Characterization of Inner $\ast$-Automorphisms of $W^\ast$-Algebras

By

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1. Introduction

For some problems in physics I would like to have a characterization of $\ast$-automorphisms of a C$^*$-algebra which can be realized by unitary operators in the enveloping von Neumann algebra. Looking at this problem I realized that one should first treat the same problem, as an "exercise", for $W^\ast$-algebras. The simplification is due to the fact that an inner automorphism lies also on a one-parametric group, while, for a permanently weakly inner automorphism this property is not known.

The technique employed here is a further development of a method I have used in a recent paper [5] in order to give a new and constructive proof of the theorem of Kadison [6] and Sakai [9] on derivations and my own result [3] on groups with semibounded spectrum. The same technique as used in [5] has been developed independently by Arveson [1] and also by Pedersen [8]. The advantage of this method is due to the fact that it gives a rather explicit construction of the spectral resolution of the unitary operator we are looking for.

The technique is derived from the concept of creation- and annihilation-operators which is used in physics. These operators define a shift operation on the spectrum of the unitary operator we are looking for. The aim is, of course to construct a spectral resolution. This means one has to associate to every projection in the center of the invariant elements a subset of the torus. The difficulty is due to the fact that an inner automorphism defines the unitary operator only up to a unitary in the center. This means the mapping from the projections to the
subset of the torus is not unique. But assume there exists such a map
then we can form for two projections the difference set which is unique.
That this can be done is the content of section 2. These difference sets
contain also enough information in order to give a classification of
automorphisms, section 3. Having associated to every pair a difference
set one only needs to fix one point of the spectrum in order to localize
the projections on the torus. That this can be done and gives rise to
a unitary operator with the correct properties will be shown in section
4. In section 5, we exploit these techniques in order to give several
necessary and sufficient conditions for an automorphism to be inner,
including a generalization of the result of Kadison and Ringrose [6].
Some further applications are also discussed. In the last section we
give a brief outline of the extension of our method to general locally
compact abelian groups.

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2. Notations and Preliminary

By the couple \((R, \alpha)\) we denote a von Neumann algebra \(R\) and
\(\alpha\) -automorphism \(\alpha\) of \(R\) which is automatically a normal automorphism
of \(R\). The group \(\{a^n, n \in \mathbb{Z}\}\) is a representation of the additive group
\(\mathbb{Z}\). Its dual-group we denote by \(T\). Since \(T\) is isomorphic to the unit
circle we identify it also with \([0, 2\pi]\) so that 0 means the identity
of \(T\). A closed interval we write often as \([a, b]\) with the prescription
that one shall start at \(e^{ia}\) on \(T\) and shall proceed in the positive direc-
tion until we get to the point \(e^{ib}\). This means e.g. that \([a, b]\) and
\((b, a)\) are complementary sets. Then for \(\varepsilon > 0\) the sets \([-\varepsilon, \varepsilon]\) are sym-
metrical, intervals around the identity of \(T\).
2.1. Definitions: We introduce the following notations:

a) \( R_0 = \{ x \in R; \forall x = x \} \) which is the set of invariant elements.

b) \( Z \) denotes the center of \( R \).

\( Z_0 \) denotes the center of \( R_0 \).

c) \( P_0 \) denotes the set of projections in \( Z_0 \) (we introduce for this a separate symbol since we work exclusively with this set).

d) Let \( x \in R \) and \( f \in L^1 \) then we denote

\[ x(f) = \sum_n a^n x f(n). \]

e) Let \( I \) be a closed interval of \( T \) then we denote by \( l_I = \{ f \in L^1; \text{support } \mathcal{F}^{-1} f = I \} \), where \( \mathcal{F} \) denotes the Fourier-operator.

f) Let \( E \) be a projection in \( R_0 \) then we denote by \( \phi_I E \) the union of the ranges of \( \{ x(f)E; x \in R, f \in l_I \} \) i.e. \( \phi_I E \) is the smallest projection \( G \) in \( R \) such that \( G x(f)E = x(f)E \) holds for all \( x \in R \) and \( f \in l_I \).

The operator \( \phi_I \) assigns to every projection \( E \in R_0 \) a unique different one. This operation is not linear, and for some pairs \( I \) and \( E \), \( \phi_I E \) might be zero. Before investigating its properties in detail we show:

2.2. Lemma: Let \( G \) be a projector in \( R_0 \) and \( E \) its central support in \( R_0 \), then we have for all \( I \subset T \)

\[ \phi_I G = \phi_I E \in P_0 \]

Proof: In order to prove the first relation, we remark that \( R_0 \) contains the identity. Hence we get \( R \cdot R_0 = R \). Furthermore the elements in \( R_0 \) are invariant. This implies for \( x \in R, y \in R_0 \) the relation \( (xy)(f) = x(f) \cdot y \). Thus we get:

\[ \{(x \cdot y)(f)G\} = \{x(f) \cdot y \cdot G\} x \in R \quad y \in R_0. \]

Since we have \( R_0 G \mathcal{H} = E \mathcal{H} \) (where \( \mathcal{H} \) denotes the Hilbert-space on which \( R \) is acting) we get

\[ \phi_I G = \phi_I E. \]

In order to prove the second relation, notice that \( G \in R_0 \) and is therefore invariant. So we get:
Since \( \alpha \) is an automorphism of \( R \) follows \( \phi_iG \) is invariant. Let now \( H \in R_0 \) then we get:

\[
Hx(f)G = (Hx)(f) \cdot G \in \{ x(f)G ; \ x \in R \}.
\]

This means \( \phi_iG \in R^0 \) is invariant under \( R_0 \). But since it belongs to \( R_0 \), the projector \( \phi_iG \) belongs to the center of \( R_0 \).

This lemma shows that, with respect to the operations \( \phi_I \), we can restrict ourselves to \( Z_0 \). Since we only work with projections a separate notation for them is justified.

2.3. \textbf{Theorem}: Consider \( \phi_I \) as a mapping from \( P_0 \) into \( P_0 \) then it has the following properties:

a) \( E_1 \geq E_2 \) implies \( \phi_iE_1 \geq \phi_iE_2 \).

\( \beta \) Put \( E_1 \cup E_2 = E_1 + E_2 - E_1 \cdot E_2 \) then we get

\[
\phi_i(E_1 \cup E_2) = (\phi_iE_1) \cup \phi_iE_2.
\]

\( \gamma \) Let \( I \) be an ordered index set, let \( E_\gamma \) be increasing i.e. \( E_\beta \subseteq E_\gamma \) for \( \beta < \gamma \) and let \( E = \text{str lim} \ E_\gamma \) then we have

\[
\phi_iE = \text{str lim} \ \phi_iE_\gamma.
\]

b) \( \alpha \) The relation \( I_1 \subset I_2 \) implies

\[
\phi_iI_1 \leq \phi_iI_2.
\]

\( \beta \) Let \( I^o \) denote the inner points of \( I \). Assume \( I_1 \cap I_2 \neq \emptyset \) and \( (I_1 \cup I_2)^o = I_1^o \cup I_2^o \) then we get

\[
\phi_{I_1 \cup I_2}E = \phi_{I_1}E \lor \phi_{I_2}E
\]

\( \gamma \) Let \( I_1 \) and \( I_2 \) be two arbitrary intervals, then we get

\[
\phi_i;\phi_{I_2}E \leq \phi_{I_1 + I_2}E.
\]

(Here we have written the group action of \( T \) with \(+\) sign.)

\( \alpha \) Assume \( I^o \ni 0 \) then we get:
\[ \phi I \subseteq E. \]

\( \beta \) Let \( I = T \) then we get:

\[ \phi T \subseteq P_0 \cap Z. \]

Furthermore \( \phi T \) is the carrier of \( E \) in \( P_0 \cap Z \).

d. \( \alpha \) Let \( F \subseteq P_0 \cap Z \) then we get:

\[ \phi F(F \cdot E) = F \cdot \phi F E. \]

\( \beta \) Let \( E_1 \) be such that \( E_1 \cdot \phi I = 0 \) then follows:

\[ E \cdot \phi I E_1 = 0. \]

**Proof:** a. \( \alpha \) Is trivial.
\( \beta \) From \( \alpha \) we get \( \phi I \subseteq E \leq \phi I \subseteq \phi (E_1 \vee E_2) \). Assume \( H \subseteq P_0 \) and \( H \cdot \phi I E_1 = H \cdot \phi I E_2 = 0 \) then we get for \( x, \in R, y \in l \) : \( H x(f) \{ E_1 \triangledown + E_2 \triangledown \} = 0. \) But this implies \( H x(f)(E_1 \vee E_2) \triangledown = 0. \) Consequently we have \( \phi I (E_1 \vee E_2) \leq (\phi I E_1) \vee (\phi I E_2) \).

\( \gamma \) Since \( E_1 \subseteq E \) we get from \( \alpha \) the relation \( \phi I E \leq \phi I E. \) Assume \( H \subseteq P_0 \) is such that \( H \phi I E = 0 \) for all \( \gamma \in \Gamma \) then follows \( H x(f)E_1 = 0 \) for all \( \gamma, x \in R, y \in l \) and consequently \( H x(f)E = 0. \) This implies \( H \cdot \phi I E = 0 \) and thus \( \phi I E \leq \text{str lim}_{\gamma \in E} \phi I E_\gamma. \) But by the first line \( \text{str lim}_{\gamma \in E} \phi I E_\gamma \leq \phi I E. \)

b. \( \alpha \) This follows directly from the relation \( l_1 I \subseteq l_2 I \) for \( I_1 \subseteq I_2 \).

\( \beta \) First we have from b. \( \alpha \) the relations \( \phi l_{1, 1} l_{1, 2} E \subseteq \phi l_1 E, j = 1, 2 \) and thus \( \phi l_{1, 1} l_{1, 2} E \supseteq \phi l_1 E \vee \phi l_2 E. \) If \( f \in l_{1, 1} l_{1, 2} \) then we find functions \( f_j \in l_{1, j}, j = 1, 2 \) with \( f = f_1 + f_2 \) since the two intervals are over lapping. So we get for \( x \in R \) the equation \( x(f)E = x(f_1)E + x(f_2)E \)

and from this \( \phi l_{1, 1} l_{1, 2} E \leq \phi l_1 E \vee \phi l_1 E \)

\( \gamma \) Assume \( f \in l_{1, 1}, g \in l_{1, 2} \) and \( x, y \in R \) then we find

\[ x(f) y(g) = \sum_{n,m} f(n) g(m) (x^n x) (x^m y) \]
\[
\begin{align*}
&= \sum_{n,m} f(n)g(n+m)x^n(xz^m) \\
&= \sum_{m} \sum_{n} f(n)g(n+m)x^n(xz^m) .
\end{align*}
\]

Since \( f \in l_1^1 \) and \( g \in l_2^1 \) follows that for every fixed \( m \) the function \( f(n)g(n+m) \in l_{1+T}^1 \). This implies

\[
(1 - \phi_{1+T}E)\gamma \phi \gamma = \sum_{m} (1 - \phi_{1+T}E) \sum_{n} f(n)g(n+m)x^n(xz^m)E = 0.
\]

This implies

\[
\phi_{1+T}E \leq 1(f)E = E
\]

c. \( \alpha \) Since zero is an inner point of \( l_1 \) exists a function \( f \in l_1^1 \) such that \( \sum_{n} f(n) = 1 \). From this we get \( 1(f) = 1 \). And consequently

\[
\phi E \geq 1(f)E = E
\]

\( \beta \) If \( I = T \) then we get from c. \( \alpha \) \( \phi T E \geq E \). On the other hand the function \( f = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \)

is an element of \( l_1^1 \) and consequently we get \( \{ x(f) ; x \in R , f \in l_1 \} = R \). This implies \( \phi T E = R E \) and hence \( \phi T E \) is the \( P_0 \cap Z \) carrier of \( E \).

d. \( \alpha \) Let \( F \in P_0 \cap Z \) then we get for \( x \in R , f \in l_1^1 \) and \( E \in P_0 \)

\[
x(f)F \cdot E \cdot x(f) = F x(f) E x(f) \] and consequently

\[
\phi F E = F \phi E.
\]

\( \beta \) Let \( E_1 , E \in P_0 \) and assume \( E_1 \phi T E = 0 \), then we have for \( x \in R , f \in l_1^1 \)

\[
E_1 x(f) E = 0 .
\]

Taking the adjoint of this equation we get \( E(x(f)^*) = 0 \). Now \( \{ x(f) \} = x*(\tilde{f}) \) which implies \( E x*(\tilde{f}) E_1 = 0 \). Since \( f \to \tilde{f} \) maps \( l_1^1 \) onto \( l_{-1}^1 \) we get \( E \phi_{-1} E_1 = 0 \).
This ends the proof of the theorem.

We should emphasize again that $\phi_1$ is not a one-to-one map. One gets a good idea of $\phi_1$ by choosing a unitary operator $u \in R$ and defining $\alpha$ by the relation $\alpha x = uxu^*$. If $E_I$ is a spectral projection of $u$ belonging to the interval $I \subset T$ then $\phi_1 E_I$ is majorized by the spectral projection belonging to the interval $I + I_1$. Therefore in the case where the spectrum of $u$ has a gap there exist pairs $(E, I)$ such that $\phi_1 E = 0$. Moreover it can happen that $\phi_1$ is the null-operator, for instance if $\alpha$ generates the group $\mathbb{Z}_2$ and $I$ does not contain the points 0 and $\pi$ then $\phi_1 = 0$.

It is our aim to assign to every projector in $P_0$ a set of $T$. If this could be done, then the operator $\phi_1$ gives us information about the relative positions of these sets belonging to two projections. Since we know the operators $\phi_1$ we start with these relative positions.

2.4. Definition: Denote by $I_\varepsilon$ the interval $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$, $\varepsilon > 0$. Let $E_1$, $E_2 \in P_0$ then we define

$$n(E_1, E_2) = \{b \in T; \exists \varepsilon > 0 \text{ with } E_2 \phi_{b+I_\varepsilon} E_1 = 0\}$$

2.5. Proposition: The sets $n(E_1, E_2)$ have the following properties:

a) $n(E_1, E_2)$ is open.

b) $n(E_2, E_1) = n^{-1}(E_1, E_2)$.

c) $E_1 \subseteq E_1$ and $E_2 \subseteq E_2$ implies

$$n(E_1, E_2) \subseteq n(E_1, E_2).$$

d) For $F \in P_0 \cap Z$ we get

$$n(Fe_1, E_2) = n(E_1, Fe_2) = n(Fe_1, Fe_2).$$

e) $n(E_1 \lor E_2, E_3 \lor E_4) = n(E_1, E_3) \cap n(E_1, E_4) \cap n(E_2, E_1) \cap n(E_2, E_4)$

f) Let $E_2$ be an increasing family of projections and define $E = \text{str. lim } E_2$, then we get

$$n(E_1, E) = \left(\bigcap_{\varepsilon} n(E_1, E_2)\right)^\circ$$

g) Let $K$ be a connected component of $n(E_1, E_2)$, and let $I$ be the
closure of $K$. Assume $I \neq T$ then we get

$$E_2 \cdot \phi_i E_1 = 0$$

h) Let $F_1$ and $F_2$ be the $P_0 \cap Z$ carriers of $E_1, E_2$ then $n(E_1, E_2) = T$ if and only if $F_1 \cdot F_2 = 0$

Proof: a) Let $b \in n(E_1, E_2)$ then exists $\varepsilon > 0$ with $E_2 \phi_{b+I_\varepsilon} E_1 = 0$. If $b' \in (b + I_\varepsilon)^0$ then exists $\varepsilon' > 0$ with $b' + I_{\varepsilon'} \subset b + I_\varepsilon$. Hence we get from 2.3. b. a)

$$E_2 \phi_{b' + I_{\varepsilon}} E_1 \leq E_2 \phi_{b + I_\varepsilon} E_1 = 0.$$ 

This means $b' \in n(E_1, E_2)$.

b) Let $b \in n(E_1, E_2)$, then exists $\varepsilon > 0$ with $E_2 \phi_{b+I_\varepsilon} E_1 = 0$. Thus we get from 2.3. d. b)

$$E_1 \phi_{(b + I_\varepsilon)^-} E_2 = 0.$$ 

Since $(b + I_\varepsilon)^- = b^{-1} + I_\varepsilon$ we get $b^{-1} \in n(E_2, E_1)$ and consequently $n^{-1}(E_1, E_2) \subseteq n(E_2, E_1)$ and $n^{-1}(E_2, E_1) \subseteq n(E_1, E_2)$ which implies $n^{-1}(E_1, E_2) = n(E_2, E_1)$

c) This follows directly from 2.3. a. a)

d) From 2.3. d. a) follows:

$$E_2 \phi_i F E_1 = E_2 F \phi_i E_1 = FE_2 \phi_i F E_1,$$ which implies d)

e) Because of b) it is sufficient to prove the relation

$$n(E_1, E_3 \lor E_4) = n(E_1, E_3) \cap n(E_1, E_4).$$

c) implies that the left hand side is contained in the intersection on the right. So it remains to prove the converse. Let $b \in n(E_1, E_3) \cap n(E_1, E_4)$ then exists $\varepsilon > 0$ with $E_3 \phi_{b+I_\varepsilon} E_1 = E_4 \phi_{b+I_\varepsilon} E_1 = 0$ and consequently $E_3 \lor E_4 \phi_{b+I_\varepsilon} E_1 = 0$. This means $n(E_1, E_3) \cap n(E_1, E_4) \subset n(E_1, E_3 \lor E_4)$.

f) Since $E \supseteq E_\varepsilon$ follows from c) $n(E_1, E) \subset n(E_1, E_\varepsilon)$, and hence $n(E_1, E) \subset \{ \cap n(E_1, E_\varepsilon) \}^\varepsilon$. Assume $b$ is contained in the right hand side, exists $\varepsilon > 0$ such that $b + I_\varepsilon$ is also contained in it. Thus we get $E_\varepsilon \phi_{b+I_\varepsilon} E_1 = 0$ for all $\varepsilon$ and consequently $E \phi_{b+I_\varepsilon} E_1 = 0$. This means
Let $K$ be a connected component of $n(E_1, E_2)$. Let $I_1$ be a connected closed subset of $K$, then $I_1$ is compact. Hence exists a finite number of points $b_i \in I_1$ and $e_i$ such that $\cup b_i + I_{e_i}$ covers $I_1$ and $E_2 \phi_{b_i + t_{e_i}} E_1 = 0$. This implies by 2.3. b. £) that $E_2 \phi_{b_i + t_{e_i}} E_1 = 0$. From this we get $E_2 x(f) E_1 = 0$ for all $x \in R$ and all $f \in l_1^1$ such that $\text{supp } \mathcal{F}^{-1} f \subset I^\circ$. Since these functions are dense in $l_1^1$ and $f \rightarrow x(f)$ is continuous in $f$ in the $l^1$ topology follows $E_2 x(f) E_1 = 0$ for $f \in l_1^1$. This implies $E_2 \phi_{x_1} = 0$.

h) From 2.3. c. £) follows $\phi_T E_1 = F_1$. If $n(E_1, E_2) = T$ then we get from g) $\phi_T E_1 E_2 = F_1 E_2 = 0$.

This ends the proof of the proposition.

3. Classification of $*$-Automorphisms

In this section we give a complete classification of automorphisms.

A. The class $K_\infty$

Our investigation is based on the sets $n(E_1, E_2)$. The first obstruction which might happen is that these sets are empty. This leads us to the first

3.1. Definition: We say $(R, \alpha)$ belongs to the class $K_\infty$ if for all pairs $E_1, E_2 \in P_0$ we have $n(E_1, E_2) = \emptyset$ or $= T$.

Equivalently $(R, \alpha)$ belongs to $K_\infty$ if for all pairs $E_1, E_2 \in P_0$ the set $n(E_1, E_2)$ is invariant under the whole rotation-group of the torus.

The second definition we have put in, since it is in analogy to the defining relations of the other classes. Next we consider an arbitrary automorphisms $(R, \alpha)$.

3.2. Proposition: Let $(R, \alpha)$ be arbitrary, then exists a unique projection $F_\infty \in P_0 \cap Z$ such that $(F_\infty R, \alpha)$ belongs to $K_\infty$ and for every $F \in P_0 \cap Z$ with $F \cdot F = 0$ follows $(FR, \alpha)$ does not belong to $K_\infty$. This means $((1 - F_\infty) R, \alpha)$ contains no invariant part which belongs to $K_\infty$. 

\[ n(E_1, E_2) \Rightarrow \{ \cap n(E_1, E_a) \}^\circ. \]
Proof: Let \( \mathcal{F} = \{ F \in P_0 \cap Z; (FR, \alpha) \in K_\infty \} \).

Assume first \( F \in \mathcal{F} \) and \( F_1 \in P_0 \cap Z \) with \( F_1 \leq F \) then follows for \( E_1, E_2 \leq F_1 \) that also \( E_1, E_2 \leq F \). Hence \( n(E, E_2) \) must be \( \emptyset \) or \( T \). This means \( F_1 \in \mathcal{F} \). Let now \( F_1, F_2 \in \mathcal{F} \) and \( E_1, E_2 \leq F_1 \lor F_2 \) then follows from 2.5. e)

\[
n(E_1, E_2) = n(E_1, F_1 E_2) \cap n(E_1, F_2 E_2)
\]

This implies \( n(E_1, E_2) \) is \( \emptyset \) or \( T \) since it holds for the right hand side. Consequently \( F_1 \lor F_2 \in \mathcal{F} \). Let \( F_\alpha \in \mathcal{F} \) be an increasing family of projections and define \( F = \operatorname{strlim}_\alpha F_\alpha \). Assume \( E_1, E_2 \leq F \) then we get by 2.5. f) \( n(E_1, E_2) = \bigcap_\alpha n(E_1, E_\alpha E_2) \). Since now \( n(E_1, F_\alpha E_2) = n(F_\alpha E_1, F_\alpha E_2) \) is \( \emptyset \) or \( T \) follows \( n(E_1, E_2) \) is \( \emptyset \) or \( T \). Hence \( F \) belongs to \( \mathcal{F} \). Using now Zorn's lemma we see that \( \mathcal{F} \) contains a unique maximal element \( F_\infty \) such that \( F \cdot F_\infty = F \) for all \( F \in \mathcal{F} \). From the construction follows furthermore that \( (1 - F_\infty)R \) contains no invariant part belonging to \( K_\infty \).

Next we want to give a better characterization of the part which does not belong to \( K_\infty \).

3.3. Lemma: For any pair \((R, \alpha)\) the following statements are equivalent.

1. \( F_\infty = 0 \).
2. For every \( F \in P_0 \cap Z \) exists a pair of projection \( E_1, E_2 \leq F \) such that \( n(E_1, E_2) \neq T \) and \( n(E_1, E_2) \neq \emptyset \).
3. For every \( E \in P_0 \) exists a projection \( E_1 \leq E, E_1 \in P_0 \) and a projection \( E_2 \in P_0 \) such that \( n(E_1, E_2) \neq T \) and \( n(E_1, E_2) \neq \emptyset \).
4. For every \( E \in P_0 \) exists a projection \( E_1 \leq E, E_1 \in P_0, E_1 \neq 0 \) such that \( n(E_1, E_1) \neq \emptyset \).

Remark that \( n(E_1, E_1) \) is also not \( T \). Since \( E_1 \neq 0 \) \( n(E_1, E_1) \) must not contain 0 by 2.5. g)
**Proof:** \(1. \Rightarrow 2.\) Assume 1 holds but not 2. Then not 2 implies there exists an element \(F \in P_0 \cap \mathbb{Z}\) that for any pair \(E_1, E_2 \leq F\) we must have \(n(E_1, E_2) = T\) or \(\emptyset\). But this means \((F, \alpha)\) belongs to \(K_\infty\), contradicting the assumption that \(F_\infty = 0\).

\(2. \Rightarrow 3.\) Assume 2 holds and let \(E \in P_0\) be given. Denote by \(F(E)\) the support of \(E\) in \(P_0 \cap \mathbb{Z}\), then from 2 follows that there exist \(G\) and \(E_2 \in P_0\) with \(G, E_2 \leq F(E)\). Then \(n(G, E_2) \neq T\) and \(n(G, E_2) \neq \emptyset\). Multiplying all projection by \(F(E_2)\) we then can assume \(F(E) = F(E_2)\). Since \(n(G, E_2) \neq T\) follows \(F(G) \neq 0\). This implies \(n(G, E) \neq T\). This implies for \(b \notin n(G, E)\) and \(\varepsilon > 0\) that \(E_1 = E\phi_{b+I_2}\neq 0\). Choose now \(a\) and \(\varepsilon\) in such a way that \(b + a + I_2 \in n(G, E_2)\) then we get:

\[
E_2 \phi_{a+I_2}E_1 = E_2 \phi_{a+I_2}(E\phi_{b+I_2}G) \leq E_2 \phi_{a+I_2} \phi_{b+I_2}G \leq E_2 \phi_{a+b+I_2}G = 0
\]

by 2.5. c), 2.3. b. \(\gamma\), and 2.5. g). This implies \(n(E_1, E_2) \neq \emptyset\). Since \(E_1 \neq 0\) and \(F(E_1) \leq F(E_2)\) by construction follows from 2.5. h) that \(n(E_1, E_2) \neq T\).

\(3. \Rightarrow 4.\) Assume 3, and let \(E\) be given, then exists a pair \(G \leq E\) and \(E_2\) such that \(n(G, E_2) \neq T\) and \(\neq \emptyset\). By 2.5. b) this also holds for \(n(E_2, G)\). Let \(b \notin n(E_1, G)\), then \(E_1 = G\phi_{b+I_2}E_2 \neq 0\). Choose \(a\) and \(\varepsilon\) such that \(a + b + I_2 \in n(E_2, G)\) then we get again by 2.5. c), 2.3. b. \(\gamma\), 2.5. g) and 2.5. h)

\[
E_1 \phi_{a+I_2}E_1 = (G\phi_{b+I_2}E_2)\phi_{a+I_2}(G\phi_{b+I_2}E_2) \leq G\phi_{a+I_2}(G\phi_{b+I_2}E_2) \\
\leq G\phi_{a+I_2} \phi_{b+I_2}E_2 \leq G\phi_{a+b+I_2}E_2 = 0.
\]

This implies \(n(E_1, E_2) \neq 0\). Since \(E_1 \neq 0\) we have also \(n(E_1, E_1) \neq T\).

\(4. \Rightarrow 1.\) Is trivial.

**B.** The classes \(K_n\), \(n \in \mathbb{N}\).

Before defining these classes we need some preparation and motivation. We start with the technical part

**3.4. Lemma:** Denote by \(n(E_1, E_2)\) those points of \(n(E_1, E_2)\) which have distance greater than \(\varepsilon\) from the boundary of \(n(E_1, E_2)\). Assume \(b \notin n(E_1, E_2)\) then \(E' = E_2\phi_{b+I_2}E_1 \neq 0\). From this we get \(n(E', E') \neq T\) and
\[ n(E', E') \supset n(E_2, E_2) \cup n_{E_1}(E_1, E_1) \cup \{ n_{E_1}(E_2, E_2) - b \} \cup \{ n_{E_1}(E_2, E_1) + b \} \]

**Proof:** (i) Assume \( a \in n(E_2, E_2) \) and \( \varepsilon' \) such that \( a + I_{E'} \subset n(E_2, E_2) \) then we get:

\[
E' \cdot \phi_{a + I_{E'}} E' = E_2 \cdot (\phi_{b + I_{E_1}} E_1) \cdot \phi_{a + I_{E'}} (E_2 \phi_{b + I_{E}} E_1)
\]

\[ \leq E_2 \cdot \phi_{a + I_{E'}} E_2 = 0. \]

(ii) Assume \( a + I_{E'} \subset n_{E_2}(E_1, E_1) \), then we find:

\[
E' \cdot \phi_{a + I_{E'}} E' = E_2 (\phi_{b + I_{E_1}} E_1) \phi_{a + I_{E'}} (E_2 \phi_{b + I_{E}} E_1) \leq (\phi_{b + I_{E_1}} E_1) \phi_{a + I_{E'}} \phi_{b + I_{E}} E_1
\]

\[ \leq (\phi_{b + I_{E_1}} E_1) \cdot \phi_{a + b + I_{E'}, E} E_1. \]

In order to show that this vanishes we make use of 2.3. d. \( \beta \).

\( a + I_{E'} \subset n_{E_2}(E_1, E_1) \) implies:

\[ 0 = E_1 \cdot \phi_{a + I_{E'} + E_1} \geq E_1 \cdot \phi_{b - I_{E'} + I_{E}} \phi_{a + b + I_{E'} + E} E_1 \]

and hence

\[ (\phi_{b + I_{E_1}} E_1) \phi_{a + b + I_{E'} + E} E_1 = 0 \]

(iii) Let \( a + I_{E'} \subset \{ n_{E_1}(E_1, E_2) - b \} \), then follows \( a + b + I_{E'} + E \subset n(E_1, E_2) \).

From this we get:

\[
E' \cdot \phi_{a + I_{E'}} E' = E_2 (\phi_{b + I_{E_1}} E_1) \phi_{a + I_{E'}} (E_2 \phi_{b + I_{E}} E_1)
\]

\[ \leq E_2 \phi_{a + I_{E'}} \phi_{b + I_{E}} E_1 \leq E_2 \cdot \phi_{a + b + I_{E'} + E} E_1 = 0 \]

(iv) The last term follows from (iii) by the relation \( n(E_1, E_1) = n^{-1}(E_1, E_1) \).

It is our aim to classify the sets \( n(E, E) \). In order to see what kind of situations might occur we look first at extremal cases.

**3.5. Lemma:** Assume \( E \neq 0 \) and \( n(E, E) \neq \emptyset \). Suppose furthermore that for every \( E_1 \) with \( n(E_1, E) \neq \emptyset \) we get for all \( b \neq n(E_1, E) \) and all \( \varepsilon > 0 \) the relation \( \{ n(E_1, E) - b \} \subset n(E, E) \), then we have:

(i) There exists \( n \in \mathbb{N} \) such that

\[ T \setminus n(E, E) = \left\{ 0, \frac{2\pi}{n}, \ldots, \frac{m}{n}, \ldots, \frac{n-1}{n} \cdot 2\pi \right\}. \]
(ii) For every $E_1$ we get $n(E_1, E)$ is invariant under rotations by the angles $\frac{m}{n}2\pi$, $m = 0, 1, \ldots, n-1$.

Proof: (i) For proving the first statement we put $E_1 = E$. Since $T$ is compact there exists a connected component $I = (a, b)$ of $n(E, E)$ with maximal length. Since $a$ does not belong to $n(E, E)$ we get from the assumption: $((a + e, b - e) - a) \subseteq n(E, E)$. Since $n(E, E)$ is symmetric follows also $((b - a - e)^{-1}, -e) \subseteq n(E, E)$. Since $e$ was arbitrary follows $0$ is an isolated point of $T \setminus n(E, E)$. This implies $(0, c) \subseteq n(E, E)$ with $c = b - a$. Since $c$ and hence $c^{-1}$ does not belong to $n(E, E)$ follows that also $(0, c) + c = (c, 2c) \subseteq n(E, E)$. By repeating the argument we get

$$(mc, (m+1)c) \subseteq n(E, E) \quad m = 1, 2, \ldots$$

Since $T$ is compact, exists a smallest $n$ such that $nc = 2\pi$. This implies $T \setminus n(E, E)$ contains at most the points $\left\{ \frac{m}{n}2\pi; m = 0, 1, \ldots, n-1 \right\}$. None of these points can belong to $n(E, E)$, since by construction there exists no component of $n(E, E)$ with length greater than $c$. Since $E \neq 0$ follows from 2.3. c. a) that $0 \notin n(E, E)$ which implies that $n$ must be at least one.

(iii) Now let $E_1$ be arbitrary. If $n(E_1, E) = T$ then the statement is correct. Assume now $b \notin n(E_1, E)$ then follows from the assumption $n_\varepsilon(E_1, E) - b \subseteq n(E, E)$ or equivalently $n_\varepsilon(E_1, E) - b \not\subseteq \left\{ \frac{m}{n}2\pi \right\}$ or $n_\varepsilon(E_1, E) \supseteq b + \left\{ \frac{m}{n}2\pi \right\}$. Since $\varepsilon$ was arbitrary follows $b \notin n(E_1, E)$ implies also $b + \frac{m}{n}2\pi \notin n(E_1, E)$, consequently $T \setminus n(E_1, E)$ is invariant and thus also $n(E_1, E)$ is invariant under the rotations by $\frac{m}{n}2\pi$.

From this lemma we extract the following

3.6. Definition: A pair $(R, \alpha)$ belongs to the class $K_n$, $n \in \mathbb{N}$, if the following two conditions are fulfilled:

(i) For every pair $E_1, E_2 \in P_0$ the set $n(E_1, E_2)$ is invariant under the
rotations by the angles $\frac{m}{n}2\pi$, $m=0,1,\ldots,n-1$.

(ii) For every $F \in P_0 \cap Z$ and every $0 < a < \frac{\pi}{n}$ exists an $E \in P_0$ with $E \leq F$ and $n(E, E) = (a, \frac{2\pi}{n} - a)$.

3.7. Proposition: Let $(R, \alpha)$ be arbitrary, then exists a unique projection $F_n \in P_0 \cap Z$ such that $(F_n R, \alpha)$ belongs to the class $K_n$, but for no projection $F \in P_0 \cap Z$ with $F \cdot F_n = 0$, $(F R, \alpha)$ belongs to $K_n$.

Proof: Define $\mathcal{F} = \{F \in P_0 \cap Z; (F R, \alpha)$ belongs to the class $K_n\}$. Now we show:

(i) $F \in \mathcal{F}$ and $F_1 \leq F$ with $F_1 \in P_0 \cap Z$ then $F_1 \in \mathcal{F}$. This is clear from the definition of $K_n$.

(ii) $F_1, F_2 \in \mathcal{F}$ implies $F_1 \lor F_2 \in \mathcal{F}$. Let $E_1, E_2 \leq F_1 \lor F_2$ then follows $n(E_1, E_2) = n(F_1 E_1, E_2) \cap n((E_2 - F_1 F_2) E_1, E_2)$. Since both sets on the right hand side have the invariance property it follows also for $n(E_1, E_2)$. Now let $F < F_1 \lor F_2$, then $F \cdot F_1$ or $FF_2$ is not zero, say $FF_1$. If $F \in P_0 \cap Z$ then also $F \cdot F_1$. Hence exists for a given $0 < a < \frac{\pi}{n}$ a projection $E$ with $n(E, E) = (a, \frac{2\pi}{n} - a)$. Consequently $F_1 \lor F_2 \in \mathcal{F}$.

(iii) Let now $F_n \in \mathcal{F}$ be an increasing family and let $F = \text{str lim} F_n$. We want to show that $F$ belongs to $\mathcal{F}$. Take $E_1, E_2 \leq F \Rightarrow$ then we have by 2.5. (i) $n(E_1, E_2) = n(E_1, E_2) \cap n((E_1 - F_n) E_2)$. Since all $n(E_1, F_n E_2)$ have the invariance property, it also follows for $n(E_1, E_2)$.

Next let $F_1 \in P_0 \cap Z$ such that $0 \neq F_1 \leq F$. Then exists $\alpha$ such that $F_1 F_n \neq 0$, and hence we find $E$ with $0 \neq E \leq F_1 F_n \leq F_1$ such that $n(E, E) = (a, \frac{n}{2} - a)$ with $0 < a < \frac{\pi}{n}$. This implies $F \in \mathcal{F}$.

Now by Zorn's lemma $\mathcal{F}$ contains a unique maximal element $F_n$ such that $F F_n = F$ for all $F \in \mathcal{F}$. For any $F \in P_0 \cap Z$ with $F F_n = 0$ the pair $(F R, \alpha)$ does not belong to $K_n$ by construction of the family $\mathcal{F}$.

Before we discuss the meaning of the different classes we show that the classification is complete.

3.8. Theorem: Let $R$ be a von Neumann algebra and $\alpha$ a $*$-auto-
morphism of $R$. Then exist projections $F_n, n \in \mathbb{N}$ and $F_{\infty}$ belonging to $P_0 \cap Z$ which are mutually orthogonal such that

$$F_{\infty} + \sum_{n \in \mathbb{N}} F_n = 1,$$

and such that $(F_n R, \varphi)$ belongs to $K_n$ for $n = 1, 2, \ldots$ and $\infty$.

Proof: First we will define a decomposition, and afterwards we will show in two separate lemmas that this decomposition coincides with the decomposition into the classes $K_n$ and $K_\infty$.

We define families $\mathcal{F}_n$ by the following conditions: $F \in P_0 \cap Z$ belongs to $\mathcal{F}_n$ if there exists an element $0 \neq E \in P_0$ such that

(i) The $P_0 \cap Z$ support of $E$ is $F$.

(ii) In $n(E, E)$ exists a connected component $I$ with length $l(I) > 2\pi/n + 1$.

From the construction follows $\mathcal{F}_n \supset \mathcal{F}_{n-1}$. Next we define projection $H_n \in P_0 \cap Z$ by the relation $H_n = \vee F$. Since $\mathcal{F}_{n+1} \supset \mathcal{F}_n$ follows $H_{n+1} \geq H_n$. Therefore exists $H_\infty = \text{str lim} \ H_n$.

Now define $G_\infty = 1 - H_\infty$ and $G_n = H_n - H_{n-1}$. From the construction follows $G_\infty + \sum G_n = 1$. It remains to prove that $G_\infty$ and $G_n$ coincides with $F_\infty$ and $F_n$ defined in 3.2. and 3.7. We start with $G_\infty$. Let $0 \neq E \leq G_\infty$ then we have $n(E, E) = 0$. Because, if $n(E, E) \neq 0$ then its $P_0 \cap Z$ support $F(E)$ would belong to some family $\mathcal{F}_n$. Now lemma 3.3 implies that for $E_1, E_2 \leq G_\infty$ follows $n(E_1, E_2) = 0$ or $\emptyset$. But this shows $G_\infty \leq F_\infty$. Conversely if $F \in P_0 \cap Z$ with $G_\infty F = 0$ then exists a number $n$ such that $F \cdot H_n \neq 0$ and by definition of $H_n$ exists a projector $F_1 \in \mathcal{F}_n$ with $F \cdot F_1 \neq 0$. But by definition of $F_1$ exists $E$ with $F(E) = F_1$ and $n(E, E) \neq 0$ hence also $n(E, FE) \neq 0$ and consequently $F \cdot F_\infty = 0$. This implies $F_\infty \leq G_\infty$ and thus $F_\infty = G_\infty$.

For proving the identity between $G_n$ and $F_n$ we need some preparations which we put into the form of two lemmas.

3.9. Lemma: (i) Let $F$ belong to $\mathcal{F}_n$ and let $\frac{2\pi}{n} > a > \frac{2\pi}{n + 1}$, then there exists a projection $0 \neq E \in P_0$ with its $P_0 \cap Z$ support $\leq F$ and $n(E, E)$ contains a connected component $I$ with $l(I) \geq a$

(ii) $H_n$ belongs to $\mathcal{F}_n$, and there exists $E$ with support $E = H_n$ and
n(E, E) contains a connected component I with \( l(I) \geq a \)

**Proof:** (i) This part is based on lemma 3.4 and 3.5. Let \( F \in \mathcal{F}_n \).

For every \( 0 \neq E \in P_0 \) with \( E \leq F \) define \( l(E) \) to be the maximum of the length of the connected components of \( n(E, E) \). Define \( \bar{l}(F) = \sup_{0 \neq E \leq F} l(E) \). Since \( F \) belongs to \( \mathcal{F}_n \) follows \( l(E) > \frac{2\pi}{n+1} \). By definition of \( \bar{l}(F) \) exists for any \( \varepsilon > 0 \) a projector \( 0 \neq E \leq F \) with \( l(E) \leq \bar{l}(F) + \varepsilon \). Assume this component \( I \) is \( \left( c + \frac{\varepsilon'}{2}, c + \bar{l}(F) - \frac{\varepsilon'}{2} \right) \) with \( 0 \leq \varepsilon' \leq \varepsilon \), then follows from the fact that there exists no \( E \neq 0 \) with \( l(E) > l(F) \) that also \( (\varepsilon', l(F) - \varepsilon') \subset n(E, E) \). If \( b \notin n(E, E) \) would belong to this interval we would find \( \varepsilon_1 > 0 \) with \( \varepsilon' < b - \varepsilon_1 \) and \( b + \varepsilon_1 < \bar{l}(F) - \varepsilon' \) and hence \( \left( c + \frac{\varepsilon'}{2}, c + \bar{l}(F) - \frac{\varepsilon'}{2} \right) + I_{\varepsilon_1} + b \) is the interval \( \left( c + b + \frac{\varepsilon'}{2} + \varepsilon_1, c + \bar{l}(F) + b - \frac{\varepsilon'}{2} - \varepsilon_1 \right) \). But we have \( c + b + \frac{\varepsilon'}{2} + \varepsilon_1 < c + \bar{l}(F) - \frac{\varepsilon'}{2} \) and \( c + \bar{l}(F) + b - \frac{\varepsilon'}{2} - \varepsilon_1 > c + \bar{l}(F) + \frac{\varepsilon'}{2} \). This would imply \( E' = E \phi_{b + I_{\varepsilon_1}, E \neq 0} \) and \( l(E') > \bar{l}(F) \).

Repeating this argument we find

\[
n(E, E) \Rightarrow (k \bar{l}(F) + (k+1)\varepsilon, (k+1)\bar{l}(F) - (k-1)\varepsilon)
\]

Since such relation holds for all \( \varepsilon > 0 \) exists a number \( m \) with \( m \bar{l}(F) = 2\pi \) or \( \bar{l}(F) = \frac{2\pi}{m} \). Since \( \bar{l}(F) > \frac{2\pi}{n+1} \) follows \( m \leq n \).

(ii) For any given a with \( \frac{2\pi}{n+1} < a < \frac{2\pi}{n} \) define a family of pairs

\[
\mathcal{F}_a = \{(F, E); F \in P_0 \cap Z \text{ and } F \leq H_n, E \in P_0 \text{ and its } P_0 \cap Z \text{ carrier } F(E) \text{ is equal to } F, \text{ and } n(E, E) \subset \left( \frac{\pi}{n} - \frac{a}{2}, \frac{\pi}{n} + \frac{a}{2} \right) \}
\]

From the construction given in the proof of (i) follows that \( \mathcal{F}_a \) is not empty and, moreover, every \( F \leq H_n \) contains \( F_1 \neq 0 \) such that \( (F_1, E) \in \mathcal{F}_a \) for a suitable \( E \).

In \( \mathcal{F}_a \) we introduce a semi-ordering by the relations \( (F_1, E_1) < (F_2, E_2) \) if \( F_1 \leq F_2 \) and \( E_1 = F_1E_2 \). Let now \( \{(F_a, E_a)\} \) be a strictly ordered increasing set and define \( F = \text{str lim } F_a \) and \( E = \text{str lim } E_a \).
Since $E \geq E_a$ follows $F(E) \geq F_a$ and hence $F(E) \geq F$. But since $E \leq F$ follows $F(E) = F$. Furthermore we get $E_a = F_a E$ and from this $n(E_a, E_a) = n(E, F_a E) = n(E, E_a)$. Using 2.5. f) we get:

$$n(E, E) = \left( \bigcap_{a} n(E_a, E) \right) \circ \left( \bigcap_{a} n(E_a, E) \right) \circ \left( \frac{\pi - a}{2}, \frac{\pi}{2} + \frac{a}{2} \right).$$

This means every strictly ordered family has an upper bound, and consequently by Zorn’s lemma exist maximal elements in $a$. Let now $(F, E)$ such a maximal element and assume $F \neq H_n$ then $H_n - F \neq 0$ and there exists $F_1 \neq 0$ such that $(F_1, E_1) \in a$ for an appropriate $E$. Since $F + F_1 = 0$ follows $(F + F_1, E + E_1)$ is strictly greater than $(F, E)$. This contradicts the maximality and therefore we get $H_n = F$ for every maximal element $(F, E)$. This proves the lemma. In addition we remark that for any given pair $(F_1, E_1)$ there exists a maximal element $(H_n, E)$ such that $E_1 = F_1 E$.

Next we turn to the projection $G_n = H_n - H_{n-1}$. We know that it fulfills the second part of the definition 3.6. In order to show that $G_n$ and $F_n$ coincide it remains to check the periodicity condition. This we will do in a separate lemma.

3.10. Lemma: Let $F$ belong to $P_0 \cap Z$ and assume for every $\frac{2\pi}{n} > a > \frac{2\pi}{n+1}$ exists $E \in P_0$ with $F(E) = F$ and $n(E, E)$ has a connected component $I$ with length $l(I) \geq a$. Assume furthermore for every $0 \neq E_1 \leq F$ no connected component $I$ of $n(E_1, E_1)$ can have length $l(I) > \frac{2\pi}{n}$. Then $n(E_1, E_2)$ is invariant under rotations by the angles $\frac{m}{n} 2\pi$, $m = 0, 1, \ldots, n-1$, provided $E_1, E_2 \leq F$.

Proof: The proof is based on a repeated use of lemma 3.4. This lemma does not say anything if $n = 1$, therefore we will assume $n > 1$. (a) Assume the conditions of the lemma are fulfilled and give $\varepsilon > 0$, then exists a projection $E \in P_0$, $F(E) = F$ such that $n(E, E) \geq \left( \frac{m}{n} 2\pi + \varepsilon, \frac{m+1}{n} 2\pi - \varepsilon \right)$, $m = 0, 1, \ldots, n-1$. 


Proof: Let \( \left\lceil \frac{n}{2} \right\rceil \) be the smallest entire number greater or equal to \( \frac{n}{2} \), and let \( \varepsilon_1 < \frac{\frac{n}{2}}{\left\lceil \frac{n}{2} \right\rceil} \). By the remark at the end of the proof of the last lemma exists a projection \( E \in P_0 \) with \( F(E) = F \) and \( n(E, E) \supseteq (\varepsilon_1, \frac{2\pi}{n} - \varepsilon_1) \). Assume \( b \in \left( \frac{2\pi}{n} + 2\varepsilon_1, \frac{2\pi}{n} - 2\varepsilon_1 \right) \) then exists \( \delta > 0 \) such that \( b + I_\delta \) is contained in the same interval. Since \( n(E, E) \) is symmetric we have \( n(E, E) \supseteq \left( -\frac{2\pi}{n} + \varepsilon_1, -\varepsilon_1 \right) \) . Now \( \left( -\frac{2\pi}{n} + \varepsilon_1, -\varepsilon_1 \right) + b \) is the interval \( \left( -\frac{2\pi}{n} + \varepsilon_1 + \delta + b, -\varepsilon_1 - \delta + b \right) \). By choice of \( b \) we have \( -\frac{2\pi}{n} + \varepsilon_1 + \delta + b < \frac{2\pi}{n} - \varepsilon_1 \) and \( -\varepsilon_1 - \delta + b < \frac{2\pi}{n} + \varepsilon_1 \). Hence we get \( \left( \left( \varepsilon_1, \frac{2\pi}{n} - \varepsilon_1 \right) \cup \left( -\frac{2\pi}{n} + \varepsilon_1, -\varepsilon_1 \right) \right) + b \supseteq \frac{2\pi}{n} \). Using 3.4 and the assumptions of this lemma we get \( E \phi_{- b + 1, E} = 0 \) hence \( n(E, E) \supseteq \left( \frac{2\pi}{n} + 2\varepsilon_1, \frac{2\pi}{n} - 2\varepsilon_1 \right) \). If we repeat this argument we loose at every step \( \varepsilon_1 \). Hence we get

\[
n(E, E) \supseteq \left( \frac{m}{n} \cdot 2\pi + (m + 1)\varepsilon_1, \frac{m + 1}{n} \cdot 2\pi - (m + 1)\varepsilon_1 \right).
\]

Since \( n(E, E) \) is symmetric we get the desired result by the choice of \( \varepsilon_1 \).

(\( \beta \)) Assume next there exists \( E_1, E_2 \in P_0, E_1, E_2 \leq F \) such that \( n(E_1, E_2) \) is not invariant under rotations by the angles \( \frac{m}{n} \cdot 2\pi \) then exists \( 0 \neq E' \in P_0, E' \leq F \) and \( \varepsilon > 0 \) and \( m \leq \left\lceil \frac{n}{2} \right\rceil \) such that \( n(E', E') \supseteq \frac{m}{n} \cdot 2\pi + I_\varepsilon \).

Proof: Since \( n(E_1, E_2) \) not is periodic exists \( b \notin n(E_1, E_2) \) and \( m \) such that \( b + \frac{m}{n} \cdot 2\pi \in n(E_1, E_2) \). Since \( n(E_1, E_2) \) is open exists \( \varepsilon > 0 \) such that \( b + \frac{m}{n} \cdot 2\pi + I_\varepsilon \subset n(E_1, E_2) \). Take \( \varepsilon_1 < \frac{\varepsilon}{3} \) and defien \( E'E_2 = \phi_{b + I_\varepsilon} E_1 \) which is unequal to zero. Lemma 3.4 implies \( n(E', E') \supseteq \left( n_{\varepsilon_1}(E_1, E_2) - b \right) \supseteq \frac{m}{n} \cdot 2\pi + I_{\varepsilon_1} \) by choice of \( \varepsilon_1 \). Since \( n(E', E') \) is symmetrical follows \( n(E', E') \supseteq \frac{n - m}{n} \cdot 2\pi + I_{\varepsilon_1} \).

(\( \gamma \)) Assume the conclusion of the lemma is false, then by (\( \beta \)) exists \( E' \neq 0, E' \leq F, \varepsilon > 0 \) and \( m \) with \( n(E', E') \supseteq \frac{m}{n} \cdot 2\pi + I_\varepsilon \). Choose \( \varepsilon_1 < \frac{\varepsilon}{5} \) then by (\( \alpha \)) exists \( E \neq 0 \) with \( F(E) = F \) and \( n(E, E) \supseteq \left( \frac{m}{n} \cdot 2\pi + \varepsilon_1, \frac{m + 1}{n} \cdot 2\pi - \right. \)
Since \( E' \neq 0 \), \( E' \leq F = F(E) \) follows by 2.5. h) that \( n(E', E) \neq T \). Hence exists \( b \notin n(E', E) \) and consequently \( E' \phi_{b+\varepsilon_1} E' = E_1 \neq 0 \). But lemma 3.4. implies

\[
n(E_1, E_1) \supset n(E, E) \cup n_{2\varepsilon_1}(E', E') = \left( \frac{m-1}{n} \cdot 2\pi + \varepsilon_1, \frac{m+1}{n} \cdot 2\pi - \varepsilon_1 \right)
\]

by construction of \( \varepsilon_1 \). The length of this interval is \( \frac{2}{n} \cdot 2\pi - 2\varepsilon_1 \) which is greater than \( \frac{2\pi}{n} \) if \( \varepsilon_1 \) is sufficiently small. But this contradicts the assumptions of the lemma which therefore holds.

Proof of the Theorem (continuation). From lemma 3.9 and 3.10 follows that \((G_n, a) \in K_n\). Since \( G_n \) is the largest element in \( P_0 \cap Z \) fulfilling 3.9 and 3.10 follows \( G_n = F_n \) and hence the Theorem 3.8 is proved.

4. Construction of Spectral Resolutions and Interpretation of the Classes

In this section we want to give an interpretation of the different classes in terms of inner and outer automorphisms. The main tool for this is the construction of spectral resolutions. We first will state the result and afterwards prepare the proof in several lemmas.

4.1. Theorem: Let \( R \) be a von Neumann algebra and \( \alpha \) be a \(*\)-automorphism of \( R \) then \((R, \alpha) \in K_n \) if and only if for any \( F \in P_0 \cap Z \) the automorphisms \( \alpha^i \) restricted to \( FR \) are inner for \( i \equiv 0 \mod n \) and outer for \( i \not\equiv 0 \mod n \). \((R, \alpha) \in K_\infty \) if and only if for every \( F \in P_0 \cap Z \) \( \alpha^i \) restricted to \( FR \) is outer for all \( i \neq 0 \).

In order to simplify the discussion assume for the moment that \( R \) is a factor and \((R, \alpha) \in K_1 \). In this case we want to construct a unitary operator \( U \in R \) which implements \( \alpha \). Our aim is to construct the spectral resolution of \( U \) which is located on \( T \) since \( U \) is unique only up to a phase-factor we have to fix an arbitrary point of \( T \). Defining a spectral resolution located on \( T \) means that we have to assign to every \( E \in P_0 \) a subset of \( T \). If this can be done then a filter
of projection can also be used for fixing a point of \( T \). Therefore our aim is to construct such a filter.

4.2. Lemma: Let \((R, \alpha)\) belong to class \( K_n \) and let \( 0 < \epsilon < \frac{n}{n} \), then exists a decreasing family of projections \( E_\epsilon \in P_0 \) such that

\( (\alpha) \ E_\epsilon_1 \leq E_\epsilon_2 \) for \( \epsilon_1 < \epsilon_2 \)

\( (\beta) \) The \( P_0 \cap Z \) support of \( E \) which we denote by \( F(E) \) is equal to 1

\( (\gamma) \ n(E_\epsilon, E_\delta) = \left( \frac{m}{n} \cdot 2\pi + \epsilon, \frac{m+1}{n} \cdot 2\pi - \epsilon \right), \ m = 0, 1, \ldots, n-1 \)

A family of projections with these properties will be called a point-fixing family \( f \).

Proof: Since by Definition 3.6 all \( n(E_1, E_2) \) are periodic it is sufficient to look at the interval \([0, \frac{2\pi}{n}]\). Since it is not required that all \( E_\epsilon \) are different for different \( \epsilon \), it is sufficient to construct \( E_\epsilon \) for some sequence tending to zero, e.g. on a geometric progression. Given \( \epsilon_0 \) then we know from lemm 3.9. the existence of a projector \( E_{\epsilon_0} \) with the properties \((\beta)\) and \((\gamma)\). Assume we have constructed \( E_\epsilon \) for \( \epsilon = \epsilon_0 2^{-i}, \ i = 1, 2, \ldots, n \) then we have to construct it for \( i = n+1 \). We write \( \epsilon_0 \cdot 2^{-n} = \delta \) for abbreviation. Let \( b = \sup \{a; 0 \leq a < \delta, a \notin n(E_\delta, E_\delta)\} \) and define \( E_1 = E_\delta \cdot \phi_{b + 1/2} E_\delta) \neq 0 \).

From lemma 3.4., we get:

\[ n(E_1, E_1) = n(E_\delta, E_\delta) \cup \{n_{b/2}(E_\delta, E_\delta) - b\} \cup \{n_{b/2}(E_\delta, E_\delta) + b\} \]

This means

\[ n(E_1, E_1) = \left( \frac{b}{2}, \frac{2\pi}{n} - \frac{b}{2} \right) \quad \text{with} \quad \frac{b}{2} \leq \epsilon_0 2^{-(n+1)}. \]

Now let \( F(E_1) \) be the \( P_0 \cap Z \) support of \( E_1 \) and \( F_2 = 1 - F_1 \) then we have:

\[ 0 = F_2 \cdot E_1 = F_2 E_\delta \cdot \phi_{b + 1/2} F_2 E_\delta. \]

This implies

\[ n(F_2 E_\delta, F_2 E_\delta) = \left( \frac{b}{2}, \frac{2\pi}{n} - \frac{b}{2} \right). \]

Putting \( E_{\delta/2} = E_1 + F_2 E_\delta \), we have \( n(E_{\delta/2}, E_{\delta/2}) = \left( \frac{\delta}{2}, \frac{2\pi}{n} - \frac{\delta}{2} \right) \) and

\[ n(E_{\delta/2}, E_{\delta/2}) = \left( \frac{\delta}{2}, \frac{2\pi}{n} - \frac{\delta}{2} \right) \]

and
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\[ F(E_{b/2}) = 1. \] This proves the lemma

4.3. Definition: Let \( f \) be a point-fixing family and \( E \in P_0 \), then we define the support of \( E \), symbol \( S_f(E) \), by the relation

\[ S_f(E) = \text{complement of } \{ \bigcup_{\varepsilon > 0} n(E_{\varepsilon}, E) \} \]

Since we choose the family \( f \) once and for all, we write simply \( S(E) \).

4.4. Proposition: Let \((R, a)\) belong to \( K_n \) then the support \( S(E) \) has the following properties:

a. \( S(E) \) is periodic with periodicity \( \frac{2\pi}{n} \)

b. \( S(E_\varepsilon) \subset \text{period } [-\varepsilon, \varepsilon] \), where period \([a, b]\) is defined as

\[ \bigcup_{0 \leq m \leq n-1} \left[ \frac{m}{n} 2\pi + a, \frac{m}{n} 2\pi + b \right] \]

c. \( S(\phi_\beta \cdot I_{\varepsilon}, E_{\varepsilon}) \subset \text{period } [+b - \varepsilon - \varepsilon_1, +b + \varepsilon + \varepsilon_1] \)

d. Let \( E = E_1 \lor E_2 \) then \( S(E) = S(E_1) \cup S(E_2) \)

e. \( S(E) = \emptyset \) if and only if \( E = 0 \)

f. The relation \( S(E_1) \cap S(E_2) = \emptyset \) implies \( E_1, E_2 = 0 \)

g. Let \( E_\alpha \) be increasing and \( E = \text{str. lim}_\alpha E_\alpha \) then we get \( S(E) = \text{closure } \{ \cup S(E_\alpha) \} \)

h. Let \( G \subset T \) be a closed set, then exists a unique maximal projection \( E_G \) such that \( S(E_G) \subset G \) and such that for any projection \( E_1 \) with \( (1 - E_G)E_1 \neq 0 \) follows \( S(E_1) \) is not a subset of \( G \).

Proof: a. This follows directly from Definition 3.6 and the definition of \( S(E) \)

b. For \( \varepsilon_1 < \varepsilon \) we have \( E_{\varepsilon_1} \leq E_\varepsilon \). This implies (see 2.5. c)) \( n(E_{\varepsilon_1}, E_\varepsilon) \subset n(E_\varepsilon, E_\varepsilon) \) and consequently \( S(E_\varepsilon) \subset \text{period } [-\varepsilon, \varepsilon] \)

c. Assume \( a \notin \text{period } [+b - \varepsilon - \varepsilon_1, +b + \varepsilon + \varepsilon_1] \) then exists \( \delta > 0 \) such that \( a + I_\delta \) does not intersect with the same set. This also implies \( a - b + I_\delta \cap [-\varepsilon, \varepsilon] = \emptyset \). For \( \varepsilon_2 < \varepsilon \) we get:

\[ E_{\varepsilon_2} \phi_{a + I_\delta} \circ \phi_{b + I_\delta}, E_\varepsilon \leq E_{\varepsilon} \cdot \phi_{-a + I_\delta} \circ \phi_{b + I_\delta}, E_\varepsilon \leq E_\varepsilon \phi_{-a + b + I_\delta + \varepsilon_1} E_\varepsilon = 0, \]

since \( -a + b + I_\delta + \varepsilon_1 \subset n(E_{\varepsilon}, E_\varepsilon) \). This implies (see 2.3. d. \( \beta \)) \( a \in \)
n(\(E_{\varepsilon_2}, \phi_{b+\varepsilon_1}E_{\varepsilon}\)) for all \(\varepsilon_2 < \varepsilon\). This proves the statement by the definition of \(S(E)\).

d. From 2.5. e) we have \(n(E_{\varepsilon}, E) = n(E_{\varepsilon_1}, E) \cap n(E_{\varepsilon_2}, E_2)\), and hence by the monotony in \(\varepsilon\) we find:

\[
\bigcup_{\varepsilon} n(E_{\varepsilon}, E) = \bigcup_{\varepsilon} \{n(E_{\varepsilon_1}, E_1) \cap n(E_{\varepsilon_2}, E_2)\} = \{\bigcup_{\varepsilon} n(E_{\varepsilon_1}, E_1)\} \cap \{\bigcup_{\varepsilon} n(E_{\varepsilon_2}, E_2)\}.
\]

Consequently we have \(\{\bigcup_{\varepsilon} n(E_{\varepsilon}, E)\} = \{\bigcup_{\varepsilon} n(E_{\varepsilon_1}, E_1)\} \cap \{\bigcup_{\varepsilon} n(E_{\varepsilon_2}, E_2)\}\) which is equivalent to statement d.

e. If \(E = 0\) then \(n(E_{\varepsilon}, E) = T\) for all \(\varepsilon\) and hence \(S(E) = \emptyset\). If \(S(E) = \emptyset\) then follows \(\bigcup_{\varepsilon} n(E_{\varepsilon}, E) = T\). Since \(T\) is compact and \(n(E_{\varepsilon}, E)\) is open exists \(\varepsilon_0\) such that \(n(E_{\varepsilon_0}, E) = T\). Since \(F(E_{\varepsilon}) = 1\) we get \(E = 0\) by 2.5. b)

f. We get from d the relation \(S(E_1, E_2) \subseteq S(E_i)\) \(i = 1, 2\). Hence \(S(E_1) \cap S(E_2) = \emptyset\) implies \(S(E_1) \cap S(E_2) = \emptyset\) and consequently \(S(E_1) \cap S(E_2) = \emptyset\) follows from e.

g. From \(E_\alpha \leq E\) follows \(S(E_\alpha) \subseteq S(E)\) and hence \(\text{cl}\{\bigcup S(E_\alpha)\} \subseteq S(E)\). Assume now \(b \notin \text{cl}\{\bigcup S(E_\alpha)\}\) then exists \(\delta > 0\) such that \(b + I_\delta \cap \text{cl}\{\bigcup S(E_\alpha)\} = \emptyset\) which implies \(b + I_\delta \cap S(E_\alpha) = \emptyset\) for all \(\alpha\). From C. follows \(S(\phi_{b+I_\delta/2}E_{\delta/2}) \cap S(E_\alpha) = \emptyset\) and consequently from f. \(E_\alpha \phi_{b+I_\delta/2}E_{\delta/2} = 0\) for all \(\alpha\). But this implies \(E \phi_{b+I_\delta/2}E_{\delta/2} = 0\) and therefore \(b \in n(E_{\delta/2}, E)\) which implies \(b \notin S(E)\). This gives the relation \(S(E) \subseteq \text{cl}\{S(E) \subseteq \text{cl}\{\bigcup S(E_\alpha)\}\}\) which proves g.

h. Define \(F_G(E; S(E) \subseteq G)\). Then d., g., and Zorn's lemma implies that \(F_G\) contains a unique maximal element. The definition of \(F_G\) implies that also the second statement is fulfilled.

Statement h. of the last proposition permits us to define a spectral resolution.

4.5. Definition: Let \((R, \s) \in K_\alpha\) then we define for \(\lambda \in [0, 2\pi]\) the spectral projector \(E_\lambda\) to be the unique maximal element belonging to the set period \([0, \frac{\lambda}{n}]\) which is described in 4.4. h.

4.6. Lemma: \(E_\lambda\) has the following properties:

a. \(E_\lambda \leq E_\mu\) for \(\lambda < \mu\)
b. \( \text{str lim } E_\lambda = 1 \)

c. \( E_\lambda = \text{str lim } E_\mu \)

d. Assume \( I_1 \) and \( I_2 \subset I_1 \) are two connected intervals, define \( E(I) \) as usual then we have for all \( x \in \mathbb{R} \): support \( \mathcal{F}^{-1}\{E(I)nx^nxE(I_2)\} \subset \text{closure } I_1 - I_2 \). The index \( l \) indicates that we have to take the Fourier transform with respect to \( l \).

**Proof:**

a. \( \lambda < \mu \) implies period \([0, \mu/n] \subset \text{period } [0, \lambda/n] \) and hence by definition of the spectral family \( E_\lambda \leq E_\mu \)

b. Let \( E' = 1 - \text{str lim } E_\lambda \), and let \( b + I_e \subset \left(0, \frac{2n}{n - 1}\right) \), and \( E_{t_2} \subset f_\epsilon \), the point-fixing family. Then we have by 4.4. c. \( S(\phi_{b + I_{t_2}})E_{t_2} \subset \text{period } \{b + I_e\} \), and hence \( \phi_{b + I_{t_2}}E_{t_2} \leq E_\lambda \) for some \( \lambda \).

This implies \( E'\phi_{b + I_{t_2}} = 0 \) and consequently \( S(E') = \text{period } \{0\} \). From the maximality condition imposed on \( E_\lambda \) follows \( E' = 0 \)

c. We have \( S(\text{str lim } E_\lambda) \subset \bigcap_{\lambda - \mu = 0} \text{period } \left[0, \frac{\lambda}{n}\right] = \text{period } \left[0, \frac{\mu}{n}\right] \). But this implies \( \text{str lim } E_\lambda \leq E_\mu \). Since the other inclusion is trivial, follows c.

d. As in b. it is easy to check that \( S(E(I)) \) is contained in \( \text{cl} \left\{ \text{period } \frac{1}{n} I \right\} \).

This implies (4.4. c. and f.) \( E(I_1)\phi_{b + I_e}E(I_2) = 0 \) if \( \text{cl} \left\{ \text{period } \frac{1}{n} I_1 \right\} \cap \text{cl} \left\{ b + I + \frac{1}{n} I_2 \right\} = \emptyset \), and hence \( n(E(I_2), E(I_1)) \subset \{ \text{complement } \text{period } \left(\frac{1}{n}I_1 - \frac{1}{n}I_2\right) \} \).

Let us denote by \( C \) the closure of the complement of \( I_1 - I_2 \), and take \( f \in l_C \). Define \( f^n(l) = \lceil f \left( \frac{l}{n} \right) \rceil \) for \( l \equiv 0 \mod n \). From the relation

\[
(\mathcal{F}^{-1} f^n)(a) = (\mathcal{F}^{-1} f)(n \cdot a)
\]

follows \( f^n(l) \in l^n_C \). Since \( \frac{1}{n}C = n(E(I_2), E(I_1)) \) we get by use of 2.5. g) and the definition of \( \phi \) the equation

\[
E(I_1)x(f^n)E(I_2) = 0, \quad x \in \mathbb{R} \quad \text{and} \quad f \in l_C.
\]

This implies statement d.

We are now prepared to prove the main part of theorem 4.1.
4.7. Proposition: Let \((R, \alpha)\) belong to \(K_n\) and let \(E_\lambda, 0 \leq \lambda < 2\pi\) be the spectral family defined in 4.5. Define the unitary operator \(u \in R\) by the equation \(u = \int_0^{2\pi} e^{i\lambda} dE_\lambda\), then we get for every \(x \in R\)

\(\alpha^n x = u x u^*\).

More precisely \(u \in Z_0\).

Proof: Define \(h = \int_0^{2\pi} \lambda dE_\lambda\) and approximate it by the operators

\[ h_N = \sum_{i=1}^N \frac{i}{N} 2\pi (E_{\frac{i}{N}2\pi} - E_{\frac{i-1}{N}2\pi}) \]

then we have \(\|h - h_N\| \leq \frac{2\pi}{N}\). Define \(u_N = e^{ih_N}\) then \(u^l - u_N^l\) is bounded on the real axis by two and hence we get from Schwarz lemma and the Phragmen-Lindelöf theorem the estimate (see [11], 5.2. and 5.6.)

\[ \|u^l - u_N^l\| = \|1 - e^{il(h-h_N)}\| \leq |l| \frac{2\pi}{N}. \]

Next consider the expression \(u^l_N u^n x u_N^l, x \in R\), as function of \(l\). Replacing \(u_N\) by its definition we get:

\[ u^l_N u^n x u_N^l = \sum_{j,k=1}^N e^{-i\frac{l}{2\pi} (E_{\frac{j}{N}2\pi} - E_{\frac{j-1}{N}2\pi})} x_{j,k} x_{k,j} e^{i\frac{l}{2\pi} \frac{2\pi}{N}}. \]

Using lemma 4.6. d. we find

\[ \text{supp} \mathcal{F} \{ e^{-i\frac{l}{2\pi} (E_{\frac{j}{N}2\pi} - E_{\frac{j-1}{N}2\pi})} x_{j,k} x_{k,j} e^{i\frac{l}{2\pi} \frac{2\pi}{N}} \} \subset \left[ -\frac{2\pi}{N}, \frac{2\pi}{N} \right]. \]

Since this holds for every term of the sum its also true for the whole sum. The Fourier integral defines an interpolation

\[ (u^l_N u^n x u_N)(z) = \int_{-\frac{2\pi}{N}}^{\frac{2\pi}{N}} e^{i\alpha z} \mathcal{F}^{-1} (u^l_N u^n x u_N)(\alpha) d\alpha \]

which is an entire function of exponential type \(\frac{2\pi}{N}\). Since it is bounded
by \( \|x\| \) on all entire values of \( z \), there exists a constant \( K \) depending only on \( N \) such that the interpolation is bounded by \( K \|x\| \) (Cartwright’s theorem, see Boas [2] theorem 10.3.2.). This constant is majorized by \( 4 + 2e \log(1/1 - \frac{2}{N}) \) and hence we can replace \( K \) by 5 if we choose \( N \geq 14 \).

From this we get \( F_{N}(z) = x - (u_{N}^{N} x u_{N})(z) \) is an entire function of exponential type \( \frac{2\pi}{N} \) with \( \|F_{N}(z)\| \leq 6 \|x\| \) for real \( z \) and \( N \geq 14 \) which vanishes at the origin. Hence from Schwarz lemma and the Phragmén-Lindlöf theorem we get the estimate

\[
\|F_{N}(z)\| \leq |z| \|x\| 6e^{\frac{2\pi}{N}}
\]

provided \( N \geq 14 \) and \( |z| \leq \frac{N}{2\pi} \). This shows \( F_{N}(z) \) converges to zero uniformly on every compact. Since \( u_{N} \) approaches \( u \) we get the desired result.

Let us now prove the converse statement.

4.8. Lemma: Let \((R, \alpha)\) be given and assume for every \( F \in P_{0} \cap z \alpha^{n} \) restricted to \( F \cdot R \) is an inner automorphism but for any \( 0 < m < n \) \( \alpha^{m} \) restricted to \( F \cdot R \) is not inner, then \((R, \alpha)\) belongs to the class \( K \).

Proof: Let \( f \in l^{1} \) then we define the functions \( f_{k}, k = 0, \ldots, n - 1 \) by the equation,

\[
f_{k}(l) = \sum_{m=0}^{n-1} \frac{1}{n} \exp \left\{ \frac{m}{n} 2\pi (l-k) \right\} f(l) = \frac{1}{n-1} \frac{1 - e^{i2\pi(l-k)}}{1 - e^{i2\pi l/n}} f(l)
\]

\[
= \begin{cases} f(l) & \text{for } l = k \mod n \\ 0 & \text{elsewhere.} \end{cases}
\]

This implies \( f(l) = \sum_{k=0}^{n-1} f_{k}(l) \). Now we compute

\[
\sum_{l} e^{-i\alpha l} f(k+n) = \sum_{l} e^{-i\alpha (k+n)} f(k+n) = e^{i\alpha} \sum_{l} e^{-i\alpha} f_{k}(l)
\]

\[
= e^{i\alpha} \sum_{l} e^{-i\alpha} \frac{1}{n} \sum_{m=0}^{n-1} e^{i2\pi (l-k) f(l)}
\]
Let now \( f \in l_1^1 \) then \( f(k + nl) \) belongs for fixed \( k \) and \( n \) to \( l_{n,l}^1 \). This implies for \( x \in \mathbb{R} \)

\[
\sum_i f(l) x^i = \sum_{k=0}^{n-1} \sum_i f(k + nl) x^i x
\]

and hence we find for \( E \in P_0 \) the relation

\[
\phi_x^x E \leq \phi^{n, x}_n E,
\]

where the upper index indicates the automorphism with which the map \( \phi_x \) is constructed. By assumption \( \alpha^n \) is an inner automorphism, and hence exists a unitary operator \( u = \int_0^{2\pi} e^{it} dE \) implementing \( \alpha^n \). Let now \( I_1 \) and \( I_2 \) be two intervals then we get

support \( \mathcal{F}^{-1}(E(I_1) u^i x u^{-1} E(I_2)) = \text{closure} I_1 - I_2 \)

and hence \( n^*(E_{I_1}, E_{I_2}) \supset (\frac{\epsilon}{n}, 2\pi - \epsilon) \). From this follows \( n^*(E_{I_1}, F_{I_2}) \subset \left( \frac{\epsilon}{n}, 2\pi - \frac{\epsilon}{n} \right) \). This means for every \( \epsilon > 0 \) exists an \( E \neq 0 \) with \( P_0 \cap Z \)

support = 1 and \( n(E, E) \) contains a connected interval of length \( \geq 2\pi - \frac{2\pi}{n} - 2\epsilon \). On the other hand there can not exist an \( E \neq 0 \) such that \( n(E, E) \) contains an interval of length greater then \( \frac{2\pi}{n} \) otherwise \( \alpha \) restricted to \( F(E)R \) has the property that \( \alpha^n \) is inner for some \( m < n \). Because theorem 3.8 permits us to decompose \( F(E) \) is \( \Sigma f(E) \) when \( k \) divides \( n \) and proposition 4.7. allows us to construct unitary operators in \( F_k(E)R \) implementing \( \alpha^k \). But this contradicts the assumptions of this lemma. Finally lemma 3.10 shows that all conditions for defining \( K_n \) are fulfilled.

**Proof of the theorem**: Let \( n \) be a number and \( d(n) \) all its divisors including 1 but not \( n \). Let \((R, \alpha) \) be arbitrary then exists a unique maximal projection \( H_n \) in \( P_0 \cap Z \) such that \( \alpha^n \) restricted to \( H_n R \) is inner. Define \( G_n = H_n - \bigvee_{k \in d(n)} H_k \). From the definition follows that all the
$G_n$ are mutual orthogonal. Define $G_\infty = 1 - \sum_n G_n$. Compare this decomposition with that given by theorem 3.8. Proposition 4.7. and lemma 4.8. shows that $G_n$ and $F_n$ coincide. Hence $G_\infty$ and $F_\infty$ coincide by construction. But $G_\infty$ can also be defined to be the maximal element of $P \cap Z$ such that for every $F \leq G_\infty$ no power of $\alpha$ restricted to $F \cdot R$ is inner. But this proves the theorem.

5. Characterization of Inner $\ast$-Automorphisms

In this section we want to give characterizations of inner automorphisms of a von Neumann algebra. Moreover we want to apply our technique to various problems related to automorphisms and derivations. To this end we introduce the following

5.1. Notations: Let $\beta$ be a linear operator acting on a Banach-Space $B$ then we denote

a. $\|\beta\|$ the usual operator norm

$\|\beta\| = \sup \{ \|\beta x\| ; \; x \in B \; \|x\| \leq 1\}$

b. $\text{sp}(\beta)$ the spectrum of $\beta$ which is the complement of the points $z \in \mathbb{C}$ such that $(\beta - z)$ has a bounded inverse. If $\alpha$ is a $\ast$-automorphism of a $W^*$ or $C^*$ algebra then $\text{sp}(\alpha)$ is located on the unit circle.

c. $\rho(\beta)$ denote the spectral norm of $\beta$

$\rho(\beta) = \sup \{|z| ; \; z \in \text{sp}(\beta)\}$

It is well known that $\rho(\beta) \leq \|\beta\|$ and that $\rho(\beta) = \lim_{n \to \infty} \|x^n\|^{1/n}$.

d. Let $B_1$ be a closed linear subspace of $B$ invariant under $\beta$, then we denote by $\beta|_{B_1}$ the restriction of $\beta$ to $B_1$.

e. Let $\alpha$ be a $\ast$-automorphism of a $C^*$- or $W^*$-algebra then its spectrum is located on the unit circle $T$ we denote by $g(\alpha) = T \setminus \text{sp}(\alpha)$ and call it the gap of the spectrum. Since $\text{sp}(\alpha)$ is closed follows $g(\alpha)$ is open.

Giving characterizations of inner automorphisms we will include in the list also the result of Kadison and Ringrose [7]. The main results of this section are as follows
5.2. Theorem: Let $\alpha$ be a $*$-automorphism of a $W^*$ algebra $R$, then the following statements are equivalent:

a. $\alpha$ is an inner automorphism.
b. $\alpha$ lies in the connected component of the $*$-automorphism group of $R$ furnished with the norm-topology.
c. There exists an invariant projection $E \in R$ with central support 1 such that

$$\rho((\alpha - 1)|_{ERE}) < \sqrt{3}$$
d. There exists an invariant projection $E' \in R$ with central support 1 such that

$$\|(\alpha - 1)|_{E'RE'}\| < 2$$

Both numbers $< \sqrt{3}$ in c and $< 2$ in d cannot be improved.

5.3. Theorem: For $*$-automorphisms we have the following relation between $\rho(\alpha - 1)$ and $\|\alpha - 1\|$. 

a. If $\rho(\alpha - 1) < \sqrt{3}$ then we get

$$\|\alpha - 1\| = \rho(\alpha - 1)$$
b. If $\sqrt{3} \leq \rho(\alpha - 1) \leq 2$ then $\|\alpha - 1\|$ is a double-valued function of $\rho(\alpha - 1)$ and takes the values

$$\|\alpha - 1\| = \begin{cases} \text{either } \rho(\alpha - 1) \\ \text{or } 2 \end{cases}$$

Both values are taken by special examples.

5.4. Theorem: A. Let $\alpha$ be a $*$-automorphism of a $W^*$-algebra $R$ then we have

(i) $\text{sp}(\alpha)$ is invariant under the substitution

$$z \longrightarrow z^{-1}$$

(ii) If $(R, \alpha)$ belongs to the class $K_n$ then it is also invariant under the substitution
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\[ z \rightarrow e^{i2\pi \frac{m}{n}}z \quad m=0, 1, \ldots n-1 \]

(iii) If the spectrum of \( \alpha \) has a gap, then, in the standard decomposition (theorem 3.8) only those \( F_n \) can be different from zero for which

\[ \exp\left\{i2\pi \frac{m}{n}\right\} \in \text{sp}(\alpha) \quad \text{for all } m=0, 1, \ldots, n-1 \]

(iv) Assume \( \|\alpha-2\|<2 \), then exists a unique unitary operator in \( R \) with the properties: For every central projection \( F \)

\[ \text{sp}(uF) \subseteq \text{sp}\{x|_{FRF}\} \cap \{\text{Im } z \geq 0\} \]

such that \( \alpha x = uxu^* \) for every \( x \in R \).

B. Let \( \delta \) be a real derivation of \( R \) then we have

(i) \( \text{sp} \delta \) is invariant under the substitution

\[ z \rightarrow -z \]

(ii) There exists a unique positive operator \( A \in R \) such that for every central projection \( F \) we have

\[ \text{sp}(FA) \subseteq \text{sp}\left\{\delta \frac{\delta}{i|_{FRF}}\right\} \cap \{z \geq 0\} \]

This last result generalizes a result of Pedersen [8].

For the proofs of these three theorems, we need some preparations. We start with

5.5. Lemma: Let \( \alpha \) be a \( \ast \)-automorphism of a \( W \)-*-algebra \( R \) and let \( \phi_I \) be the map defined in 2.2., then \( \phi_I=0 \) if \( I \subseteq g(\alpha) \) and only if \( I^\circ \subseteq g(\alpha) \) (\( I^\circ \) the interior points of \( I \)).

Proof: Assume first \( I \subseteq g(\alpha) \). Since \( I \) is closed and \( g(\alpha) \) is open exists a constant \( M \) such that \( \|(\alpha-z)^{-1}\| \leq M \) for all \( z \) in some compact set including \( I \).

Now let \( f \in I \) and \( \hat{f} = \mathcal{F}^{-1}f \) then we have \( \text{supp}\hat{f} \subseteq I \) and \( |\hat{f}(a)| \leq \|f\| \) and furthermore \( f(n) = \int e^{-ins}\hat{f}(a)da \). This gives
\[\sum_n f(n)\alpha^n = \sum_n e^{i\alpha f(n)} d\alpha^n = \sum_{-\infty}^0 e^{i\alpha f(a)} d\alpha^n + \sum_{0}^\infty e^{i\alpha f(a)} d\alpha^n\]

Since \(f \in l_1\) follows \(\int e^{-i\alpha+\nu f(a)} d\alpha = f^+(n)\) converges to \(f^+(n)\) for \(n \geq 0\) and \(r \to +0\) in \(l_1\) and \(\int e^{-i\alpha+\nu f(a)} d\alpha = f^-(n)\) converges to \(f^-(n)\) for \(n < 0\) and \(r \to +0\) in \(l_1\). Hence exists for any \(\epsilon > 0\) a number \(r_0(\epsilon) > 0\) such that \(f^+(n) - f^+(n) \leq \epsilon \|f^+(n)\|\) and \(f^-(n) - f^-(n) \leq \epsilon \|f^-(n)\|\) for \(0 \leq r \leq r_0\). Inserting now \(f^+_r\) for \(f^+\) and \(f^-_r\) for \(f^-\) then we can interchange the summation with the integration. So we get

\[\|\sum f(n)\alpha^n\| - \epsilon \|f\| \leq \sum_{n \to -\infty} e^{-i\alpha+\nu f(a)} d\alpha^n + \sum_{0}^\infty e^{-i\alpha+\nu f(a)} d\alpha^n\]

\[= \left| \int d\alpha f(a) e^{-i\alpha-r\alpha^{-1}} (1 - e^{i\alpha-r\alpha^{-1}})^{-1} + \int d\alpha f(a) (1 - e^{-i\alpha-r\alpha^{-1}})^{-1} \right|\]

\[= \left| \int d\alpha f(a) \{e^{-i\alpha-r}(\alpha - e^{-i\alpha-r})^{-1} - e^{i\alpha+r}(\alpha - e^{i\alpha+r})^{-1} \} \right|\]

\[= \left| \int d\alpha f(a) e^{-i\alpha+r}(e^{-r} - \alpha)(\alpha - e^{i\alpha+r})^{-1} \right| \leq 2 \sinh r \|f\| L(I) M^2\]

where \(L(I)\) denotes the length of the interval \(I\). This implies

\[\|\sum f(n)\alpha^n\| \leq \|f\| + 2 \sinh r \cdot M(M+1)L(I)\].

Since \(\epsilon\) and \(r\) are arbitrarily small follows \(\sum f(n)\alpha^n = 0\).

Let now conversely \(f_0 = 0\) then follows \(f_1 = 0\) and hence by definition of \(f_1\) follows \(\sum f(n)\alpha^n = 0\) for all \(f \in l_1\) and all \(x \in \mathbb{R}\), and hence \(\sum f(n)\alpha^n = 0\) for all \(f \in l_1\). Hence we get with the same calculation

\[\left| \int d\alpha f(a) \{e^{i\alpha-r}(\alpha - e^{-i\alpha-r})^{-1} - e^{i\alpha+r}(\alpha - e^{-i\alpha+r})^{-1} \} \right| \leq \|f\| \]

for \(r < r_0(\epsilon)\). This implies by the edge of the wedge theorem that \(e^{i\alpha-r}(\alpha - e^{-i\alpha-r})^{-1}\) is analytic in \(I^0\). Hence \(I^0 \subset g(x)\).

From this we immediately get
5.6. **Corollary**: Let \( \alpha \) be a \(*\)-automorphism then we have

(i) \( \text{sp}(\alpha) \) is invariant under the substitution \( z \rightarrow z^{-1} \)

(ii) If \((R, \alpha) \subseteq K_n\) then \( \text{sp}(\alpha) \) is invariant under the substitution

\[
    z \longrightarrow e^{i2\pi \frac{m}{n}} z \quad m = 0, 1, \ldots, n - 1
\]

and \( \exp \left\{ i2\pi \frac{m}{n} \right\} \in \text{sp}(\alpha) \) for \( m = 0, 1, \ldots, n - 1 \)

(iii) If \( \rho(\alpha - 1) < \sqrt{3} \) then \( \alpha \) is inner and there exists an example with \( \rho(\alpha - 1) = \sqrt{3} \) and \( \alpha \) is outer.

**Proof**: Let \( z \in g(\alpha) \) then exists an closed interval \( I \subseteq g(\alpha) \) such that \( z \in I^\circ \). Since we have \( \phi_t = 0 \) follows for every \( E \in P_0 \) that \( E\phi_t 1 = 0 \) and hence by Theorem 2.3 we have \( \phi_{t^{-1}} E = 0 \). Since this holds for all \( E \in P_0 \) follows \( \phi_{t^{-1}} = 0 \). Since \( z \in I^\circ \) follows \( z^{-1} \in (I^{-1})^\circ \) and hence by lemma 5.5 we get \( z^{-1} \in g(\alpha) \).

Now let \((R, \alpha) \subseteq K_n\) then we have from lemma 3.10, for all \( E_1, E_2 \in P_0 \) that \( n(E_1, E_2) \) is invariant under the rotations by the angles \( 2\pi \frac{m}{n}, m = 0, 1, \ldots, n - 1 \). Now let \( z \in g(\alpha) \) then for some \( \varepsilon > 0 \) we have \( z + I_\varepsilon \subseteq g(\alpha) \) hence \( \phi_{z+I_\varepsilon} E = 0 \) which means \( n(E_1, E_2) \subseteq g(\alpha) \) for all \( E_1, E_2 \). This implies \( \phi_{z+e^{i2\pi \frac{m}{n}}+I_\varepsilon} E = 0 \) for all \( E \) and hence \( z^{-1} \in g(\alpha) \).

Theorem 2.3 implies \( \phi_t E \geq E \) if \( 1 \subseteq I^\circ \) hence \( 1 \in \text{sp} \alpha \) and since \((\alpha, R) \subseteq K_n\) follows from the previous result \( e^{i2\pi \frac{m}{n}} \in \text{sp}(\alpha) \).

Let now \( \rho(\alpha - 1) < \sqrt{3} \) then \( \text{sp}(\alpha) \) is contained in the interval \([e^{-i1}, e^{+i1}]\) with \( \rho^2(\alpha - 1) = 2 - 2 \cos l \) which means \( l < \frac{2\pi}{3} \). Let now \( \left\lfloor \frac{n}{2} \right\rfloor \) the smallest entire number \( \geq \frac{n}{2} \) then we have for \( n \geq 2, \left\lfloor -\frac{n}{2} \right\rfloor \leq \frac{2}{3} \) and \( \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{1}{2} \) which implies \( \pi \leq 2\pi \left\lfloor \frac{n}{2} \right\rfloor /n \leq \frac{4\pi}{3} \).

This means in the standard decomposition (Theorem 3.8) can appear only the term \( F_1 \) this means \( \alpha \) is inner.
Let now $M$ be a factor of type II or III and $\alpha$ be the mapping $x \otimes y \otimes z \rightarrow z \otimes x \otimes y$. By a recent result of Sakai [10] this defines an outer automorphism of $M \otimes M \otimes M$. Since $\alpha^3 = 1$ follows the spectrum of $\alpha$ are exactly the three third roots of 1. This implies $\rho(\alpha - 1) = \sqrt{3}$.

As next we show

**5.7. Lemma:** Let $\alpha$ be a $\ast$-automorphism of a $W^*$-algebra $R$ and $E$ be an invariant projection with central support equal to one. Suppose $\alpha|E\mathcal{E}$ is an inner automorphism then follows $\alpha$ is inner.

**Proof:** Let identify $R$ with some normal faithful representation of $R$ on some Hilbert space $H$. By the assumption exists an operator $v \in E\mathcal{E}$ with $vv^* = v^*v = E$ and $vxv^* = \alpha x$ for every $x \in E\mathcal{E}$. Define an operator $u$ on $H$ by the equation

$$u \sum_i x_i y_i f_i = \sum_i (\alpha x_i) y_i f_i$$

$x_i \in R$, $y_i \in R'$ and $f_i \in E \cdot H$

We get the estimate (since $f_i = E f_i$):

$$\|u \sum_i x_i y_i f_i\|^2 = \sum_{ij} (f_j, v^* \alpha(x_j^* x_i)v y_j^* y_i f_i)$$

$$= \sum_{ij} (f_j, v^* \alpha(Ex_j^* x_i)E v y_j^* y_i f_i)$$

$$= \sum_{ij} (f_j, Ex_j^* x_i) y_j^* y_i f_i = \| \sum_i x_i y_i f_i \|^2.$$  

This means $u$ is an isometric operator. Since $E$ has central support 1 follows that the domain and the range is dense in $H$ which means the closure of $u$ is a unitary operator.

From the defining equation of $u$ one sees that $u \in R$ and furthermore the relation $(\alpha x) u = u x$. Since $u$ is unitary follows that $u$ implements the automorphism.

As last step for the demonstration of Theorem 5.2 we prove the following

**5.8. Lemma:** Let $\alpha$ be an inner $\ast$-automorphism of a $W^*$-algebra and $\varepsilon > 0$ arbitrary, then exists an invariant projection $E$ with central
support one such that \(\|(x-1)\|_{E}\| \leq c\).

**Proof:** We start from the construction of section 4. Since we have a point fixing family follows that for every \(\delta > 0\) that \(E([-\delta, +\delta])\) has central support 1. Choose now \(\delta = \frac{\epsilon}{4}\) and \(E = E([-\delta, +\delta])\) then \(uE = E \cdot e^{iAE}\) with \(\|AE\| \leq \delta\). Now \(uE\) implements the automorphism \(\alpha_{E}\). Let now \(\tilde{\alpha}x = e^{itAE}xe^{-itAE}\) for \(x \in E\). \(\tilde{\alpha}\) is an entire analytic function with \(\|\tilde{\alpha}\| \leq \epsilon e^{2\delta|\text{Im}\, t|}\). Now \(\tilde{\alpha}_t - 1\) is zero for \(t = 0\) and fulfills the estimate \(\|\tilde{\alpha}_t - 1\| \leq 2e^{2\delta|\text{Im}\, t|}\). Using now Schwarz lemma for the circle \(|2\delta t| \leq 1\) we get \(\|\tilde{\alpha}_t - 1\| \leq 4e \cdot \delta|\text{Im}\, t|, \|2\delta t| \leq 1\) for \(t = 1\) we have \(\tilde{\alpha}_1 = (x-1)\|_{E}\) and hence \(\|(x-1)\|_{E}\| \leq 4e\delta = c\).

**Proof of theorem 5.2.** If \(a\) is fulfilled, then lemma 5.8 shows that \(d\) is fulfilled and also \(c\) since \(\rho((x-1)\|_{E}\| \leq \|(x-1)\|_{E}\). If \(c\) holds, then corollary 5.6 shows that \(\alpha_{E}\) is inner and consequently \(\alpha\) is inner by lemma 5.7. If \(d\) holds then \(\alpha_{E}\) is inner by the result of Kadison and Ringrose ([7] theorem 7) and hence \(a\) is inner by lemma 5.7. If \(a\) is an automorphism fulfilling \(b\), then \(a\) is inner by either \(c\) or \(d\). If \(a\) holds then \(\alpha x = e^{i\lambda x}e^{-i\lambda}\) with some \(\lambda \in R\) and \(\|\lambda\| < T\). Hence \(\alpha x = e^{i\lambda x}e^{-i\lambda}\) is a norm continuous one-parametric group of inner automorphisms. This means \(b\) is fulfilled. Since there exist an outer automorphism with \(\rho(x-1) = \sqrt{3}\) (See the example given at the end of the proof of corollary 5.6. We must also have \(\|x-1\| = 2\) which means that the region of validity of 5.2 can not be improved.

Our next aim is the proof of theorem 5.4, but we first need some preparation.

**5.9. Lemma:** Let \(\rho(x-1) < \sqrt{3}\) (this implies \(a\) is inner by Theorem 5.2) and define for \(-\frac{\pi}{3} < \mu < \frac{2\pi}{3}\) the interval

\[I_\mu = \left[\frac{2\pi}{3} + \mu, -\mu\right]\]

and \(E_\mu = \text{maximal projection } E \in P_0\) such that \(\phi_{1\mu}E_\mu = 0\). Theorem 2.3 implies that this maximal projection exists and is unique. \(E_\mu\) has the following properties:

- \(\mu < 0\) implies \(E_\mu = 0\)
b. $\mu_1 > \mu_2$ implies $E_{\mu_1} \geq E_{\mu_2}$

c. For every central projection $F$ define $I_F > 0$ by $2(1 - \cos I_F) = \rho^2 ((\alpha - 1)_{FRF})$, then $F \cdot E_{\mu} = F$ for $\mu > I_F$

d. For $\mu > 0$ we have central support of $E_{\mu} = 1$

e. $\mu > 0$ then we have $T \cap n(E_{\mu}, E_{\mu}) \subset [-\mu, +\mu]$

The properties b, d and e show that $\{E_{\mu}\}$ is a point-fixing family.

Let $S(E)$ be the support of $E$ defined by $\{E_{\mu}\}$ (see Definition 4.3) then we have

f. For every $E \in P_0$ and every central projection $F$ we have

$$S(E \cdot F) \subset \text{sp} \{x_{|FRF}\} \cap \{\text{Im } z \geq 0\}$$

**Proof:**

a. For $\mu < 0$ follows $1 \leq I^0_{\mu}$ and hence by Theorem 2.3. c. d) we have $\phi_{1\mu} E \geq E$ for all $E \in P_0$ which implies $E_{\mu} = 0$

b. $\mu_1 > \mu_2$ implies $I_{\mu_1} < I_{\mu_2}$ and hence by 2.3. $\phi_{1\mu_1} E_{\mu_2} \leq \phi_{1\mu_2} E_{\mu_2} = 0$ which implies $E_{\mu_2} \leq E_{\mu_1}$

c. Let $\mu > I_F$ then follows $I_{\mu} \subset (x_{|FRF})$ and hence by Lemma 5.5 $\phi_{1\mu} F E = 0$ for all $E$ which implies $E_{\mu} \geq F$

d. Let $\mu > 0$ and $F_{\mu}$ the central carrier of $E_{\mu}$. Assume $F_{\mu} \neq 1$ then follows $\phi_{1\mu} E \neq 0$ for all $0 \neq E \leq (1 - F)$. This implies $\phi_{1\mu - 1}(1 - F_{\mu}) = (1 - F_{\mu})$ (otherwise if there exists an $E \leq (1 - F_{\mu})$ with $E \phi_{1\mu - 1}(1 - F_{\mu}) = 0$ would imply $0=(1 - F_{\mu}) \phi_{1\mu} E = \phi_{1\mu} E$, contradicting the assumption). Since $\phi_{1\mu - 1}(1 - F_{\mu}) = \phi_{[\mu, 2\pi]}(1 - F_{\mu}) \vee \phi_{[\frac{2\pi}{3}, \frac{4\pi}{3}]}(1 - F_{\mu})$ by 2.3, and the second expression vanishes for sufficiently small $\varepsilon$ by Lemma 5.5, we get $\phi_{[\mu, 2\pi]}(1 - F_{\mu}) = (1 - F_{\mu})$. Assume now $\mu < l_{1 - F\mu}$ and $2l_{1 - F\mu} - \mu + \frac{3\varepsilon}{2} \leq \frac{4\pi}{3}$ then we have $[\mu, \frac{2\pi}{3}] + (l_{1 - F\mu} - \mu + \varepsilon + \frac{1}{2}) \subset g(\alpha)$ and hence by 5.5 and theorem 2.3.

$0 = \phi_{[\mu, \frac{2\pi}{3}] + l_{1 - F\mu} - \mu + \varepsilon + \frac{1}{2}}(1 - F_{\mu}) \geq \phi_{l_{1 - F\mu} - \mu + \varepsilon + \frac{1}{2}} \phi_{[\mu, \frac{2\pi}{3}] + \varepsilon + \frac{1}{2}}(1 - F_{\mu}) = \phi_{l_{1 - F\mu} - \mu + \varepsilon + \frac{1}{2}}(1 - F_{\mu})$. This implies by Lemma 5.5 $l_{1 - F\mu} - \mu + \varepsilon \subset g(\alpha_{(1 - F_{\mu})R(1 - F_{\mu})})$. This contradicts the definition of $l_{1 - F_{\mu}}$.

e. Let $\frac{2\pi}{3} > b > \mu$ and $b - \varepsilon > \mu$ then $\phi_{-b + \varepsilon} E_{\mu} \leq \phi_{-b} E_{\mu} = 0$. If $\frac{2\pi}{3} \leq b + \varepsilon$ then $b \in g(\alpha)$ and hence $\phi_{b + \varepsilon} = 0$. Since $n(E_{\mu}, E_{\mu})$ is invariant under $b \rightarrow -b$ follows $(\mu, 2\pi - \mu) \subset n(E_{\mu}, E_{\mu})$. 

f. Let $b<0$ and $\varepsilon<\frac{|b|}{2}$ then $\phi_{b+1}\mid E\mid\frac{|b|}{2}\leq\phi_{\frac{|b|}{2}}\mid E\mid\frac{|b|}{2}=0$ and hence $b \notin S(E)$ for any $E$. Let now $b+I\subset g(\mathcal{A}_{F|R|F})$ then $\phi_{b+1}F=0$ and hence $b \notin S(EF)$. This means $S(FE)\subset \text{sp}(\mathcal{A}_{F|R|F}) \cap \{|\text{Im} z| \geq 0\}$.

**Proof of theorem 5.4.** Part A (i), (ii) and (iii) are proved in corollary 5.6. Assume now $\|\alpha-1\|<2$, then exists according to Kadison and Ringrose [7] (Theorem 7) a unique real derivation with $\|\delta\|<\frac{\pi}{2}$ such that $\alpha=e^\delta$. If we consider $\alpha_t=e^{t\delta}$ then for $t=\frac{2}{3}$ we have $\rho(\alpha_t-1)<\sqrt{3}$. Now according to Lemma 5.9 exists unitary operators $u_t$ with $\text{sp} u_t \subset \text{sp} \mathcal{A}_{t} \cap \{|\text{Im} z| \geq 0\}$ which implement the automorphisms. By construction of $u_t$ follows that $u_t$ is a one-parametric group and hence the result extends also to $t=1$. This proves the first part of A (iv). Let $\delta$ be a derivation then we can pass to the one-parametric group $e^{t\delta} = \alpha_t$ for $|t|<\frac{2\pi}{3} \|\delta\|^{-1}$ we have $\rho(\alpha_t-1)<\sqrt{3}$ and thus part B (i) and the first part of (ii) are consequences of A (i) and the first part of A (iv). In order to prove the uniqueness it is sufficient to do this for the derivation. Let $\delta$ be a real derivation then exists a positive operator $d$ with $\text{sp} \text{id} F \subset \text{sp} \{\delta_{F|R|F}\}$ and $\delta = \text{id} \text{id}$ since $d$ is positive follows $\|\delta_{F|R|F}\| \leq \|d\cdot F\| \leq \rho(\delta_{F|R|F}) \leq \|\delta_{F|R|F}\|$. This shows that $\|\delta_{F|R|F}\| = \|dF\| = \rho(\delta_{F|R|F})$. Assume there exists a second positive operator $d'$ with $\delta = \text{id} \text{id}'$ and $\text{sp} \text{id}' F \subset \text{sp} \{\delta_{F|R|F}\}$ then we have $d-d' \in \mathbb{Z}$. Assume the spectrum of $d-d'$ contains a point $x \neq 0$ then exists a spectral projection $F$ with $\text{sp} F(d-d') = \left[\frac{x}{2}, \frac{3}{2}, x\right]$. Assume first $x>0$ follows from $F(d-d') = \left[\frac{x}{2}, \frac{3}{2}, x\right]$. Assume $\text{sp} F(d-d') \subset \left[\frac{x}{2}, \frac{3}{2}, x\right]$. Assume next $x$ is negative then we have $F(d-d') = Fd'$ and hence $\text{sp} F(d-d') = \left[\frac{x}{2}, \frac{3}{2}, x\right]$ which is also absurd and hence $d=d'$.

**Proof of theorem 5.3.** Let $\delta$ be a real derivation with $\|\delta\|<\pi$. The function $t\mapsto e^{t\delta} - e^{-t\delta}$ is an entire function of $t$ and we get $\|e^{t\delta} - e^{-t\delta}\| \leq 2 \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \|t\delta\|^{2n+1} = 2 \sin \|t\delta\|$. Since $e^{t\delta}$ is a $*$-automorphism
finds \( \phi \beta E = \beta \phi ? E \) and hence for \( E_1, E_2 \in P_0 \) 
\( n^\nu(E_1, E_2) = n^\nu(\beta E_1, \beta E_2) \).
for real \( t \) we have \( \| e^{it\delta} - e^{-it\delta} \| \leq 2 \) for \( t \) real. Using that \( e^{it\delta} x - e^{-it\delta} x \)
is selfadjoint for \( x \) selfadjoint we get (see Boas [2] 6.2.6) \( \| e^{it\delta} - e^{-it\delta} \| \leq 
2 \cosh \| \delta \| \text{Im} t \). Since this function vanishes at the origin we can define

\[
F(t) = \frac{e^{it\delta} - e^{-it\delta}}{\sin t \| \delta \|}
\]
in the strip \( |\text{Re} t| < \frac{\pi}{2 \| \delta \|} \). Since \( |\sin x + iy| = \sqrt{\cosh^2 y - \cos^2 x} \)
follows \( \| F(t) \| \) is bounded in the strip \( |\text{Re} t| \leq \frac{\pi}{2 \| \delta \|} \) and \( \| F(t) \| \leq 2 \) for \( |\text{Re} t| = 
\frac{\pi}{2 \| \delta \|} \). This implies (see e.g. Titchmarsh [11] 5.6.5.) \( \| F(t) \| \leq 2 \) in 
\( |\text{Re} t| \leq \frac{\pi}{2 \| \delta \|} \). From this we get:

\[
\| e^\delta - 1 \| = \| e^\delta (e^{\delta/2} - e^{-\delta/2}) \| \leq \| e^{\delta/2} - e^{-\delta/2} \| = \| F\left(\frac{1}{2}\right)\| \cdot \sin \| \delta \| / 2 \leq 2 \sin \| \delta \| / 2.
\]

From theorem 5.4 we know \( \| \delta \| = \rho(\delta) \). Since we have \( \rho^2(e^{\delta} - 1) = 2 - 2 \cos \rho(\delta) = 4 \sin^2 \rho(\delta) / 2 \) we find \( \| e^\delta - 1 \| \leq \rho(e^\delta - 1) \) which implies that both expressions are equal. If now \( \rho(\alpha - 1) < \sqrt{3} \) then \( \alpha \) lies on a one parametric group \( e^{it\delta} \) with \( \rho(\delta) < \frac{\pi}{2} \). Hence we have \( \| \alpha - 1 \| = \rho(\alpha - 1) \). If \( \sqrt{3} \leq \rho(\alpha - 1) \leq 2 \) then \( \| \alpha - 1 \| \) might be 2. If \( \| \alpha - 1 \| < 2 \) then by the result of Kadison and Ringrose we have \( \alpha = e^{\delta} \) with \( \| \delta \| < \pi \) and hence \( \| \alpha - 1 \| = \rho(\alpha - 1) \). Let now \( M \) be a factor of type II or III and \( \alpha_0 \)
the outer automorphism of \( M \otimes M \otimes M \) at the end of corollary 5.6. Let \( 2 \pi / 3 \leq l \leq \pi \) and \( d \) be a positive invariant (under \( \alpha_0 \)) operator with \( \text{sp} d = [0, l] \) then we have \( \| ad id \| = l \). Let \( \alpha_1 = e^{a d id} \) then we have \( \rho(\alpha_1 - 1) = \rho(\alpha_0 \alpha_1 - 1) = 2 \sin l / 2 \) but \( \| \alpha_1 - 1 \| = 2 \sin l / 2 \) and \( \| \alpha_0 \alpha_1 - 1 \| = 2 \), since \( \alpha_0 \) is an outer automorphism.

### 6. Some Final Remarks

**A.** If \( \alpha \) and \( \beta \) are two \( \ast \)-automorphisms of the same \( W^\ast \)-algebra \( R \)
and one defines \( \gamma = \beta \alpha \beta^{-1} \) and denotes the quantities used in this paper
with an upper index denoting the automorphism which we are considering, then one finds easily from the definitions \( R_0^\nu = \beta R_0^\nu \), for \( E \in P_0^\nu \) one
This means for a fixed $W^*$-algebra $R$ the classes $\text{Aut}_n = \{ \alpha, (R, \alpha) \in K_n \}$ are invariant sets.

B. With a little care most of our technique can also be used for arbitrary locally compact abelian groups which we will denote by $G$. We want to give a brief discussion of this case.

a. Since we are looking for continuous group-representations it is according to an earlier paper [4] necessary that $G$ acts strongly continuous on the pre-dual $R_*$ of $R$, i.e. for every $\varepsilon > 0$ and every $\varphi \in R_*$ exists a neighbourhood $U$ of the group identity with $\| (\alpha - 1) \varphi \| < \varepsilon$, $\alpha \in U$. If this is the case integration with respect to the invariant Haar-measure of expressions $\int f(\varphi) \alpha \varphi \times d\varphi$ are well defined as weak integral.

b. Let $\hat{G}$ be the dual group of $G$ then we can define $\phi_\alpha E$ for elements in $R_0$ for sets $S$ with the properties (i) $S = S^\circ$ and (ii) for every $\varphi \in S^\circ$ exists a function $f \in L^1_0$ with support $S^\circ f \subset S$ and $(S^\circ f) g \neq 0$. With this provision and a little care about the union of two such sets all results of section 2 are true in this general situation.

c. For the definition of the classes it is easier to start from the group $G$ itself. Let $H$ be a closed subgroup of $G$ then we give the following

Definition: A group $\alpha(G)$ of $*$-automorphisms belong to the class $K_H$ if and only if for every Projection $F \in P_0 \cap Z$ the subgroup of automorphisms $\alpha_h$ such that $\alpha_h|_{FRF}$ is inner, and we can find a continuous group-representation $\nu(h) \in R$ of $H$, implementing $H$, is exactly $H$.

d. Since in general a group $G$ has continuously many different subgroups, a standard decomposition of $R$ is in general not possible or at least not so easy. Therefore we will stick to the case where $R$ is a factor then $\alpha(G)$ lies always in one of the classes.

e. For $E_1, E_2 \in P_0$ we can investigate the expressions $n(E_1, E_2)$ on $\hat{G}$. Let now $\hat{H}(E_1, E_2)$ be the subgroup of $\hat{G}$ such that $\varphi \in n(E_1, E_2)$ implies $\varphi \in n(E_1, E_2)$ for all $\varphi \in \hat{H}(E_1, E_2)$ and $\hat{H}_0 = \bigcap_{E_1, E_2} \hat{H}(E_1, E_2)$. If $R$ is a factor then $\hat{H}_0$ is well defined and is a closed subgroup of
G. This then gives a second definition of classes $K_B$ and one would like to show that these classifications are isomorphic.

f. If $\alpha(G)$ belongs to $K_{B_0}$ then the definition of a point-fixing family is possible on $\hat{G}/\hat{H}_0$ and hence we get also a spectral measure on $\hat{G}/\hat{H}_0$.

g. Let $H$ be the subgroup of $G$ such that $(h, \hat{h})=1$ for all $\hat{h} \in \hat{H}_0$ then $\hat{G}/\hat{H}_0$ is isomorphic to $\hat{H}$. From this the following statement is immediate. If $\alpha(G)$ belongs to $K_H$, then it need not belong to $K_B$ in general. The converse statement seems to be true, but, I don't know how to prove in general. Having a spectral measure on $\hat{G}/\hat{H}_0$ it is of cause easy to construct a group-representation $u(h)$ of $H$. But to show that $u(h)$ implements $\alpha_h$ I know to do only in special cases. This is based on the fact that our proof uses entire analytic functions. So we can give the proof for the cases $\hat{G}/\hat{H}_0=\mathbb{R}, \mathbb{T}, \mathbb{Z}, \mathbb{Z}_n$ and finite direct products of these special cases.

h. These special cases just mentioned cover most of the interesting problems. This leads e.g. to the conclusion that derivations are inner and the case with semibounded spectrum treated in [5] leads to inner automorphisms.

i. Passing from one single automorphism to an abelian group of automorphisms, then the following possibility does occur and complicates the situation: Every single automorphism is inner, but we have only a continuous group-representation up to a multiplier. Take for example a representation of the C. C. R. of one degree of freedom, then the Weyl operators induce an automorphism-group which is isomorphic to $\mathbb{R}^2$. (I learned this example from A. Connes.)

References


