On the Cohomology of the Classifying Spaces of $PSU(4n+2)$ and $PO(4n+2)$

By

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§0. Introduction

The quotients of $SU(m)$ and $SO(2m)$ by their centers $\Gamma_m = \left\{ e^{\frac{2\pi i}{m}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; 0 \leq j < m \right\}$ and $\Gamma_2 = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ are denoted by $PU(m)$ and $PO(2m)$ respectively.

The purpose of this paper is to determine the module structure of the cohomology mod 2 of the classifying spaces $BPU(4n+2)$ and $BPO(4n+2)$.

The method is first to determine the $E_2$-term of the Eilenberg-Moore spectral sequence by constructing an injective resolution for $H^*(G; \mathbb{Z}_2)$, $(G = SU(4n+2)/\Gamma_2$, $PO(4n+2))$. Then by making use of naturality of the Eilenberg-Moore spectral sequence we show that the spectral sequence with $\mathbb{Z}_2$-coefficient collapses for these $G$.

Our results are

**Theorem.** As a module

$$H^*(BPU(4n+2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, a_3, x_8, y(I)]/R,$$

where $1 \leq l \leq 2n$ and $R$ is an ideal generated by $a_3y(I)$, $y(I)^2 + \sum x_{8l+8}...a_4^2 + ...x_{8l+8}$ and $y(I)y(J) + \sum f_l y(I_l)$.

**Theorem.** As a module

$$H^*(BPO(4n+2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, x_{4l+4}, y'(I)]/R,$$
where $1 \leq l \leq 2n$ and $R$ is an ideal generated by $a_{2}y'(I)$, $y'(I)^{2} + \Sigma x_{4_{i_{1}}+4 \ldots a_{2}l_{1} \cdots x_{4_{l_{1}}+4}$ and $y'(I)y'(J) + \Sigma f_{j_{1}}^{(l)}y'(I)_{j}$.

In the above theorems $I$ runs over all sequences of integers $(i_{1}, \ldots, i_{r})$ satisfying $1 \leq r \leq 2n$ and $1 \leq i_{1} < \ldots < i_{r} \leq 2n$. (For details see §5.)

The paper is organized as follows:

In the first section we show that there exists a sort of "stability" in $H^{*}(BG; \mathbb{Z}_{2})$. §2 is used to calculate $H^{*}(U(n)/\Gamma_{p}; \mathbb{Z}_{p})$. In §3 we determine the $E_{2}$-term of the Eilenberg-Moore spectral sequence, Cotor $H^{*}(G; \mathbb{Z}_{2})(\mathbb{Z}_{2}, \mathbb{Z}_{2})$, for $G = PO(4n+2), PU(4n+2)$. In the next section, §4, we show that the Eilenberg-Moore spectral sequence (with $\mathbb{Z}_{2}$-coefficient) collapses for these $G$. §5 is devoted to showing that the elements $a_{1}$'s in the above theorems, namely Theorems 4.9 and 4.12, are in the trangression image. In the last section, the generators $x_{8l+8}$ and $x_{4l+4}$ in Theorems 4.9 and 4.12 are shown to be represented by certain exterior power representations.

Throughout the paper the map $BH \rightarrow BG$ induced from a homomorphism $H \rightarrow G$ of groups is denoted by the same symbol.

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§1. Quotients of $SU(n)$ and $SO(n)$

**Notation.** $I_{n} = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \end{pmatrix} \in U(n)$ the identity matrix,

$C(n) = \{ zI_{n}; |z| = 1 \text{ and } z \in C \}$,

$\Gamma_{n} = \{ wI_{n}; w^{m} = 1 \text{ and } w \in C \} \subset C(n)$.

Then $C(n)$ (resp. $\Gamma_{n}$) is the center of the unitary group $U(n)$ (resp. $SU(n)$). In particular we have the inclusions

$\Gamma_{2} = \left\{ \pm \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \right\} \subset SO(2n) \subset SU(2n)$.

Hereafter we use the following

**Notation.**
Denote by $\pi$ the natural projections $SU(m) \to G(m)$ and $SO(2n) \to PO(2n)$.

Consider the $k$-fold diagonal map:

\[
\Delta_k: SU(n) \rightarrow (SU(n))^k \rightarrow SU(nk),
\]

\[
\Delta_k: SO(n) \rightarrow (SO(n))^k \rightarrow SO(nk),
\]

where $\Delta_k$ is the diagonal embedding:

\[
\Delta_k(A) = \begin{pmatrix}
A & \cdots & 0 \\
0 & \ddots & A
\end{pmatrix}.
\]

For the identity matrix $I_n$ then we have

\[
\Delta_k(I_n) = I_{nk} \quad \text{and} \quad \Delta_k(-I_n) = -I_{nk}.
\]

So for even $n$ there exist maps $G(n) \to G(nk)$ and $PO(n) \to PO(nk)$ such that the following diagrams commute:

\[
\begin{align*}
SU(n) & \xrightarrow{\Delta_k} SU(nk) & SO(n) & \rightarrow SO(nk) \\
\pi & \downarrow & \pi & \downarrow \\
G(n) & \rightarrow G(nk) & PO(n) & \rightarrow PO(nk)
\end{align*}
\]

We denote them by the same symbol:

\[
\Delta_k: G(n) \rightarrow G(nk), \quad \Delta_k: PO(n) \rightarrow PO(nk).
\]

Notation.

\[
C(n, k) = SU(nk)/\Delta_k SU(n),
\]

\[
R(n, k) = SO(nk)/\Delta_k SO(n).
\]
So we have fiberings:

(1.1) \[ SU(n) \xrightarrow{\Delta_k} SU(nk) \xrightarrow{\rho} C(n, k) . \]
(1.2) \[ SO(n) \xrightarrow{\Delta_k} SO(nk) \xrightarrow{\rho} R(n, k) , \]

**Remark 1.3.**

(1) \( C(n, k) \) is homeomorphic to \( G(nk)/\Delta_k G(n) \) for \( l|n \).
(2) \( R(2n, k) \) is homeomorphic to \( PO(2nk)/\Delta_k PO(2n) \).

Now recall from [4] and [5] the following

**Proposition 1.4.**

(1) \[ H^*(SU(n); \mathbb{Z}) \cong \Lambda(u_3, \ldots, u_{2n-1}) , \]
\[ H^*(U(n); \mathbb{Z}) \cong \Lambda(u_1, u_3, \ldots, u_{2n-1}) , \]
where \( \deg u_{2i-1} = 2i - 1 \) and \( u_{2i-1} \) is universally transgressive with \( \tau(u_{2i-1}) = c_i \) the \( i \)-th universal Chern class.

(2) \[ H^*(SO(n); \mathbb{Z}) \cong \Delta(v_1, \ldots, v_{n-1}) , \]
where \( \deg v_{i-1} = i - 1 \) and \( v_{i-1} \) is universally transgressive with \( \tau(v_{i-1}) = w_i \) the \( i \)-th universal Stiefel-Whitney class.

Then

**Proposition 1.5.** (1) For any integer \( k > 0 \) and any prime \( p \) with \( (k, p) = 1 \), we have
\[ H^*(C(n, k); \mathbb{Z}_p) \cong \Lambda(\bar{x}_{2n+1}, \ldots, \bar{x}_{2nk-1}) \]
where \( \deg \bar{x}_{2i+1} = 2i + 1 \) and \( \rho^* \bar{x}_{2i+1} = u_{2i+1} . \)
(2) For any odd integer \( k > 0 \) we have
\[ H^*(R(n, k); \mathbb{Z}_2) \cong \Delta(\bar{z}_n, \ldots, \bar{z}_{nk-1}) \]
where \( \deg \bar{z}_i = i \) and \( \rho^* \bar{z}_i = v_i . \)
Proof. (1) The map $\Delta_k: SU(n) \to SU(nk)$ induces a map $\Delta_k: BSU(n) \to BSU(nk)$ which gives the $k$-fold Whitney sum of complex vector bundles. Thus

\[(1.6) \quad \Delta_k^*(c) = \sum_{i_1 + \cdots + i_k = i} c_{i_1} \cdots c_{i_k} = kc_i + \text{(decomposables)}.
\]

For the Serre cohomology spectral sequence with $\mathbb{Z}_p$-coefficient $\{E^*_r, \ast\}$ of the fibering

$$SU(nk) \longrightarrow C(n, k) \longrightarrow BSU(n),$$

we have

$$E^*_2 = \mathbb{Z}_p[c_2, \ldots, c_n] \otimes \Lambda(u_3, \ldots, u_{2nk-1}),$$

and

$$E^*_\infty \cong \mathcal{H}(H^*(C(n, k); \mathbb{Z}_p)).$$

Then it follows from Proposition 1.4 and (1.6) that

$$d_2(1 \otimes u_{2i-1}) = kc_i \otimes 1 \quad \text{for} \quad 2 \leq i \leq n$$

and all other differentials are trivial. So we get

$$\mathcal{H}(H^*(C(n, k); \mathbb{Z}_p)) \cong E^*_2 \cong E^*_2 \cong \Lambda(u_{2n+1}, \ldots, u_{2nk-1}).$$

Since $(k, p) = 1$, (1.6) implies that $\Delta_k^*: H^*(SU(nk); \mathbb{Z}_p) \to H^*(SU(n); \mathbb{Z}_p)$ is epimorphic, and hence $SU(n)$ is totally non-homologous to zero in the fibering (1.1). Thus $\rho^*: H^*(C(n, k); \mathbb{Z}_p) \to H^*(SU(nk); \mathbb{Z}_p)$ is monomorphic.

(2) is proved quite similarly. Q.E.D.

Theorem 1.7. (1) Let $p$ be a prime, $k$ an integer with $(k, p) = 1$ and $l | n$. Then $\Delta_k^*: H^i(BG(nk); \mathbb{Z}_p) \to H^i(BG(n); \mathbb{Z}_p)$ is isomorphic for $i \leq 2n$ and monomorphic for $i \leq 2n+1$.

(2) Let $k$ be an odd integer. Then $\Delta_k^*: H^i(BPO(2kn); \mathbb{Z}_2) \to H^i(BPO(2n); \mathbb{Z}_2)$ is isomorphic for $i \leq n-1$ and monomorphic for $i \leq n$.

Proof. Proposition 1.5 applied with the Serre exact sequence (Proposition 5 of [12]) for the fibrings

$$C(n, k) \longrightarrow BG(n) \longrightarrow BG(nk).$$
\( R(2n, k) \longrightarrow BPO(2n) \longrightarrow BPO(2nk) \)
gives the results. Q.E.D.

**Notation.** For each rational number \( k \), define \( v_p(k) \) to be the exponent of \( p \) when \( k \) is expressed as a product of powers of distinct primes.

**Corollary 1.8.** (1) If \( v_p(n) = v_p(m) \), then as algebras there hold
\[
H^*(BG(n); \mathbb{Z}_p) \cong H^*(BG(m); \mathbb{Z}_p) \quad \text{for} \quad * \leq 2 \min(m, n).
\]

(2) If \( v_2(m) = v_2(n) \), then as algebras there hold
\[
H^*(BPO(2n); \mathbb{Z}_2) \cong H^*(BPO(2m); \mathbb{Z}_2) \quad \text{for} \quad * \leq \min(m, n).
\]

In the below we denote by \( \phi \) the diagonal map in \( H^*(G; \mathbb{Z}_p) \) induced from the multiplication on a group \( G \). Put \( \bar{\phi} = (\eta \otimes \eta) \circ \phi \), where \( \eta: H^*(G; \mathbb{Z}_p) \to \sum_{i \geq 0} H^i(G; \mathbb{Z}_p) \) is the natural projection.

Now we recall from [3] and [5] the following facts:

**Proposition 1.9.** Let \( n = pn' \) with \( (p, n') = 1 \) and \( l \mid n \). Then
\[
H^*(G(n); \mathbb{Z}_p) \cong \mathbb{Z}_p[y]/(y^{p^l}) \otimes A(x_1, \ldots, x_{2p^{l-1}}, \ldots, x_{2n-1}),
\]
where \( \deg y = 2 \) and \( \deg x_{2i-1} = 2i-1 \).

**Proposition 1.9'.** There exist generators \( y \in H^1(G(4n+2); \mathbb{Z}_2) \) and \( x_{2i+1} \in H^{2i+1}(G(4n+2); \mathbb{Z}_2) \), \( 2 \leq i \leq 4n+1 \), such that
\[
(1) \quad H^*(G(4n+2); \mathbb{Z}_2) \cong \Lambda(y, y^2, x_5, \ldots, x_{8n+3}),
\]
\[
(2) \quad \bar{\phi}(y) = 0, \quad \bar{\phi}(x_{4j+1}) = 0 \quad \text{for} \quad 1 \leq j \leq 2n,
\]
\[
\bar{\phi}(x_{4j+3}) = x_{4j+1} \otimes y^2 \quad \text{for} \quad 1 \leq j \leq 2n,
\]
\[
(3) \quad Sq^{2k}x_{2i-1} = (k, i-k-1)x_{2i+2k-1}.
\]

**Remark 1.9".** \( \bar{\phi}(x_{4j+3} + \text{decomp.}) \neq 0 \).

**Proposition 1.10.** There exist generators \( y \in H^1(PO(4n+2); \mathbb{Z}_2) \) and
z_i \in H^i(PO(4n+2); \mathbb{Z}_2), 2 \leq i \leq 4n+1, such that

\begin{enumerate}
  \item \[H^*(PO(4n+2); \mathbb{Z}_2) \cong \Delta(y, z_2, \ldots, z_{4n+1}),\]
  \item \[\overline{\phi}(y) = 0, \quad \overline{\phi}(z_{2k}) = 0 \quad \text{for} \quad 1 \leq k \leq 2n,
  \quad \overline{\phi}(z_{2k+1}) = z_{2k} \otimes y \quad \text{for} \quad 1 \leq k \leq 2n,\]
  \item \[Sq^i z_k = (k-j) z_{j+k}.\]
\end{enumerate}

**Notation.** \(PS(X; \ p)\) = the Poincaré series of \(X\) over \(\mathbb{Z}_p\), i.e.,

\[PS(X; \ p) = \sum_{i=0}^{\infty} \{\text{rank } H^i(X; \ \mathbb{Z}_p)\} t^i.\]

Using this expression we obtain from Propositions 1.5, 1.9 and 1.10:

\[PS(G_l(n); \ p) \cdot PS(C(n, k); \ p) = PS(G_l(nk); \ p) \quad \text{for } (k, p) = 1,\]

\[PS(PO(2n); 2) \cdot PS(R(n, k); 2) = PS(PO(2nk); 2).\]

Thus we have

**Proposition 1.11.** (1) The cohomology Serre spectral sequence with \(\mathbb{Z}_p\)-coefficient for the fibering \(G_l(n) \to G_l(nk) \to C(n, k)\) collapses if \((k, p) = 1\).

(2) The cohomology Serre spectral sequence with \(\mathbb{Z}_2\)-coefficient for the fibering \(PO(2n) \to PO(2nk) \to R(2n, k)\) collapses.

Now we choose generators in \(H^*(G(4n+2); \mathbb{Z}_2)\) and \(H^*(PO(4n+2); \mathbb{Z}_2)\) appropriately.

**Lemma 1.12.** In Proposition 1.9' we may choose generators \(y, x_{2i+1}, 2 \leq i \leq 4n+1\), of \(H^*(G(4n+2); \mathbb{Z}_2)\) by using the correspondent generators in \(H^*(G(4n-2); \mathbb{Z}_2)\) and in \(H^*(C(4n-2, 2n+1); \mathbb{Z}_2)\) as follows:

\[y = \Delta^{4}_{2n-1} \circ \Delta^{4}_{2n+1}^{-1}(y),\]
\[x_{2i+1} = \Delta^{4}_{2n-1} \circ \Delta^{4}_{2n+1}^{-1}(x_{2i+1}), \quad 2 \leq i \leq 4n-3,\]
\[x_{2i+1} = \Delta^{4}_{2n-1} \circ \rho^*(x_{2i+1}), \quad 4n-2 \leq i \leq 4n+1.\]
Proof. This is clear from Proposition 1.11. Q.E.D.

Similarly

**Lemma 1.13.** In Proposition 1.10 we may choose generators $y$, $z_i$, $2 \leq i \leq 4n+1$, of $H^*(PO(4n+2); \mathbb{Z}_2)$ by using the correspondent generators in $H^*(PO(4n-2); \mathbb{Z}_2)$ and in $H^*(R(4n-2, 2n+1); \mathbb{Z}_2)$ as follows:

\[
y = \Delta^{4n}_{2n-1} \circ \Delta^{4n+1}_{2n+1}^{-1}(y),
\]

\[
z_i = \Delta^{4n}_{2n-1} \circ \Delta^{4n+1}_{2n+1}^{-1}(z_i), \quad 2 \leq i \leq 4n-3,
\]

\[
z_i = \Delta^{4n}_{2n-1} \circ p^*(z_i), \quad 4n-2 \leq i \leq 4n+1.
\]

**Proposition 1.14.** (1) In $H^*(C(4n-2, 2n+1); \mathbb{Z}_2) \cong A(\bar{x}_{8n-3}, \bar{x}_{8n-1}, \bar{x}_{8n+1}, \bar{x}_{8n+3}, \ldots)$ there hold $Sq^4 \bar{x}_{8n-3} = \bar{x}_{8n+1}$ and $Sq^4 \bar{x}_{8n-1} = \bar{x}_{8n+3}$.

(2) In $H^*(R(4n-2, 2n+1); \mathbb{Z}_2) \cong A(\bar{z}_{4n-2}, \bar{z}_{4n-1}, \bar{z}_{4n}, \bar{z}_{4n+1}, \ldots)$ there hold $Sq^2 \bar{z}_{4n-2} = \bar{z}_{4n}$ and $Sq^2 \bar{z}_{4n-1} = \bar{z}_{4n+1}$.

**Proof.** (1) and (2) follow from (3) of Proposition 1.9' and (3) of Proposition 1.10 respectively. Q.E.D.

**Remark.** See [9] for the results of the symplectic case.

### §2. Quotients of $U(n)$

In this section let $p$ be a prime and $n$ an integer such that $(n, p) = 1$. Then obviously

\[(2.1) \quad H^*(BPU(n); \mathbb{Z}_p) \cong H^*(BSU(n); \mathbb{Z}_p).
\]

The following are easily obtained:

\[(2.2) \quad H^*(BC(n); \mathbb{Z}_p) \cong \mathbb{Z}_p[x] \quad \text{with} \quad \deg x = 2.
\]

\[(2.3) \quad H^*(B\Gamma_p; \mathbb{Z}_p) \cong \begin{cases} 
\mathbb{Z}_2[t] & \text{with} \quad \deg t = 1 \quad \text{for} \quad p = 2 \\
\mathbb{Z}_p[\mu] \otimes A(\lambda) & \text{with} \quad \deg \mu = 2, \quad \deg \lambda = 1, \\
\delta \lambda = \mu & \text{for} \quad p: \text{odd},
\end{cases}
\]
where $\delta$ is the Bockstein operator.

Consider the cohomology Serre spectral sequence with $\mathbb{Z}_p$-coefficient associated with the fibering:

\[(2.4) \quad BC(n) \xrightarrow{i'} BU(n) \to BPU(n),\]

where $i'$ is induced from the natural inclusion $C(n) \subset U(n)$. The map $i'^*$ is epimorphic since the spectral sequence collapses by (2.2) and by the fact that $H^3(BPU(n); \mathbb{Z}_p) = H^3(BSU(n); \mathbb{Z}_p) = 0$. Let $j: \Gamma_p \subset C(n)$ be the inclusion. Then

\[(2.5) \quad \text{Im } j^* \cong \begin{cases} \mathbb{Z}_p[\mu] & \text{for } p: \text{odd} \\ \mathbb{Z}_2[t^2] & \text{for } p = 2. \end{cases}\]

Putting $i = i' \circ j$ and choosing $\mu$ (or $t$) suitably we get

\[(2.6) \quad i^*(c_1) = \begin{cases} \mu & \text{for } p: \text{odd} \\ t^2 & \text{for } p = 2. \end{cases}\]

Let $\{E_n^{*,*}\}$ be the cohomology Serre spectral sequence with $\mathbb{Z}_p$-coefficient associated with the fibering $U(n) \xrightarrow{\pi} U(n)/\Gamma_p \to B\Gamma_p$. Since the generators in $H^*(U(n); \mathbb{Z}_p) \cong A(u_1, u_3, \ldots, u_{2n-1})$ are universally transgressive, they are transgressive with respect to this fibering. In particular we have

\[\tau(u_1) = i^*(c_1)\]

where $\tau$ is the transgression.

Therefore $E_3^{*,b} = 0$ if $a \geq 2$, and hence

\[(2.7) \quad E_3 \cong E_\infty \cong \begin{cases} A(\lambda) \otimes A(u_3, u_5, \ldots, u_{2n-1}) & \text{for } p: \text{odd} \\ A(t) \otimes A(u_3, u_5, \ldots, u_{2n-1}) & \text{for } p = 2. \end{cases}\]

**Proposition 2.8.** $H^*(U(n)/\Gamma_p; \mathbb{Z})$ is $p$-torsion free and hence it is torsion free.

Proof is left to the reader.

It follows from this proposition
Theorem 2.9. Let \((n, p) = 1\). Then
\[ H^*(U(n)/\Gamma_p; \mathbb{Z}_p) \cong \Lambda(\bar{\lambda}, u'_3, \ldots, u'_{2n-1}) \]
such that

1. \(\bar{\lambda}\) and \(u'_{2i-1}\) are universally transgressive (and hence they are primitive),
2. \(\deg \bar{\lambda} = 1\) and \(\deg u'_{2i-1} = 2i - 1\),
3. \(\pi^*(u'_{2i-1}) = u_{2i-1}\) for the projection \(\pi: U(n) \to U(n)/\Gamma_p\).

Proof. (1) and (2) follow from (2.7) and the Borel's theorem (Theorem 13.1 of [4]). (3) is clear, since \(\pi^*(u'_{2i-1}) \neq 0\) by (2.7) and since \(\pi^*(u'_{2i-1})\) are universally transgressive. Q. E. D.

§3. The \(E_2\)-term of the Eilenberg-Moore Spectral Sequence

Put \(A = H^*(G(4n+2); \mathbb{Z}_2)\) for simplicity and regard \(A\) as a coalgebra over \(\mathbb{Z}_2\), where the coalgebra structure \(\bar{\phi}\) is given by Proposition 1.9'.

Let \(L\) be a \(\mathbb{Z}_2\)-submodule of \(A^+ = \sum_{i>0} H^i(G(4n+2); \mathbb{Z}_2)\) generated by \(\{y, y^2, x_{4j+1}, x_{4j+3}\}, 1 \leq i, j \leq 2n\). Let \(s: L \to sL\) be the suspension. We express the corresponding elements as \(sL = \{a_2, a_3, a_{4j+2}, b_{4j+4}\}, 1 \leq i, j \leq 2n\). Let \(\iota: L \to A\) be the inclusion and \(\theta: A \to L\) the projection such that \(\theta \circ \iota = 1_L\). Define \(\bar{\theta}: A \to sL\) by \(\bar{\theta} = s \circ \theta\) and \(\tau: sL \to A\) by \(\tau = \iota \circ s^{-1}\). Consider the tensor algebra \(T(sL)\). Denote by \(I\) the ideal of \(T(sL)\) generated by \(\text{Im}((\bar{\theta} \otimes \bar{\theta}) \circ \bar{\phi}) \circ \text{Ker} \bar{\theta}\). Put \(\bar{X} = T(sL)/I\). Then \(\bar{X} \cong \mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{4j+4}], 1 \leq i, j \leq 2n\).

The map \(\bar{d} = (\bar{\theta} \otimes \bar{\theta}) \circ \bar{\phi} \circ \bar{\tau}\) on \(sL\) can be extended over \(\bar{X}\), since \(\bar{d}(I) \subset I\). Further, \(\bar{d}\) satisfies \(\bar{d} \circ \bar{d} = 0\) on \(\bar{X}\). So \(\bar{X}\) is a differential algebra. Now we construct the twisted tensor product \(X = A \otimes \bar{X}\) with respect to \(\bar{\theta}\) following Brown (cf. [7], [8] or [13]). Then \(X = A \otimes \bar{X}\) is a dif-
ferential $A$-comodule with the differential operator $d = 1 \otimes \overline{d} + (1 \otimes \psi) \circ (1 \otimes \theta \otimes 1) \circ \phi \otimes 1$, where $\phi$ is the diagonal structure in $A$ and $\psi$ is the multiplication in $\overline{X}$. More concretely,

\[ dy = a_2, \quad dy^2 = a_3, \]
\[ dx_{4j+1} = a_{4j+2}, \quad 1 \leq j \leq 2n, \]
\[ dx_{4i+3} = b_{4i+4} + x_{4i+1}a_3, \quad 1 \leq i \leq n. \]

Now we define weight in $X$ as follows:

\[
\begin{array}{cccc}
A: & y & y^2 & x_{4j+1} & x_{4i+3} \\
\phi & & & & \\
\overline{X}: & a_2 & a_3 & a_{4j+2} & b_{4i+4} \\
\text{weight} & 0 & 0 & 0 & 1
\end{array}
\]

The weight of a monomial is a sum of the weight of each element. Put $F_i = \{ x \mid \text{weight } x \leq i \}$. Then

$E_0X = \sum_i F_i / F_{i-1}$

\[ \cong A(y, y^2, x_{4j+1}, x_{4i+3}) \otimes \mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}], \]

where the induced differential operator $d_0$ is given by

\[ d_0y = a_2, \quad d_0y^2 = a_3, \quad d_0x_{4j+1} = a_{4j+2}, \quad d_0x_{4i+3} = b_{4i+4}. \]

Thus $E_0X$ is acyclic and hence $X$ is acyclic. Namely $X = A \otimes \overline{X}$ is an injective resolution for $A$ over $\mathbb{Z}_2$. Therefore by definition

\[ H^*(\overline{X}; \overline{d}) = \text{Cotor}^A(\mathbb{Z}_2, \mathbb{Z}_2). \]

As described above the differential operator $\overline{d}$ in $\overline{X} = \mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}]$ is given by

\[ \overline{d}a_i = 0 \quad \text{for} \quad i = 2, 3, 4j+2 \quad (1 \leq j \leq 2n), \]
\[ \overline{d}b_{4i+4} = a_{4i+2}a_3 \quad (1 \leq i \leq 2n). \]

For simplicity we put $P = \mathbb{Z}_2[a_{4j+2}; 1 \leq j \leq 2n]$ and $Q = \mathbb{Z}_2[b_{4i+4}; 1 \leq i \leq 2n]$. 

Let $C$ be a submodule of $X$ generated by \{\(b_{5i}^{i+4}, e_{i}=0\) or 1\}. Then as a module

\[ X \cong \mathbb{Z}_2[a_2] \otimes Q \otimes \mathbb{Z}_2[a_3] \otimes P \otimes C. \]

We remark that as a chain complex, $X$ may be thought of as a tensor product of $(\mathbb{Z}_2[a_2] \otimes Q)$ with a trivial differential operator $d_0$ and $(\mathbb{Z}_2[a_3] \otimes P \otimes C)$ with a differential operator $d_1$ such that $d_1(a_3) = d_1(a_{4i+2}) = 0$ and $d_1(b_{4i+4}) = a_3a_{4i+2}$. Therefore

\[ H(X; d) \cong H(\mathbb{Z}_2[a_2] \otimes Q; d_0) \otimes H(\mathbb{Z}_2[a_3] \otimes P \otimes C; d_1) \]

For $f \in P \otimes C$ there hold $d_1(f) = a_3f$ for some $\tilde{f} \in P \otimes C$. Then we define $\tilde{d}_1 : P \otimes C \to P \otimes C$ by $\tilde{d}_1(f) = \frac{d_1(f)}{a_3}$.

**Lemma 3.1.** The chain complex $(P \otimes C; \tilde{d}_1)$ is acyclic.

**Proof.** Consider the Koszul resolution of the exterior algebra $\Lambda(b_{4i+4}; 1 \leq i \leq 2n)$. Q.E.D.

**Proposition 3.2.** Let $f \in \mathbb{Z}_2[a_3] \otimes P \otimes C$. Then $d_1f = 0$ iff there exists an element $g \in \mathbb{Z}_2[a_3] \otimes P \otimes C$ such that $d_1(g) = a_3f$, or else $f = 1 \otimes 1 \otimes 1$.

**Proof.** Sufficiency is clear, since $X$ is a polynomial algebra.

(Necessity) It suffices to prove necessity for an element $f \in \mathbb{Z}_2 \otimes P \otimes C \cong P \otimes C$. Suppose $d_1(f) = 0$. Then $a_3\tilde{f} = 0$, and hence $\tilde{f} = 0$. So by definition $\tilde{d}_1(f) = 0$, from which we deduce that $f = 1 \otimes 1 \otimes 1$ or else by Lemma 3.1 that there is an element $g \in P \otimes C$ such that $\tilde{d}_1(g) = f$. Thus $a_3f = d_1(g)$. Q.E.D.

Let $I = (i_1, \ldots, i_r)$ be a sequence of integers satisfying

\[ 1 \leq r \leq 2n \quad \text{and} \quad 1 \leq i_1 < \cdots < i_r \leq 2n. \]

We put $y(I) = \frac{1}{a_3} \tilde{d}(b_{4i_1+4} \cdots b_{4i_r+4})$.

It follows from Proposition 3.2 that a system of generators of
Ker \overline{d} over \mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{2i+4}] is \{1, y(I)\}, where I runs over all sequences satisfying (3.3).

**Theorem 3.4.** For \( A = H^*(G(4n+2); \mathbb{Z}_2) \)

\[
\text{Cotor}^4(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, a_3, x_{8i+8}, y(I)]/R,
\]

where \( x_{8i+8} = \{b_{4i+4}\} \) for \( 1 \leq i \leq 2n \) and I runs over all sequences satisfying (3.3). Further R is the ideal generated by \( a_3y(I), y(I)^2 + \sum_{j=1}^{k} x_{8i+8} \cdot a_{4i+2} \cdots x_{8i+8} \) and \( y(I)y(J) + \sum f_i y(I_i) \), where \( f \) is a polynomial of \( a_2, a_3, x_{8i+8} \) s.

**Remark 3.5.** \( y(\{i\}) = a_{4i+2} \). For \( I = (i_1, \ldots, i_r) \), \( r \geq 2 \), \( y(I) \) can be defined inductively. Put \( I' = (i_1, \ldots, i_{r-1}) \). Suppose that \( y(I') = \{\frac{1}{a_3} \overline{d}(b_{4i_1+4} \cdots b_{4i_{r-1}+4})\} \) is defined. Then \( y(I) = y(I', i_r) = (b_{4i_1+4} \cdots b_{4i_{r-1}+4})a_{4i_r+2} + y(I')b_{4i_r+4} = <y(I'), a_3, a_{4i_r+2}> \), the Massey product.

**Remark 3.6.** The relation \( y(I)y(J) + \sum f_i y(I_i) \) can be obtained by calculation on the cochains, since \( \{1, y(I)\} \) is a system of generators over \( \mathbb{Z}_2[a_2, a_3, x_{8i+8}] \).

Now we consider the case \( A = H^*(PO(4n+2); \mathbb{Z}_2) \). By a similar argument to the before we have \( X = \mathbb{Z}_2[a_2, a_{2j+1}, b_{2i+2}]/R, 1 \leq i, j \leq 2n \), where \( R \) is the ideal generated by \( [a_{2k+1}, b_{2k+2}] + a_{4k+1}a_2, 1 \leq k \leq n \) and \( [r, s] \) for other pairs of generators \( (r, s) \) \( ([x, y] = xy + yx) \). We define weight in \( X = A \otimes \overline{X} \), the twisted tensor product with respect to \( \overline{\theta} \):

\[
\begin{array}{ccc}
A & y & z_2j & z_{2i+1} \\
\downarrow & & & \\
X & a_2 & a_{2j+1} & b_{2i+2} \\
\text{weight} & 0 & 0 & 1
\end{array}
\]

Put \( F_i = \{x \mid \text{weight } x \leq i\} \) as before. Then \( E_{0}X = \Sigma F_i/F_{i-1} \)

\[
\cong A(y) \otimes (z_2j, z_{2i+1}) \otimes \mathbb{Z}_2[a_2, a_{2j+1}, b_{2i+2}].
\]
where the induced differential operator is given by $d_0y=a_2$, $d_0z_{2j}=a_{2j+1}$ and $d_0z_{2i+1}=b_{2i+2}$. It shows that $E_0X$ and hence $X$ is acyclic.

The differential operator $\partial$ in $\mathcal{X}$ is given by $\partial a_j=0$ for any $j$ and $\partial b_{2i+2}=a_{2i+1}a_2$. By a similar, although a little bit complicated, calculation to the before, we obtain the following.

For a sequence of integers $I=(i_1,\ldots, i_r)$ satisfying (3.3) we put $y'(I)=\frac{1}{a_2}\partial(b_{2i_1+2}\ldots b_{2i_r+2})$.

**Theorem 3.7.** For $A=H^*(PO(4n+2); \mathbb{Z}_2)$ 

\[
\text{Cotor}^4(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, x_{4l+4}, y'(I)]/R, 
\]

where $x_{4l+4}=b_{2l+2} + a_2b_{4l+2}$ for $1 \leq l \leq n$ and $=b_{2l+2}$ for $n+1 \leq l \leq 2n$ and $I$ runs over all sequences satisfying (3.3). Further $R$ is the ideal generated by $a_2y'(I)$, $y'(I)^2 + \sum_{j=1}^4 x_{4i_1+4} \cdots a_{2l+1} \cdots x_{4i_r+4}$ and $y'(I)y'(J) + \sum_i f_i y'(I_i)$.

**Remark 3.8.** $y'(\{i\})=a_{2i+1}$, For $I=(i_1,\ldots, i_r)$, $y'(I)$ can also be defined inductively, i.e.,

\[
y'(I)=\langle y'(I'), a_2, a_{2i_r+1}, \rangle, \text{ where } I=(I', i_r).
\]

The following results can easily be obtained.

**Proposition 3.9.**

1. \(\text{Cotor}^{H^*(U(2n+1); \mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{c}_1,\ldots, \bar{c}_{2n+1}],\) \text{with deg } \bar{c}_i = 2i.

2. \(\text{Cotor}^{H^*(U(2n+1)/\mathbb{Z}_2; \mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a'_2, \bar{c}_2,\ldots, \bar{c}_{2n+1}],\) \text{with deg } a'_2 = 2 \text{ and deg } \bar{c}_i = 2i.

3. \(\text{Cotor}^{H^*(SO(4n+2); \mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{w}_2, \bar{w}_3,\ldots, \bar{w}_{4n+2}],\) \text{with deg } \bar{w}_i = i.

4. \(\text{Cotor}^{H^*(Sp(2n+1); \mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{q}_1,\ldots, \bar{q}_{2n+1}],\) \text{with deg } \bar{q}_i = 4i.
(5) \[ \text{Cotor}^H_{(PSp(2n+1); \mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a'_2, a'_3, \ldots, \bar{q}'_2, \ldots, \bar{q}'_{2n+1}], \]
with \( \deg a'_i = i \) and \( \deg \bar{q}'_i = 4i \).

(6) \[ \text{Cotor}^H_{(SU(4n+2); \mathbb{Z}_2)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{c}_2, \ldots, \bar{c}_{2n+1}], \]
with \( \deg \bar{c}_i = 2i \).

§ 4. Collapsing of the Eilenberg-Moore Spectral Sequence

Let \( G \) be a topological group. In 1959 Eilenberg-Moore constructed a new type of spectral sequence \( \{E_r(G), d_r\} \) such that

(1) \[ E_2(G) \cong \text{Cotor}^H_{(G, \mathbb{Z}_p)}(\mathbb{Z}_p, \mathbb{Z}_p), \]
(2) \[ E_\infty(G) \cong \mathcal{O}H^*_B(G; \mathbb{Z}_p). \]

Furthermore, this spectral sequence satisfies naturality for a homomorphism \( f: G \to G' \). We denote by \( f^*: E_r(G') \to E_r(G) \) the induced homomorphism.

In this section we will show that the Eilenberg-Moore spectral sequence collapses for various \((G, p)\). In particular, we will show that for \( G = G(4n+2) \) and \( PO(4n+2) \) the Eilenberg-Moore spectral sequence with \( \mathbb{Z}_2 \)-coefficient collapses.

The following directly follows from Theorem 2.9:

**Proposition 4.1.** Let \((n, p) = 1\). Then the Eilenberg-Moore spectral sequence collapses for \((G, p) = (U(n)/\Gamma_p, p)\).

By Kono [9] \( H^*(PSp(2n+1); \mathbb{Z}_2) \) is transgressively generated and hence we have

**Proposition 4.2.** The Eilenberg-Moore spectral sequence collapses for \((G, p) = (PSp(2n+1), 2)\).

The following result will be used below. The proof is easy and left to the reader.

**Proposition 4.3.** (1) The Eilenberg-Moore spectral sequence col-
lapses for $G = U(2n+1)/\Gamma_2$, $SO(4n+2)$, $U(2n+1)$, $PSp(2n+1)$, $SU(4n+2)$ and $Sp(2n+1)$.

(2) The elements $\overline{c}_i$ and $\overline{w}_i$ in Proposition 3.9 represent $c_i$ and $w_i$ respectively. The elements $\overline{q}_i$ and $\overline{c}_i$ do $q_i$ and $c_i$ in $H^*(BG; \mathbb{Z}_2)$ such that $\pi^*(c_i) = c_i + (\text{decomp.})$ and $\pi^*(q_i) = q_i + (\text{decomp.})$, where $\pi$ is the covering homomorphism.

For simplicity we use the following

**Notation.**

\begin{align*}
A_1 &= H^*(U(2n+1); \mathbb{Z}_2), \\
A_2 &= H^*(U(2n+1)/\Gamma_2; \mathbb{Z}_2), \\
A_3 &= H^*(SO(4n+2); \mathbb{Z}_2), \\
A_4 &= H^*(PO(4n+2); \mathbb{Z}_2), \\
B_1 &= H^*(Sp(2n+1); \mathbb{Z}_2), \\
B_2 &= H^*(PSp(2n+1); \mathbb{Z}_2), \\
B_3 &= H^*(SU(4n+2); \mathbb{Z}_2), \\
B_4 &= H^*(G(4n+2); \mathbb{Z}_2).
\end{align*}

**Case I.** $H^*(PO(4n+2); \mathbb{Z}_2)$.

Consider the commutative diagram

\[
\begin{array}{ccc}
U(2n+1) & \xrightarrow{i} & SO(4n+2) \\
\downarrow{\pi} & & \downarrow{\pi} \\
U(2n+1)/\Gamma_2 & \xrightarrow{i} & PO(4n+2)
\end{array}
\]

where $\pi$ is the projection and $i$'s are the standard maps (cf. §6).

**Lemma 4.4.** The elements $a_2' \in \text{Cotor}^A_4(\mathbb{Z}_2, \mathbb{Z}_2)$ and $a_2 \in \text{Cotor}^A_4(\mathbb{Z}_2, \mathbb{Z}_2)$ are permanent cycles and $i^*(a_2) = a_2'$.

**Proof.** Recall $H^*(B\mathbb{Z}_2^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[t]$. In the commutative diagram
the elements $a_2$ and $a'_2$ represent the transgression images of $t$, and hence they are permanent cycles. For dimensional reason we have $i^*(a_2) = a'_2$.

Q.E.D.

The following relations among the elements in Theorem 3.7 and Proposition 3.9 are easily checked to be true:

(4.5.1) $\pi^*(x_{4i}) = \bar{w}_{2i} + W_i$, where $W_i$ is a sum of monomials containing elements of lower degree,

(4.5.2) $i^*(\bar{w}_{2i}) = \bar{c}_i + (\text{decomp.})$, (see § 6),

(4.5.3) $\pi^*(\bar{c}_i) = \bar{c}_i + (\text{decomp.})$,

(4.5.4) $\pi^*(a_2) = \pi^*(a'_2) = 0$.

Therefore

(4.6) $i^*(x_{4i}) = \bar{c}_{2i} + \gamma_i$, where $\gamma_i$ is a sum of monomials containing elements of lower degree.

Let $E_r(1)$ be the Eilenberg-Moore spectral sequence with $\mathbb{Z}_2$-coefficient for $PO(4n+2)$ and $\{E_r(2), d_r\}$ be the cartesian product of the Eilenberg-Moore spectral sequences of $U(2n+1)/\mathbb{Z}_2$ and $SO(4n+2)$, i.e.,

$E_r(2) = \text{Cotor}^{A_2}(\mathbb{Z}_2, \mathbb{Z}_2) \times \text{Cotor}^{A_3}(\mathbb{Z}_2, \mathbb{Z}_2)$ and $d_r = 0$

for all $r \geq 2$. Then the map $i^* \times \pi^*$ induces a homomorphism between the spectral sequences:

$E_r(1) \longrightarrow E_r(2)$ for $r \geq 2$.

**Lemma 4.7.** $i^* \times \pi^* : E_2(1) \to E_2(2)$ is injective.
Proof. Let $f_1$ be a sum of monomials containing $a_2$ and $f_2$ a sum of those not containing $a_2$. Suppose $(i^* \times \pi^*)(f_1 + f_2) = 0$ from which $\pi^*(f_1 + f_2) = \pi^*(f_2) = 0$ and hence $f_2 = 0$ by (4.5.1). Meanwhile $(i^* \times \pi^*)(f_1 + f_2) = 0$ implies $i^*(f_1 + f_2) = 0$, which implies $i^*(f_1) = 0$, and hence $f_1 = 0$ by (4.6). Thus $i^* \times \pi^*$ is injective. Q.E.D.

Thus we have shown

**Theorem 4.8.** The Eilenberg-Moore spectral sequence with $\mathbb{Z}_2$-coefficient collapses for $G = \text{PO}(4n + 2)$.

In fact, Lemma 4.7 indicates that all differentials in $E_2(1)$ are trivial. An immediate corollary is

**Theorem 4.9.** As a module

$$H^*(\text{BPO}(4n + 2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, x_{4l+4}, y'(I)]/R,$$

where $1 \leq l \leq 2n$, $I$ runs over all sequences satisfying (3.3) and $R$ is the ideal generated by $a_2y'(I)$, $y'(I)^2 + \sum_{j=1}^{r} x_{4l_j+4} \cdots a_{2l_j+1} x_{4l+4}$ and $y'(I)y'(J) + \sum_{i} f_i y'(I_i)$.

**Case II.** $H^*(G(4n + 2); \mathbb{Z}_2)$.

Consider the commutative diagram

$$
\begin{array}{ccc}
\text{Sp}(2n+1) & \longrightarrow & \text{SU}(4n+2) \\
\downarrow \pi & & \downarrow \pi \\
\text{PSp}(2n+1) & \longrightarrow & G(4n+2)
\end{array}
$$

where $\pi$ is the projection and $i$'s are the standard maps.

**Lemma 4.4'.** The elements $a_i \in \text{Cotor}^{B*}(\mathbb{Z}_2, \mathbb{Z}_2)$ and $a'_i \in \text{Cotor}^{B*}(\mathbb{Z}_2, \mathbb{Z}_2)$ are permanent cycles and $i^*(a_i) = a'_i$ for $i = 2, 3$.

Proof is similar to that of Lemma 4.4.

The following relations among the elements in Theorem 3.4 and Proposition 3.9 are easily checked to be true:
(4.10.1) \[ \pi^k(\xi_{8l+8}) = \bar{\xi}_{2l+2} + v_l, \]

(4.10.2) \[ \pi^k(\bar{q}_l) = \bar{q}_l + (\text{decomp.}), \]

where \( v_l \) is a sum of monomials containing elements of lower degree,

(4.10.3) \[ \pi^k(a_i) = \pi^k(a_l) = 0 \quad \text{for} \quad i = 2, 3, \]

(4.11) \[ i^*(\zeta_{2l}) = \bar{q}_l + (\text{decomp.}). \]

**Lemma 4.7'.** Let \( f \in \text{Cotor}^B_*(Z_2, Z_2) \) such that \( \deg f \) is odd. Then \( i^*(f) = 0 \) iff \( f = 0. \)

**Proof.** \[ i^*(\xi_{8l+8}) = \bar{q}_{l+1} + Q_l, \]

where \( Q_l \) is a sum of monomials containing elements of lower degree. Thus the elements \( i^*(\xi_{8l+8}), 1 \leq l \leq 2n, i^*a_3 \) and \( i^*a_2 \) are algebraically independent. Q.E.D.

**Theorem 4.10.** The Eilenberg-Moore spectral sequence with \( Z_2 \)-coefficient collapses for \( G = G(4n + 2). \)

**Proof.** Recall that \( a_2 \) and \( a_3 \) are permanent cycles. All generators of \( \text{Cotor}^B_*(Z_2, Z_2) \) except \( a_3 \) are of even degree. So \( d_r(\alpha) \) is of odd degree for \( \alpha \in \{ y(I), x_{8l+8} \} \). By naturality \( i^*d_r(\alpha) = d_i^*i^*(\alpha) = 0. \) Hence by Lemma 4.7' \( d_r(\alpha) = 0. \) Thus all generators survive into \( E_{\infty}. \) Q.E.D.

Immediate corollaries are

**Theorem 4.11.** As a module

\[ H^*(BG(4n + 2); Z_2) \cong Z_2[a_2, a_3, x_{8l+8}, y(I)]/R, \]

where \( x_{8l+8} = \{ b_{2i+4} \} \) for \( 1 \leq l \leq 2n \) and \( I \) runs over all sequences satisfying (3.3) and \( R \) is the ideal generated by \( a_3y(I), y(I)^2 + \sum_{j=1}^I x_{8l+8} \ldots a_{2l+2} \ldots a_{8l+8} \) and \( y(I)y(J) + \sum_I f_Iy(I_i). \)

**Theorem 4.12.** As a module

\[ H^*(BPU(4n + 2); Z_2) \cong Z_2[a_2, a_3, x_{8l+8}, y(I)]/R \]
with the same $l, I$ and $R$ as in Theorem 4.11.

§5. Some Generators in $\text{H}^*(\text{BG}(4n+2); \mathbb{Z}_2)$ and $\text{H}^*(\text{BPO}(4n+2); \mathbb{Z}_2)$

Let $G$ be a compact, connected Lie group and $H$ its closed subgroup. Let $EG$ and $EH$ be the total spaces of the universal $G$- and $H$-bundles respectively. Then the following diagram is commutative:

\[
\begin{array}{ccc}
H & \rightarrow & EH \\
\downarrow j & & \downarrow j \\
G & \rightarrow & EG \\
\downarrow p & & \downarrow p \\
G/H & \rightarrow & BH \\
\end{array}
\]

Then naturality of the transgression implies

**Lemma 5.1.** Let $k$ be a commutative field.

1. If $x \in H^*(G/H; k)$ is transgressive with respect to the bottom fiber-in, then $p^*(x) \in H^*(G; k)$ is universally transgressive.

2. If $x \in H^*(G; k)$ is universally transgressive, so is $j^*(x) \in H^*(H; k)$.

3. Suppose $H^i(G/H; k) = 0$ for $i < n$. Let $x \in H^i(G; k)$ and $i < n-1$. If $j^*(x)$ is universally transgressive, so is $x$.

Recall the following:

(5.2) $G(2) = \text{SO}(3)$,

(5.3) $H^*(\text{SO}(3); \mathbb{Z}_2) = \mathbb{Z}_2[a]/(a^4)$ where $a$ is universally transgressive.

Now we prove

**Proposition 5.4.** The elements $a$ and $x_{4j+1}$ of $H^*(G(4n+2); \mathbb{Z}_2)$, $1 \leq j \leq 2n$, are all universally transgressive.

**Proof.** Proof is induction on $n$. The case $n=0$ is clear from (5.2) and (5.3). Suppose as the inductive hypothesis that the elements $a$ and $x_{4j+1}, 1 \leq j \leq 2n-1$, are universally transgressive in $H^*(G(4n-2);$
COHOMOLOGY OF THE CLASSIFYING SPACES

It follows from Proposition 1.5 and (2), (3) of Lemma 5.1 that the elements \( a \) and \( x_{4j+1}, 1 \leq j \leq 2n-1 \) are universally transgressive. Clearly the element \( x_{8n-3} \) is transgressive with respect to the fibering \( C(4n-2, 2n+1) \rightarrow BG(4n-2) \rightarrow BG((4n-2)(2n+1)) \), and hence so is \( x_{8n+1} \), since \( x_{8n+1} = Sq^4 x_{8n-3} \) by Proposition 1.14. Thus by (1) of Lemma 5.1 the elements \( x_{8n-3} \) and \( x_{8n+1} \) are universally transgressive. Q.E.D.

It follows from (2.2) and (2.3) that \( H^*(BG(2); \mathbb{Z}_2) \cong H^*(BSO(3); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, a_3] \), where \( a_2 = \tau(a) \) and \( a_3 = \tau(a^2) \) with \( \deg a_i = i \). As \( \Delta^*_2 : H^i(BG(4n+2); \mathbb{Z}_2) \rightarrow H^i(BG(2); \mathbb{Z}_2) \) is isomorphic for \( i \leq 4 \) by (1) of Theorem 1.7, we denote by \( a_2 = \tau(a) \) and \( a_3 = \tau(a^2) \) the generators of \( H^i(BG(4n+2); \mathbb{Z}_2) \cong \mathbb{Z}_2 \) for \( i = 2, 3 \).

Lemma 5.5. \( Sq^1 a_3 = 0 \) and \( Sq^2 a_3 = a_2 a_3 \) in \( H^*(BG(4n+2); \mathbb{Z}_2) \).

Proof. We obtain the above formula by virtue of the Wu formula, since \( a_i \) is the inverse image of \( \Delta^*_2 \) of the \( i \)-th Stiefel-Whitney class. Q.E.D.

Proposition 5.6. There exist elements \( a_{4j+2}, 1 \leq j \leq 2n \), in \( H^*(BG(4n+2); \mathbb{Z}_2) \) such that

\[
\begin{align*}
(1) \quad & \deg a_{4j+2} = 4j + 2, \\
(2) \quad & a_{4j+2} = \tau(x_{4j+1}) \mod (\text{decomp.}), \\
(3) \quad & a_3 a_{4j+2} = 0.
\end{align*}
\]

Proof. Proof is induction on \( n \). The case \( n = 0 \) is clear from (2.2) and (2.3). Suppose that the assertion is true for \( BG(4n-2) \). By Theorem 1.7 the homomorphism \( \Delta^*_2 : H^i(BG(4n+2); \mathbb{Z}_2) \rightarrow H^i(BG(2); \mathbb{Z}_2) \) is injective for \( \deg \leq 8n-4e+1 \) with \( e = \pm 1 \). Put \( a_i = \Delta^*_{2n-1} \circ \Delta^*_{2n+1}(a_i) \) for \( i \leq 8n-6 \). Then \( a_i \) satisfies the properties (1), (2), (3) by the inductive hypothesis. For the transgression \( \tau \) of the fibering

\[
(5.7) \quad C(4n-2, 2n+1) \rightarrow BG(4n-2) \rightarrow BG((4n-2)(2n+1))
\]

we put \( a_{8n-2} = \Delta^*_2 \tau(x_{8n-3}). \) The element \( x_{8n-1} \in H^*(G(4n+2); \mathbb{Z}_2) \) is not universally transgressive, since it is not primitive by Proposition
1.9'. So the corresponding element $\tilde{x}_{8n-1}$ of $H^*(C(4n-2, 2n+1); \mathbb{Z}_2)$ is not transgressive in the fibering (5.7). That is, in the cohomology Serre spectral sequence $\{E_r^*, d_r\}$ with $\mathbb{Z}_2$-coefficient of (5.7) we have $d_3(1 \otimes \tilde{x}_{8n-1}) = a_3 \otimes \tilde{x}_{8n-3}$, from which we get $a_3\tau(\tilde{x}_{8n-3}) = 0$. Applying $\Delta_{2n-1}^*$ we obtain $a_3a_{8n-2} = 0$. Thus the element $a_{8n-2}$ satisfies (1), (2), (3). Next, we put

$$a_{8n+2} = \Delta_{2n-1}^* \tau(\tilde{x}_{8n+1}) + a_2 Sq^2 a_{8n-2} + a_3 Sq^1 a_{8n-2}.$$ 

Then

$$a_3a_{8n+2} = a_3(Sq^4 a_{8n-2} + a_2 Sq^2 a_{8n-2} + a_3 Sq^1 a_{8n-2})$$

$$= Sq^4(a_3 a_{8n-2})$$

$$= 0.$$ 

So the element $a_{8n+2}$ satisfies (1), (2), (3). Q.E.D.

Quite similarly one can prove

**Proposition 5.8.** There exist elements $a_2, a_{2j+1}, 1 \leq j \leq 2n$, in $H^*(BPO(4n+2); \mathbb{Z}_2)$ such that

(1) $\deg a_2 = 2$, $\deg a_{2j+1} = 2j + 1$,

(2) $a_2 = \tau(y), a_{2j+1} = \tau(z_{2j}), 1 \leq j \leq 2n$,

(3) $a_2 a_{2j+1} = 0$.

**Remark 5.9.** The elements $a_i$ in Theorems 4.9, 4.11 and 4.12 are thus the transgression images of some generators in $H^*(G(4n+2); \mathbb{Z}_2)$, $H^*(PU(4n+2); \mathbb{Z}_2)$ or $H^*(PO(4n+2); \mathbb{Z}_2)$. The relations among them are given in Propositions 5.6 and 5.8.

§ 6. Exterior Power Representations

To begin with we recall the definition of the exterior power representation (p. 90 of [14]).

Let $G$ be a group and $k$ a commutative field. Denote by $GL(n, k)$ the general linear group. Let $A = (a_{ij}) : G \rightarrow GL(n, k)$ be a matrix rep-
cohomology of the classifying spaces

Representation. For a pair of sequences of $r$ integers $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_r)$ such that

\[(*) \quad 1 \leq i_1 < \cdots < i_r \leq n,\]
\[1 \leq j_1 < \cdots < j_r \leq n,\]
we define

\[a_{IJ}(x) = \det \begin{pmatrix} a_{i_1 j_1}(x) & \cdots & a_{i_1 j_r}(x) \\ \vdots & \ddots & \vdots \\ a_{i_r j_1}(x) & \cdots & a_{i_r j_r}(x) \end{pmatrix} \quad \text{for } x \in G.

**Definition 6.1.** Let $1 \leq r \leq n$. We define a representation $A^{(r)}(x) : G \rightarrow GL(\mathbb{F}, k)$ by

\[A^{(r)}(x) = \begin{pmatrix} J \\ \vdots \\ \vdots \\ a_{i_r j_r}(x) \end{pmatrix}

where $I$ and $J$ run over all sequences satisfying $(*)$. We call $A^{(r)}$ the exterior power representation of degree $r$ of $G$.

If $G$ is a topological group and $k = \mathbb{R}$ or $\mathbb{C}$ and if $A : G \rightarrow GL(n, k)$ is continuous, so is $A^{(r)}$, namely, $A^{(r)}$ is a representation of $G$.

When $G$ is a compact group and $k = \mathbb{C}$ (resp. $\mathbb{R}$), we may suppose

\[A^{(r)} : G \rightarrow U((\mathbb{F})) \quad \text{(resp. } A^{(r)} : G \rightarrow O((\mathbb{F})))

by making use of the $G$-invariant Hermitian (resp. Riemannian) metric (see [2]).

**Proposition 6.2.** Let $G$ be a subgroup of $GL(n, k)$. Let $A : G \rightarrow GL(n, k)$ be an inclusion. For $G \ni x = \begin{pmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix} \in GL((\mathbb{F}), k)$ if $r$ is even.

**Proof.** By definition

\[a_{IJ}(x) = \begin{cases} (-1)^r & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}

Q.E.D.
In the below we regard the identity map $\lambda: G = U(n) \to U(n)$ (or the inclusion $\lambda: SU(n) \to U(n)$) as an $n$-dimensional complex representation.

**Corollary 6.3.** Let $n$ be even. Then there exists a map $\lambda^{(2)}$ such that the right diagram commutes:

$$
\begin{array}{ccc}
SU(n) & \xrightarrow{\lambda^{(2)}} & U((\frac{n}{2})) \\
\downarrow \pi & & \downarrow \lambda^{(2)} \\
G(n) & & 
\end{array}
$$

Let $t_k$ be a generator of $H^2(BT^n; \mathbb{Z})$ corresponding to the torus

$$
T^1 = \left\{ \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}; \ 0 \leq \theta < 2\pi \right\} \subset T_n \subset U(n).
$$

Then according to Borel-Hirzebruch (p. 492 of [6]) the total Chern class $c(\lambda^{(2)})$ of the second exterior power representation $\lambda^{(2)}$ is given by

$$
c(\lambda^{(2)}) = \prod_{1 \leq i < j \leq n} (1 + t_i + t_j) \in H^*(BU(n); \mathbb{Z}).
$$

**Remark 6.5.**

$t_1 + \cdots + t_n = 0$ if $G = SU(n)$.

Let $x_i, 1 \leq i \leq n$, be indeterminates with $\deg x_i = 1$. Express

$$
\prod_{1 \leq i < j \leq n} (1 + x_i + x_j) = \beta_1 + \cdots + \beta_n + \text{(higher terms)},
$$

where $\beta_k$ is a homogeneous term of degree $k$. Denoting by $\sigma_k$ the $k$-th elementary symmetric function, we have $\beta_k = a_k \sigma_k(x_1, \ldots, x_n) + \text{(decomp.)}$ for some integer $a_k$.

**Lemma 6.6.** If $n$ is odd, $a_i$ is odd for $2 \leq i \leq n$.

(A proof will be given at the end of the section.)

Let $i: Sp(n) \to SU(2n)$ be the usual inclusion map defined by

$$
q_{ij} = x_{ij} + jy_{ij} \mapsto c_{ij} = \begin{pmatrix} x_{ij} & -y_{ij} \\ y_{ij} & \bar{x}_{ij} \end{pmatrix},
$$
where \( \alpha_{i,j}, \beta_{i,j} \in \mathbb{C} \).

Let \( s_i \) be a generator of \( H^2(BT^n; \mathbb{Z}) \) corresponding to the torus

\[
T^1 = \left\{ i \begin{pmatrix} 1 & \cdots & e^{i\theta} & 0 \\ 1 & \cdots & 1 & \vdots \\ \vdots & \ddots & 1 & \vdots \\ 0 & \cdots & 1 & 1 \end{pmatrix} \in Sp(n) : 0 \leq \theta < 2\pi \right\} \subset T^n \subset Sp(n).
\]

Then

\[
i^*(t_{2i-1}) = s_i \quad \text{and} \quad i^*(t_{2i}) = -s_i.
\]

Consider the composite of the maps

\[
BSp(n) \overset{i}{\longrightarrow} BSU(2n) \overset{j^{(2)}}{\longrightarrow} BU(\left(\frac{n}{2}\right)).
\]

**Proposition 6.8.** The mod 2 reduction of \( i^*c(\lambda^{(2)}) \) is given by

\[
i^*c(\lambda^{(2)}) = \prod_{1 \leq i < j \leq n} (1 + s_i^4 + s_j^4) \in H^*(BSp(n); \mathbb{Z}_2).
\]

**Proof.**

\[
i^*c(\lambda^{(2)}) = i^*\left( \prod_{1 \leq i < j \leq n} (1 + s_i + t_j) \right) \quad \text{by (6.4)}
\]

\[
= \prod_{1 \leq i < j \leq n} (1 + s_i + s_j)^4 \quad \text{by (6.7)}
\]

\[
= \prod_{1 \leq i < j \leq n} (1 + s_i^4 + s_j^4).
\]

Q.E.D.

Next we consider the commutative diagram:

\[
\begin{array}{ccc}
BSp(2n+1) & \overset{i}{\longrightarrow} & BSU(4n+2) \\
\downarrow \pi & & \downarrow \pi \\
BPSp(2n+1) & \overset{i}{\longrightarrow} & BG(4n+2)
\end{array}
\]

For the mod 2 reduction of the Chern class \( c_{4i} \in H^8(BU(\left(\frac{4n+2}{2}\right)); \mathbb{Z}_2) \) we put

\[
x_{8i} = \overline{\lambda}^{(2)*}(c_{4i}) \in H^8(BG(4n+2); \mathbb{Z}_2), \quad 2 \leq i \leq 2n + 1.
\]

Then by the commutativity of the diagram (6.9)

\[
i^*\pi^*x_{8i} = i^*\pi^*\overline{\lambda}^{(2)*}(\Sigma c_{4i})
\]
Apply Lemma 6.6 and we obtain

\[ i^*\pi^*x_{8i} = \sigma(s_i^4, \ldots, s_{2n+1}^4) + \text{(decomp.)}, \]

Denoting by \( q_i \) the mod 2 reduction of the \( i \)-th symplectic Pontrjagin class, we have

\[ i^*\pi^*x_{8i} = q_i^2 + P, \]

where \( P \) is a sum of monomials containing \( q_j (j < i) \).

On the other hand, since \( i^*: H^m(BSU(4n+2); \mathbb{Z}_2) \to H^m(BSp(2n+1); \mathbb{Z}_2) \) is trivial for \( m \equiv 0 \) (mod 4), we have

\[ i^*\pi^*(a_2) = i^*\pi^*(a_3) = i^*\pi^*(a_{4j+2}) = 0, \]

and hence

\[ i^*\pi^*(y(I)) = 0. \]

Thus we have shown

**Theorem 6.10.** There exist non-decomposable elements \( x_{8i+8} \in H^{8i+8}(BG(4n+2); \mathbb{Z}_2), 1 \leq i \leq 2n, \) such that \( i^*\pi^*(x_{8i+8}) = q_i^2 + P, \) where \( P \) is a sum of monomials containing \( q_j (j < i+1) \).

Now we turn to the orthogonal case.

Let \( \lambda: SO(n) \to O(n) \) be the natural inclusion and regard it as a real representation. As before we consider its exterior power representation \( \lambda^{(2)}: SO(n) \to O(\mathbb{Z}_2) \). The total Stiefel-Whitney class is then given as

\[ w(\lambda^{(2)}) = \prod_{1 \leq i < j \leq 2n} (1 + t_i + t_j), \]

where \( t_i \) is a generator of \( H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2) \) corresponding to

\[ \mathbb{Z}_2 = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}; \varepsilon = \pm 1 \in (\mathbb{Z}_2)^n \subset O(n). \]
Remark. \[ t_1 + \cdots + t_n = 0. \]

Let \( i: U(n) \rightarrow SO(2n) \) be the inclusion defined by the correspondence \[ b + c, \sqrt{-1} \mapsto \begin{pmatrix} b & -c \\ c & b \end{pmatrix}. \] Let \( s_i \) be a generator of \( H^1(B(Z_2^n); Z_2) \) corresponding to

\[
Z_2 = \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & \varepsilon
\end{pmatrix}; \quad \varepsilon = \pm 1 \subseteq (Z_2)^n \subseteq U(n).
\]

Then

\[ i^* (t_{2i-1}) = i^* (t_{2i}) = s_i. \]

Let \( w_i \) be the Stiefel-Whitney class. Then

\[ i^* (w_{2i-1}) = 0, \]

\[ i^* (w_{2i}) = c_i, \] the mod 2 reduction of the \( i \)-th Chern class.

Consider the following commutative diagram

\[
\begin{array}{ccc}
BU(2n+1) & \xrightarrow{i} & BSO(4n+2) \\
\downarrow \pi & & \downarrow \pi \\
B(U(2n+1)/\Gamma_2) & \xrightarrow{i} & BPO(4n+2)
\end{array}
\]

where \( \pi \) is the natural projection and \( \lambda^{(2)} \) the one induced from \( \lambda^{(2)}. \) Then

\[ i^* \pi^* \lambda^{(2)}(\sum_{i=0}^l w_i) = i^* (w(\lambda^{(2)})) \quad \text{with} \quad l = \binom{4n+2}{2}, \]

where \( w(\lambda^{(2)}) = \prod_{1 \leq i < j \leq 4n+2} (1 + t_i + t_j). \)

So by Lemma 4.6 we have

\[ i^* \pi^* \lambda^{(2)}(\sum_{i=0}^l w_i) = \prod_{1 \leq i < j \leq 2n+1} (1 + s_i^* + s_j^*). \]
Thus by a similar argument to the unitary case we have

**Theorem 6.12.** There exist non-decomposable elements \( x_{4j+4} \in H^{4j+4}(BPO(4n+2); \mathbb{Z}_2) \), \( 1 \leq j \leq 2n \), such that \( i^*\pi^* x_{4j+4} = c_{j+1}^2 + P \), where \( P \) is a sum of monomials containing \( c_k (k < j+1) \).

First we consider the case \( G = G(4n+2) \). The projection \( \pi: SU(4n+2) \rightarrow G(4n+2) \) induces \( \pi^*: \text{Cotor}^B_4(Z_2, Z_2) \rightarrow \text{Cotor}^{BG}(Z_2, Z_2) \) on the \( E_2 \)-level of the Eilenberg-Moore spectral sequence. By naturality we have

\[
\pi^* x_{8i+8} = \pi^* b_{4i+4}^2 = c_{2i+2}^2 \quad \text{for} \quad 1 \leq i \leq 2n,
\]

which survives in the \( E_\infty(SU(4n+2)) \)-term, since \( E_2(SU(4n+2)) \cong E_\infty(SU(4n+2)) \cong \mathcal{S} H^*(BSU(4n+2); \mathbb{Z}_2) \) by Proposition 4.3. On the other hand, since \( q_{i+1} = i^* c_{2i+2} \), it follows from Theorem 6.10 that for \( \pi^*: \text{H}^*(BG(4n+2); \mathbb{Z}_2) \rightarrow \text{H}^*(BSU(4n+2); \mathbb{Z}_2) \) we have

\[
\pi^* x_{8i+8} = c_{2i+2}^2 + P'. \quad 1 \leq i \leq 2n,
\]

where \( P' \) is a sum of monomials containing \( c_j (j < i+1) \).

Thus we obtain

**Theorem 6.13.** The element \( x_{8i+8} \in \text{Cotor}^B_4(Z_2, Z_2) \) survives in the \( E_\infty(G(4n+2)) \)-term and represents \( x_{8i+8} \in \text{H}^*(BG(4n+2); \mathbb{Z}_2) \).

Similarly,

**Theorem 6.13'.** The element \( x_{4i+4} \in \text{Cotor}^A_4(Z_2, Z_2) \) survives in the \( E_\infty(PO(4n+2)) \)-term and represents \( x_{4i+4} \in \text{H}^*(BPO(4n+2); \mathbb{Z}_2) \).

**Proof of Lemma 6.6.** Let \( m \) be an odd integer. We regard the identity map \( \lambda: U(m) \rightarrow U(m) \) as an \( m \)-dimensional complex representation as before. Let \( t_k \) be a generator of \( H^2(T^m; \mathbb{Z}) \) corresponding to the torus

\[
T^1 = \begin{bmatrix}
1 & 1 \\
\vdots & e^{i\theta} \\
1 & 1 \\
\end{bmatrix}, \quad 0 \leq \theta < 2\pi \subset T^m \subset U(m).
\]
Then by (6.4) the total Chern class of the exterior representation of
degree 2 of $\lambda$ is given by

$$c(\lambda^{(2)}) = \prod_{1 \leq i < j \leq m} (1 + t_i + t_j) \in H^*(BU(m); \mathbb{Z}).$$

We will show that the integer $a_k$ is odd by taking $x_i = t_i$ and
$\beta_i = c_i(\lambda^{(2)})$, the $i$-th Chern class of $\lambda^{(2)}$.

Let $\Phi^k$ be the Adams operation on representations and $ch_q$
the Chern character. Denote by $\lambda^2$ the tensor product $\lambda \otimes \lambda$.

**Lemma 6.14.** (1) $ch_q(\Phi^2(\lambda)) = 2^q ch_q(\lambda)$.

(2) $\Phi^2(\lambda) = \lambda^2 - 2\lambda^{(2)}$.

(3) $ch_q(\lambda^2) = 2mch_q(\lambda) + (\text{decomp.})$.

(4) Let $m \geq 3$. For $\eta = \lambda$ or $\lambda^{(2)}$

$$ch_q(\eta) = \frac{(-1)^q}{(q-1)!} c_q(\eta) + (\text{decomp.}).$$

*Proof.* (1), (2), (3) follow directly from the definition (also see [1]).
(4) follows from the Newton formula. Q.E.D.

By this lemma we have

$$ch_i(\lambda^{(2)}) = \frac{1}{2} \{ ch_i(\lambda^2) - ch_i(\Phi^2(\lambda)) \}$$

$$= \frac{1}{2} \{ 2(n - 2^{i-1})ch_i(\lambda) \} + (\text{decomp.})$$

$$= (n - 2^{i-1})ch_i(\lambda) + (\text{decomp.}).$$

Now by (4) we obtain

$$c_i(\lambda^{(2)}) = (n - 2^{i-1})c_i(\lambda) + (\text{decomp.})$$

$$= (n - 2^{i-1})\sigma_i(t_1, \ldots, t_n) + (\text{decomp.}),$$

where $(n - 2^{i-1})$ is odd if $i \geq 2$. Q.E.D.
References