ON V-HARMONIC FORMS IN COMPACT LOCALLY CONFORMAL KÄHLER MANIFOLDS WITH THE PARALLEL LEE FORM

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Introduction. A locally conformal Kähler manifold (l.c. K-manifold) has been studied by I. Vaisman [8]. Especially when its Lee form is parallel, the manifold seems to have properties exceedingly similar to that of a Sasakian manifold. In this paper, we consider certain forms which correspond to C(C*)-harmonic forms of a Sasakian manifold and with it we have some informations on the Betti number of the manifold by a decomposition of such forms. The main result is that in a 2m-dimensional compact l.c. K-manifold with the parallel Lee form, the following relation holds good between the p-th (p < m) Betti number \( b_p \) and the dimension \( a_p \) of the vector space of certain \( p \)-forms which are defined in §2:

\[
\begin{align*}
 b_p &= a_p - a_{p-2}, \\
 a_p &= b_p + b_{p-2} + \cdots + b_{p-2r} ,
\end{align*}
 r = \left\lfloor \frac{b}{2} \right\rfloor.
\]

§ 1. Preliminaries. A locally conformal Kähler manifold is characterized as a Hermitian manifold \( M^{2m}(\varphi, g) \), \( 2m \) = the dimension, such that

\[

\nabla_k \varphi_{ji} = -\alpha_j \varphi_{ki} + \alpha_i \varphi_{jk} - \alpha_i \varphi_{ki} + \alpha_i \varphi_{jk}, \quad (\varphi_{ji} = \varphi_{ji} g_{kj})
\]

with a closed 1-form \( \alpha \) which is called the Lee form, ([2], [8]). Moreover, we assume \( \nabla \alpha = 0, |\alpha| = 1 \) and \( M \) is compact throughout this paper.

In this manifold, the following formulas are valid:

\[

\begin{align*}
 \nabla_k \varphi_{ji} &= -\beta_j g_{ki} + \beta_i g_{kj} - \alpha_j \varphi_{ki} + \alpha_i \varphi_{kj} , \\
 \beta_j &= \alpha^i \varphi_{ij} , \\
 J_{ji} &= \nabla_j \beta_i = -\beta_j \alpha_i + \alpha_j \beta_i - \varphi_{ji} , \quad (=-\nabla_i \beta_j) , \\
 \alpha^i J_{ri} &= \beta^i J_{ri} = 0 , \\
 J_i J^l &= \beta_i \beta^l + \alpha_i \alpha^l - \delta_i^l , \\
 \nabla_k \nabla_j \beta_i &= -\beta^l R_{ijkl} \\
 &= \beta_j g_{ki} - \beta_i g_{kj} + (\alpha_j \beta_i - \beta_j \alpha_i) \alpha_k ,
\end{align*}
\]

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Furthermore, by virtue of Ricci's identity, we have

\[ \alpha R_{rj} = 0, \]

from which

\[ \frac{1}{2} R_{hhr^2} J_{rk} = -R_{h} J_{rj} + (2m-3) J_{hj}, \]

\[ R J_{rj} + J_{rj} J_{th} = 0, \]

\[ R_{k} J_{rj} - R_{h} J_{rj} = -(J_{rj} t_{r} J_{h}) r, \]

\[ R_{hhr^2} J_{r} J_{h} = R_{rj} t_{r} J_{h} J_{h}. \]

The exterior product of 1 or 2-form \( \omega \) and p-form \( u = \frac{1}{p!} u_{1 \ldots p} dx_{1} \wedge \ldots \wedge dx_{p} \) is given as

\[ (\omega \wedge u)_{i_{1} \ldots i_{p+1}} = \sum_{k=1}^{n} (-1)^{k+i} u_{i_{k} u_{i_{1} \ldots i_{p+1}}} \]

(\( \omega \): 1-form),

\[ (\omega \wedge u)_{i_{1} \ldots i_{p+2}} = \sum_{k<l} (-1)^{k+l+1} u_{i_{k} i_{l} u_{i_{1} \ldots i_{p+2}}} \]

(\( \omega \): 2-form),

where \( u_{i_{1} \ldots i_{p}} \) means \( i_{k} \) is omitted, and the inner product for p-forms \( u, v \) is

\[ (u, v) = \frac{1}{p!} \int_{M} u_{1 \ldots p} v_{1 \ldots p} d\sigma. \]

In general, the star operator * in a Hermitian manifold satisfies for a p-forms \( u, v \)

\[ **u = (-1)^{p} u, \quad (*u, *v) = (u, v), \]

\[ \delta u = -*\delta *u, \quad \Delta * = *\Delta, \]

where

\[ (du)_{i_{0} \ldots i_{p}} = \sum_{k=0}^{p} (-1)^{k} \nabla_{i_{k}} u_{i_{1} \ldots i_{p}}, \quad (\delta u)_{i_{1} \ldots i_{p}} = -\nabla_{i_{1} \ldots i_{p}}, \]

\[ (\Delta u)_{i_{1} \ldots i_{p}} = (\delta du + d\delta u)_{i_{1} \ldots i_{p}} \]

\[ = -\nabla_{i_{1} \ldots i_{p}} u_{i_{1} \ldots i_{p}} + \sum_{k} R_{i_{k}} u_{i_{1} \ldots i_{p}} \]

and \( u_{i_{1} \ldots i_{p}} \) means that \( r \) appears at the \( k \)-th position.

Let operators \( e(\omega), i(\omega) \) with respect to a 1-form \( \omega \) and \( L, A \) be as follows
for a $p$-form $u$:

$$e(\omega)u = \omega \wedge u, \quad i(\omega)u = *e(\omega)*u,$$

$$Lu = d\beta \wedge u = (e(\beta)d + d e(\beta))u,$$

$$Au = (-1)^p *L*u = (i(\beta)\delta + \delta i(\beta))u.$$

Explicitly, these are written as

$$(i(\omega)u)_{i_2 \ldots i_p} = \omega^{r_{i_2 \ldots i_p}},$$

$$(Lu)_{i_1 \ldots i_p} = 2 \sum_{k<l} (-1)^{k+l+1} \nabla_{i_k} \beta_{i_l} u_{i_2 \ldots \hat{i_k} \ldots \hat{i_l} \ldots i_{p+2}},$$

$$(Au)_{i_3 \ldots i_p} = \nabla \beta^{i_1} u_{r_1 i_2 \ldots i_p}.$$

It should be remarked that

$$(e(\omega)u, v) = (u, i(\omega)v), \quad (Lu, v) = (u, A\nu).$$

Besides under the condition $\nabla \alpha = 0$, it is valid for $\omega = \alpha, \beta,$

$$(Le(\omega) = e(\omega)L, \quad Ai(\omega) = i(\omega)A, \quad \ldots \quad Li(\omega) = i(\omega)L, \quad Ae(\omega) = e(\omega)A.$$ (1.6)

Since $\omega = \alpha, \beta$ are Killing, the Lie derivative

$$\theta(\omega) = i(\omega)d + d i(\omega)$$

satisfies the relations ([1]):

$$\theta(\omega) = -(e(\omega)\delta + \delta e(\omega))$$

and then $\theta(\omega)$ commutes with $i(\omega), e(\omega), d$ and $\delta$ for $\omega = \alpha, \beta$ respectively. In the following we often write briefly $e, i$ (resp. $e', i'$) instead of $e(\beta), i(\beta)$ (resp. $e(\alpha), i(\alpha)$).

We notice here that

$$ei' + ie = \text{identity},$$

(1.7)

$$ei'' = -i'e, \quad e' i = -ie', \quad ii' = -i'i,$$

and

$$\Delta e - e \Delta = \delta L - L \delta, \quad \Delta i - i \Delta = d A - A d,$$

(1.8)

$$\Delta e' - e' \Delta = 0, \quad \Delta i' - i' \Delta = 0$$

because, for any Killing vector $\omega$, the following relation holds good:
\[(\Delta e(\omega) - e(\omega) \Delta) u = \delta(d\omega \wedge u) - d\theta(\omega) u + \theta(\omega) du - d\omega \wedge \delta u.\]

We remark also \(\nabla_\beta J_{st} = \nabla_\alpha J_{st} = 0\), and from which for \(\omega = \alpha, \beta\)
\[i(\omega)\nabla_\beta = -\nabla_\beta i(\omega), \quad i(\omega)\nabla_\alpha = -\nabla_\alpha i(\omega),\]
where we denote \(\nabla_\alpha u_{1 \ldots p} = \omega^r \nabla_r u_{1 \ldots p}^r\).

\[\nabla_\omega \nabla_\alpha \ldots u = \omega^r \nabla_r u_\ldots.\]

In this paper the following formulas are used frequently:

**Lemma 1.1.** In a l.c. K-manifold \(M^{2m}\) with the parallel Lee form, \(|\alpha| = 1\), the followings hold good for any \(p\)-form \(u\).

1. \((AL^k - L^k A)u = 4k(m - p - k) L^{k-1} u + 4k(e'i' + ei) L^{k-1} u.\)
2. \((\delta L - L \delta) u = 2(\delta \nabla_\beta \nabla_\beta d) u + 4((m - p) e - e' e') u.\)
3. \((d A - A d) u = 2(-\delta \nabla_\beta \nabla_\beta d) u + 4((p - m)i - e' i') u.\)

Proof. (i) and (i)' are known by the mathematical induction. (ii): Putting
\[\Gamma u = \sum_{k=0}^{\infty} (-1)^k J_{1k} r \nabla_r u_{1 \ldots k \ldots p},\]
we get by straightforward computations
\[\frac{1}{2}(\delta L u - L \delta u) = \Gamma u + ee'i'u + (2m - p - 2)e u,\]
\[d \nabla_\beta u - \nabla_\beta d u = \Gamma u - ee'i'u + pe u.\]

(ii) is known by the dual of (ii) and the property \(*\nabla_\beta u = \nabla_\beta *u.\) q.e.d.

§ 2. \(V\)-harmonic forms. At first we get

**Lemma 2.1.** If \(u\) is harmonic \(p\)-form, then
1. \(i(\alpha) u \) and \(e(\alpha) u\) are harmonic,
2. \(\nabla_\alpha u = 0\),
3. \((\Lambda A) u = 0\) (effective) and \(i(\beta) u = 0\) provided that \(p < m\).

(i) and (ii) are evident if we notice that \(\theta(\omega) u = 0\) for a Killing vector \(\omega\), a harmonic \(p\)-form \(u\), and \(\nabla_\alpha = \theta(\alpha)\).

Proof of (iii): In general for any \(p\)-form \(u\), from (1.8) and Lemma 1.1, it follows that
\[\Delta e i u = 2(d \nabla_\beta i - \nabla_\beta d i) u + e(\Delta i + 4(m - p) e i + 4 ee'i'i) u.\]
\[ (i\epsilon u, \Delta \epsilon u) = 2(i\epsilon u, d\nabla_{\beta}i u - \nabla_{\beta} d\epsilon u) \]
\[ = 2(\Lambda u - i\delta u, \nabla_{\beta}i u) - 2(i\epsilon u, \nabla_{\beta} d\epsilon u) \]
\[ = -2(i\delta u, X_{\beta}i u) - 2(i\epsilon u, X_{\beta} d\epsilon u), \]
where we have used \((\Lambda u, l\beta i u) = ^{\Lambda u, H\beta i u} = ^{\lambda}.\)

Hence, for a form \(u\) which satisfies
\[ (2.1) \quad d\epsilon u = \delta e u = 0, \]
the equality
\[ (2.2) \quad (i\epsilon u, \Delta \epsilon u) = 0 \]
holds good. We notify beforehand that this fact will be used after again in the proof of Lemma 2.5.

Now, let \(u\) be harmonic. By virtue of \(\theta(\beta)u = 0\), (2.1) is satisfied and then from (2.2) it follows \(d\epsilon u(= L i u) = 0\). So, making use of Lemma 1.1, we can obtain
\[ (-\Lambda i u, i u) = 4((m - p)i u + e^{i'i u, i u}), \]
which implies \(i u = 0\) under \(p < m\), and then \(\Lambda u = (\delta i + i\delta)u = 0\). \text{q.e.d.}

**Definition.** A form \(u\) is called \(V\)-harmonic if it satisfies
\[ du = 0 \quad \text{and} \quad \delta u = e(\beta)\Lambda u. \]
As a harmonic \(p\)-form \((p < m)\) is effective, the following is trivial:

**Proposition 2.2.** A \(p\)-form \((p < m)\) is harmonic if and only if it is effective \(V\)-harmonic.

Corresponding to a well known property between a harmonic form and a Killing vector, we can get

**Proposition 2.3.** For any \(V\)-harmonic form \(u,\)
\[ \theta(\beta)u = 0 \]
holds.

This property follows immediately from the Lemma:

**Lemma 2.4.** For any \(V\)-harmonic form \(u, d\theta(\beta)u = 0\) is valid.

**Proof.** If \(u\) is \(V\)-harmonic, taking account of (1.8), we get
\[ \delta d\epsilon u = \Delta i u - d\delta i u = i \Delta u + d i u \]
\[ =i \delta u + d \delta u = \theta(\beta) e \Lambda u. \]

Then it follows that

\[(d i u, d i u) = (i u, d d i u) = (i u, \theta(\beta) e \Lambda u) = 0\]

because of \[ e \theta(\beta) = \theta(\beta) e \]. q. e. d.

Next we shall consider orthogonal property to \( \beta \) of V-harmonic form. For it, we provide

**Lemma 2.5.** For any V-harmonic form \( u \), it is valid that

(i) \( \delta e(\beta) u = 0 \),

(ii) \( i(\beta) u \) is V-harmonic,

(iii) \( Li(\beta) u = 0 \).

*Proof.* (i) follows from \( \delta e u = -\theta(\beta) u - e \delta u = 0 \).

(ii): \( d i u = 0 \) is Lemma 2.4. Next, taking account of (1.6), we have

\[ \delta i u = \Lambda u - i \delta u = \Delta u - i e \Lambda u = e \Lambda i u. \]

(iii): By virtue of (i) and (ii), the equality (2.1) holds good, and then (2.2) as mentioned before. Hence on account of \( \delta e i u = -\theta(\beta) i u - e \delta i u = 0 \) (ii)), we can get

\[ (d e i u, d e i u) = -(d i u, d e i u) = -(i e u, \Delta e i u) = 0, \]

which implies \( L i u = 0 \). q. e. d.

**Theorem 2.6.** In a compact l. c. K-manifold \( M^m(\phi, g, \alpha) \) with the parallel Lee form, a V-harmonic \( p \)-form \( u \) \((p < m)\) is orthogonal to \( \beta \), i.e., \( i(\beta) u = 0 \).

*Proof.* By virtue of Lemma 2.5 and Lemma 1.1, we get

\[ -\Lambda i u = 4((m - p) i + e' i' i) u, \]

and then

\[ (i u, -\Lambda i u) = -(\Lambda i u, i u) \]

\[ = 4(m - p)(i u, i u) + 4(i' i u, i' i u). \]

This equality implies \( i u = 0 \) for \( m > p \). q. e. d.

**Proposition 2.7.** If a \( p \)-form \( u \) \( (p < m) \) is V-harmonic, then so is \( \Lambda u \).
Proof. Let \( u \) be \( V \)-harmonic. About the codifferential, we know obviously 
\[ \partial \Delta u = \Lambda \partial u = e \Lambda (Au). \]

We shall prove now \( d \Delta u = 0 \). On account of Lemma 1.1 and (1.7), we have 
\[ d \Delta u = 2(-\partial \nabla_\beta u + \nabla_\beta \partial u), \]
from which 
\[ (d \Delta u, d \Delta u) = 2(Au, \partial \nabla_\beta e Au) \]
\[ = 2(Au, -\theta(\beta) \nabla_\beta Au - e \partial \nabla_\beta Au) \]
\[ = 2(\theta(\beta) Au, \nabla_\beta Au) - 2(\Delta Au, \partial \nabla_\beta Au) \]
\[ = 0, \]
where we have used \( \nabla_\beta e = e \nabla_\beta \) and the properties of Lie derivative: 
\[ \theta(\beta) A = \Lambda \theta(\beta), (\theta(\beta) v, w) = -(v, \theta(\beta) w) \] for any forms \( v, w \). q.e.d.

We can also state a \( V \)-harmonic form with the Laplacian as follows:

**Proposition 2.8.** A \( p \)-form \( u \) \((p < m)\) is \( V \)-harmonic if and only if \( i(\beta) u = 0 \) and \( Au = \Lambda Au \).

**Proof.** Necessity follows from \( \Delta u = d \Delta u = d e Au = \Lambda \Delta u \) (Prop. 2.7) and Theorem 2.6.

Now we prove the sufficiency. Since 
\[ (d u, d u) + (\partial u - e Au, \partial u - e Au) \]
\[ = (d u, d u) + (\partial u, \partial u) - 2(e Au, \partial u) + (e Au, e Au) \]
\[ = (u, \Delta u) - 2(Au, Au - \partial i u) + (Au, Au - e i Au), \]
we know, under the assumption \( u = 0 \) and \( \Delta u = \Lambda \Delta u \), the right hand side is zero. Hence \( d u = 0, \partial u = e Au \), which prove our Theorem. q.e.d.

The following Proposition provides examples of \( V \)-harmonic forms actually.

**Proposition 2.9.** The \( 2k \)-form \( L^k \cdot 1 \) is \( V \)-harmonic for any \( k \).

**Proof.** \( d(L^k \cdot 1) = 0 \) is trivial. So we shall prove \( \delta L^k \cdot 1 = eAL^k \cdot 1 \) by induction.

For \( k = 1 \), as \( \delta L^1 \cdot 1 = 4(m-1)\beta, eAL^1 \cdot 1 = 4(m-1)\beta \), it is satisfied. Now we assume \( \delta L^{k-1} \cdot 1 = eAL^{k-1} \cdot 1 \). Taking account of Lemma 1.1 and \( \nabla_\beta d \beta = 0, i' L' \cdot 1 = 0 \), we can obtain 
\[ \delta L^k \cdot 1 = \delta L \cdot L^{k-1} \cdot 1 \]
\[ = L \delta L^{k-1} \cdot 1 + 4(m-2k+1)e L^{k-1} \cdot 1 \]
§ 3. Decomposition to harmonic forms. The purpose of this section is to study the Betti number relating with $V$-harmonic forms.

At first we consider a relation between $\Delta A$ and $A\Delta$. On account of Lemma 1.1 and $\Lambda \delta = \delta \Lambda$, we have for any $p$-form $u$

\[
A\Delta u = A(\delta d + d\delta)u
\]

\[
= \delta d Au - 2\delta(\nabla_{\beta} \delta + 2(m-p)\iota - 2ie'i')u
\]

\[
+ \delta d Au - 2(-\delta \nabla_{\beta} \delta + 2(p-1-m)i\delta - 2ie'i\delta)u
\]

\[
= \Delta Au - 4(p-m)Au + 4(i\delta + di'e'i' + e'i'i\delta)u.
\]

Since

\[-\delta e'i'i'u = \theta(\alpha)ii'u + e'i'i'u
\]

\[
= \nabla_{\alpha}ii'u + e'i'Au - e'i'i\delta u,
\]

taking account of $\delta i' = -i'\delta$ and $\theta(\alpha) = \nabla_{\alpha}$, we can get finally

\[
A\Delta u - \Delta Au = 4\{m-p+1\delta + \nabla_{\alpha}ii'u + e'i'i\delta u\}.
\]

**LEMMA 3.1.** For any $p$-form $u$, we have

\[
(A\Delta - \Delta A)u = 4\{m-p+1\delta + \nabla_{\alpha}ii'L - e'i'i'L + \nabla_{\alpha}e(\alpha)e(\delta)\} u
\]

Taking the dual of above formula and on account of $*\nabla_{\alpha} = \nabla_{\alpha}$ we can get

**LEMMA 3.2.** For any $p$-form $u$,

\[
(L\Delta - \Delta L)u = 4\{(p-m+1)L - e(\alpha)i(\alpha)L + e(\delta)d - \nabla_{\alpha}e(\alpha)e(\delta)\} u
\]

holds good.

Especially if $u$ is a $V$-harmonic $p$-form ($p < m$), Lemma 3.2 implies

\[
\Delta Lu = LLu - 4((p-m+1)L - e'i'L + \nabla_{\alpha}e'eu)
\]

\[
= L(LAu - 4(e'eu - 4e'eu) - 4((p-m+1)L - e'i'L + \nabla_{\alpha}e'eu)
\]

\[
= L\Lambda(Lu - 4\nabla_{\alpha}e'eu,
\]

which means $Lu$ is also $V$-harmonic if $\nabla_{\alpha}e'eu = 0$.

From this fact, we know that the $(2p+1)$-form $(2p+1 < m)\alpha_{\wedge}d\beta_{\wedge,\ldots,\wedge}d\beta$ is $V$-harmonic, because $e'\alpha_{\wedge}d\beta_{\wedge,\ldots,\wedge}d\beta = 0$ and $\alpha$ is $V$-harmonic.
Proposition 3.3. If \( u \) is a harmonic \( p \)-form, then \( L^k u \) is \( V \)-harmonic, where \( 2+p \leq 2k+p \leq m+2 \).

Proof. It is sufficient to notice
\[
\nabla a' e' e L^k u = 0,
\]
which follows from Lemma 2.1 (ii), \( \nabla a' e' e = e' e \nabla a \) and \( \nabla a d = 0 \).
q. e. d.

Theorem 3.4. In a compact l.c. K-manifold \( M^{2m}(\varphi, g, \alpha) \) with the parallel Lee form, any \( V \)-harmonic \( p \)-form \( u \) \((p < m)\) can be represented uniquely as
\[
u = \sum_{k=0}^{p} L^k \phi_{p-2k}, \quad r = \left[ \frac{p}{2} \right],
\]
where \( \phi_{p-2k} \) is harmonic \((p-2k)\)-form.

Conversely, \( p \)-forms \((p < m)\) of the type in the right hand side are \( V \)-harmonic.

Proof. We shall prove it by the mathematical induction. At first the case \( p = 0 \) and 1 are trivial because a \( V \)-harmonic form is harmonic necessarily.
We assume now its validity for \((p-2)\)-form. Let \( u \) be a \( V \)-harmonic \( p \)-form \((p < m)\). Since \( Au_p \) is \( V \)-harmonic by virtue of Proposition 2.7, there exist harmonic \((p-2-2k)\)-forms \( \phi_{p-2-2k} \) such that
\[
Au_p = \sum_{k} L^k \phi_{p-2-2k}.
\]
Now we put
\[
\nu_{p-2} = \sum_{k} L^k \phi_{p-2-2k},
\]
where
\[
\phi_{p-2-2k} = \frac{\psi_{p-2-2k}}{4(k+1)(m-p+k+1)} - \frac{e' e' \phi_{p-2-2k}}{4(k+1)(m-p+k+1)(m-p+2+k)}.
\]
From Lemma 2.1, \( \phi_{p-2-2k} \) are also harmonic. By virtue of Lemma 1.1 and Lemma 2.1, it follows
\[
AL\nu_{p-2} = \sum_{k} AL^{k+1} \left( \phi_{p-2-2k} - \frac{e' e' \phi_{p-2-2k}}{m-p+2+k} \right) / (4(k+1)(m-p+k+1))
\]
\[
= \sum_{k} \left[ 4(k+1)(m-p+k+1) L^k \phi_{p-2-2k} + 4(k+1)e' e' L^k \phi_{p-2-2k} \right.
\]
\[
= \sum_{k} \left[ 4(k+1)(m-p+k+1) L^k \phi_{p-2-2k} + 4(k+1)e' e' L / (m-p+2+k) \right]
\]
\[
= \sum_{k} L^k \phi_{p-2-2k}
\]
namely, \( AL\nu_{p-2} = Au_p \). Now we define a \( p \)-form \( \phi_p \) as
\[ \phi_p = u_p - L v_{p-2}. \]

Since \( L v_{p-2} = \sum k \phi_{p-2-2k} \) is \( V \)-harmonic because of Proposition 3.3, \( \phi_p \) is \( V \)-harmonic. Moreover as \( A \phi_p = A u_p - A L v_{p-2} = 0 \), \( \phi_p \) is harmonic. Then \( u_p = \phi_p + L v_{p-2} = \phi_p + \sum k \phi_{p-2-2k} \) is the desired representation.

The uniqueness comes from the following Lemma:

**Lemma 3.5.** For harmonic \( p, q \)-form \( \omega, \zeta \) \((p, q < m)\), we have
\[
(L^k \omega, L^h \zeta) = 0 \quad (k \neq h).
\]

**Proof.** As \( i \omega = A \omega = 0 \), making use of Lemma 1.1 and (1.6), we know for \( h < k \)
\[
A^h L^k \omega = \lambda L^{k-h} \omega + \mu L^{k-h} e' \omega, \quad (\lambda, \mu = \text{const.}).
\]

Then from the property \( (L \omega, \zeta) = (\omega, A \zeta) \), the Lemma is proved. q.e.d.

From Proposition 3.3, Theorem 3.4 and Lemma 3.5, we can get

**Corollary 3.6.** If \( u \) is a \( V \)-harmonic \( p \)-form \((p < m)\), then so is \( Lu \). Moreover the operator \( L \) is injective.

By virtue of Theorem 3.4 and Corollary 3.6, we can now obtain the desired result:

**Theorem 3.7.** In a compact 2m-dimensional l.c. K-manifold with the parallel Lee form, we have for \( p < m \),
\[
a_p = b_p + b_{p-2} + \cdots + b_{p-2r}, \quad r = \left\lfloor \frac{b}{2} \right\rfloor
\]
\[
b_p = a_p - a_{p-2},
\]
where \( a_p \) is the dimension of the vector space \( V_p \) of all \( V \)-harmonic \( p \)-forms and \( b_p \) is the \( p \)-th Betti number.

§ 4. \( V^* \)-harmonic forms. In this section we shall consider a dual form of a \( V \)-harmonic form.

**Definition.** A form \( u \) is called \( V^* \)-harmonic if it satisfies
\[
d u = i(\beta)L u, \quad \delta u = 0.
\]

For example, \( \beta \wedge d \beta \wedge \ldots \wedge d \beta \) is \( V^* \)-harmonic. From the definition, we know easily

**Proposition 4.1.** A \( p \)-form \( u \) is \( V^* \)-harmonic if and only if the \((2m-p)\)-form \(* u \) is \( V \)-harmonic.
Since $\beta \wedge d\beta \wedge \cdots \wedge d\beta$ is $V^*$-harmonic $(2p+1)$ for any $p$, by virtue of Proposition 4.1, $e_{2m-2p-1} \geq 1$ is valid for any $p$. Hence combining with Proposition 2.9, we can say

**Theorem 4.2.** In a compact $2m$-dimensional c. K-manifold with the parallel Lee form, we have $a_k \geq 1$ for any $k=0, 1, \ldots, 2m$.

**Lemma 4.3.** For a $V^*$-harmonic $p$-form $u$, we have

(i) $e(\beta)u = 0 \quad (p > m)$,

(ii) $\theta(\beta)u = 0 \quad (\forall p)$,

(iii) $\Lambda e(\beta)u = 0 \quad (\forall p)$.

*Proof.* (i) follows from $i*u = 0$, $(2m-p < m)$.

(ii) follows from Proposition 2.3, i.e., $\theta(\beta)*u = \theta(\beta)^*u = 0$ for any $V$-harmonic $(2m-p)$-form $*u$.

(iii) follows from Lemma 2.5 (iii), i.e., $\Lambda e(\beta)*u = (\Lambda e(\beta))^*u = 0$ for any $V$-harmonic $(2m-p)$-form $*u$.

Next we shall consider a decomposition of $V^*$-harmonic forms. For it, we provide some Lemmas.

**Lemma 4.4.** For any $p(\neq m)$, we have

$$H_p = V_p \cap V^*_p,$$

where $V^*_p$ is the vector space of $V^*$-harmonic $p$-forms.

*Proof.* From the definition, $H_p \supseteq V_p \cap V^*_p$ is trivial. We shall prove $H_p \subseteq V_p \cap V^*_p$. For $p < m$, it holds good evidently because of $i*u = \Lambda u = 0$ ($u \in H_p$). As for $p > m$, taking account of that $e\Lambda u = -\Lambda L*u$ and $*u$ is harmonic for a harmonic form $u$, we have also $e\Lambda u = 0$ and $iLu = 0$. Hence the Lemma is proved.

**Lemma 4.5.** (i) $e(\beta)$ is a homomorphism of $V_p \cup V^*_p \to V^*_p$. Especially, $e(\beta)|_{V_p}$ is injective for $p < m$.

(ii) $i(\beta)$ is a homomorphism of $V_p \cup V^*_p \to V_{p-1}$. Especially, $i(\beta)|_{V^*_p}$ is surjective for $p < m+1$.

*Proof.* For $u \in V_p$, $deu = Lu = iLeu + eLu = iLeu$ because $Liu = 0$ for any $p$ (Lemma 2.5), and $\delta eu = -\theta(\beta)u - e\delta u = 0$ because $\theta(\beta)u = 0$ (Prop. 2.3). Then $eu \in V^*_{p+1}$.

For $u \in V^*_p$, $deu = Lu - edu = Lu - eiLu = iLeu$, and as above, by virtue of Lemma 4.2, $\delta eu = 0$. Then $eu \in V^*_{p+1}$.
Especially, for \( u \in V_p \ (p < m) \), if \( eu = 0 \), then \( 0 = ieu = u - eiu = u \), namely, \( e(\beta) \) is 1:1.

In a similar way, (ii) can be verified. Especially for non-zero \( u \in V_{p-1} \), from (i), \( eu \ni V_p^* \), and \( ieu = u - eiu = u \) for \( p - 1 < m \). q. e. d.

**Lemma 4.6.** It is valid that for \( p < m \)

\[ H_p = i(\beta)e(\beta)V_p^* \]

Therefore \( b_p = 0 \) if any only if \( e(\beta)V_p^* \subset \{0\} \) for \( p < m \).

**Proof.** It is sufficient to notice that for \( u \in V_p^* \), \( V_p^* \ni ieu = u - eiu \in V_p^* \) and for \( u \in H_p \ (\subset V_p^*(p < m)) \), \( u = ieu + eiu - ieu \). q. e. d.

**Lemma 4.7.** If \( p < m \), we have \( V_p^* = H_p \oplus e(\beta)V_{p-1} \).

Hence \( a_p^p = b_p + a_{p-1} = b_p + \sum_{k=0}^{r} b_{p-1-2k} \left( r = \left[ \frac{p-1}{2} \right] \right) \) holds good where \( a_p^p \) is the dimension of \( V_p^p \).

**Proof.** \( H_p \cap eV_{p-1} = \{0\} \) follows from \( ieu = u \) for \( u \in V_{p-1} \ (p \leq m) \) which oppose to \( iH_p = \{0\} \ (p < m) \). Next, for \( u \in V_p^* \), from the previous Lemmas, \( u = ieu + eiu \in H_p \oplus eV_{p-1} \) is valid. Moreover from \( V_p^* \supset eV_{p-1} \), \( V_p^* \supset H_p \oplus eV_{p-1} \) is valid also, which completes the proof. q. e. d.

Making use of Lemma 4.7 and Theorem 3.4, we can obtain the following:

**Theorem 4.8.** In a compact l.c. K-manifold \( M^{*m}(\varphi, g, \alpha) \) with the parallel Lee form, any \( V^* \)-harmonic \( p \)-form \( u \ (p < m) \) is decomposed uniquely in the following form.

\[ u = \phi_p + \sum_{k=0}^{r} e(\beta)L^k\phi_{p-1-2k}, \quad r = \left[ \frac{p-1}{2} \right], \]

where \( \phi_k \) is harmonic \( k \)-form.

Conversely, \( p \)-forms \( (p < m) \) of the type in the right hand side are \( V^* \)-harmonic.

**Remark.** Recently, Ogawa and Tachibana [6] obtain the fact that if a connected compact orientable Riemannian manifold admits a parallel vector field, then \( \sum_{k=0}^{p} (-1)^k b_{p-k} \geq 0 \) holds good. Hence in our manifold now, as \( a_p - a_{p-1} \)

\[ \geq \sum_{k=0}^{p} (-1)^k b_{p-k} \] because of Theorem 3.7, we can see the relation \( a_p \geq a_{p-1} \).
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