Abstract

It is known that a complete linear system on a projective variety in a projective space is generated from the linear system of the projective space by restriction if its degree is sufficiently large. We obtain a bound of degree of linear systems on weighted projective spaces when they are generated from those of the projective spaces. In particular, we show that a weighted projective 3-space embedded by a complete linear system is projectively normal. We treat more generally \( \mathbb{Q} \)-factorial toric varieties with the Picard number one, and obtain the same bounds for them as those of weighted projective spaces.

Introduction

Let \( X \) be a nondegenerate projective variety of dimension \( n \) in \( \mathbb{P}^r \). It is well known that the homomorphism

\[
H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \rightarrow H^0(X, \mathcal{O}_X(k))
\]

is surjective for large enough \( k \). We say that \( X \) is \( k \)-normal if this homomorphism is surjective. It is of interest to find an explicit bound \( k_0 \) such that all nonsingular, nondegenerate, projective varieties of dimension \( n \) and degree \( d \) in \( \mathbb{P}^r \) are \( k \)-normal for all \( k \geq k_0 \). This was done for curves in \( \mathbb{P}^3 \) by Castelnuovo [C], and for reduced irreducible curves in \( \mathbb{P}^r \), \( r \geq 3 \) by Gruson, Lazarsfeld and Peskine [GLP]. They showed that the best possible \( k_0 = d + 1 \cdot r \). This suggests the equality

\[
k_0 = d + n - r.
\]

According to Mumford [M1], [M2], we say that \( X \) is \( k \)-regular if \( H^i(\mathbb{P}^r, \mathcal{I}_X(k - i)) = 0 \) for all \( i \geq 1 \), where \( \mathcal{I}_X \) is the sheaf of ideals of \( X \) in \( \mathbb{P}^r \). It is easy to see that \( X \) is \( (k + 1) \)-regular if and only if \( X \) is \( k \)-normal and \( H^i(X, \mathcal{O}_X(k - i)) = 0 \) for all \( i \geq 1 \). Eisenbud and Goto [EG] conjectured that \( X \) is \( k \)-regular for all \( k \geq d + n - r + 1 \). For nonsingular surfaces, Pinkham [P] obtained a bound, and Lazarsfeld [L] obtained the full conjecture. Kwak [Kw1], [Kw2] obtained a good bound for \( n = 3, 4 \).

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In this paper we obtain a bound of $k$-normality for a class of toric varieties containing weighted projective spaces. A weighted projective space of dimension $n$ is a quotient of the projective $n$-space by a finite abelian group. We treat a class of toric varieties that are quotients of the projective $n$-space by finite abelian groups, in other words, a class of $\mathbb{Q}$-factorial toric varieties with the Picard number one. These toric varieties are defined by integral simplices (see [F], [Od]). We use combinatorics of polytopes corresponding to toric varieties. Herzog and Hibi [HH] also obtain a result on the Castelnuovo regularity of affine semigroup rings defined by integral simplices.

A projective toric variety of dimension one is the projective line. It is known [Ko] that an ample line bundle on a toric surface $X$ is normally generated, i.e., it is very ample and $X$ is $k$-normal for all $k \geq 1$. In general, it is known [NO] that for an ample line bundle $L$ on a projective toric variety $X$ of dimension $n (> 1)$ the multiplication map

$$H^0(X, L^n) \otimes H^0(X, L) \to H^0(X, L^{n+1})$$

is surjective for all $i \geq n - 1$.

**Theorem 1.** Let $X$ be a projective toric variety of dimension $n$ which is a quotient of the projective $n$-space by a finite abelian group, and let $L$ a very ample line bundle on $X$. Then we have that

$$H^0(X, L^{n+i}) \otimes H^0(X, L) \to H^0(X, L^{n+i+1})$$

is surjective for all $i \geq \lfloor n/2 \rfloor$. In particular, any weighted projective $3$-space embedded by a very ample line bundle is projectively normal.

**Theorem 2.** Let $X$ be a projective toric variety of dimension $n$ $(n > 3)$ which is a quotient of the projective $n$-space by a finite abelian group embedded by a very ample line bundle in $\mathbb{P}^r$. Then $X$ is $k$-normal for all $k \geq n - 1 + \lfloor n/2 \rfloor$.

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1. Polarized toric varieties

First we mention the fact about toric varieties needed in this paper following Oda’s book [Od], or Fulton’s book [F].

Let $N$ be a free $\mathbb{Z}$-module of rank $n$, $M$ its dual and $\langle , \rangle : M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_\mathbb{R} := N \otimes \mathbb{R}$ and $M_\mathbb{R} := M \otimes \mathbb{R}$. Let $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic torus over the complex number field $\mathbb{C}$, where $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$. Then $M = \text{Hom}_{\mathbb{R}}(T_N, \mathbb{C}^*)$ is the character group of $T_N$. For $m \in M$ we denote $\epsilon(m)$ as the character of $T_N$. Let $\Delta$ be a complete finite fan of $N$ consisting strongly convex rational polyhedral cones $\sigma$, that is, with a finite number of elements $v_1, v_2, \ldots, v_s$ in $N$ we can denote
\[ \sigma = R_{\geq 0}v_1 + \cdots + R_{\geq 0}v_s \]

and it satisfies that \( \sigma \cap \{-\sigma\} = \{0\} \). Then we have a complete toric variety \( X = T_N \text{emb}(\Delta) := \bigcup_{\sigma \in \Delta} U_{\sigma} \) of dimension \( n \) (see Section 1.2 [Od], or Section 1.4 [F]). Here \( U_{\sigma} = \text{Spec} C[\sigma^\vee \cap M] \) and \( \sigma^\vee \) is the dual cone of \( \sigma \) with respect to the paring \( \langle , \rangle \). For the origin \( \{0\} \), the affine open set \( U_{\{0\}} = \text{Spec} C[M] \) is the unique dense \( T_N \)-orbit. We note that a toric variety is always normal.

Let \( L \) be an ample \( T_N \)-invariant invertible sheaf on \( X \). Then the polarized variety \((X, L)\) corresponds to an integral convex polytope. We call the convex hull \( \text{Conv}\{u_0, u_1, \ldots, u_r\} \) in \( M_R \) of a finite subset \( \{u_0, u_1, \ldots, u_r\} \subset M \) an integral convex polytope in \( M_R \). The correspondence is given by the isomorphism

\[
H^0(X, L) \cong \bigoplus_{m \in \mathbb{P}^1 \setminus M} C(e(m)),
\]

where \( e(m) \) are considered as rational functions on \( X \) because they are functions on an open dense subset \( T_N \) of \( X \) (see Section 2.2 [Od], or Section 3.5 [F]).

Let \( P_1 \) and \( P_2 \) be integral convex polytopes in \( M_R \). Then we can consider the Minkowski sum \( P_1 + P_2 := \{x_1 + x_2 \in M_R; x_i \in P_i \ (i = 1, 2)\} \) and the multiplication by scalars \( rP_1 := \{rx \in M_R; x \in P_1\} \) for a positive real number \( r \). If \( l \) is a natural number, then \( IP_1 \) coincides with the \( l \) times sum of \( P_1 \), i.e., \( IP_1 = \{x_1 + \cdots + x_l \in M_R; x_i \in P_1\} \). The \( l \) times twisted sheaf \( L^{\otimes l} \) corresponds to the convex polytope \( IP_1 := \{lx \in M_R; x \in P_1\} \). Moreover the multiplication map

\[
H^0(X, L^{\otimes l}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (l+1)})
\]

transforms \( e(u_1) \otimes e(u_2) \) for \( u_1 \in IP \cap M \) and \( u_2 \in P \cap M \) to \( e(u_1 + u_2) \) through the isomorphism (1.1). Therefore the equality \( IP \cap M + P \cap M = (l + 1)P \cap M \) means the surjectivity of (1.2). For the case of dimension two Koelman \([Ko]\) proved that \( IP \cap M + P \cap M = (l + 1)P \cap M \) for all natural number \( l \). Nakagawa and Ogata generalize this in the higher dimension.

**Proposition 1.1 (Nakagawa-Ogata \([NO]\)).** Let \( P \) be an integral polytope of dimension \( n \) (>1). Then

\[
iP \cap M + P \cap M = (i + 1)P \cap M
\]

for all \( i \geq n - 1 \).

For a proof see Proposition 1.2 in \([NO]\).

In this article we assume that \( L \) is very ample, that is, the global sections of \( L \) defines an embedding of \( X \) into the projective space \( \mathbb{P}(H^0(X, L)) \cong \mathbb{P}' \). Since \( H^0(P', C_{P'}(1)) \cong H^0(X, L) \), the \( k \)-normality of \( X \) implies the surjectivity of the multiplication map \( \text{Sym}^k H^0(X, L) \to H^0(X, L^{\otimes k}) \). We denote the subset of \( kP \cap M \) consisting of sums of \( k \) elements in \( P \cap M \) by \( \sum^k P \cap M \). Then the \( k \)-normality means the equality

\[
\sum^k P \cap M = kP \cap M.
\]
Next we may explain how to describe a weighted projective space as a toric variety according to Fulton’s book [F]. Let \( q_0, q_1, \ldots, q_n \) be positive integers with \( \gcd\{q_0, q_1, \ldots, q_n\} = 1 \). Then we define the weighted projective \( n \)-space with the weight \((q_0, q_1, \ldots, q_n)\) as the quotient \( \mathbb{P}(q_0, q_1, \ldots, q_n) := \mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^* \), where the action of \( t \in \mathbb{C}^* \) is defined by \( t \cdot (x_0, x_1, \ldots, x_n) = (t^{q_0} x_0, t^{q_1} x_1, \ldots, t^{q_n} x_n) \). We know that the space can be expressed as the quotient of the projective \( n \)-space by an action of a finite abelian group as \( \mathbb{P}(q_0, q_1, \ldots, q_n) \cong \mathbb{P}^n/(\mathbb{Z}/(q_0) \times \mathbb{Z}/(q_1) \times \cdots \times \mathbb{Z}/(q_n)) \). Let \( m := \text{lcm}\{q_0, q_1, \ldots, q_n\} \) and \( d_i = m/q_i \) for \( i = 0, 1, \ldots, n \). Set \( u_0 = (d_0, 0, \ldots, 0), \ u_1 = (0, d_1, 0, \ldots, 0), \ldots, u_n = (0, \ldots, d_n) \) in \( M := \mathbb{Z}^{n+1} \). Let \( P = \text{Conv}\{u_0, u_1, \ldots, u_n\} \) be a convex hull of this \( n+1 \) points in \( M \). Let \( H \) be the affine hyperplane containing \( P \), and let \( M := H \cap M \). Then \( P \subset M = H \) is an integral convex polytope of \( M \). The integral convex polytope \( P \) defines a polarized toric variety \( (\mathbb{P}(q_0, q_1, \ldots, q_n), \mathcal{O}(m)) \). We can easily see that on \( \mathbb{P}(1,6,10,15) \) the invertible sheaf \( \mathcal{O}(30) \) is ample, but not very ample.

In this paper we treat an integral \( n \)-simplex \( P \) in \( M = \mathbb{Z}^n \), which corresponds not only to a weighted projective space but also to a toric variety defined as a quotient of the projective \( n \)-space by a finite abelian group. For example, set \( n = 3 \) and \( P = \text{Conv}\{(0,0,0),(1,0,0),(0,1,0),(3,3,4)\} \). Then the corresponding toric variety \( X \) is isomorphic to \( \mathbb{P}^3/\langle \zeta \rangle \), where \( \zeta \) is a primitive 4-th root of unity, and the corresponding embedding is \( X \cong \{z_0z_1z_2z_3 = z_4^3\} \subset \mathbb{P}^4 \).

2. \textit{k-normality}

Let \( n \) be an integer greater than two and \( M = \mathbb{Z}^n \). Let \( P = \text{Conv}\{u_0, u_1, \ldots, u_n\} \) be an integral \( n \)-simplex with its vertices \( u_0, u_1, \ldots, u_n \in M \). We assume that \( L \) is very ample for the polarized toric variety \((X, L)\) corresponding to \( P \). We may say that \( P \) is \textit{very ample} when \( L \) is very ample.

\textbf{Lemma 2.1.} \textit{Let} \( P = \text{Conv}\{u_0, u_1, \ldots, u_n\} \textit{be a very ample integral} \( n \)-\textit{simplex. Let} \( s \) \textit{be an integer greater than one and let} \( x \in sP \cap M \). \textit{Then for any} \( u_i \) \textit{there exist} \( x_1, \ldots, x_{2s-1} \in P \cap M \) \textit{with} \( (s-1)u_i + x = x_1 + \cdots + x_{2s-1} \).

\textit{Proof.} Since \( sP = \text{Conv}\{su_0, su_1, \ldots, su_n\} \), any \( x \in sP \) can be expressed uniquely as a linear combination \( x = \sum_{i=0}^n \lambda_i u_i \) with \( 0 \leq \lambda_i \leq 1 \). We may write as \( x = \sum_{i=0}^n \mu_i (s u_i) \) with \( 0 \leq \mu_i \leq 1 \). For simplicity we may take \( u_i \) as \( u_0 \). By an affine transformation of \( M \) we may put \( u_0 \) as the origin. Then \( x = \sum_{i=0}^n \lambda_i u_i \) is contained in \( tP \) if and only if \( \sum_{i=1}^n \lambda_i \leq t \). Now since \( x \in sP \), we have \( \sum_{i=1}^n \lambda_i \leq s \). Since \( P \) is very ample, the equality \((1.3)\) holds for a sufficiently large \( k \). Hence, for \((k-s)u_0 + x \in kP \cap M \) there exist \( x_1, \ldots, x_k \in P \cap M \) such that \((k-s)u_0 + x = x_1 + \cdots + x_k \). If \( x_1 + x_2 \notin P \), then by setting \( y_1 = x_1 + x_2 \) we have \((k-1-s)u_0 + x = y_1 + x_3 + \cdots + x_k \) with \( y_1 \in P \cap M \). If we write as \( x_1 + x_2 = \sum_{i=0}^n \lambda'_i u_i \) and if \( x_1 + x_2 \notin P \), then \( \sum_{i=1}^n \lambda'_i > 1 \). Hence, if \( x_i + x_j \notin P \) for every \( i \) and \( j \), then \( \sum_{i=1}^n \lambda_i > \frac{k}{2} \). This implies \( k < 2s \).
PROPOSITION 2.2. Let \( P = \text{Conv}\{u_0, u_1, \ldots, u_n\} \) be an integral \( n \)-simplex. If \( P \) is very ample, then we have
\[
lP \cap M = (l - 1)P \cap M + P \cap M
\]
for all \( l > n/2 \).

In particular, if \( P \) is a very ample integral 3-simplex, then it is normally generated.

**Proof.** Set \( l \geq 2 \). Assume that \( lP \cap M \neq (l - 1)P \cap M + P \cap M \). Take \( x \) in \( lP \cap M \) but not in \((l - 1)P \cap M + P \cap M \). We can express uniquely as \( x = \sum_{i=0}^{n} \lambda_i u_i \) with \( \lambda_i \geq 0 \) and \( \sum_{i=0}^{n} \lambda_i = l \). From Lemma 2.1 there exist \( x_1, \ldots, x_{2l-1} \in P \cap M \) such that \((l - 1)u_0 + x = x_1 + \cdots + x_{2l-1} \). Move \( u_0 \) to the origin. Set \( P_{lP} = (l - 1/n)u_0 \); \( u_1, \ldots, u_{n} \). We shall find \( x \) for all \( l \). \( \lambda_i \) are generated by the assumption. Since \( U \) is contained in \( \text{Conv}(x_1, \ldots, x_{2l-1}) \), then \( x \) is not contained in \( (l - 1)P \cap M \). Hence we have \( \lambda_i < 1/2 \) for \( i = 0, 1, \ldots, n \). Thus we have \( l = \sum_{i=0}^{n} \lambda_i < (n + 1)/2 \). The inequality \( n/2 < l < (n + 1)/2 \) does not hold. Hence we have \( lP \cap M = (l - 1)P \cap M + P \cap M \).

LEMMA 2.3. Let \( P = \text{Conv}\{u_0, u_1, \ldots, u_n\} \) be an integral \( n \)-simplex. For \( l \geq n + 1 \) we have
\[
lP = \bigcup_{i=0}^{n} \{u_i + (l - 1)P\}.
\]

PROPOSITION 2.4. Let \( n \geq 4 \) and let \( P = \text{Conv}\{u_0, u_1, \ldots, u_n\} \) a very ample integral \( n \)-simplex. For \( l \geq n - 1 + [n/2] \) we have
\[
\sum_{i=0}^{l} P \cap M = lP \cap M.
\]

**Proof of Proposition 2.4.** Set \( t = [n/2] \). Then \( l \geq n - 1 + t \). Take \( x \in lP \cap M \). We shall find \( x_1, \ldots, x_l \in P \cap M \) with \( x = x_1 + \cdots + x_l \). If we successively \( l - n \) times apply Lemma 2.3, then we can find nonnegative integers \( a_0, a_1, \ldots, a_n \) with \( \sum_{i=0}^{n} a_i = l - n \) and an \( x' \in P \cap M \) such that \( x = \sum_{i=0}^{n} a_i u_i + x' \). By applying Proposition 2.2 \( n - t \) times to \( x' \in nP \cap M \), there exist \( x_1, \ldots, x_{n-t} \in P \cap M \) and an \( y \in tP \cap M \) such that \( x' = y + x_1 + \cdots + x_{n-t} \). If we could find \( x_{n-t+1}, \ldots, x_l \in P \cap M \) with \( \sum_{i=0}^{n} a_i u_i + y = x_{n-t+1} + \cdots + x_l \), then we complete the proof. It is obtained by the following lemma.

LEMMA 2.5. Set \( t = [n/2] \). For nonnegative integers \( a_0, a_1, \ldots, a_n \) with \( \sum_{i=0}^{n} a_i = t - 1 \) and \( y \in tP \cap M \) there exist \( y_1, y_2, \ldots, y_{2l-1} \in P \cap M \) such that
\[
\sum_{i=0}^{n} a_i u_i + y = y_1 + \cdots + y_{2l-1}.
\]
Proof. Take $a_i$ to be positive. From Lemma 2.1 there exist $y_1, \ldots, y_{2t-1} \in P \cap M$ such that $(t-1)u_i + y = y_1 + \cdots + y_{2t-1}$. Move $u_i$ to the origin. If sum of any two among $y_j$’s is contained in $P$, then we may write as $y = y_1 + \cdots + z_{t-1} + y_{2t-1}$ with $z_j = y_{2j-1} + y_{2j}$ for $j = 1, \ldots, t - 1$. Thus $y$ is in $\sum P \cap M$. In this case we proved the lemma since $\sum_{i=0}^n a_i u_i \in \sum (t-1) P \cap M$. If $y_1 + y_2 \notin P$, then $z := y_3 + \cdots + y_{2t-1}$ is contained in $(t-1)P$. Thus we have $u_i + y = y_1 + y_2 + z$ in $(t+1)P \cap M$. Next we consider $\sum_{j \neq i} a_j u_j + (a_i - 1) u_i + z$ for $z \in (t-1)P \cap M$. By induction we obtain a proof.

References


