ON THE BOUNDARY LIMITS OF POLYHARMONIC FUNCTIONS IN A HALF SPACE

Dedicated to Professor Mitsuru Ozawa on the occasion of his 60th birthday

BY YOSHIHIRO MIZUTA

1. Introduction and statement of result.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space ($n \geq 2$), and set

$$R^+_n = \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > 0 \}.$$ 

For $\xi \in \partial R^+_n$, $\gamma \geq 1$ and $a > 0$, define

$$T_{\gamma}(\xi, a) = \{(x', x_n) \in R^+_n; |(x', 0) - \xi| < a x_n^{1/\gamma}\}.$$ 

Recently Cruzeiro [2] proved the existence of $\lim u(x)$ as $x \to \xi$, $x \in T_{\gamma}(\xi, a)$, for a harmonic function $u$ with gradient in $L^p(R^+_n)$. In this note we are concerned with polyharmonic functions in $R^+_n$, and our purpose is to give a generalization of her result to the polyharmonic case.

For a nonnegative integer $m$, denote by $\Delta^m$ the Laplace operator iterated $m$ times; in particular, $\Delta^0$ denotes the identity operator. A function $u \in C^m(R^+_n)$ is said to be polyharmonic of order $m$ in $R^+_n$ if

$$\Delta^m u = 0 \quad \text{on} \quad R^+_n.$$ 

For $u \in C^m(R^+_n)$ and $x = (x_1, \ldots, x_n) \in R^+_n$, define

$$|\nabla_m u(x)| = \left( \sum_{|\lambda|=m} |D^\lambda u(x)|^2 \right)^{1/2},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes a multi-index with length $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and $D^\lambda = (\partial/\partial x_1)^{\lambda_1} \cdots (\partial/\partial x_n)^{\lambda_n}$.

**Theorem.** Let $m$ be a positive integer and $u$ be a function which is polyharmonic of order $m+1$ in $R^+_n$ and satisfies

$$\int_G |\nabla_m u(x)|^p x_n^\alpha \, dx < \infty, \quad p > 1, \quad \alpha < mp - 1,$$

for any bounded open set $G$ in $R^+_n$. Suppose $(\alpha + 1)/p$ is not a positive integer.

Received April 21, 1983

192
(i) If $n - mp + \alpha > 0$, then for each $\gamma > 1$ there exists a set $E_r \subset \partial R^n_+$ such that $H_{\gamma(n - mp + \alpha)}(E_r) = 0$ and

$$\lim_{x \to \xi, x \in \partial R^n_+} u(x)$$

exists and is finite for any $\alpha > 0$ and any $\xi \in \partial R^n_+ - E_r$.

(ii) If $n - mp + \alpha = 0$, then there exists a set $E \subset \partial R^n_+$ such that $B_{1/p, p}(E) = 0$ and (2) exists and is finite for any $\alpha > 0$, any $\gamma > 1$ and any $\xi \in \partial R^n_+ - E$.

(iii) If $n - mp + \alpha < 0$, then $\lim_{x \to \xi, x \in \partial R^n_+} u(x)$ exists and is finite for any $\xi \in \partial R^n_+$.

Here $H_l$ denotes the $l$-dimensional Hausdorff measure, and $B_{l/p, p}$ the Bessel capacity of index $(l, p)$ (see Meyers [4]). Note the following results (cf. [4]):

(a) If $H_{n-l}(E) < \infty$, then $B_{l/p, p}(E) = 0$ for any $p > 1$;
(b) If $B_{l/p, p}(E) = 0$ for some $p > 1$, then $H_{l'}(E) = 0$ for any $l' > n - l$.

In the case where $(\alpha + 1)/p$ is a positive integer, we have the next theorem.

**Theorem.** Let $u$ be a function which is polyharmonic of order $m+1$ in $R^n_+$ and satisfies (1) for any bounded open set $G$ in $R^n_+$, where $p > 1$ and $(\alpha + 1)/p$ is a positive integer smaller than $m$.

(i) If $n - mp + \alpha > 0$, then for each $\gamma > 1$ there exists a set $E_r \subset \partial R^n_+$ such that $E_r$ has Hausdorff dimension at most $\gamma(n - mp + \alpha)$ and (2) exists and is finite for any $\alpha > 0$ and any $\xi \in \partial R^n_+ - E_r$.

(ii) If $n - mp + \alpha = 0$, then there exists a set $E \subset \partial R^n_+$ such that $E$ has Hausdorff dimension 0 and (2) exists and is finite for any $\alpha > 0$, any $\gamma > 1$ and any $\xi \in \partial R^n_+ - E$.

(iii) If $n - mp + \alpha < 0$, then $\lim_{x \to \xi, x \in \partial R^n_+} u(x)$ exists and is finite for any $\xi \in \partial R^n_+$.

If $\lim_{x \to \xi, x \in \partial R^n_+} u(x)$ exists and is finite for any $\alpha > 0$, then $u$ is said to have a nontangential limit at $\xi$. If $u$ is a function which is polyharmonic of order $m+1$ in $R^n_+$ and satisfies (1) with $p > 1$ and $\alpha < mp - 1$ for any bounded open set $G$ in $R^n_+$, then $u$ has a nontangential limit at any $\xi \in \partial R^n_+$ except for those in a set $E$ with $B_{m-\alpha/p, p}(E) = 0$; this result is best possible as to the size of the exceptional set in the following sense: If $E \subset \partial R^n_+$, $B_{m-\alpha/p, p}(E) = 0$ and $-1 < \alpha < mp - 1$, then we can find a harmonic function $u$ in $R^n_+$ which satisfies (1) with $G = R^n_+$ such that $\lim_{x \to \xi, x \in \partial R^n_+} u(x) = \infty$ for any $\xi \in E$ (see [8; Theorems 1 and 2]).

Thus (ii) of the theorem gives an improvement of [8; Theorem 1], and also the best possible result as to the size of the exceptional set.

2. **Lemmas.**

First we prepare several properties of polyharmonic functions. Let $B(x, r)$ denote the open ball with center at $x$ and radius $r$. For $E \subset R^n$, denote the closure of $E$ by $\overline{E}$.
LEMMA 1. Let \( u \) be a function which is polyharmonic of order \( m+1 \) in \( \mathbb{R}^n \). Then there exist constants \( c_i \) independent of \( u \) such that
\[
\sum_{i=1}^m c_i r^{2i-2} \Delta^i u(x) = \sum_{i=1}^m c_i r^{2i-2} \Delta^i u(x) \quad \text{whenever} \quad B(x, r) \subset \mathbb{R}^n.
\]

\textbf{Proof.} By a result in [9; p. 189], there exist harmonic functions \( v_i \) in \( B(x, r') \) such that
\[
\Delta u(y) = \sum_{i=1}^m |y-x|^{2i-2} v_i(y) \quad \text{on} \quad B(x, r'),
\]
where \( B(x, r') \subset \mathbb{R}^n \). Then we note that \( \Delta^i u(x) = c_i' v_i(x) \), so that
\[
\sum_{i=1}^m c_i' r^{2i-2} v_i(x) = \sum_{i=1}^m c_i r^{2i-2} \Delta^i u(x)
\]
for \( r \) with \( 0 < r < r' \). The constants \( c_i' \) and \( c_i \) depend only on \( i \) and the dimension \( n \).

LEMMA 2. Let \( u \) be a function which is polyharmonic of order \( m+1 \) in \( \mathbb{R}^n \), and let \( B(x, r) \subset \mathbb{R}^n \). Then for each nonnegative integer \( i \), \( i \leq m \), there exist constants \( a_i \) independent of \( u, x \) and \( r \) such that
\[
(3) \quad \Delta^i u(x) = r^{-n-2i} \sum_{0 \leq j \leq m} a_j^{(i)} \int_{B(x, r)} (y-x)^2 D^i u(y) \, dy.
\]

\textbf{Proof.} In view of [3; (15)],
\[
\Delta^i u(x) = \sum_{k=0}^{m-1} a_k \rho^k \int_{\partial B(0,1)} \left( \frac{\partial}{\partial \rho} \right)^k \Delta^i u(x+\rho \sigma) \, dS(\sigma)
\]
with constants \( a_k \). We introduce a differential operator
\[
\nu = \sum_{j=1}^n (y_j-x_j) \frac{\partial}{\partial y_j}.
\]
Letting \( I \) denote the identity operator, we note that
\[
\nu^k \Delta^i = \Delta^i (\nu-2iI)^k,
\]
so that
\[
\rho^{n-2} \Delta^i u(x) = \sum_{k=0}^{m-1} a_k \int_{\partial B(x, \rho)} \nu^k \Delta^i u(y) \, dS(y) = \sum_{k=0}^{m-1} a_k \int_{\partial B(x, \rho)} \Delta^i (\nu-2iI)^k u(y) \, dS(y).
\]
Integrating both sides with respect to \( \rho \) over the interval \( (0, r) \), we obtain
\[ \Delta^k u(x) = r^{-n} \sum_{k=0}^{m-1} a'_k \int_{B(x,r)} \Delta^k (\nu - 2iI)^k u(y) dy \]

\[ = r^{-n-L} \sum_{k=0}^{m-1} a'_k \int_{\partial B(x,r)} \nu \Delta^{k-1} (\nu - 2iI)^k u(y) dS(y) \]

\[ = r^{-n-L} \sum_{k=0}^{m-1} a'_k \int_{\partial B(x,r)} \Delta^{k-1} (\nu - 2(i-1)I)(\nu - 2iI)^k u(y) dS(y). \]

Repeating this process, we finally obtain

\[ \Delta^k u(x) = r^{-n-2z} \sum_{k=0}^{m-1} a'_k \int_{B(x,r)} \nu (\nu - 2I) \cdots (\nu - 2(i-1)I)(\nu - 2iI)^k u(y) dy, \]

which is of the form (3).

The following fact can be proved easily (cf. [6; Lemma 5]).

**Lemma 3.** Let \( u \) be a function in \( C^1(\mathbb{R}^n) \) such that

\[ \int_G |\nabla u(x)|^p x_n dx < \infty, \quad p > 1, \]

for any bounded open set \( G \) in \( \mathbb{R}^n \). Then

\[ \int_G |u(x)|^p x_n dx < \infty \]

for any bounded open set \( G \) in \( \mathbb{R}^n \), where \( \beta = \alpha - p \) if \( \alpha > p - 1 \) and \( \beta > -1 \) if \( \alpha = p - 1 \).

By [6; Lemma 4] we have

**Lemma 4.** Let \( k \) be a positive integer, \( p > 1 \) and \( \beta < p - 1 \). Let \( u \) be a function in \( C^k(\mathbb{R}^n) \) such that

\[ \int_G |\nabla^k u(x)|^p x_n dx < \infty \]

for any bounded open set \( G \) in \( \mathbb{R}^n \). If we set

\[ A = \{ \xi \in \partial B^*_\infty; \int_{B(\xi,1) \cap \mathbb{R}^n_+} |\xi - x|^{k-n} |\nabla^k u(y)| dy = \infty \} \]

then \( B_{k-\beta/p,p}(A) = 0 \).

**Lemma 5.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that

\[ \int_G f(y) dy < \infty \]

for any bounded open set \( G \) in \( \mathbb{R}^n \), and define

\[ B_\beta = \{ \xi \in \partial B^*_\infty; \int_{B(\xi,1) \cap \mathbb{R}^n_+} (|\xi - y'| |y'|^2 + y_n^2)^{-(1+\beta)/2} f(y) y_n^\beta dy = \infty \}, \]
where \( t \geq 0 \) and \( \gamma \geq 1 \). Then \( H_{\gamma}(B_{\delta}) = 0 \) for any \( \delta > 0 \); in case \( t = 0 \), this implies that \( B_{\delta} \) is empty.

**Proof.** Suppose \( H_{\gamma}(B_{\delta}) > 0 \). Then by [1; Theorems 1 and 3 in §I] we can find a nonnegative measure \( \mu \) with compact support in \( \partial R^n \) such that \( \mu(B) > 0 \) and

\[
\mu(B(x, r)) \leq r^t
\]

for any \( x \) and \( r \). Then

\[
\int (|\xi' - y' - y_\alpha^2|^{(t+\gamma)/2} d\mu(\xi) \leq \text{const. } y_\alpha^n \text{ for } y \in R^n.
\]

Hence

\[
\int_\Omega (|\xi' - y' - y_\alpha^2|^{(t+\gamma)/2} d\mu(\xi) f(y) y_\alpha^n d\, y \leq \text{const. } \int_G f(y) \, dy < \infty,
\]

which is a contradiction. Here \( G = \bigcup_{\xi \in \text{supp } \mu} B(\xi, 1) \cap R^n \).

**Lemma 6.** Let \( k \) be a positive integer, \( p > 1 \) and \( \beta < p - 1 \). Let \( K \) be a Borel measurable function on \( R^n \) such that \( |\nabla_i K(x)| \leq |x|^{k-l-n} \) on \( R^n \setminus \{0\} \) for \( l = 0, 1, \ldots, k-1 \), and define

\[
u(x) = \int K(x-y) f(y) \, dy
\]

for a nonnegative measurable function \( f \) on \( R^n \) such that \( \int |x-y|^{k-n} f(y) \, dy \equiv \infty \) and \( \int_G f(y)^p |y_\alpha|^\beta \, dy < \infty \) for any bounded open set \( G \subset R^n \). Set

\[
E_{l, \gamma} = \{ \xi \in \partial R^n \cap \text{supp } K; \limsup_{x \to \xi, x \in \cap B(x, r)} \int_B |\nabla_i u(y)|^p y_\alpha^{l-p-n} \, dy > 0 \text{ for some } a > 0 \}
\]

for \( \gamma \geq 1 \) and \( l = 1, \ldots, k-1 \). Then \( H_{t(n-kp+\beta)}(E_{l, \gamma}) = 0 \) if \( n-kp+\beta > 0 \), and \( E_{l, \gamma} \) is empty if \( n-kp+\beta \leq 0 \).

**Proof.** Define

\[
E_l = \{ \xi \in \partial R^n \cap \text{supp } K; \limsup_{x \to \xi, x \in \cap B(x, r)} \int_B f(y)^p |y_\alpha|^\beta \, dy > 0 \}
\]

for \( \gamma \geq 1 \). Then, in view of [7; Lemma 2], we see that \( H_{t(n-kp+\beta)}(E_l) = 0 \) if \( n-kp+\beta > 0 \) and \( E_l \) is empty if \( n-kp+\beta \leq 0 \).

Let \( l \) be a positive integer such that \( l < k \). Then for almost every \( x \),

\[
|\nabla_i u(x)| \leq \int |x-y|^{k-l-n} f(y) \, dy = U_i(x) + U_2(x) + U_3(x),
\]

where
ON THE BOUNDARY LIMITS OF POLYHARMONIC FUNCTIONS

\[ U_1(x) = \int_{B(z, \varepsilon_n/2)} |x-y|^{k-t-n} f(y) \, dy, \quad 0 < c < 1/3, \]
\[ U_2(x) = \int_{B(\xi, 2|x-\xi|) \cap B(x, \varepsilon_n/2)} |x-y|^{k-t-n} f(y) \, dy, \]
\[ U_3(x) = \int_{R^n \setminus B(\xi, 2|x-\xi|)} |x-y|^{k-t-n} f(y) \, dy. \]

We first note from Hölder's inequality that

\[ \lim_{r \to 0} r^{k-n} \int_{B(\xi, r)} f(y) \, dy = 0 \]

if \( \xi \in \partial R_+ - E \) and hence if \( \xi \in \partial R_+ - E_j \). Setting \( \varepsilon(\eta) = \sup_{0 < r < \eta} \int_{B(\xi, r)} f(y) \, dy \) for \( \eta > 0 \), we have

\[ U_3(x) \leq \text{const.} \int_{R^n \setminus B(\xi, 2|x-\xi|)} |y-\xi|^{k-t-n} f(y) \, dy \]
\[ \leq \text{const.} \left\{ \int_{R^n \setminus B(\xi, \eta)} |y-\xi|^{k-t-n} f(y) \, dy + \varepsilon(\eta) |x-\xi|^{-t} \right\}. \]

Consequently, \( \lim \sup \int_{x \in R^n \setminus B(x, \varepsilon_n/2)} U_3(x)^p x_n^{p-n} \, dx \leq \text{const.} \varepsilon(\eta)^p \). This implies that

\[ \lim_{x \to \xi, x \in R_+^n \setminus B(\xi, \varepsilon_n/2)} U_3(x)^p x_n^{p-n} \, dx = 0. \]

By Hölder's inequality,

\[ U_3(x) \leq \text{const.} x_n^{(k-t)/p} \left\{ \int_{B(z, \varepsilon_n/2)} |x-y|^{k-t-n} f(y)^p \, dy \right\}^{1/p}, \]

so that

\[ \int_{B(z, \varepsilon_n/2)} U_3(x)^p x_n^{p-n} \, dx \]
\[ \leq \text{const.} x_n^{(k-t)/p} \int_{B(z, (1+\varepsilon)\varepsilon_n/2)} f(y)^p \left\{ \int_{B(z, \varepsilon_n/2)} |x-y|^{k-t-n} \, dx \right\} \, dy \]
\[ \leq \text{const.} x_n^{k-p-\beta-n} \int_{B(\xi, 2|x-\xi|)} f(y)^p |y_n|^{-p} \, dy. \]

Therefore if \( n-kp+\beta > 0 \) and \( \xi \in \partial R_+ - E_j \), then

\[ \lim_{x \to \xi, x \in R_+ \setminus B(\xi, \varepsilon_n/2)} U_3(x)^p x_n^{p-n} \, dx = 0; \]

if \( n-kp+\beta \leq 0 \), then

\[ \lim_{x \to \xi, x \in R_+ \setminus B(\xi, \varepsilon_n/2)} U_3(x)^p x_n^{p-n} \, dx = 0. \]
Letting $\eta=2|x-\xi|$ and $M=\int_{B(\xi, \gamma)} f(y) |y_n|^{\beta} \, dy$, we have by [7; Lemma 5],

$$U_\eta(x)^p \leq \text{const.} \begin{cases} x_n^{(k-l)p-\beta-n} M & \text{if } (k-l)p-\beta-n < 0, \\
\log(\eta x_n^{-1}+2)]^{p-1} M & \text{if } (k-l)p-\beta-n = 0, \\
x_n^{(k-l)p-\beta-n} M & \text{if } (k-l)p-\beta-n > 0. \\
\end{cases}$$

If $z \in T_\gamma(\xi, a) \cap B(\xi, 1)$ and $x \in B(z, z_n/2)$, then there exists $a'>0$ such that $x \in T_\gamma(\xi, a')$. Hence we obtain

$$\int_{B_2(\xi, z_n/2)} U_\eta(x)^p x_n^{k-\beta-n} dx \leq \text{const.} \begin{cases} x_n^{k-\beta-n} M & \text{if } (k-l)p-\beta-n < 0, \\
\log(\eta x_n^{-1}+2)]^{p-1} M & \text{if } (k-l)p-\beta-n = 0, \\
x_n^{k-\beta-n} M & \text{if } (k-l)p-\beta-n > 0, \\
\end{cases}$$

which tends to zero as $z \to \xi$, $z \in T_\gamma(\xi, a)$, if $k-\beta-n < 0$ and $\xi \in E_\gamma$, and as $z \to \xi$ if $k-\beta-n \geq 0$. Thus we proved that $E_{1, \gamma} \subset E_\gamma$ if $n-kp+\beta > 0$ and $E_{1, \gamma}$ is empty if $n-kp+\beta \leq 0$. The proof is now complete.

**Corollary.** Let $k$, $p$ and $\beta$ be as in the lemma. Let $u$ be a function in $C^k(\mathbb{R}^n)$ such that $\int_G |\nabla_k u(x)|^p x_n^{k-\beta-n} dx < \infty$ for any bounded open set $G$ in $\mathbb{R}^n$, and define $E_{1, \gamma}$ as in the lemma. Then $H_{T_{1-n-kp+\beta}}(E_{1, \gamma}) = 0$ if $n-kp+\beta > 0$ and $E_{1, \gamma}$ is empty if $n-kp+\beta \leq 0$.

**Proof.** Let $q=p$ if $\beta \leq 0$ and $1 < q < p/(\beta+1)$ if $\beta > 0$. By Hölder's inequality we have

$$\int_G |\nabla_k u(x)|^q dx < \infty$$

for any bounded open set $G$ in $\mathbb{R}^n$. By Theorem 5 and its proof in [10; Chap. VI], we can find a function $v \in L_{k, \text{loc}}(\mathbb{R}^n)$ such that $v = u$ a.e. on $\mathbb{R}^n$,

$$\int_G |\nabla_k v(x)|^q dx < \infty \quad \text{and} \quad \int_G |\nabla_k v(x)|^p x_n^{k-\beta-n} dx < \infty$$

for any bounded open set $G$ in $\mathbb{R}^n$, where the derivatives are taken in the sense of distributions.

We shall show that $H_{T_{1-n-kp+\beta}}(E_{1, \gamma} \cap B(0, r)) = 0$ if $n-kp+\beta > 0$ and $E_{1, \gamma} \cap B(0, r)$ is empty if $n-kp+\beta \leq 0$ for any $r > 0$. Let $r > 0$ be fixed, and take a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi = 1$ on $B(0, 2r)$. Set $w = \phi v$. Then by [5; Theorem 4.1],

$$w(x) = \sum_{i=1}^{2^n} a_i \int \frac{(x-y)^i}{|x-y|^n} D^i w(y) \, dy \quad \text{a.e. on } \mathbb{R}^n.$$

Since $w$ is considered to be continuously $k$ times differentiable on $\mathbb{R}^n$, the right hand side is also continuously $k$ times differentiable on $\mathbb{R}^n$ and the equality is
considered to hold at every point of $R^p_+$. Further,

$$\int_{B(0, r)} |\nabla_k w(y)|^p |y_n|^{\delta} dy < \infty.$$  

Thus the proof of Lemma 6 shows that $H_7(u^{n-k+\beta})\cap B(0, r) = 0$ if $n - kp + \beta > 0$ and $E_{1, 7} \cap B(0, r)$ is empty if $n - kp + \beta \leq 0$. By noting the arbitrariness of $r$, we conclude the proof.

3. Proof of the theorem.

Let $u$ be as in the theorem. If $\alpha < p - 1$, we let $k = 1$, and if $\alpha \equiv p - 1$, then we let $k$ be a positive integer such that $(k - 1)p - 1 < \alpha < kp - 1$. Define $\beta = \alpha - (k - 1)p$. Then $\beta < p - 1$, and, in view of Lemma 3,

$$\int_G |\nabla_{m-k} u(x)|^q |x_n|^{-p} dx < \infty$$

for any bounded open set $G$ in $R^p_+$ and $l = 0, 1, \ldots, k - 1$.

Let $q = p$ if $\beta \leq 0$ and $1 < q < p/(\beta + 1)$ if $\beta > 0$. By Hölder’s inequality we have

$$\int_G |\nabla_{m-k+1} u(x)|^q dx < \infty$$

for any bounded open set $G$ in $R^p_+$. As in the proof of the corollary to Lemma 6, we can find a function $v \in L^q_{loc}(R^n)$ such that $v = u$ a.e. on $R^n$,

$$\int_G |\nabla_{m-k+1} v(y)|^q dy < \infty$$

and

$$\int_G |\nabla_{m-k+1} v(y)|^p |x_n|^{\delta} dy < \infty$$

for any bounded open set $G$ in $R^n$.

Define

$$A = \left\{ \xi \in \partial R^p_+ ; \int_{B(\xi, \rho)} |\xi - y|^{m-k+1-n} |\nabla_{m-k+1} v(y)| dy = \infty \right\},$$

$$E_{l, 7} = \left\{ \xi \in \partial R^p_+ ; \lim_{x \to \xi, x \in F_{l, 7}(\xi, n)} \int_{B(x, \rho_n)} |\nabla_k u(y)|^p |x_n|^{\delta} dy > 0 \text{ for some } \alpha > 0 \right\},$$

$$F_\gamma = \left\{ \xi \in \partial R^p_+ ; \lim_{r \to 0} r^{-\gamma} \int_{B(\xi, r)} |\nabla_{m-k+1} v(y)|^p |y_n|^{\delta} dy > 0 \right\} \text{ for } \gamma > 0,$$

$$F_0 = \left\{ \xi \in \partial R^p_+ ; \lim_{r \to 0} (\log r^{-1}) |\nabla_{m-k+1} v(y)|^p |y_n|^{\delta} dy > 0 \right\}$$

and

$$E_7 = A \cup \left( \bigcup_{l=1}^m E_{l, 7} \right) \cup F_7(m - p + \alpha) \text{ for } n - m p + \alpha \geq 0.$
We shall show below that \( \lim_{x \to \xi, x \in F_{\gamma}(\xi, a)} u(x) \) exists and is finite for any \( \xi \in \partial R_n^+ - E_{\gamma} \) and any \( a > 0 \); in case \( n - m p + \alpha < 0 \), our proof below shows that \( u(x) \) has a finite limit as \( x \to \xi, x \in R_n^+ \), for any \( \xi \in \partial R_n^+ \).

By Lemma 4, \( B_{m-n p + \alpha}(A) = 0 \). In view of Lemma 5,

\[
\int_{F_{\gamma}(\xi, a) \cap B(\xi, r)} |\nabla u(x)|^q x^{l-p-n} dx < \infty, \quad l = m - k + 1, \ldots, m,
\]

for any \( a > 0 \) and any \( \xi \in \partial R_n^+ \), except for a set \( B_r \) such that \( H_{n-m p + \alpha}(B_r) = 0 \) if \( n - m p + \alpha \geq 0 \), and \( B_r \) is empty if \( n - m p + \alpha < 0 \), so that \( H_{n-m p + \alpha}(E_{l, r}) = 0 \) if \( n - m p + \alpha \geq 0 \) and \( l = m - k + 1, \ldots, m \).

The corollary to Lemma 6 implies that \( H_{n-m p + \alpha}(E_{l, r}) = 0 \) if \( n - m p + \alpha > 0 \) and \( l = 1, \ldots, m - k \), and \( E_{l, r} \) is empty if \( n - m p + \alpha \leq 0 \) and \( l = 1, \ldots, m - k \). Thus, with the aid of [7; Lemmas 2 and 3], we see that \( H_{n-m p + \alpha}(E_{l, r}) = 0 \) if \( n - m p + \alpha > 0 \), and \( B_{n-m p + \alpha}(E_{l, r}) = 0 \) if \( n - m p + \alpha = 0 \), where \( E_{l, r} = A \cup F_r \).

Let \( \xi \in \partial R_n^+ - E_{\gamma} \), and take a function \( \phi \in C_c^\infty(R^n) \) such that \( \phi = 1 \) on \( B(\xi, 2) \). Write \( m - k + 1 = 2s + s^* \), where \( s \) and \( s^* \) are nonnegative integers such that \( 0 \leq s^* \leq 1 \). Setting \( w = \phi u \), we have the following integral representation (cf. [5; Theorems 4.1 and 4.2]):

\[
w(x) = U(x; w) = \begin{cases} \sum_{j=1}^{s^*} \int_{\partial K_{2s+1}(x-y)} \left( \frac{\partial}{\partial y_j} \Delta^s w(y) \right) \, dy, & \text{if } s^* = 0, \\ \sum_{j=1}^{s^*} \int_{\partial K_{2s+2}(x-y)} \left( \frac{\partial}{\partial y_j} \Delta^{s^*} w(y) \right) \, dy, & \text{if } s^* = 1, \end{cases}
\]

holds for almost every \( x \in R^n \), where \( K_{2s}(x) = C_s |x|^{2l-n} \) if \( 2l < n \) or \( n \) is odd, and \( K_{2s}(x) = C_s |x|^{2l-n} \log |x| \) if \( 2l \geq n \) and \( n \) is even; the constants \( C_s \) are chosen so that \( U(x; \phi) = \phi \) for any \( \phi \in C_c^\infty(R^n) \). Since \( w \) is infinitely differentiable on \( R_n^+ \), \( U(x; w) \) is continuous on \( R_n^+ \) and \( w(x) = U(x; w) \) holds for any \( x \in R_n^+ \).

We shall prove the theorem only in the case \( s^* = 1 \); the case \( s^* = 0 \) can be proved similarly. Write \( U(x; w) = U_1(x) + U_2(x) + U_3(x) \), where

\[
U_1(x) = \sum_{j=1}^{s^*} \int_{B(x, x_{n/2})} \frac{\partial K_{2s+1}(x-y)}{\partial y_j} \left( \frac{\partial}{\partial y_j} \Delta^s w(y) \right) \, dy,
\]

\[
U_2(x) = \sum_{j=1}^{s^*} \int_{B(x, x_{n/2})} \frac{\partial K_{2s+2}(x-y)}{\partial y_j} \left( \frac{\partial}{\partial y_j} \Delta^{s^*} w(y) \right) \, dy,
\]

\[
U_3(x) = \sum_{j=1}^{s^*} \int_{R^n - B(x, x_{n/2})} \frac{\partial K_{2s+2}(x-y)}{\partial y_j} \left( \frac{\partial}{\partial y_j} \Delta^{s^*} w(y) \right) \, dy.
\]

Since \( \xi \in A \) by our assumption, \( \int |\nabla K_{2s+1}(\xi - y)| \, dy < \infty \), so that Lebesgue’s dominated convergence theorem implies that \( \lim_{x \to \xi, x \in R_n^+} U_3(x) \) exists and is finite as \( x \to \xi, x \in R_n^+ \).
Define $W(x) = \int_{B(\xi, 1) \cap \mathbb{R}^n} |\nabla_{2s+1} w(y)|^p |y_n|^{\beta} dy$. As in [7; Lemma 5], we have
\[
|U_\alpha(x)|^p \leq \text{const.} \begin{cases} 
\frac{x^m n^{p-a-n} W(x)}{n^{m-p-a-n}} & \text{if } n - m p + \alpha > 0, \\
\left( \log \left( \frac{|x - \xi| + 2}{x_n} \right) \right)^{p-1} W(x) & \text{if } n - m p + \alpha = 0, \\
|x - \xi|^{m p - a - n} \left[ \log (|x - \xi|^{-1} + 2) \right]^{p-1} W(x) & \text{if } n - m p + \alpha < 0.
\end{cases}
\]
Since $w(x) = v(x)$ on $B(\xi, 1)$, $x \in T_j(\xi, a)$, so that $U_\alpha(x) = 0$ for any $\alpha > 0$.

Set $k_s(r) = K_{2s+2}(x)$, where $r = |x|$. If $n - m p + \alpha \geq 0$, then $2s+1 = m - k + 1 < n$.

First suppose $2s+2 \leq n$. Then
\[
U_j(x) = -\sum_{j=1}^n k_j(x_n/2) \int_{\partial B(x, x_n/2)} \frac{\partial}{\partial y_j} \Delta^i u(y) \frac{y_j - x_j}{y - x} dS(y)
+ \int_{B(x, x_n/2)} K_{2s+4}(x-y) \Delta^{i+1} u(y) dy
= -\int_{B(x, x_n/2)} \left\{ k_j(x_n/2) - K_{2s+4}(x-y) \right\} \Delta^{i+1} u(y) dy
= -\sum_{i=1}^{2s+2} c_i \Delta^{i+1} u(x) \int_0^{x_n/2} \left( k_j(x_n/2) - k_j(r) \right) r^{n-1+s-i} dr
= -\sum_{i=1}^{2s+2} c_i \Delta^{i+1} u(x) x_n^{2s+2}
= x_n^{-n} \sum_{\alpha \leq i \leq m} c_i \int_{B(x, x_n/2)} (y-x)^j D^j u(y) dy
\]
by Lemmas 1 and 2, where $x \in B(\xi, 1) \cap \mathbb{R}^n$, so that $u(x) = w(x)$ there. Hence it follows from Hölder's inequality that
\[
|U_j(x)| \leq \text{const.} \sum_{i=1}^m \left( \int_{B(x, x_n/2)} |\nabla_i u(y)| \frac{y_n^{m-p-n} dy}{y_n^{1/p}} \right)^{1/p},
\]
which tends to zero as $x \to \xi$, $x \in T_j(\xi, a)$, since $\xi \in \bigcup_{i=1}^m E_{i,j}$. Thus the proof of the theorem is complete.

4. Further results and remarks.

Let $D$ be a special Lipschitz domain as defined in Stein [10; Chap. VI]. Then similar results can be shown to hold for $u$ which is polyharmonic of order $m+1$ in $D$ and satisfies
if we replace $T_f(\xi, a)$ by the set \{\(x \in D; \ |x-\xi| < ad(x)^{1/\alpha}\)\}. Here \(d(x)\) denotes the distance from \(x\) to the boundary \(\partial D\).

Finally we give an open problem: If \(u\) is a function which is polyharmonic of order \(m+1\) in \(\mathbb{R}^n\) and satisfies (1) with \(p>1\) and \(\alpha=m \cdot p-1\) for any bounded open set \(G\) in \(\mathbb{R}^n\), then does there exist a set \(E\) such that \(H_{n-1}(E)=0\) and \(u\) has a nontangential limit at any \(\xi \in \partial \mathbb{R}^n_+=E\)? By a well known result [10; Theorem 4 in Chap. VII], this is true for a harmonic function \(u\) in \(\mathbb{R}^n\) satisfying (1) with \(1<p\leq 2\) and \(\alpha=p-1\) for any bounded open set \(G\) in \(\mathbb{R}^n\). In view of the proofs of [8; Theorem 1] and our theorem, we have the following result: If \(u\) is a function which is polyharmonic of order \(m+1\) in \(\mathbb{R}^n\) and satisfies (1) with \(p>1\) and \(\alpha=m \cdot p-1\) for any bounded open set \(G\) in \(\mathbb{R}^n\), then there exists a set \(E \subset \partial \mathbb{R}^n_+\) such that \(H_{n-1}(E)=0\) and

\[ C(\xi; u, l_\xi) = C(\xi; u, T_f(\xi, a)) \]

for any \(a > 0\) and any \(\xi \in \partial \mathbb{R}^n_+-E\), where \(C(\xi; u, F) = \bigcap_{r>0} u(F \cap B(\xi', r))\) for a set \(F \subset \mathbb{R}^n\) and \(l_\xi = \{\xi+(0, \ldots, 0, t); t > 0\}\).

References