§ 1. Introduction.

Let \( f(z) \) and \( g(z) \) be entire functions. Then we have the well-known inequality

\[
\log M(r, f(g)) \leq \log M(M(r, g), f).
\]

And it follows from Clunie [2] that if \( g(0) = 0 \), then for \( r \geq 0 \)

\[
\log M(r, f(g)) \leq \log M(c(\rho)M(pr, g), f),
\]

where \( 0 < \rho < 1 \) and \( c(\rho) = \frac{(1 - \rho)^2}{4\rho} \). Furthermore, these inequalities (1) and (2) are best possible. We next wish to have similar estimations of \( T(r, f(g)) \).

As an immediate consequence of (1) and well-known inequalities \( T(r, f) \leq \log^+ M(r, f) \leq 3T(2r, f) \), we have

\[
T(r, f(g)) \leq 3T(2M(r, g), f).
\]

The inequality (3), however, is not sharp.

The main purpose of this paper is to give an upper estimation of \( T(r, f(g)) \) and prove the following:

**Theorem 1.** Let \( f(z) \) and \( g(z) \) be entire functions. If \( M(r, g) > \frac{(2 + \varepsilon)/\varepsilon}{|g(0)|} \) for any \( \varepsilon > 0 \), then we have

\[
T(r, f(g)) \leq (1 + \varepsilon)T(M(r, g), f).
\]

In particular, if \( g(0) = 0 \), then

\[
T(r, f(g)) \leq T(M(r, g), f)
\]

for all \( r > 0 \).

Since \( T(r, f(x^n)) = T(r^n, f(x)) \) for any meromorphic function \( f(z) \), Theorem 1 is best possible. In the above example \( g(z) \) is a polynomial. However, we shall

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prove that

**Theorem 2.** Let \( f(z) \) be a transcendental entire function of order zero and \( g(z) \) a transcendental entire function of lower order zero. Suppose that for any \( 0 < \sigma < 1 \) there are two numbers \( \alpha > 1 \) and \( r_0 > 1 \) such that

\[
\frac{T(r^\sigma, f)}{T(r, f)} > \sigma^\alpha
\]

holds for all \( r > r_0 \). Then we have

\[
\limsup_{r \to \infty} \frac{T(r, f(g))}{T(M(r, g), f)} = 1.
\]

It is clear that there exist entire functions satisfying (6). For instance, it follows from a result of Clunie [1] that there is an entire function \( f(z) \) satisfying \( T(r, f) \sim (\log r)^\beta \) \( (r \to \infty) \) with a constant \( \beta > 1 \) and so \( f(z) \) satisfies (6) with a suitable number \( \alpha > 1 \).

We shall now give some lower estimations of \( T(r, f(g)) \). Firstly, for certain classes of entire functions, we shall show the following theorem, which we can deduce from \( \cos \pi \lambda \)-theorem (cf. Kjellberg [4], [5]) and the argument of the proof of Theorem 2:

**Theorem 3.** Let \( f(z) \) be a transcendental entire function of order zero satisfying (6) and \( g(z) \) a transcendental entire function of lower order \( \lambda \) \( (\lambda < 1/2) \). Then we have

\[
\limsup_{r \to \infty} \frac{T(r, f(g))}{T(M(r, g), f)} \geq (\cos \pi \lambda)^\alpha.
\]

In general we shall prove

**Theorem 4.** Let \( f(z) \) and \( g(z) \) be transcendental entire functions, \( K (> 0) \) an arbitrary number and \( \beta(r) \) unbounded, strictly increasing, continuous function of \( r (> 0) \) satisfying

\[
\beta(r) \geq r \quad \text{and} \quad \log \beta(r) = o(T(\xi'r, g)) \quad (r \to \infty),
\]

where \( \xi' \) is a constant satisfying \( 0 < \xi' < 1 \). Then there is an unbounded increasing sequence \( \{r_v\} \) such that

\[
T(r_v, f(g)) + O(1) \geq N(r_v, 0, f(g)) \geq K \left( \frac{N(\beta(r_v), 0, f)}{\log \beta(r_v)} - O(1) \right) \quad (v \to \infty)
\]

When \( g(z) \) is of finite order, from a result of Valiron [7] and Edrei-Fuchs [3] and the argument of the proof of Theorem 4 we can deduce
THEOREM 5. Let \( f(z) \) be a transcendental entire function, \( g(z) \) a transcendental entire function of finite order, \( c \) a constant satisfying \( 0 < c < 1 \) and \( \alpha \) a positive number. Then we have for all \( r \geq R_0 \),

\[
T(r, f(g)) + O(1) \geq N(r, 0, f(g)) \\
\geq (\log(1/c))(N(M((cr)^{1/(1+\alpha)}), g), 0, f) - O(1) - O(1) \quad (r \to \infty).
\]

\[\text{§ 2. Proof of Theorem 1.}\]

Let \( u(z) \) be the harmonic function in the disk \( \{|z| < r\} \) which has the boundary values \( \log^+ |f(g(re^{i\theta}))| \) on the circumference \( \{|z| = r\} \). We define \( u^*(z) \) by

\[
u^*(z) = u(z) \text{ in } \{|z| < r\}
\]

\[
u^*(z) = \log^+ |f(g(z))| \text{ in } \{r \leq |z| < \infty\}.
\]

Then it is clear that \( u^*(z) \) is a subharmonic function in \( \{|z| < \infty\} \). Let \( v(w) \) be the harmonic function in the disk \( \{|w| < M(r, g)\} \) with the boundary values \( \log^+ |f(M(r, g)e^{i\phi})| \) on \( \{|w| = M(r, g)\} \). We denote by \( D_z \) the component of the set \( \{z \in D_z \text{; } g(z) = w, \ |w| < M(r, g)\} \), which contains the origin. Then we have \( \{|z| < r\} \subset D_z \). Further \( v(g(z)) \) is harmonic in \( D_z \) and \( v(g(z)) = \log^+ |f(g(z))| = u^*(z) \) on the boundary of \( D_z \). Hence it follows from the maximum principle that \( u^*(z) \leq v(g(z)) \) in \( D_z \). In particular we have

\[
(2.1) \quad u^*(0) \leq v(g(0)).
\]

By Gauss' mean value theorem we have

\[
u^*(0) = u(0) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(g(re^{i\theta}))| \, d\theta = T(r, f(g)),
\]

\[
u(0) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(M(r, g)e^{i\phi})| \, d\phi = T(M(r, g), f).
\]

Hence, if \( g(0) = 0 \), (5) follows from (2.1), (2.2) and (2.3). If \( g(0) \neq 0 \) and \( M(r, g) > (2+\varepsilon)/\varepsilon \cdot |g(0)| \), then it follows from Harnack's inequality that

\[
\nu(g(0)) \leq \frac{M(r, g) + |g(0)|}{M(r, g) - |g(0)|} \nu(0) < (1 + \varepsilon) \nu(0),
\]

which, together with (2.1), proves (4).

Thus the proof of Theorem 1 is complete.
§ 3. Proof of Theorem 2.

In the first place we shall prove the following:

**Lemma 1.** Let \( g(z) \) and \( f(z) \) be two entire functions. Suppose that \( |g(z)| > R > |g(0)| \) on the circumference \( \{ |z| = r \} \) for some \( r > 0 \). Then we have

\[
T(r, f(g)) \geq \frac{R - |g(0)|}{R + |g(0)|} T(R, f)
\]

**Proof.** Let \( u(z) \) be the harmonic function in the disk \( \{ |z| < r \} \) which has boundary values \( \log T(f(g(re^{i\theta}))) \) on the circumference \( \{ |z| = r \} \). Let \( v(w) \) be the harmonic function in the disk \( \{ |w| < R \} \) which has the boundary values \( \log T(f(Re^{i\phi})) \) on \( \{ |w| = R \} \). We define \( v^*(w) \) by

\[
v^*(w) = v(w) \quad \text{in} \quad \{ |w| < R \},
\]

\[
= \log T(f(w)) \quad \text{in} \quad \{ |w| \geq R \}.
\]

Then we deduce that \( v^*(w) \) is subharmonic in \( \{ |w| < \infty \} \) and so \( v^*(g(z)) \) is subharmonic in \( \{ |z| < \infty \} \). Since \( |g(z)| > R \) for \( |z| = r \), it follows from the definitions of \( u(z) \) and \( v^*(w) \) that \( v^*(g(z)) = \log T(f(g(z))) = u(z) \) on the circumference \( \{ |z| = r \} \). Hence by virtue of the maximum principle we have \( u(z) \geq v^*(g(z)) \) in \( \{ |z| \leq r \} \) and in particular

\[
(3.1) \quad u(0) \geq v^*(g(0)).
\]

Since \( R > |g(0)| \), by Harnack’s inequality we obtain

\[
(3.2) \quad v^*(g(0)) = v(0) = \frac{R - |g(0)|}{R + |g(0)|} v(0).
\]

On the other hand by Gauss’ mean value theorem we have

\[
u(0) = T(r, f(g)) \quad \text{and} \quad v(0) = T(R, f),
\]

which, together with (3.1) and (3.2), proves our Lemma.

We are now ready to prove our Theorem 2. We deduce from Theorem 1 that

\[
(3.3) \quad \limsup_{r \to \infty} \frac{T(r, f(g))}{T(M(r, g), f)} \leq 1.
\]

Since \( g(z) \) is of lower order zero, it follows from a result of Kjellberg [5] that there is an increasing, unbounded, positive sequence \( \{ r_n \} \) such that

\[
\min_{|z| = r_n} \log |g(z)| \sim \log M(r_n, g) \quad (n \to \infty).
\]
Hence for any $\varepsilon > 0$ we have

$$|g(z)| > M(r_n, g)^{1-\varepsilon} \text{ for } |z| = r_n, r_n > r_0.$$ 

We may assume that $M(r_n, g)^{1-\varepsilon} > |g(0)|$ and (6) is valid for $r = M(r_n, g)$. Hence our Lemma 1 and (6) yield

$$T(r_n, f(g)) \geq \frac{M(r_n, g)^{1-\varepsilon} - |g(0)|}{M(r_n, g)^{1-\varepsilon} + |g(0)|} T(M(r_n, g), f)$$

and consequently

$$\limsup_{n \to \infty} \frac{T(r_n, f(g))}{T(M(r_n, g), f)} \geq \liminf_{n \to \infty} \frac{T(r_n, f(g))}{T(M(r_n, g), f)} \geq (1-\varepsilon)^n.$$ 

Since $\varepsilon$ is arbitrary, (7) follows from this and (3.3).

Thus the proof of Theorem 2 is complete.


We first need the following lemma, which we can deduce from the proof of Lemma 1 in Clunie [2] (cf. [4, Lemma 2]):

LEMMA 2. Let $g(z)$ be a transcendental entire function, $K$ a positive number and $\alpha(r)$ and $\beta(r)$ two unbounded, strictly increasing, continuous functions satisfying

$$\alpha(r) = r, \quad \beta(r) \geq r$$

and

$$\log \beta(\eta \alpha(r)) = o(T(\xi r, g)) \quad (r \to \infty),$$

where $\eta$ and $\xi$ are constants satisfying $\eta > 1$ and $0 < \xi < 1$. Let $c$ satisfy $\xi < c \leq 1$. Then there are a positive number $R_0$ and an unbounded increasing sequence $\{r_v\}_{v=1}^\infty$ with $r_v > R_0$ and $r_v \to \infty$ (as $v \to \infty$) such that for $v \geq 1$ and for all $r$ in $r_v \leq r \leq \alpha(r_v)$ and all $w$ satisfying $\beta(R_0) = R_v \leq |w| \leq \beta(r)$ we have

$$n(\alpha(r), w, g) > K.$$ 

We also need the following well-known inequalities:

LEMMA 3. Let $f(z)$ be a meromorphic function and $c$ a constant satisfying $0 < c < 1$. Then there are two positive constants $r_0$ and $R_0$ such that for all $r \geq R_0$

$$n(\alpha(r), 0, f) \log (1/c) \leq N(r, 0, f) \leq n(\alpha(r), 0, f)(\log r - \log r_0).$$

Now we shall prove Theorem 4. Choose two constants $\eta$ and $\xi$ such that
\[ \eta \geq 1, \ 0 < \xi < 1 \text{ and } \xi' = \xi / \eta. \] Then (8) yields
\[
\log \beta(\eta r) = \log \beta(\xi r / \xi') = o(T(\xi r, g)) \quad (r \to \infty),
\]
which shows that (4.1) is true with \( \alpha(r) = r \). Hence Lemma 2 implies that there is an unbounded increasing sequence \{r_\nu\} such that for all \( w \) satisfying \( R_1 < |w| \leq \beta(r_\nu) \) we have
\begin{equation}
(4.2)
\tag{4.2}
 n(cr_\nu, w, g) > K \log \left( \frac{1}{c} \right).
\end{equation}

Let \( \{w_\mu\} \) be the zeros of \( f(z) \). Then taking Lemma 3 and (4.2) into account we have
\[
N(r_\nu, 0, f(g)) \geq n(cr_\nu, 0, f(g)) \log \left( \frac{1}{c} \right) = \sum_{\mu} n(cr_\nu, w_\mu, g) \log \left( \frac{1}{c} \right)
\geq K \left( n(\beta(r_\nu), 0, f) - n(R_1, 0, f) \right)
\geq K \left( \frac{N(\beta(r_\nu), 0, f)}{\log \beta(r_\nu) - \log r_\nu} - n(R_1, 0, f) \right).
\]

Using this and Nevanlinna's first main theorem, we obtain Theorem 4.

**References**


[3] Edrei, A. and W.H.J. Fuchs, On the zeros of \( f(g(z)) \) where \( f \) and \( g \) are entire functions. J. Analyse Math. 12 (1964), 243-255.


