1}) \). If \( b_r \neq t_0 \), we may assume that the subdivision is chosen so that say \( a_i < b_r < a_{i+1} \). Then the index \( i(I_r) \) can be interpreted as the index of a corresponding quadratic form \( I_r \) on a corresponding vector space \( L_r \) of broken \( B \)-Jacobi fields or Jacobi fields along \( \gamma_r \). This vector space \( L_r \) is to be constructed using the subdivision \( a < a_1 < \cdots < a_t < b_r \) of \([a, b_r]\). Since a broken \( B \)-Jacobi field or a Jacobi field is uniquely determined by its values at the break points \( \gamma(a_m) \), this vector space \( L_r \) is isomorphic to the direct sum

\[
N_r = \begin{cases} 
N(a_1) \times \cdots \times N(a_{j-1}) \times N(a_{j+1}) \times \cdots \times N(a_1) & \text{if } b_r > t_0 \\
N(a_1) \times \cdots \times N(a_i) & \text{if } b_r < t_0 
\end{cases}
\]

by a map \( \mathcal{N}_r : L_r \to N_r \) defined to be

\[
\mathcal{N}_r(Y) := \begin{cases} 
(Y_1(a_1), \ldots, Y(a_{j-1}), Y(a_{j+1}), \ldots, Y(a_1)) & \text{if } b_r > t_0 \\
(Y_1(a_1), \ldots, Y(a_i)) & \text{if } b_r < t_0 
\end{cases}
\]

Note that this vector space \( N_r \) is independent of \( r \). Evidently, by Lemma 1.2, the quadratic form \( B_r := I_r \circ \mathcal{N}_r^{-1} \) on \( N_r \) varies continuously with \( r \).

Now \( B_r \) is negative definite on a subspace \( \mathcal{V} \subset N_r \) of dimension \( i(B_r) \). For all \( r' \) sufficiently close to \( r \) it follows that \( B_{r'} \) is negative definite on \( \mathcal{V} \). Therefore \( i(B_{r'}) \geq i(B_r) \). But if \( r' = r - \varepsilon < r \) then we also have \( i(B_{r-\varepsilon}) \leq i(B_r) \) by Assertion (2). Hence \( i(B_{r-\varepsilon}) = i(B_r) \). \( \square \)

**Assertion (4).** For all sufficiently small \( \varepsilon > 0 \) we have

\[
i(I_{r+\varepsilon}) = i(I_r) + n(I_r).
\]
Proof that \( i(I_{r+\varepsilon}) \leq i(I_r) + n(I_r) \). Let \( B_r \) and \( N_r \) be as in the proof of Assertion (3). Since \( \dim N_r < \infty \) we see that \( B_r \) is positive definite on some subspace \( \mathcal{V} \subset N_r \). For all \( r' \) sufficiently close to \( r \), it follows that \( B_{r'} \) is positive definite on \( \mathcal{V}' \). Hence
\[
i(B_{r'}) \leq \dim N_r - \dim \mathcal{V}' = a(B_r) = i(B_r) + n(B_r).
\]
\[ \square \]

Proof that \( i(I_{r+\varepsilon}) \geq i(I_r) + n(I_r) \). Let \( V \in N_r \), with \( V(a_i) \neq 0 \), and denote by \( V_{b_i} \in L_r \) the broken \( B \)-Jacobi field or Jacobi field which coincides with \( V(a_m) \) at \( a_m, m = 1, \ldots, i \), and which vanishes at the point \( b_r \in (a_i, a_{i+1}) \). We claim that
\[
B_r(V, V) = I_r(V_{b_r}, V_{b_r}) > I_{r+\varepsilon}(V_{b_{r+\varepsilon}}, V_{b_{r+\varepsilon}}) = B_{r+\varepsilon}(V, V).
\]
In fact, if we denote by \( W_{b_r} \) the vector field defined along \( \gamma_{r+\varepsilon} \) by
\[
W_{b_r}(t) = \begin{cases} V_{b_r}(t), & t \in [a, b_r] \\ 0, & t \in [b_r, b_{r+\varepsilon}] \end{cases},
\]
we have, from Lemma 1.6,
\[
I_r(V_{b_r}, V_{b_r}) = I_{r+\varepsilon}(W_{b_r}, W_{b_r}) > I_{r+\varepsilon}(V_{b_{r+\varepsilon}}, V_{b_{r+\varepsilon}}),
\]
where the last inequality is strict, since \( W_{b_r} \mid [a_i, b_{r+\varepsilon}] \) is neither a \( B \)-Jacobi field nor Jacobi field. Therefore, if \( V \in N_r \) and \( B_r(V, V) = I_r(V_{b_r}, V_{b_r}) \leq 0 \), then \( B_{r+\varepsilon}(V, V) = I_{r+\varepsilon}(V_{b_{r+\varepsilon}}, V_{b_{r+\varepsilon}}) < 0 \). Hence, if \( B_r \) is negative definite on a subspace \( \mathcal{V} \subset N_r \), \( B_{r+\varepsilon} \) will still be negative definite on the direct sum of \( \mathcal{V} \) with the null space of \( B_r \). Therefore
\[
i(B_{r+\varepsilon}) \geq i(B_r) + n(B_r). \quad \square
\]

The index Theorem 2.2 clearly follows from the Assertion (1), (2), (3) and (4). \( \square \)

3. Comparison theorem

Let \((M, g)\) (resp. \((\overline{M}, \overline{g})\)) be Riemannian manifold with Riemannian submanifold \(B_1\) (resp. \(\overline{B}_1\)) for \(\lambda = 1, 2\), and \(\psi\) (resp. \(\overline{\psi}\)) isometry from \(B_1\) to \(B_2\) (resp. \(\overline{B}_1\) to \(\overline{B}_2\)). Let \((M, g) = (M_1, g_1) \cup \psi(M_2, g_2)\) and \((\overline{M}, \overline{g}) = (\overline{M}_1, \overline{g}_1) \cup \overline{\psi}(\overline{M}_2, \overline{g}_2)\) be glued Riemannian spaces. We put \(B := B_1 \simeq B_2\) and \(\overline{B} := \overline{B}_1 \simeq \overline{B}_2\) and assume that \(\dim \overline{B} > 0\) if \(\dim B > 0\). Let \(\gamma \in \Omega_{t_0}\) (resp. \(\overline{\gamma} \in \Omega_{t_0}\)) be a \(B\)-geodesic (resp. \(\overline{B}\)-geodesic) with \(\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B\) (resp. \(\overline{\gamma}'(t_0 + 0) \notin T_{\overline{\gamma}(t_0)}\overline{B}\)). We assume that \(\gamma(t_0)\) (resp. \(\overline{\gamma}(t_0)\)) is not conjugate point to \(\gamma(a)\) (resp. \(\overline{\gamma}(a)\)). For \(\lambda = 1, 2\), let \(R^\lambda\) (resp. \(\overline{R}^\lambda\)) be the Riemannian curvature tensor of Riemannian manifold \(M_\lambda\) (resp. \(\overline{M}_\lambda\)). We define operators\(R_i^\lambda: \{\gamma'(t)\}^\perp \to \{\gamma'(t)\}^\perp\) and \(\overline{R}_i^\lambda: \{\overline{\gamma}'(t)\}^\perp \to \{\overline{\gamma}'(t)\}^\perp\) by
\[
R_i^\lambda v = R^\lambda(v, \gamma'(t))\gamma'(t) \quad \text{for } v \in \{\gamma'(t)\}^\perp
\]
and
\[
\overline{R}_i^\lambda \overline{v} = \overline{R}^\lambda(\overline{v}, \overline{\gamma}'(t))\overline{\gamma}'(t) \quad \text{for } \overline{v} \in \{\overline{\gamma}'(t)\}^\perp,
\]
where
\[ \{ \gamma'(t) \}^\perp = \{ v \in T_{\gamma(t)} M \lambda | g_{\lambda}(v, \gamma'(t)) = 0 \} \]
and
\[ \{ \bar{\gamma}'(t) \}^\perp = \{ \bar{v} \in T_{\bar{\gamma}(t)} \bar{M} \lambda | \bar{g}_{\lambda}(\bar{v}, \bar{\gamma}'(t)) = 0 \}. \]

Similarly, a bar is used to distinguish objects in \( \bar{M} \) from the corresponding objects in \( M \). We put \( \Gamma_2(\gamma') := T_{\gamma(t_0)} B \oplus \text{Span} \{ \text{nor}_2 \gamma'(t_0 + 0) \} \), \( \Gamma_2^\perp(\gamma') := \{ v \in \Gamma_2(\gamma') | g_2(v, \gamma'(t_0 + 0)) = 0 \} \) and \( A := A_{\gamma'(t_0 - 0), \gamma'(t_0 + 0)} | \Gamma_2^\perp(\gamma') \).

We assume that \( \dim M_\lambda \geq 2 \) and \( \dim \bar{M}_\lambda \geq 2 \). Then the following assertion holds:

**Proposition 3.1.** We assume that \( \dim M_\lambda \leq \dim \bar{M}_\lambda \) \(( \lambda = 1, 2) \) and the following conditions hold:

1. For any \( t \in [a, b] \),
   
   (the maximal eigenvalue of \( R_\gamma^\perp \)) \leq \ (the minimal eigenvalue of \( \bar{R}_{\bar{\gamma}}^\perp \))

2. If \( \dim B > 0 \), then
   
   (the minimal eigenvalue of \( A \)) \geq \ (the maximal eigenvalue of \( \bar{A} \)).

Then \( i(I_\gamma^0, \perp) \leq i(\bar{I}_{\bar{\gamma}}^0, \perp) \) holds. In particular, if one of two inequalities (1) and (2) is strict, then \( a(I_\gamma^0, \perp) = i(I_\gamma^0, \perp) + n(I_\gamma^0, \perp) \leq i(\bar{I}_{\bar{\gamma}}^0, \perp) \) holds.

**Proof.** For \( Y \in T_{\gamma(t_0)} \Omega_{t_0}(\gamma(a), \gamma(b)) \), let \( e^-_1, \ldots, e^-_{m_1} := \gamma'(t_0 - 0) \) be an orthonormal basis of \( T_{\gamma(t_0)} M_1 \) and \( e^+_1, \ldots, e^+_{m_2} := \gamma'(t_0 + 0) \) an orthonormal basis of \( T_{\gamma(t_0)} M_2 \) such that \( e^-_i = Y(t_0 - 0) / \| Y(t_0 - 0) \|_1 \) and \( e^+_i = Y(t_0 + 0) / \| Y(t_0 + 0) \|_2 \) if \( Y(t_0 - 0) \neq 0 \). Let \( e^-_i(t) \) (resp. \( e^+_i(t) \)) be the vector field along \( \gamma \mid [a, t_0] \) (resp. \( \gamma \mid [t_0, b] \)) obtained by parallel translation of \( e^-_i \) (resp. \( e^+_i \)) along \( \gamma \mid [a, t_0] \) (resp. \( \gamma \mid [t_0, b] \)) for \( i = 1, \ldots, m_1 \) (resp. \( i = 1, \ldots, m_2 \)). We can denote \( Y(t) \) by

\[
Y(t) = \sum_{i=1}^{m_1-1} y^-_i(t) e^-_i(t), \quad t \in [a, t_0]
\]

and

\[
Y(t) = \sum_{i=1}^{m_2-1} y^+_i(t) e^+_i(t), \quad t \in [t_0, b].
\]

Let \( \bar{e}_1, \ldots, \bar{e}_{m_1} := \bar{\gamma}'(t_0 - 0) \) (resp. \( \bar{e}^+_1, \ldots, \bar{e}^+_{m_2} := \bar{\gamma}'(t_0 + 0) \)) be an orthonormal basis of \( T_{\bar{\gamma}(t_0)} \bar{M}_1 \) (resp. \( T_{\bar{\gamma}(t_0)} \bar{M}_2 \)) such that if \( \bar{e}^-_1 \in \bar{\Gamma}_1(\bar{\gamma}') \) and \( \bar{e}^+_1 = \bar{\mathcal{O}}(\bar{e}^-_1) \) if \( Y(t_0 - 0) \neq 0 \). Let \( \bar{e}^-_i(t) \) (resp. \( \bar{e}^+_i(t) \)) be the vector field along \( \bar{\gamma} \mid [a, t_0] \) (resp. \( \bar{\gamma} \mid [t_0, b] \)) obtained by parallel translation of \( \bar{e}^-_i \) (resp. \( \bar{e}^+_i \)) along \( \bar{\gamma} \mid [a, t_0] \) (resp. \( \bar{\gamma} \mid [t_0, b] \)) for \( i = 1, \ldots, m_1 \) (resp. \( i = 1, \ldots, m_2 \)). If we put

\[
\bar{Y}(t) = \sum_{i=1}^{m_1-1} y^-_i(t) \bar{e}^-_i(t), \quad t \in [a, t_0]
\]
and

$$\bar{Y}(t) = \sum_{i=1}^{m-1} y^i(t) \varepsilon^+(t), \quad t \in [t_0, b],$$

then it holds that \( \bar{Y} \in T^+_{\bar{y}} \Omega_{\bar{y}_0}(\bar{y}(a), \bar{y}(b)) \), since \( \bar{Y}(t_0 + 0) = y^1(t_0 + 0) \varepsilon^+ = y^1(t_0 - 0) \bar{Q}(\varepsilon^-) = \bar{Q}(\bar{Y}(t_0 - 0)) \) if \( Y(t_0) \neq 0 \). Furthermore, by the definition, we have that \( \|\bar{Y}(t)\|_{\bar{\gamma}} = \|Y(t)\|_{\gamma} \) and \( \|\bar{Y}'(t)\|_{\bar{\gamma}} = \|Y'(t)\|_{\gamma} \). From the assumption (1) and (2), we get

$$g_\bar{\gamma}(\bar{R}^\bar{\gamma}_t Y(t), Y(t)) \leq g_\bar{\gamma}(\bar{R}^\bar{\gamma}_t \bar{Y}(t), \bar{Y}(t))$$

and

$$g_\gamma(A(Y(t_0 + 0)), Y(t_0 + 0)) \geq g_\gamma(A(\bar{Y}(t_0 + 0)), \bar{Y}(t_0 + 0)).$$

Then we have that

$$I_\gamma(Y, Y) \geq I_{\bar{\gamma}}(\bar{Y}, \bar{Y}). \quad (3.1)$$

Let \( \mathcal{W} \) be the subspace of \( T^+_{\gamma} \Omega_{\gamma_0}(\gamma(a), \gamma(b)) \) on which \( I^+_\gamma \) is negative definite and \( \bar{\mathcal{W}} := \{ \bar{Y} | Y \in \mathcal{W} \} \). If \( Y \in \mathcal{W} \), then \( I_{\bar{\gamma}}(\bar{Y}, \bar{Y}) < 0 \). Hence, \( \bar{I}_{\bar{\gamma}} \) is negative definite on \( \bar{\mathcal{W}} \) and we have \( i(I^+_\gamma) \leq i(I^+_{\bar{\gamma}}) \).

If one of two inequalities (1) and (2) is strict, then it holds that

$$I_\gamma(Y, Y) > I_{\bar{\gamma}}(\bar{Y}, \bar{Y}). \quad (3.2)$$

Let \( \mathcal{V} \) be the subspace of \( T^+_{\gamma} \Omega_{\gamma_0}(\gamma(a), \gamma(b)) \) on which \( I^+_{\gamma} \) is negative semi-definite and \( \bar{\mathcal{V}} := \{ \bar{Y} | Y \in \mathcal{V} \} \). If \( Y \in \mathcal{V} \), then \( I_{\bar{\gamma}}(\bar{Y}, \bar{Y}) < 0 \). Hence, \( \bar{I}_{\bar{\gamma}} \) is negative definite on \( \bar{\mathcal{V}} \) and we have \( a(I^{-1}_{\gamma}) \leq a(I^{-1}_{\bar{\gamma}}) \).

The condition that \( \dim M_\lambda \leq \dim \bar{M}_\lambda (\lambda = 1, 2) \) is necessary. We give an example which shows that:

**Example 3.** Let \( S^m(1) \) be the \( m \)-sphere of constant curvature 1 and \( \gamma \) a geodesic on \( S^m(1) \). Let \( e_1(t), e_2(t), \ldots, e_{m-1}(t), \gamma'(t) \) be a parallel orthonormal frame along \( \gamma \). Let \( \tau \) be the geodesic through \( \gamma(0) \) with \( \tau'(0) = e_1(0) \). We put \( M_\lambda := S^m(1) \quad (\lambda = 1, 2) \), \( B := \{ \tau(t) | t \in R \} \), \( \psi = \text{id}_B \) and \( M = M_1 \cup_{\psi} M_2 \). Then \( \gamma : [-\pi/2, \pi] \rightarrow M \) is a \( B \)-geodesic. We set \( a := -\pi/2 \), \( t_0 := 0 \) and \( b := \pi/2 \). Then \( \gamma(b) \) is a \( B \)-conjugate point to \( \gamma(a) \), its multiplicity is \( m - 1 \) and \( i(I^+_\gamma) = m - 1 \). For \( \bar{m} < m \), we set \( \bar{M}_\lambda := S^\bar{m}(1), \bar{B}, \bar{\psi}, \bar{M} = \bar{M}_1 \cup_{\bar{\psi}} \bar{M}_2 \) and \( \bar{\gamma} \) as above. Then, we have that \( i(I^{0,\perp}_\gamma) > i(I^{0,\perp}_{\bar{\gamma}}) \).

In [11], the following assertion is given without the assumption that \( \dim M_\lambda \leq \dim \bar{M}_\lambda (\lambda = 1, 2) \):

**Corollary 3.2.** We assume that \( \dim M_\lambda \leq \dim \bar{M}_\lambda (\lambda = 1, 2) \) and the following conditions hold:

1. For any \( t \in [a, b] \),

   \[ \text{(the maximal eigenvalue of } R^\lambda \text{)} \leq \text{(the minimal eigenvalue of } \bar{R}^\lambda \text{)} \]
(2) If \( \dim B > 0 \), then
\[
(\text{the minimal eigenvalue of } A) \geq (\text{the maximal eigenvalue of } A).
\]

(3) \( \bar{\gamma}(t) \) is not a conjugate point to \( \bar{\gamma}(a) \) for any \( t \in (a, t_0) \) and also \( \bar{\gamma}(t) \) is not a \( \bar{B} \)-conjugate point to \( \bar{\gamma} (a) \) for any \( t \in (t_0, b] \).

Then \( \gamma(t) \) is not a conjugate point to \( \gamma(a) \) for any \( t \in (a, t_0) \) and also \( \gamma(t) \) is not \( B \)-conjugate point to \( \gamma(a) \) for any \( t \in (t_0, b] \).

Proof. By the assumption (3), \( i(I_{\gamma}^{0, -}) = 0 \) holds. Hence we have that \( i(I_{\gamma}^{0, -}) = 0 \) from Proposition 3.1. \( \square \)

REFERENCES


