A Critical Review of Thickener Design Methods †

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Abstract

In this work we analyze, in the light of their physical foundation, the different methods of thickener design that have been proposed in the literature. We distinguish three types of methods: those based on macroscopic balances, those based on kinematic models and those based on dynamic models. This classification permits the analysis of thickener design procedures with a clear perspective of their applicability and limitations.

1. Introduction: Definition, equipments and operation.

Thickening is the process of separating parts of the liquid of a suspension so as to obtain a denser product and a flow of pure liquid. The objective of the process may be focused on obtaining a thicker pulp or on recovering the liquid of a suspension. In the first case we refer to thickening and in the second, we use the term clarification.

The mechanism of thickening is sedimentation under the force of gravity. The process is performed industrially in a thickener, a cylindrical vessel where the suspension is allowed to settle. See Fig 1. The suspension is fed from the top and center of the tank and two outlets are provided: a cone discharges the thickened pulp at the bottom and center of the thickener and an overflow weir at the top and periphery of the tank eliminates the clear liquid. A raking mechanism, supported by a shaft at the axis of the tank, conveys the settled material to the center discharge cone. Small thickeners may have flat bottoms, but usually their floor is slightly tilted toward the center to aid in the discharge of the product.

According to Coe and Clevenger¹ four distinct zones can be distinguished in a continuous thickener. At the top, there is a zone of clear liquid labeled zone I. This liquid, that has been separated from the suspension, is recovered at the overflow. When the feed material contains very fine particles, zone I may be turbid unless a chemical reagent is added to flocculate such particles. In this latter case a sharp interface forms at the bottom of zone I, its depth...

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depending on the amount of floculant added. It is precisely by the addition of floculant that the depth of the clear liquid is controlled in industrial thickeners and, for a safe operation, it is maintained at a minimum of 0.5 to 1.0 m. When zone I is invaded with solid particles we say that the thickener overflows.

Beneath the clear liquid, is zone II, called the hindered settling zone. This zone contains pulp of uniform concentration settling at a constant rate. According to Coe and Clevenger this zone may have a concentration between that of the feed and that of the thickest hindered-settling pulp. Comings et al. observed that the feed is usually diluted on entering the thickener. They reported many experiments which proved that, in a normal operation, the concentration in zone II depends on the solid feed rate rather than on the solid feed concentration. The concentration in zone II is low if the feed rate is low and it increases with the feed rate, reaching a maximum value when the solids settle at the maximum possible rate in this zone. If solids are fed at a rate higher than this maximum, the concentration in zone II will continue being that corresponding to the maximum feed rate and the excess solids will not settle through zone II but will pass to zone I and will be eliminated through the overflow. Comings et al. also report that, if the feed concentration exceeds the maximum concentration mentioned for zone II, the concentration in this zone will be that of the feed and the settling capacity of zone II will increase.

Below zone II is a region characterized by a concentration gradient, designated here as zone III, and called the transition zone because it makes the transition from a pulp of constant concentration to the sediment underneath. It is not clear whether this zone really exists in all cases. Some research workers (Comings et al., Fitch and Stevenson) simply ignore this zone and show a sharp break in concentration between zone II and zone IV.

Finally at the bottom is zone IV, called the compression zone, containing a thick pulp or sediment. In this region the flocs rest directly one upon another, the ones at the top exerting pressure upon those at lower levels, creating a concentration gradient. Comings et al. divide zone IV into an upper compression zone, with the features discussed above and a rake-action compression zone, where the movement of the rake allows further concentration of the pulp resulting in an additional concentration gradient.

The concentration of the discharge is the concentration at the bottom of the compression zone. It depends on the thickness of zone IV because a thicker compression zone implies a greater weight of solids supported by the solid skeleton and therefore a higher bottom concentration. Retention time has also been mentioned by Comings et al. as the cause of differing discharge concentrations for the same thickness of compression zones.

The depth of each zone in a thickener depends on the settling characteristic of the pulp. Coe and Clevenger defined the handling capacity of each zone as the amount of solids that passes down from one zone to the other per unit area and unit time. They reasoned that the settling velocity is a function of the pulp concentration and therefore so is the handling capacity. If no solid passes through the overflow, the same solid flux should pass through all zones in the thickener at steady state, so that those zones with a smaller handling capacity will have a greater depth at the expense of the zones with higher handling capacities.

We say that a thickener overflows if solid particles pass from zone II to zone I. According to Dixon and Ecklund and Jernquist simply ignore this zone and show a sharp break in concentration between zone II and zone IV.
there are three ways in which a thickener can overflow. We have already discussed the first two of these, that is, when the feed contains very fine particles that cannot settle and when the feed rate exceeds the settling capacity of zone II. The third way corresponds to a normal operation of the thickener when the feed rate is higher than the discharge rate. In this case solid particles accumulate in the thickener and are eventually transported to zone I.

When, due to a change in the solid concentration of the feed or when the solid feed rate changes, the discharge concentration diminishes, it is possible to readjust it back to its original value by controlling the discharge volume flow rate. This, in turn, is obtained by manipulating a variable speed pump or, when the discharge is effected by gravity, by varying the outlet aperture of the thickener. The result is that, by slowing down the discharge flow rate, more solids accumulate in the compression zone increasing its depth and therefore producing a more concentrated discharge. The opposite occurs when the discharge rate is increased. These procedures are usually used as thickener control.

As a summary, Fig. 3 shows the variables that describe the operation and control of a thickener. In this figure c is the depth of the clear water zone (zone I), h is the depth of the hindered settling zone, zc is the depth of the compression zone (zone IV), qfl is the flocculant flow rate, D is the solid mass flow rate in the underflow discharge, F is the solid mass flowrate in the feed, F 3 (x) is the particle size distribution in the feed, Q is the volume flowrate of pulp, φ is the volume fraction of solids and F, O and D are the subscripts for feed, overflow and underflow respectively.

2. Macroscopic mass balance in a continuous thickener at steady state: Classical methods of thickener design.

2.1 Mishler’s method.

The first equation to predict the capacity of a thickener was developed by Mishler in 1912 and corresponds to a simple macroscopic mass balance in the equipment. Consider a thickener working at steady state, as shown in Fig. 4. Using the same variables as Mishler, a solid and water mass balance yield:

Solid \[ F = D \]  \hspace{1cm} (1)

Water \[ F \cdot D_f = D \cdot D_D + O \] \hspace{1cm} (2)

where F and D are the solid mass flow rates in the feed and discharge respectively, O is the water mass flowrate in the overflow and \( D_f \) and \( D_D \) are the dilutions of the feed and discharge. Dilution is a measure of concentration consisting in the ratio of the mass of water to the mass of solid.

The volume flowrate of water at overflow is then:

\[ Q_O = \frac{F \cdot (D_f - D_D)}{\phi \cdot (D_f)} \] \hspace{1cm} (3)

where \( \phi \cdot (D_f) \) is the water density. According to Mishler, the flowrate of water per unit thickener area \( Q_O / S \), that is the spatial water velocity in zone I of a continuous thickener, must be equal to the rate of water formed in a batch sedimentation test with the same pulp at the concentration of the feed. Since this rate is equal to the rate of descent of the water-suspension interface in the batch settling test, which we denote by \( \sigma_1(D_f) \) and is equal to the settling velocity of the solid as we will prove later on, he wrote:

\[ |\sigma_1(D_f)| = \frac{F \cdot (D_f - D_D)}{\phi \cdot (D_f)} \] \hspace{1cm} (4)

and the settling area required to treat a feedrate F is then:

\[ S = \frac{F \cdot (D_f - D_D)}{\phi \cdot |\sigma_1(D_f)|} \] \hspace{1cm} (5)
where $S$ is the thickener area, $F$ is the mass flow rate of solid in the feed, $D_e$ and $D_b$ are the pulp dilutions in the feed and in the discharge respectively and $|Q_1(D_p)|$ is the absolute value of the rate of descent of the water-suspension interface in a batch settling test performed with the suspension at the dilution $D_p$ of the feed.

Mishler used the following units: $F$ in short tons, $|\sigma|$ in ft/min and $|Q_1|$ in lb/ft$^3$ and obtained $S$ in ft$^2$.

$$S = 0.0222 \frac{F(D_p-D_b)}{|Q_1|\sigma_1(D_p)}, \text{ (ft}^2) \text{) (6)}$$

The design method consists in measuring, in the laboratory, the initial settling rate of a suspension with the concentration of the feed to the thickener and applying equation (6) to find the area $S$ of the thickener.

As we have already discussed, the concentration in the zone II of the thickener is not that of the feed and therefore equation (4) is not correct invalidating Mishler's method of thickener design.

### 2.2 Coe and Clevenger's method.

Coe and Clevenger$^{1}$ assumed that a hindered settling zone II will form in a thickener with a dilution $D_k$ having the minimum solid handling capacity. Since the dilution of this zone is not known in advance they proposed to perform a macroscopic balance in the thickener for different dilutions $D_k$ as shown in Fig. 5.

The volume flow rate of water eliminated from zone of dilution $D_k$ when the suspension passes from this zone to zone of dilution $D_b$ is:

$$Q_k = \frac{F(D_k-D_b)}{Q_f} \text{ (ft}^3\text{)} \text{) (7)}$$

and the rate of water appearance in batch settling of a suspension of dilution $D_k$ is:

$$Q_k/S = |\sigma_1(D_k)| \text{) (8)}$$

From the above equations, the solid handling capacity $F/S$ of a thickener having a dilution $D_k$ in zone II is:

$$\frac{F}{S} = \frac{Q_f|\sigma_1(D_k)|}{F(D_k-D_b)} \text{) (9)}$$

and the minimum solid handling capacity, in Coe and Clevengers units ($F$ in lb/h, $Q_f$ in lb/ft$^3$, $S$ in ft$^2$ and $|\sigma|$ in ft/h) results in:

$$\min\left(\frac{F}{S}\right) = \min D_k \left\{ \frac{62.35 |\sigma_1(D_k)|}{(D_k-D_D)} \right\}, \text{ (lb/ft}^2\text{)} \text{) (10)}$$

For future reference we will express equation (11) in terms of the solid volume fraction $\phi$. Since the dilution is given by:

$$D = \frac{Q_f(1-\phi)}{Q_s \phi} \text{) (12)}$$

the unit area becomes:

$$AU_0 = \max \left\{ \frac{D_k-D_D}{Q_f|\sigma_1(D_k)|} \right\}, \text{ (ft}^2/\text{short tons/day)} \text{) (11)}$$

For future reference we will express equation (11) in terms of the solid volume fraction $\phi$. Since the dilution is given by:
Equation (13), that gives the unit area of a thickener based on laboratory initial settling tests, we call the Coe and Clevenger’s Equation, and the design method we call the Coe and Clevenger’s Method of Thickener Design.

If the following units are selected: \( q_0 \) in g/cm\(^3\), \( |\sigma| \) in cm/s and \( A_U_0 \) in m\(^2\)/TPD, where TPD = metric tons/24 hours, we have:

\[
A_U_0 = \max \left\{ \frac{1}{1.1574 \times 10^{-3}} \frac{1}{q_0 |\sigma| (\phi_k)} \left( \frac{1}{\phi_k} - \frac{1}{\phi_D} \right) \right\} \text{ m}^2/\text{TPD (14)}
\]

According to Coe and Clevenger\(^1\), when the discharge concentration of a thickener is still in the range of hindered settling, the depth of the tank is of no consequence, except in so far as to permit ample depth of clear liquid to care for fluctuations of the feed. On the other hand, when the consistency of the pulp at the discharge is in the range where it is necessary to expel fluid by compression, sufficient capacity must be given to the tank so that the pulp in compression is retained in the thickener the necessary period of time to reach the required density.

To calculate the height of the compression zone, the time \( t^* \) to reach the desired discharge concentration \( \phi_D \) is measured in a batch test. The time interval \([0, t^*]\) is divided into \( n \) smaller intervals \( \Delta t = [t_{i-1}, t_i] \) and the height \( z_i \) of each interval is calculated from a volume balance:

\[
z_i = \frac{V_i}{S}, \quad i = 1, \ldots, n
\]

where \( V_i \) is the volume of pulp of an average pulp density \( \bar{\phi}_i \) and \( S \) is the area of the settling column.

The volume \( V_i = \frac{F}{Q_i} \Delta t_i \), where \( F \) is the mass flux of pulp. Then:

\[
z_i = \frac{F \Delta t_i}{Q_i S}, \quad i = 1, \ldots, n
\]

and the total sediment depth is:

\[
z_c = \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} \left( \frac{1}{A.U.} \right) \Delta t_i \left( \frac{\Delta t_i}{\Delta Q \bar{\phi}_i + Q_i} \right), \quad i = 1, \ldots, n
\]

where \( \bar{\phi}_i = (\phi_{i-1} + \phi_i) \) the average concentration within the interval \( i \). To this depth \( z_c \), an extra 0.5 to 1 m must be added to allow for feed space and clear liquid region.

Coe and Clevenger’s method of thickener design is correct for obtaining the unit area, but the theory does not establish that this unit area \( A_U_0 \) is just an asymptotic minimum, as will be shown later. The method, that continues to be most popular in the mining industry, has been successful because a recommendation was made by the authors to use a safety factor “to take into account changes in the character of the pulp and variations in temperature”. This safety factor has determined the verification of the method. On the other hand, Coe and Clevenger’s method used to obtain the height of the thickener is not correct because it does not take into account the compressibility of the sediment.

3. Thickener design methods based on kinematic sedimentation processes.

The establishment of Kynch theory of sedimentation\(^9\) in 1952 immediately opened a new field of research, the consideration of thickener design from a theoretical point of view and in this way finding a faster and more accurate method of thickener design. Several researchers were involved in this work, leaving their names associated to thickener design procedures. We will restrict our review to some of them, namely W.P. Talmage, B. Fitch, J.H. Wilhelm, Y. Naïde, H. Oltmann, N.J. Hassett and N. Yoshioka. In this section we will review the Kynch theory of sedimentation and those design methods based on it.

3.1 Kynch theory of batch sedimentation.

Let us consider a mixture of solid particles in a fluid that satisfy the following properties:\(^9,11,12\):
- The solid particles are all small (with respect to the container) and of the same size, shape and density,
- the solid and the fluid components of the mixture are incompressible,
- there is no mass transfer between components,
- the sedimentation velocity at any point in the suspension is only a function of the local particle concentration.

Such a mixture is called an ideal suspension\(^10\) and may be regarded as non-interactive superimposed continuous media consisting of two incompressible
The suspension concentration $\phi$ is in general a function of three space variables and time. In the case of batch settling, a settling column is defined as a vessel having a constant cross-sectional area where no wall effect is taken into account. The particle concentration is in this case constant at any cross-section of the column and the field variables are functions of only one space variable and time.

The gravity batch sedimentation of an ideal suspension in a settling column is determined by the volume fraction of solids $\phi(z, t)$ and the velocity of the solid component $v_s(\phi(z, t))$. These two field variables constitute a Kynch sedimentation process (KSP) if, for all $z$ and $t > 0$ they obey the following equations in those regions where the variables are continuous:

$$\frac{\partial \phi}{\partial t} + f'_{bk}(\phi) \frac{\partial \phi}{\partial z} = 0 \tag{17}$$

where $f'_{bk}(\phi)$ is the first derivative of the solid flux density $f_{bk}(\phi)$ with respect to the concentration $\phi$. Let us assume that the initial concentration $\phi(z, 0)$ is given by:

$$\phi(z, 0) = \begin{cases} 0, & L < z \\ \phi_0, & 0 \leq z \leq L \\ \phi_\infty, & z < 0 \end{cases} \tag{22}$$

The solution of the quasilinear hyperbolic equation (21) with initial conditions (22) may be obtained by the method of characteristics, which states that $\phi$ is constant along characteristic lines of slope $dz/dt = f'_{bk}(\phi)$ in the $z$-$t$ plane, where the values of $dz/dt$ are the speeds of the waves of constant concentration.

The characteristics starting from the $z$ axis and drawn as parallel lines in Fig. 6b have speeds given by:

$$dz = \begin{cases} f'_{bk}(0), & L \leq z \\ f'_{bk}(\phi_0), & 0 \leq z \leq L \\ f'_{bk}(\phi_\infty), & z < 0 \end{cases} \tag{23}$$

These terms can be obtained graphically from Fig 6a. The speed of the discontinuity $\sigma(0, \phi_0)$ starting from $z = L$ and $t = 0$ and separating the liquid from the suspension of initial concentration $\phi_0$, is given by:

$$\sigma(0, \phi_0) = \frac{f_{bk}(\phi_0) - f_{bk}(0)}{\phi_0} = 0 \tag{24}$$

This term may be obtained graphically as the slope of the cord drawn from point $(0, 0)$ to point $(f_{bk}(\phi_0), \phi_0)$ in Fig. 6a. Another cord can be drawn directly from $(f_{bk}(\phi_0), \phi_0)$ to $(f_{bk}(\phi_\infty), \phi_\infty)$ to obtain the discontinuity $\sigma(\phi_0, \phi_\infty)$:

$$\sigma(\phi_0, \phi_\infty) = \frac{f_{bk}(\phi_\infty) - f_{bk}(\phi_0)}{\phi_\infty - \phi_0} \tag{25}$$

The intersection of the two discontinuities with the slopes given by equations (231) and (232) defines the point $(z_1, t_1)$ in the settling plot of Fig. 6b.

Extending the characteristics originating from the $z$ axis for $0 < z < L$, we can fill the region of the $z$-$t$ plane separated by the two discontinuities. Ex-
tending now the characteristics from the z axis for $z \leq 0$ in Fig. 6b, we observe that there is a wedge with a vertex at $z = 0$, $t = 0$ and sides with slopes $\sigma(\phi_0, \phi_0')$ and $f_{\phi k}'(\phi)$. We see that the lines with decreasing slopes $f_{\phi k}'(\phi)$ for increasing concentrations from $\phi_0$ to $\phi_\infty$ will fill the wedge in the settling plot. The fan with slope $f_{\phi k}'(\phi)$ is called a rarefaction wave.

The water suspension interface, that up to the point $(z_1, t_1)$ has a slope given by equation (24), will now have increasing slopes given by $\sigma_1(0, \phi)$, with concentration $\phi$ between $\phi_0'$ and $\phi_\infty$:

$$\sigma(0, \phi) = \frac{f_{\phi k}(\phi) - f_{\phi k}(0)}{\phi - 0} = \frac{f_{\phi k}(\phi)}{\phi}$$

$$\equiv \sigma_1(\phi), \quad \phi_0' \leq \phi \leq \phi_\infty \quad (26)$$

The intersection of the discontinuity of slope $(0, \phi_\infty)$ with the characteristics of slope $f_{\phi k}'(\phi_\infty)$ defines the critical point $(z_c, t_c)$. The slope of the discontinuity starting at $(z_c, t_c)$ and the separating zones with constant concentrations $\phi = 0$ and $\phi = \phi_\infty$ is given by:

$$\sigma(\phi_\infty, 0) = \frac{f_{\phi k}(0) - f_{\phi k}(\phi_\infty)}{0 - \phi_\infty} = 0 \quad (27)$$

Finally a global mass balance gives $SL\phi_0 = Sz_\infty\phi_\infty$, from which the height of the suspension at the end of the process is calculated:

$$z_\infty = L\phi_0/\phi_\infty \quad (28)$$

Different possible solutions for Kynch’s problem, give different settling plots, and are called modes of sedimentation (MS). They are entirely determined by the constitutive equation of the flux-density function and the initial concentration. Flux density functions having one inflection point have three MS and flux-density functions having two inflection points can have a maximum of five MS(12).

### 3.2 Analysis of the batch sedimentation curve.

Let us consider a batch Kynch Sedimentation Process and draw a sedimentation curve and a characteristic line for the concentration $\phi_k$, such as that shown in Fig. 7. The line $Z-T$ is tangent to the curve at the point $(z_k, t_k)$. As we have seen, for all the regions of the settling plot where the variables are continuous it is possible to obtain the settling parameters: $\phi$, $\sigma_1(\phi)$, $f_{\phi k}(\phi)$ and $f_{\phi k}'(\phi)$ graphically. See Fig. 5 and 6.

**Settling Rate.**

From the solution of equation (21), already described, we know that the rate of fall of the water-suspension interface $\sigma_1(\phi_k)$ is given by:

$$\sigma_1(\phi_k) = \sigma(0, \phi_k) = \frac{f_{\phi k}(\phi_k)}{\phi_k}$$

$$v_s(\phi_k) = \frac{\partial x}{\partial t} \bigg|_{\phi_k}$$

therefore, $\sigma_1(\phi_k)$ is equal to the slope of the settling curve at point $(z_k, t_k)$

$$\sigma_1(\phi_k) = -\frac{Z}{T} \quad (29)$$

**Concentration.**

Let $W_0$ be the total volume of solids present in the settling column per unit cross-sectional area.
Then, the flux of solids crossing the iso-concentration wave \( \phi_k \), as it travels from \( z = 0 \) to \( z = z_k \), is:

\[
W_0 = \int_{0}^{t_k} \phi_k \left( -v_s(\phi_k) + f'_{bk}(\phi_k) \right) dt
\]  

(30)

where \( v_s(\phi_k) = \sigma_1(\phi_k) \) is the settling rate of the suspension of concentration \( \phi_k \). Since the slope \( f'_{bk}(\phi_k) \) of the characteristic of concentration \( \phi_k \) is constant and the velocity \( v_s(\phi_k) \) is also constant, we can integrate equation (30) directly:

\[
W_0 = \phi_k \left( -v_s(\phi_k) + \frac{z_k}{t_k} \right) t_k
\]

(31)

From Fig. 7 we can see that:

\[
\frac{Z}{T} = \frac{Z - z_k}{t_k}
\]

(32)

On the other hand, since at \( t = 0 \) the suspension is homogeneous and has a concentration \( \phi_0 \), the volume of solids per unit cross-sectional area present in the column is:

\[
W_0 = L\phi_0.
\]

Substituting the last two equations into equation (31) yields:

\[
\phi_k = \phi_0 \frac{L}{Z}
\]

(33)

As a conclusion we can say that, by knowing the settling curve of a batch KSP, for a given suspension having an initial concentration \( \phi_0 \) and initial height \( L \), the parameters for any other concentration \( \phi_k \) can be obtained graphically from the curve. Summarizing we can write:

\[
\phi_k = \phi_0 \frac{L}{Z}, \quad \sigma_1(\phi_k) = v_s(\phi_k) = \frac{-Z}{T}
\]

(34)

\[
f'_{bk}(\phi_k) = -\phi_0 \frac{L}{T}, \quad f'_{bk}(\phi_k) = \frac{z_k}{t_k}
\]

(35)

3.3 Design of continuous thickeners based on the batch Kynch theory.

As we have seen, Coe and Clevenger’s method of thickener design uses equation (13) to calculate the basic unit area:

\[
AU_0 = \max_{\phi_k} \left\{ \frac{1}{Q_0|\sigma_1(\phi)|} \left( \frac{1}{\phi_k} - \frac{1}{\phi_D} \right) \right\}
\]

where \( Q_0 \) is the density of the solid, \( |\sigma_1(\phi)| \) is the initial settling rate of a suspension of concentration \( \phi_k \) and \( \phi_D \) is the discharge concentration.

Coe and Clevenger suggested performing a number of laboratories tests with suspensions of concentrations ranging from that of the feed to that of the critical concentration to find \( |\sigma_1(\phi)| \). If the suspension to be thickened can be considered as an ideal suspension, that is, if \( (\phi_k, f'_{bk}(\phi_k)) \) constitutes a KSP, one properly selected sedimentation test should give all the information necessary to calculate \( AU_0 \).

See equations (34) and (35). To calculate \( \phi_k \) and \( |\sigma_1(\phi)| \) a tangent is drawn at any point in the settling curve, and \( \phi_k \) and \( |\sigma_1(\phi)| \) are calculated from equations (34a) and (34b).

A completely graphical procedure can be established by realizing that equations (34) and (35) must also hold for \( \phi_D \) (remember that the assumption is that the pulp follows Kynch theory), that is (see Fig. 8):

![Fig. 7 Analysis of the settling curve.](image)

![Fig. 8 Thickener design method based on batch Kynch theory.](image)
Substituting equations (34) and (36) into Coe and Cleveenger's equation (13) yields:

\[ AU_0 = \frac{1}{Q_s \phi_0 \Delta L} \max_k \left( \frac{T(Z - Z_n)}{Z} \right) \] (37)

By a properly selected concentration we imply an initial condition for the KSP that would give a continuous settling curve. The best concentration would be that at the inflection point in the flux density function, because it would give a Mode of Sedimentation III (see Concha and Bustos11). Obviously we don't know the concentration at the inflection point and must make a guess. Too low a value for the initial concentration will lead to a Mode of Sedimentation I, with a break from the initial to the final concentration, which would give no possibility of drawing tangents.

**Talmage and Fitch Method.**

Talmage and Fitch15) assumed in 1955 that batch sedimentation of a suspension could be represented by Kynch theory and therefore the settling velocity for a concentration \( \phi_k \) could be expressed, in relation to Fig. 8, by:

\[ |\sigma(\phi_k)| = \frac{Z}{T} = \frac{Z - Z_D}{t_u} \] (38)

Substituting this equation into (37) gives:

\[ AU_0 = \frac{1}{Q_s \phi_0 \Delta L} \max(t_u). \] (39)

From Fig. 9 we can see that the maximum value of \( t_u \) is obtained when \( t_u \) coincides with \( t_k \). We will call this time \( t_0 \):

\[ AU_0 = \frac{t_0}{Q_s \phi_0 \Delta L} \] (40)

Talmage and Fitch method of thickener design may be summarized by the following steps:

1. Perform one settling test at an "intermediate" concentration (we have already referred to this concentration as a "properly selected" concentration) and obtain all initial settling velocities \( \sigma(\phi) \) by drawing tangents to the settling curve, according to Kynch's theory.
2. Calculate the height \( Z_D \) using \( Z_D = \phi_0 \Delta L \).
3. Draw a horizontal line in the settling plot and determine the intersection with the settling curve. This point defines the time \( t_0 \).
4. Calculate the Unit area using equation (40).

Since this method has been developed to design industrial thickeners, very often the horizontal line, drawn through the point \((Z_D, 0)\), does not intersect the settling curve (settling curves of compressible suspensions). In this case, the limiting concentration is the critical concentration and a tangent must be drawn at this point to the settling curve. The intersection of this tangent with the horizontal line defines the time \( t_0 \). See Fig. 9.

The main assumption in Talmage and Fitch method is that the suspension follows Kynch theory. As it turns out, and we will prove it later in this paper, this is correct for concentrations under hindered settling conditions. Unfortunately in most cases the settling curves used for calculating a thickener area by Talmage and Fitch method are obtained well beyond the hindered settling region, that is, part of the curve is obtained in the compression region. Obviously that curve is not unique in this case because it depends on the initial height of the suspension.

**Oltmann Design Method.**

Since Talmage and Fitch's method usually gives thickener areas larger than those experimentally observed, Fitch and Stevenson3) proposed in 1976 the use of an empirical variant of the Talmage and Fitch method, the Oltmann method. Both rely upon identifying the critical compression point. In Oltmann method a straight line is drawn from point \((L, 0)\) to the critical point \((z_c, t_c)\) extending it beyond this point. The intersection of this line with the horizontal line drawn through the point \((Z_D, 0)\) gives the value of \( t_0 \), which replaces \( t_u \) in equation (40). See Fig. 10. There is no theoretical justification for this method.
3.4 Kynch theory of continuous sedimentation.

The continuous gravity sedimentation of an ideal suspension in an ideal thickener was studied by Petty [16] in 1975, by Bustos, Concha and Wendland [17] in 1989 and by Concha and Bustos [18] in 1992. It is defined by the volume fraction of solids \( \phi(z, t) \), the velocity of the solid component \( v_s(\phi(z, t)) \) and the velocity of the fluid component \( v_f(\phi(z, t)) \). These three field variables constitute a Continuous Kynch sedimentation process (CKSP) if, for all \( z \) and \( t > 0 \) where the field variables are continuous, they obey the following equations:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z} (\phi v_s) = 0 \tag{41}
\]

\[
\frac{\partial}{\partial t} (1 - \phi) + \frac{\partial}{\partial z} ((1 - \phi)v_f) = 0 \tag{42}
\]

Substituting \( f_k = \phi v_s \) in equation (41) and deriving \( f_k \) with respect to \( \phi \), adding equations (41) and (42) and defining the volume average velocity \( q(t) \) by:

\[
q = \phi v_s + (1 - \phi) v_f \tag{43}
\]

we can substitute equations (41) and (42) by:

\[
\frac{\partial \phi}{\partial t} + f'_k(\phi) \frac{\partial \phi}{\partial z} = 0 \tag{44}
\]

\[
\frac{\partial q}{\partial z} = 0 \tag{45}
\]

At lines of discontinuities in the suspension the field variables satisfy the jump balance, or Rankine-Hugoniot conditions, and the Lax entropy condition:

\[
s(\phi^+, \phi^-) \geq f_k(\phi^+) - f_k(\phi^-) \tag{46}
\]

\[
q(\phi^+) = q(\phi^-) \tag{47}
\]

Inequality (48) establishes the admissible discontinuities in the suspension.

The kinematic process is completely defined when a constitutive equation is postulated for the flux density function \( f_k(\phi) \) and initial conditions are selected for \( \phi \). From equation (43) we can write:

\[
q = v_s - (1 - \phi) u \tag{49}
\]

where \( u = v_s - v_f \) is the solid-fluid relative velocity. Equation (45) shows that the volume average velocity \( q(t) \) is independent of \( z \). Then, multiplying the previous equation by \( \phi \), we can write:

\[
f_k(\phi) = q(t)\phi + \phi(1 - \phi) u \tag{50}
\]

The value of \( q(t) \) must be defined by boundary conditions. For example, for batch sedimentation, there is no volume flow through the bottom of the settling column so that, for batch sedimentation \( q(t) = 0 \). Substituting this value in (50), leads to:

\[
f_{bk} = \phi(1 - \phi) u \tag{51}
\]

then, the continuous solid flux density function \( f_k(\phi) \) may be written in the form:

\[
f_k(\phi(z, t), t) = q(t)\phi + f_{bk}(\phi) \tag{52}
\]

Fig. (11) shows continuous Kynch flux density functions for a suspension characterized by the batch, or drift flux density function \( f_{bk}(\phi) \), and several constant values of the volume average velocity \( q(t) \).

Ideal Continuous Thickener.

The concept of an ideal continuous thickener (Shannon and Tory [19], Hassett [19], Bustos, Concha and Wendland [17], Concha and Bustos [18]) pretends to define the domain of validity of the field equations by idealizing the shape and operation of a real continuous thickener. Only the main attributes of the equipment are retained while the details are ignored.

An ideal continuous thickener (ICT) has been defined as a cylindrical vessel with no wall effect, so that the concentration of particles is constant at any cross section. In such a vessel the flow is one-dimensional and the field variables are functions of only one space variable and time. The ICT is provided with a feeding, an overflow and a discharge system. See Fig. 12.
Continuous Kynch flux-density functions for several values of the volume average velocity \( q(t) \) and a drift flux density with one inflection point.

The following additional assumptions are made for an ICT:

a) The thickener is fed at \( z = L \) through a surface source. If \( Q_F(t) \) is the feed volume flux of the suspension, \( \phi_F(t) \) its concentration and \( S \) the cross sectional area of the ICT, the solid density at \( z = L \) is given by:

\[
f_{S}(\phi(L, t)) = f_{S}(t) = -\frac{Q_F(t)\phi_F(t)}{S} \tag{53}
\]

where \( f_{S}(t) \) is the feed solid flux density. It can be controlled externally by changing \( Q_F(t) \) or \( \phi_F(t) \).

b) At \( z = 0 \) a surface sink discharges the settled suspension at a volume flow rate \( Q_D(t) \) and concentration \( \phi_D(t) \). Then, the solid flux density and concentration at the discharge are:

\[
f_{D}(\phi(0, t)) = f_{D}(t) = -\frac{Q_D(t)\phi_D(t)}{S} \tag{54}
\]

The discharge solid flux density \( f_{D}(t) \) can be controlled externally by changing \( Q_D(t) \).

c) The solid particles are restricted to the settling section (see Fig 12). If the solid particles cross from the feeding level to the clear liquid section we say that the ICT overflows.

d) The discharge concentration \( \phi_D(t) \) is restricted to values greater than \( \phi(L, t) \). If \( \phi_D(t) \leq \phi(L, t) \), we say that the ICT empties.

Bustos, Concha and Wendland\(^{17}\) and Concha and Bustos\(^{18}\) studied the transient evolution of the concentration in an ICT when certain initial conditions are established and determined the possible steady states. With the initial conditions consisting of only two constant states \( \phi_1 = \phi_L \) for \( z > c \) and \( \phi_1 = \phi_\infty \) for \( z \leq c \), where \( z = c \) is an arbitrary height in the thickener, see Fig. 13, they found by the method of characteristics that, depending on the sign of \( f'_{D}(\phi_\infty) \) and the value of \( \phi_1 \), three distinct solutions may be obtained for equation (44), which are called Modes of continuous sedimentation (MCS), for flux density functions having one inflection point. The MCS are characterized by the types of settling plots derived from the problem. In all the cases two regions of constant concentrations are established, under the clear liquid-suspension interface, separated by: MCS-I a shock wave; MCS-II a contact discontinuity and MCS-III a rarefaction wave.

Among all the possible solutions, only MCS-II leads to a steady state. If \( \phi_L = \phi_M \), \( \phi_L = \phi_M \) a contact discontinuity with a displacement velocity of \( \sigma(\phi_M, \phi_M) = f'_{D}(\phi_M) = 0 \) is formed from point \( (f_{D}(\phi_M), \phi_M) \) to \( (f_{D}(\phi_M), \phi_M) \), so that the concentration changes abruptly from \( \phi_M \) to \( \phi_M \), then increases continuously to \( \phi_\infty \) and finally decreases so that the ICT reaches a steady state at \( \phi = \phi_M \). See Fig. 14a and 14b. The value of \( \phi_L = \phi(L, t) \) can be calculated by solving equation (52) with \( \phi = \phi_L \):

\[
f_{k}(\phi_L, t) = q(t)\phi_L + f_{D}(\phi_L) \tag{55}
\]
Steady State Capacity of an ICT for ideal Suspensions.

The analysis of the continuous Kynch sedimentation process for ideal suspensions having a flux density function with one inflection point shows that the only possible steady state is a CMS-II with a discharge concentration \( c_D = c_M \) and a conjugate concentration \( c_L = c_M \). See Fig. 14. Since equation (52) must always be satisfied, for the discharge concentration and for its conjugate concentration we must have:

\[
\begin{align*}
  f_F &= q \phi_M - f_{bk}(\phi_M) \\
  f_D &= q \phi_M + f_{bk}(\phi_M)
\end{align*}
\]

At steady state \( f_E = f_D \), therefore obtaining \( q \) from equation (56) and substituting it into equation (57) yields:

\[
f_F \left( \frac{1}{\phi_M} - \frac{1}{\phi_M} \right) = f_{bk}(\phi_M) - f_{bk}(\phi_M)
\]

The term \( f_{bk}(\phi) \) can be defined as the fall of the clear liquid-suspension interface \( \sigma(\phi) \) in batch settling of a suspension of concentration \( \phi \). Then, the previous equation may be written in the form:

\[
f_F = \sigma_f(\phi_M') - \sigma_f(\phi_M) \left\{ \frac{1}{\phi_M} - \frac{1}{\phi_M} \right\}
\]

The capacity of an ICT in terms of mass flow rate per unit area \( F/S \) is: \( F/S = -\phi_k f_F \). then, from equation (58), the steady state capacity of an ICT is given by:

\[
F/S = Q_s(\sigma_f(\phi_M) - \sigma_f(\phi_M')) \left\{ \frac{1}{\phi_M} - \frac{1}{\phi_M} \right\}
\]

where \( \phi_M = \phi_D, \phi_M' = \phi_L \). The Unit Area is \( UA_0 = S/F \), so that the Unit Area of an ICT for an ideal suspension with a discharge concentration \( \phi_D = \phi_M \) is:

\[
UA_0 = \frac{1}{p_b(\sigma_f(\phi_M') - \sigma(\phi_M))} \left\{ \frac{1}{\phi_M'} - \frac{1}{\phi_M} \right\}
\]

Design of continuous thickeners based on the continuous Kynch theory.

We analyze in this section those thickener design methods in which the continuous flux density function has been mentioned explicitly. The researchers involved are N. Yoshioka, N.J. Hassett, J.H. Wilhelm and Y. Naide.

Yoshioka-Hasset Method.

Yoshioka\(^{20}\) developed in 1957 a graphical thickener design method based on the total solid flux density function. From the previous section we know that:

\[
f_k(\phi) = q \phi + f_{bk}(\phi)
\]

and at steady state \( f_k(\phi) = f_F \), so that:

\[
f_F = q \phi + f_{bk}(\phi)
\]

Solving equation (61) for \( f_{bk}(\phi) \) with \( q = q_D \), leads to:

\[
f_{bk}(\phi) = f_F - q_D \phi
\]

where \( q_D \) is the volume average velocity at the discharge. Equation (62) represents a straight line with \( q_D \) as the slope \( (q_D = -f_{bk}(\phi_M)) \) at \( \phi = \phi_M \) and \( f_F \) as the intercept of the ordinate in a plot of \( f_{bk}(\phi) \) versus \( \phi \). See Fig. 15. Therefore, the intercept of the straight line with the vertical axis in Fig. 15 gives the continuous flux-density function at steady state. The unit area, of course, is inversely proportional to the feed flux density \( UA = 1/(p_b f_F) \).

Yoshioka\(^{20}\) and Hassett\(^{21}\) independently interpreted the result of Fig. 15 still in another way. If the continuous flux-density function \( f_k(\phi) \) is plotted in-
stead of $f'_b(\phi)$ against $\phi$, Fig. 16 is obtained.

Here the solid flux-density at steady state is the horizontal line tangent to the continuous flux density at its maximum with concentration $\phi_M$.

Hasset realized that there was a problem of interpretation in this approach, because Fig. 16 shows that only two concentrations are possible in the thickener, the limiting concentration $\phi_M$ and its conjugate concentration $\phi_M^*$ (see our Fig. 14 and equations (55) and (56)). Hasset says: "Thus, the theory predicted the absence of the feed and discharge concentrations within the thickener, and shows that there must be an abrupt increase up to the discharge concentration at the moment of discharge...". It is obvious that this conclusion is absurd, because it would mean that the passage of a suspension through a series of contractions would increase its concentration making the thickener an unnecessary equipment.

The principal objection to these graphical methods of thickener design is that they use the Kynch flux-density function for values of concentration that are outside its range of validity. Remember that the Kynch batch flux-density function, is obtained through initial settling experiments and, therefore, they are valid up to the critical concentration only. Obviously the definition of flux density is valid beyond this concentration, but in this range it is not a unique function of concentration. We will discuss this fact further in a later section and will give an explanation of Hasset’s problems.

Wilhelm and Naide’s Method.

Wilhelm and Naide also use the continuous flux density function at steady state in the form:

$$f_F = q_D\phi + f'_b(\phi)$$  \hspace{1cm} (63)

Since $f'_F = 0$, the derivative of this equation at the limiting concentration $\phi = \phi_M$ yields:

$$q_D = -f'_b(\phi)|_{\phi_M}$$  \hspace{1cm} (64)

They suggest expressing the settling velocity of a suspension of uniform concentration, with the following equation (see Fig. 17):

$$v_s(\phi)|_{\text{batch}} = a\phi^{-b}$$  \hspace{1cm} (65)

where $a$ has the dimensions of L.T$^{-1}$ and $b$ is dimensionless.
Since $f_{ba}(\phi) = \phi v_s(\phi)|_{batch}$, we have:

$$f_{ba}(\phi) = -a\phi^{1-b}$$  \hspace{1cm} (66)

and

$$q_D = a(b-1)\phi^{-b}_M$$  \hspace{1cm} (67)

Substituting equations (64) and (67) into equation (63) yields for the limiting continuous flux-density $f_F(\phi_M)$:

$$f_F(\phi_M) = -ab\phi^{1-b}_M$$  \hspace{1cm} (68)

On the other hand, since $f_F = q_D\phi_D$, substituting into equation (68) yields a relationship between the discharge and the limiting concentration:

$$\phi_M = \frac{(b-1)}{b}\phi_D$$  \hspace{1cm} (69)

Then, in terms of the discharge concentration, the steady state flux density is given by:

$$f_F = -ab\left(\frac{b-1}{b}\right)^{1-b}\phi_D^{-b}$$  \hspace{1cm} (70)

and the unit area $UA = 1/(Q_s f_F)$ is given by:

$$UA = \frac{1}{Q_s ab}\left(\frac{b-1}{b}\right)^{b-1}\phi_D^{b-1}$$  \hspace{1cm} (71)

Wilhem and Naide also demonstrated that equation (71) is equivalent to the Talmage and Fitch method of thickener design. To appreciate this, consider equation (40). From Fig. 9 we see that:

$$\phi_D L = \phi_D Z_D \quad \text{and} \quad t_u = \frac{Z_D}{f'(\phi_D)}$$  \hspace{1cm} (72)

Then, substituting into the equation for $UA$ yields:

$$UA = \frac{1}{ab}\left(\frac{b-1}{b}\right)^{b-1}\phi_D^{-b}$$

which is identical to equation (71). Fig. 18 shows a simulation of the thickener capacity in terms of the discharge concentration based on Wilhem and Naide's data.

According to Wilhem and Naide, when the effect of compressive forces is negligible, for example for thickeners with shallow beds, the method described give unique results, otherwise different results are obtained for each bed height. The recommendation they give in this case, is to carry out batch experiments at similar bed heights as expected in the continuous thickener.

Wilhem and Naide's method of thickener design may be summarized by the following steps:

1. Carry out batch settling experiments with suspensions at initial concentrations between that of the feed and that of the discharge of the thickener to be designed and record the initial settling velocity $\phi_1(\phi)$.

2. Alternatively, perform one settling test at an intermediate concentration and obtain all initial settling velocities by drawing tangents to the settling curve, according to Kynch's theory.

3. Plot $\log(\phi_1(\phi))$ versus $\log(\phi)$, as in Fig. 17, and approximate the curve with one or more straight lines.

4. From each straight line in Fig. 17 calculate the parameters $a$ and $b$ of equation (65) graphically or by linear regression.

5. Using the values of $a$ and $b$, determined for each section, calculate the unit area using equation (71).

6. Plot the curve $UA$ versus discharge concentration $\phi_D$ in a log-log scale as in Fig. 18.

4. Thickener design methods based on the dynamic sedimentation process.

The sedimentation of flocculated suspensions, such as those encountered in the Mineral Processing
Industry, cannot be described by Kynch theory because the consolidation of the sediment under its own weight involves forces not taken into account in the kinematic theory.

Careful observations of batch settling of flocculated suspensions show the existence of two zones separated by a discontinuity. The upper zone is in hindered settling and lasts only up to the critical time, while the lower zone is in consolidation from the beginning up to the end of sedimentation.

4.1 Batch sedimentation of a compressible suspension.

The simplification of the general sedimentation equations, together with a dimensional analysis (Concha and Bustos, 23), leads to the definition of the sedimentation of flocculated suspensions with the following field variables: the solid concentration \( \phi(z, t) \), the solid flux density function \( f_b(z, t) \) and the excess pore pressure \( P_e(z, t) \). These three variables must satisfy the following equations in those regions where the variables are continuous:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial f_b}{\partial z} = 0
\]  
(73)

\[
f_b = f_{bk}(\phi) \left(1 + \chi(\phi) \frac{\partial \phi}{\partial z}\right)
\]  
(74)

\[
\frac{\partial P_e}{\partial z} = -\Delta \rho g \left(1 + \chi(\phi) \frac{\partial \phi}{\partial z}\right)
\]  
(75)

and at discontinuities they must satisfy the Rankine Hugoniot and the Lax entropy conditions:

\[
\sigma = \frac{f_b^+ - f_b^-}{\phi^+ - \phi^-}
\]  
(76)

\[
f_b^+ \geq \sigma (\phi^+, \phi^-) \geq f_b^-
\]  
(77)

where \( f_{bk}(\phi) \) is the extended Kynch batch flux density function and \( \chi(\phi) \) is a measure of the ratio of compressibility of the sediment in relation to the forces available to compress it, in this case, the self weight of the sediment. (The name extended comes from the fact that the Kynch function is extended beyond the region where it is valid, that is, beyond the critical concentration \( \phi_c \).) In terms of the parameter of hindered settling and flow in a porous media \( f_{bk}(\phi) \) is given by:

\[
f_{bk}(\phi) = \begin{cases} 
\frac{\Delta \rho \phi^2 (1 - \phi) g}{\mu K(\phi)}, & \phi < \phi_c \\
\frac{\Delta \rho \phi^2 g}{\mu k(\phi)}, & \phi \geq \phi_c 
\end{cases}
\]  
(78)

where \( K(\phi) \) is the coefficient of resistance of the suspension in hindered settling, \( k(\phi) \) is the permeability of the sediment, \( g \) is the acceleration of gravity and \( \mu \) is the fluid viscosity.

The compressibility of the sediment can be expressed as:

\[
\chi(\phi) = \begin{cases} 
0, & \text{for } \phi < \phi_c \\
\frac{d\sigma_e/d\phi}{\Delta \rho \phi g}, & \text{for } \phi \geq \phi_c
\end{cases}
\]  
(79)

Substituting equation (74) into equation (73) yields:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z} \left( f_{bk}(\phi) \left(1 + \chi(\phi) \frac{\partial \phi}{\partial z}\right) \right) = 0
\]  
(80)

The initial and boundary conditions for this equation are:

\[
\phi(z, 0) = \phi_0
\]  
(81)

\[
\left. \frac{\partial \phi}{\partial z} \right|_{z=0} = -\frac{1}{\chi(\phi)} \left. \frac{\partial \phi}{\partial z} \right|_{z=0}
\]  
(82)

Equation (82) is obtained by taking \( f_b = 0 \) at \( z = 0 \) in equation (74).

The position of the clear liquid-suspension interface is obtained from the jump balance equations (76):

\[
\frac{dx}{dt} = \begin{cases} 
\sigma_1 = f_{bk}(\phi)/\phi, & t < t_c \\
\sigma_2 = f_{bk}(\phi_c)/\phi_c, & t \geq t_c
\end{cases}
\]  
(83)

where \( t_c \) is the time for which the suspension, immediately under the clear liquid-suspension interface reaches the critical concentration \( \phi_c \).

Since equation (80) is parabolic, no discontinuities exist for regions where the concentration is greater than the critical concentration. In those regions where the concentration is lower than the critical value, equation (80) degenerates into the quasi-linear hyperbolic batch Kynch equation:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z} \left( f_{bk}(\phi) \right) = 0
\]  
(84)
This result implies that Kynch's theory is still valid for the sedimentation of flocculated suspensions in the region with concentrations lower than the critical value. This region is said to be in Kynch regime. It is bounded by two interfaces, the clear liquid-suspension interface, given by equation (83) and the suspension-sediment interface having a displacement velocity of:

$$\sigma_3 = \frac{f_{bk}(\phi^*) - f_b(\phi, \partial\phi/\partial z)}{\phi^* - \phi_c},$$

(85)

where $\phi^*$ is the concentration at the point of tangency obtained by drawing a straight line from the point $(\phi_0, f_{bk}(\phi))$ and tangent to the extended batch flux-density curve. See Fig. 15.

Once the concentration distribution $\phi(z, t)$ is known, the excess pore pressure is calculated using equation (75) with boundary condition $p_e(0) = 0$. Fig. 19 to 21 show the result of batch sedimentation of a flocculated suspension of copper tailings.

### 4.2 Continuous sedimentation of a compressible suspension in an Ideal Thickener at steady state.

The continuous sedimentation of a flocculated suspension may be defined by the following set of variables: the solid concentration $\phi(z, t)$, the solid flux density function $f(\phi(z, t), t)$, the volume average velocity $q(t)$ and the excess pore pressure $p_e(z, t)$. In those regions where the variables are continuous these variables must satisfy the following equations:

$$\frac{\partial \phi}{\partial t} + \frac{\partial f}{\partial z} = 0,$$

(86)

$$\frac{\partial q}{\partial z} = 0,$$

(87)

$$f = q\phi + f_{bk}(\phi) \left(1 + x(\phi) \frac{\partial \phi}{\partial z}\right),$$

(88)

$$\frac{\partial p_e}{\partial z} = -\Delta \phi g \left(1 + x(\phi) \frac{\partial \phi}{\partial z}\right),$$

(89)

At discontinuities the field variables satisfy the jump balance and the Lax entropy conditions:

$$\sigma = \frac{f^* - f^-}{\phi^* - \phi^-},$$

(90)

$$f' \geq \sigma(\phi^*, \phi^-) \geq f'^-,$$

(91)

where $f$ is the continuous flux-density function, $f' = \partial f/\partial q$, $q$ is the volume average velocity and where all the other variables have the same meaning as previously.

At steady state, equations (86) and (87) become:

$$\frac{df}{dz} = 0, \text{ for } 0 \leq z \leq L,$$

(92)

$$\frac{dq}{dz} = 0, \text{ for } 0 \leq z \leq L,$$

(93)

with boundary conditions:

$$f(L) = f_p,$$

(94)

$$q(0) = q_p,$$

(95)
and where \( f_F = -Q_F \phi_F / S \) and \( q_D = -Q_d / S \). The solution of equations (92) and (93) give, for any value of \( z \):

\[
f(z) = f_F \\
q(z) = q_D
\]

To obtain the concentration profile in the ICT, substitute the value of \( f(z) \) and \( q(z) \) from equations (96) and (97) into equation (88) and solve for \( d\phi / dz \):

\[
d\phi \frac{dz}{dz} = -\frac{1}{x(\phi)} \left( 1 - \frac{f_F - q_D \phi}{f_{hK}(\phi)} \right)
\]

with boundary condition:

\[
\phi(0) = \phi_D
\]

Since the term \( x(\phi) \) becomes zero for \( \phi < \phi_c \), see equation (79), it is necessary to divide the problem into two parts, one for values of \( z \) greater than \( z_c \) and one for values smaller than \( z_c \), where \( z_c \) is the vertical coordinate for \( \phi = \phi_c \).

**Case I:** \( z_c < z \leq L \).

In this region \( \phi < \phi_c \) and equation (98) becomes indeterminate, but from equation (88) we obtain:

\[
f_F = q_D \phi + f_{hK}(\phi), \text{ for } \phi < \phi_c
\]

Differentiating with respect to \( \phi \) yields:

\[
0 = (q_D + f_{hK}(\phi)) \frac{d\phi}{dz}, \text{ for } \phi < \phi_c
\]

Since the value between the parentheses is not zero, except for special cases, \( d\phi / dz \) must be zero, then:

\[
\frac{d\phi}{dz} = 0, \text{ for } \phi < \phi_c
\]

The concentration at \( z = L \) is \( \phi_L \), so that the solution of equation (101) is:

\[
\phi(z) = \phi_L, \text{ for } \phi < \phi_c
\]

Equation (100) is Kynch flux density function for an ideal suspension. This shows that Kynch theory is valid for the sedimentation of a flocculated suspension in an ICT at steady state, in those regions where hindered settling exists, that is, where the concentration is lower than the critical concentration. The value of this concentration \( \phi_L \), known as the conjugate concentration, can be determined from the continuous flux density curve, by solving the equation \( f_F = q_D \phi_L + f_{hK}(\phi_L) \) for \( \phi_L \).

**Case II:** \( 0 < z \leq z_c \).

The concentration profile in an ICT for values of \( 0 < z \leq z_c \), can be obtained by integrating equation (98) numerically with boundary condition (99). For example for the thickening of copper tailings\(^{25}\), the solution of this problem is given in Fig. 22.

The hold up of solids in the thickener may be calculated by integrating the concentration profile:

\[
W = \int Q_s \phi(z) S dz + Q_s \phi_L S l
\]

where \( z_c \) is the depth of the sediment layer and \( l \) is the height of the hindered settling zone.

Once the concentration profile is obtained, the excess pore pressure is obtained by integrating numerically equation (89) with boundary condition \( p_e(L) = 0 \). Then:

\[
p_e = \begin{cases} 
\Delta \phi \phi_L (L - z), & \text{for } z_i < z \leq z_c \\
\phi(z) + \Delta \phi g \int_{z}^{z_c} \frac{f_F - q_D \phi(t)}{f_{hK}(\phi(t))} dz, & \text{for } z \leq z_i
\end{cases}
\]

Fig. 23 shows the excess pore pressure profile in an ICT treating a flocculated suspension\(^{25}\).

### 4.3 Existence of a steady state in an ICT.

For gravity sedimentation to take place, it is necessary that the concentration in a thickener increases downwards or, at least, stays constant. Therefore, a necessary condition for sedimentation is that:

\[
\frac{d\phi}{dz} \leq 0
\]

Since \( x(\phi) > 0 \), \( q_D < 0 \) and \( f_{hK}(\phi) < 0 \), from equation (98) we see that, for inequality (105) to be always satisfied, the following must be true:

For every value of $f_p$, a different value for the discharge concentration is obtained as the intersection of the corresponding line $f = f_F$ and the line $f = q_0$. Values of $f = f_F$ that intersect the flux density curve twice do not give valid steady states. The corresponding concentration and excess pore pressure profiles may be seen in Fig. 22 and 23. The concentration profiles for $f_1$ to $f_3$ are normal, but that for $f_4$ is not.

This analysis leads to the conclusion that a limit exists for the capacity of a thickener, which can be obtained by calculating the maximum of the function $f_k(q_0, \phi)$. Let us consider a given and constant discharge concentration $q_D$, then a linear link exists between $f_F$ and $q_D$ through $f_F = q_0 \phi D$. Inequality (107) can be written in the form:

$$f_F \geq \max_{\phi} f_k(q_0, \phi)$$

where $f_k(q_D, \phi) = q_D \phi + f_{hk}(\phi)$. Since $q_D = f_F/\phi_D$, we can write:

$$f_k(q_D, \phi) = f_F \phi \phi_D + f_{hk}(\phi)$$

ICT treating a flocculated suspension, is that, in a flux density plot, the extended continuous Kynch flux density function lies below the line $f = f_F$. Fig. 24 shows an extended continuous Kynch flux density function, for a given value of $q_0$, together with several lines $f = f_F$. All the horizontal lines lying above the curve represent stable steady states.

4.4 Limiting capacity of an ICT treating a flocculated suspension at Steady State.

For every value of $f_p$, a different value for the discharge concentration is obtained as the intersection of the corresponding line $f = f_F$ and the line $f = q_D$. Values of $f = f_F$ that intersect the flux density curve twice do not give valid steady states. The corresponding concentration and excess pore pressure profiles may be seen in Fig. 22 and 23. The concentration profiles for $f_1$ to $f_3$ are normal, but that for $f_4$ is not.

This analysis leads to the conclusion that a limit exists for the capacity of a thickener, which can be obtained by calculating the maximum of the function $f_k(q_0, \phi)$. Let us consider a given and constant discharge concentration $q_D$, then a linear link exists between $f_F$ and $q_D$ through $f_F = q_0 \phi D$. Inequality (107) can be written in the form:

$$f_F \geq \max_{\phi} f_k(q_0, \phi)$$

where $f_k(q_D, \phi) = q_D \phi + f_{hk}(\phi)$. Since $q_D = f_F/\phi_D$, we can write:

$$f_k(q_D, \phi) = f_F \phi \phi_D + f_{hk}(\phi)$$

ICT treating a flocculated suspension, is that, in a flux density plot, the extended continuous Kynch flux density function lies below the line $f = f_F$. Fig. 24 shows an extended continuous Kynch flux density function, for a given value of $q_0$, together with several lines $f = f_F$. All the horizontal lines lying above the curve represent stable steady states.
Substituting this equation into inequality (108) and remembering that \( \sigma_{1}(\phi) = \frac{f_{b}(\phi)}{\phi} \), yields:

\[
\frac{2}{\phi} \geq \max_{\phi} \left\{ \left( -\sigma_{1}(\phi) \right) \frac{1}{\phi} - \frac{1}{\phi_{D}} \right\} \tag{109}
\]

This inequality expresses the maximum solid flux density that can be fed to an ICT at steady state\(^{25}\).

In terms of mass flow rate per unit area, \( \frac{F}{S} = -\frac{Q_{s}f_{F}}{\phi} \), the maximum capacity of the thickener is given by:

\[
\frac{F}{S} \geq \max_{\phi} \left( -\frac{Q_{s}f_{F}}{\phi} \right) \left( \frac{1}{\phi} - \frac{1}{\phi_{D}} \right) \tag{110}
\]

Finally, the Unit Area \( UA = \frac{S}{F} \) is given by:

\[
UA \geq \max_{\phi} \left( \frac{1}{\phi} - \frac{Q_{s}f_{F}}{\phi} \right) \left( \frac{1}{\phi} - \frac{1}{\phi_{D}} \right) \tag{111}
\]

This result indicates that the minimum unit area of an ICT, treating a flocculated suspension, depends on the hindered settling velocity \( \sigma_{1} \) of the suspension and not on the compressibility of the sediment. A comparison of this result with Coe and Clevenger equation (13), leads to the conclusion that, the unit area of the thickener should always be greater than the basic unit area \( UA_{0} \) of Coe and Clevenger equation. Defining the function \( G(\phi, \phi_{D}) \) by:

\[
G(\phi, \phi_{D}) = \left( \frac{1}{\phi} - \frac{Q_{s}f_{F}}{\phi} \right) \left( \frac{1}{\phi} - \frac{1}{\phi_{D}} \right) \tag{112}
\]

The basic unit area, given by equation (13) can be written in the form

\[
UA_{0} = \max_{\phi} G(\phi, \phi_{D})
\]

then, we can write\(^{25}\):

\[
UA \geq UA_{0}
\]

Plotting \( G(\phi, \phi_{D}) \) the minimum value of the unit area \( UA \) can be seen graphically as the maximum value of \( G(\phi, \phi_{D}) \). See Fig 25.

4.5 Adorjan's method of thickener design.

Adorjan\(^{27,28}\) was the first researcher to express in a clear and consistent way the sizing of an industrial thickener based on the mass and momentum balance. His results are equivalent to those deduced in the preceeding section, especially equation (111) for the limiting Unit Area and equation (98) with boundary condition (99) for calculating the thickener height.

![Figure 25: Unit area function G(\phi, \phi_{D}) versus \phi](image)
Thickener Area.

Adorjan argues that a thickener operated under limiting conditions \( f_L = f_q (q_0, \phi) \) requires very considerable pulp depth and therefore it must be operated at only a fraction of the limiting feed rate. This factor he called loading factor and defined it by:

\[
\lambda = \frac{F}{F_0}
\]

(113)

where \( F \) is the actual feed rate and \( F_0 \) the limiting feed rate. In terms of unit areas:

\[
AU = \frac{AU_0}{\lambda}
\]

(114)

Adorjan related the criteria to select \( \lambda \) to the safety factor in the design, so that a certain deviation from the design capacity would be possible. For example, selecting the arbitrary criteria:

\[
\lambda = \frac{0.95}{1 + I_c}
\]

where \( I_c \) is the fraction of increase in capacity needed, the area for the thickener is:

\[
S = \frac{F}{\lambda} AU_0
\]

(115)

where \( AU_0 \) is Coe and Clevenger's basic unit area given by equation (13).

Thickener Height.

The height of the thickener consists of three terms; \( c \) is the depth of the clear liquid (zone I), \( \ell \) the depth of the feed and hindered settling zone (zone II and III) and \( z_c \) the thickness of the sediment layer (zone IV).

\[
H = c + \ell + z_c
\]

(116)

Any criteria can be used to size zones I, II and III. For example, let us assume that \( c = 0.5 \) m, and \( \ell = 0.5 \) m, so that the total height depends only on the depth \( z_c \) of the sediment. This depth is obtained by integrating equation (98) with boundary condition (99):

\[
H = c + \ell + \int_{\phi_d}^{\phi_n} \kappa(\phi) \, d(\phi) - \frac{1}{\lambda}
\]

(117)

where \( \kappa(\phi) \) is a measure of the compressibility of the suspension, given by equation (79). For example, Fig. 26 gives the result of Adorjan's method for a specific case.

5. Parameter estimation.

We have already said that two types of motions can be distinguished in thickening: (1) hindered settling, at the beginning of the batch process or at concentrations lower than the critical value for continuous thickening, is characterized by the absence of cont-
act throughout the solid component. All interaction between the solid particles is produced through the fluid and appears as an interaction force; (2) consolidation, at the final stages of batch sedimentation or at concentrations greater than the critical value in continuous thickening, is characterized by the formation of a network of solid particles, linked by a flocculant, that produces an increase in resistance to compression with concentration. Both stages are separated by the critical concentration. This parameter will be studied in (3).

(1) Hindered settling.

Hindered settling is characterized by a single parameter, the constitutive equation for the coefficient of resistance $K(\phi)$ of the solid-fluid interaction force $m$ ($m = -\mu K(\phi) u$, where $m$ is the drag force and $u$ the solid-fluid relative velocity). During hindered settling the absence of (or negligible) solid pressure gradient makes the suspension behave ideally, that is, it follows Kynch theory. Therefore, this stage of the process can be used to determine the parameter of hindered settling.

During batch sedimentation of a compressible suspension there is an initial period of time during which the concentration is constant and, therefore, the settling velocity of the suspension is also constant. By carrying out laboratory experiments with the suspension at several initial concentrations lower than the critical concentration, the corresponding hindered settling velocity may be determined by measuring the rate of descent of the water-suspension interface. Once the data has been collected any equation can be fitted to the data. Two types of equations, with two parameters to be determined experimentally, have been successfully used: (a) Richardson and Zaki equation\(^{(4)}\) and b) Michael and Bolger’s equation\(^{(26)}\).

a) Richardson and Zaki Equation.

Richardson and Zaki\(^{(4)}\) represent the batch settling velocity of a suspension of particles by the equation:

$$\sigma_1(\phi) = u_\infty (1 - \phi)^n \quad (118)$$

on a log-log plot of $(-\sigma_1)$ versus $(1 - \phi)$, $(-u_\infty)$ corresponds to the intercept of the straight line with the ordinate and $n$ is the slope of the line. For example, for a given copper tailings\(^{(49)}\), the parameters of the Richardson and Zaki equation, as determined from Fig. 27a are: $u_\infty = -6.3 \times 10^{-4}$ m/s and $n = 12.6$.

Since in batch sedimentation $\sigma_1 = (1 - \phi) u$, from the definition of $f_{sk}$ in equation (78) and equation (118) we have:

$$K(\phi) = -\frac{\Delta \rho g}{\mu u_\infty (1 - \phi)^{n-1}} \quad (119)$$

(2) Consolidation.

The consolidation process is characterized by two constitutive properties: a) the compressibility of the sediment and b) the solid-fluid interaction force.

a) Sediment compressibility.

During the last stage of batch sedimentation the process tends to equilibrium and $f_{sk}(\phi)$ tends to zero. Then, from equations (74) we can write:

$$\chi(\phi) = -\frac{1}{\phi \partial / \partial z} \bigg|_{eq} \quad (122)$$

and therefore:
From equations (74), (75) and (78) we have:

\[
\frac{\partial p_e}{\partial z} = \frac{\mu(f - q\phi)}{\phi k(\phi)} \tag{124}
\]

where \(k(\phi)\) is the permeability of the sediment.

When batch sedimentation has reached a near equilibrium, the solid velocity is negligible and \(f \equiv 0\). If then, water is allowed to drip from the bottom of the settling column, the permeability may be obtained by measuring the excess pore pressure \(p_e(z)\) with a manometer, and using the following equation:

\[
k(\phi) = -\frac{\mu q}{\partial p_e/\partial z} \tag{125}
\]

where \(q\) is the volume average velocity of percolation through the sediment.

(3) Determination of the critical concentration.

Two methods have been proposed to determine the critical concentration of a flocculated suspension: a) Robert's method and b) Michaels and Bolger's method.

a) Robert's method.

Robert\(^{29}\) considered that hindered settling and consolidation obeyed different mechanisms and therefore it should be possible to distinguish between them by appropriately plotting experimental results of batch sedimentation tests. Robert's method consists of plotting \(\log(z_1 - z_w)\) versus time, where \(z_1\) and \(z_w\) are the height of the clear liquid suspension interface at any instant and at the end of a batch sedimentation experiment respectively. Usually the results are three straight lines intersecting at two points. The first intersection point corresponds to the time of the jump of concentration from \(\phi_0\) to \(\phi_0^*\) (see section 3.1) and the second intersection to the critical time, that is the time at which the concentration reaches the critical concentration. See Fig. 28.

Very often it is difficult to distinguish between these two intersection points and only one can be seen. In those cases it is convenient to use Robert's method for different initial concentrations, and plot the result versus the initial concentration. The critical concentration is that result obtained for an initial concentration tending to zero. See Fig 29.
b) Michaels and Bolger’s method.

A procedure to obtain the critical concentration has been proposed based on the Michaels and Bolger’s method to describe hindered settling\(^{26}\). The basis is that Michaels and Bolger’s equation is valid during hindered settling and therefore a deviation of the straight line in a plot of \((-n_{1})^{1/4.65} \) versus \(\phi \) would indicate a change in regime from hindered settling to consolidation. The critical concentration would be the concentration at which the deviation occurs. See Fig. 27.

Obviously the fitting of all the previous equations may be done by linear or non-linear regression techniques instead of doing it graphically.

### 6. Conclusions.

The review of the principal methods of thickener design available in the literature, have shown that they can be grouped into three categories: those based on a macroscopic balance, those based on a kinematic balance equation and those based on dynamic balance equations. Only the last group provides complete information for design purposes.

The principal method based on a macroscopic mass balance is Coe and Clevenger’s method of thickener design. This method leads to the same equation obtained from the more rigorous dynamic balances, but it only gives a lower limit for the unit area, that is, thickeners must have unit areas greater than this basic unit area. The success of this method, which is the most popular in the mining industry, is due to the fact that a safety factor is always used in the design. If this were not the case, problems would be encountered. The justification for this safety factor is not the variability of the material properties or the variation in temperature, as Coe and Clevenger proposed, but rather the fact that the basic unit area is just a limiting value. The final value chosen for the unit area is related to the height of the thickener. Using the basic unit area would lead to an infinite thickener height, which would diminish as the unit area is increased. In conclusion, Coe and Clevenger’s method of thickener design is valid for obtaining the limiting value \(UA_0 \) for the unit area. The method proposed by the same authors for estimating the height of the thickener is not correct and is not recommended.

No method based on kinematic balances is recommended. Those methods give no information on the critical concentration and are usually applied to pulps with discharge concentrations well in the consolidation state. The goal of research workers using these methods is to have a simpler and more rapid method to design thickeners. We believe that performing batch settling experiments for several initial concentrations to obtain a reliable flux-density curve is simple and rapid enough not justifying shortcuts, especially when the equipment to be designed is as costly as a thickener.

It is unfortunate that Adorjan’s method, the principal exponent of the dynamic methods, is almost unknown to the mining industry and to engineers responsible for the design of thickeners, in spite of the fact that it was published in 1975\(^{27}\) and 1976\(^{28}\).

The main problem associated with the dynamic methods is the experimental determination of the
material parameters of the pulps. In the original work, Adorjan proposed the use of a compression cell to determine the compressibility of the sediment. Unfortunately compression cells require much higher pressures than those produced by the self weight of the pulp in thickeners. It is more convenient to measure other variables and calculate the effective solid stress. For example the concentration gradient may be calculated in a batch or continuous test by sampling, gamma ray absorption, X-ray absorption, ultrasonic absorption and capacitance measurements and the excess pore pressure gradient is measured with a manometer. Based on this information the effective solid stress may be calculated using equation (123) and the sediment permeability using equation (125).

Acknowledgments.

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Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>parameter in Wilhelm and Naide equation</td>
</tr>
<tr>
<td>b</td>
<td>parameter in Wilhelm and Naide equation</td>
</tr>
<tr>
<td>c</td>
<td>vertical coordinate separating different values of the initial concentration in an ideal continuous thickener</td>
</tr>
<tr>
<td>f_b</td>
<td>batch flux-density function</td>
</tr>
<tr>
<td>f_b_k</td>
<td>extended batch Kynch flux-density function</td>
</tr>
<tr>
<td>f</td>
<td>continuous flux-density function</td>
</tr>
<tr>
<td>f_k</td>
<td>extended continuous flux-density function</td>
</tr>
<tr>
<td>f'_b</td>
<td>first derivative of f_b with respect to the concentration</td>
</tr>
<tr>
<td>f'</td>
<td>first derivative of f with respect to the concentration</td>
</tr>
<tr>
<td>g</td>
<td>gravity acceleration</td>
</tr>
<tr>
<td>l</td>
<td>height of the hindered settling zone in a continuous thickener</td>
</tr>
<tr>
<td>q</td>
<td>volume average velocity</td>
</tr>
<tr>
<td>q_d_h</td>
<td>volume flow rate of flocculant</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>t_u</td>
<td>parameter defined in Fig. 9</td>
</tr>
<tr>
<td>t_U</td>
<td>parameter defined in Fig. 9</td>
</tr>
<tr>
<td>u</td>
<td>solid fluid relative velocity</td>
</tr>
<tr>
<td>u_in</td>
<td>settling velocity of a single particle (floc) in an unbounded medium</td>
</tr>
<tr>
<td>v_s</td>
<td>solid component velocity</td>
</tr>
<tr>
<td>v_f</td>
<td>fluid component velocity</td>
</tr>
<tr>
<td>z</td>
<td>vertical coordinate</td>
</tr>
<tr>
<td>z_c</td>
<td>thickness of the sediment layer for a flocculated suspension</td>
</tr>
<tr>
<td>z_in</td>
<td>thickness of the sediment layer for an ideal suspension at the end of batch sedimentation</td>
</tr>
<tr>
<td>F</td>
<td>solid mass flow rate</td>
</tr>
<tr>
<td>F_0</td>
<td>limiting solid mass flow rate</td>
</tr>
<tr>
<td>F_n(x)</td>
<td>particle size distribution in the feed</td>
</tr>
<tr>
<td>ICT</td>
<td>ideal continuous thickener</td>
</tr>
<tr>
<td>CKSP</td>
<td>continuous Kynch sedimentation process</td>
</tr>
<tr>
<td>L</td>
<td>feeding level in an ideal continuous thickener</td>
</tr>
<tr>
<td>MS</td>
<td>mode of sedimentation in batch settling</td>
</tr>
<tr>
<td>MCS</td>
<td>mode of continuous sedimentation</td>
</tr>
<tr>
<td>0</td>
<td>mass flow rate of water in the overflow</td>
</tr>
<tr>
<td>Q</td>
<td>volume flow rate</td>
</tr>
<tr>
<td>S</td>
<td>cross sectional area of an ideal continuous thickener</td>
</tr>
<tr>
<td>T</td>
<td>intercept with the abscissa of the tangent to the settling plot at point z_{k}, t_{k}</td>
</tr>
<tr>
<td>UA</td>
<td>unit area = S/F</td>
</tr>
<tr>
<td>UA_0</td>
<td>basic unit area</td>
</tr>
<tr>
<td>W</td>
<td>solid hold up in an ICT</td>
</tr>
<tr>
<td>W_0</td>
<td>total volume of solids in the settling column per unit area</td>
</tr>
<tr>
<td>Z</td>
<td>intercept with the ordinate of the tangent to the settling plot at point z_{k}, t_{k}</td>
</tr>
<tr>
<td>λ</td>
<td>coordinate of the water-suspension and water-sediment interface</td>
</tr>
<tr>
<td>ξ</td>
<td>compresibility of the sediment</td>
</tr>
<tr>
<td>σ</td>
<td>displacement velocity of a surface of discontinuity</td>
</tr>
<tr>
<td>σ_f</td>
<td>velocity of fall of the water-suspension interface</td>
</tr>
<tr>
<td>σ_c</td>
<td>effective solid stress</td>
</tr>
<tr>
<td>ρ_s</td>
<td>solid component density</td>
</tr>
<tr>
<td>ρ_f</td>
<td>fluid component density</td>
</tr>
<tr>
<td>Δ Q</td>
<td>solid-fluid density difference</td>
</tr>
<tr>
<td>φ</td>
<td>concentration as volume fraction of the solid component</td>
</tr>
<tr>
<td>φ_c</td>
<td>critical concentration</td>
</tr>
<tr>
<td>φ_m</td>
<td>solid concentration of a floc</td>
</tr>
<tr>
<td>φ∞</td>
<td>maximum concentration of an ideal suspension</td>
</tr>
<tr>
<td>φ_k</td>
<td>concentration difference between the feed and the critical value</td>
</tr>
<tr>
<td>φ_0</td>
<td>initial concentration in continuous test</td>
</tr>
<tr>
<td>φ_1</td>
<td>initial concentration of a batch test</td>
</tr>
</tbody>
</table>
| φ_0    | concentration obtained by drawing a tangent to the flux-density curve from the
point of coordinates $\phi_0$ and $f(\phi_0)$  

$\phi^\infty$: concentration obtained by drawing a tangent to the flux-density curve from the point of coordinates $\phi_\infty$ and $f(\phi_\infty)$  

$\phi_L$: concentration at the coordinate $z = L$  

$\phi^+$: solid concentration at the front of a discontinuity  

$\phi^-$: solid concentration at the rear of a discontinuity  

F: subscript for a property at the feed of the ICT  

D: subscript for a property at the discharge of the ICT  

n: parameter in the Richardson and Zaki equation

References


