Propagation of Singularities of Fundamental Solutions of Hyperbolic Mixed Problems

By

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§ 1. Introduction

In this paper we shall deal with hyperbolic mixed problems with constant coefficients in a quarter-space and study the wave front sets of the fundamental solutions under the only assumption that the hyperbolic mixed problems are $\mathcal{C}$-well posed. Recently Garnir has studied the wave front sets of fundamental solutions for hyperbolic systems [2]. The author was stimulated by his work. For the detailed literatures we refer the reader to [7], [8].

Now let us state our problems, assumptions and main results. Let $\mathbb{R}^n$ denote the $n$-dimensional euclidean space and write $x = (x_1, \ldots, x_n)$ for the coordinate $x' = (x_1, \ldots, x_{n-1})$ in $\mathbb{R}^n$ and $\xi' = (\xi_1, \ldots, \xi_{n-1})$, $\xi = (\xi, \xi_n)$ for the dual coordinate $\xi = (\xi_1, \ldots, \xi_n)$. We shall also denote by $\mathcal{H}_+$ the half-space \{x = (x', x_n) \in \mathbb{R}^n; x_n > 0\}. For differentiation we will use the symbol $D = \partial_t \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}$. Let $P = P(\xi)$ be a hyperbolic polynomial of order $m$ of $n$ variables $\xi$ with respect to $\vartheta = (1, 0, \ldots, 0) \in \mathbb{R}^n$ in the sense of Gårding, i.e.

\[ P^0(-i\vartheta) \neq 0 \text{ and } P(\xi - s\vartheta) \neq 0 \text{ when } \xi \text{ is real and } \text{Im } s \leq \gamma_0, \]

where $P^0$ denotes the principal part of $P$, i.e.

\[ P(t\xi) = t^m (P^0(\xi) + o(1)) \quad \text{as } t \to \infty, \quad P^0(\xi) \neq 0. \]

Let $\Gamma' = \Gamma(P, \vartheta) \subset \mathbb{R}^n$ be the component of the set \{\(\xi \in \mathbb{R}^n; P(\xi - i\xi) \neq 0\)\} which contains $\vartheta$. We also write $\Gamma'(P) = \Gamma'(P, \vartheta)$. Put

\[ \Gamma_0 = \{\xi \in \mathbb{R}^{n-1}; \langle \xi', 0 \rangle \in \Gamma\}, \]

\[ \hat{\Gamma} = \{\xi \in \mathbb{R}^{n-1}; \langle \xi', \xi_n \rangle \in \Gamma \text{ for some } \xi_n \in \mathbb{R}\}. \]
The localization $P_{\nu}(\gamma)$ of $P(\xi)$ at $\xi^0$ and the multiplicity $m_{\nu}$ of $\xi^0$ relative to $P$ are defined by

$$\nu^m P(\nu^{-1}\xi^0 + \gamma) = \nu^{m_{\nu}} P_{\nu}(\gamma) + o(1)$$

as $\nu \to 0$, $P_{\nu}(\gamma) \neq 0$ (see [1]). We note that

$$\Gamma \subset \Gamma_{\nu} = \Gamma(P_{\nu}).$$

Now write

$$P(\xi) = \sum_{j=0}^{m} P_j(\xi') \xi'^j, \quad P_{m'}(\xi') \neq 0.$$  

Then we see that

$$P_{m'}(\xi') \neq 0 \text{ for } \xi' \in \mathbb{R}^{n-1} - i\gamma_0 \theta' - i\Gamma_0.$$  

In fact, $P_{m'}(\xi') = P_{0,1}(\xi)$ and $\Gamma_{0,1} = \Gamma_{0,1}(\xi) \times \mathbb{R}$. It easily follows that $\Gamma_0 \subset \Gamma \subset \Gamma_{0,1}$. When $\xi' \in \mathbb{R}^{n-1} - i\gamma_0 \theta' - i\Gamma_0$, we can denote the roots of $P(\xi', \lambda) = 0$ with respect to $\lambda$ by $\lambda_1(\xi'), \ldots, \lambda_m(\xi'), \ldots, \lambda_{m'}(\xi')$, which are enumerated so that $\text{Im} \lambda(\xi') \geq 0$. We consider the mixed initial-boundary value problem for the hyperbolic operator $P(D)$ in a quarter-space

$$P(D) u(x) = f(x), \quad x \in \mathbb{R}_+^n, \quad x_i > 0,$$

$$D^i x u(x)|_{x_{i=0}} = 0, \quad 0 \leq i \leq m - 1, \quad x_n > 0,$$

$$B_j(D) u(x)|_{x_n=0} = 0, \quad 1 \leq j \leq l, \quad x_i > 0.$$  

Here the $B_j(D)$ are boundary operators with constant coefficients. Put

$$P_+(\xi', \lambda) = \prod_{j=1}^{l}(\lambda - \lambda_j(\xi')) \in \mathbb{R}^{n-1} - i\gamma_0 \theta' - i\Gamma_0.$$  

Then Lopatinski's determinant for the system $\{P, B_j\}$ is defined by

$$R(\xi') = \text{det} L(\xi') \text{ for } \xi' \in \mathbb{R}^{n-1} - i\gamma_0 \theta' - i\Gamma_0,$$

where

$$L(\xi') = \left( \frac{1}{2\pi i} \oint B_j(\xi', \lambda) \lambda^{k-1} P_+(\xi', \lambda)^{-1} d\lambda \right)_{i,k=1,\ldots,l}.$$  

We impose the following assumption on $\{P, B_j\}$:

(A) The system $\{P, B_j\}$ is $\mathcal{E}$-well posed, i.e.

$$R^0(-i\theta') \neq 0, \quad R(\xi' + s\theta') \neq 0 \text{ when } \xi' \text{ is real and } \text{Im} s < -\gamma_1,$$
where $R^0(\xi')$ denotes the principal part of $R(\xi')$ and $\gamma_1 > \gamma_0$ (see [3]).

Now we can construct the fundamental solution $G(x, y)$ for $\{P, B_j\}$ which describes the propagation of waves produced by unit impulse given at position $y = (0, y_2, \cdots, y_n)$ in $R^n$. Write

$$G(x, y) = E(x - y) - F(x, y),$$

where $E(x)$ is the fundamental solution of the Cauchy problem represented by

$$E(x) = (2\pi)^{-n} \int_{R^n - \Gamma} \exp [ix \cdot \xi] P(\xi)^{-1} d\xi, \quad \eta \in \gamma \theta + \Gamma.'$$

Then $F(x, y)$ is written in the form

$$F(x, y) = (2\pi)^{-(n+1)} \int_{R^{n+1} - \Gamma} i^{-1} \sum_{j=1}^{l} \exp [i \{x' - y\} \cdot \xi']$$

$$+ x_n \xi_n - y_n \xi_{n+1}] R_{jk}(\xi') B_k(\xi', \xi_{n+1})$$

$$\times \xi_n^{-1} (R(\xi') P_+(\xi) P(\xi', \xi_{n+1}))^{-1} d\xi,$$

where $\gamma > \gamma_1, \gamma = (\theta, 0) \in R^{n+1}$ and $R_{jk}(\xi') = (k, j)$-cofactor of $L(\xi')$ (see [3], [4], [6]). $F(x, y)$ has to be interpreted in the sense of distribution with respect to $(x, y)$ in $R^n \times R^n$. We put

$$\tilde{F}(\tilde{z}) = F(x', z_n, 0, -z_{n+1}), \quad \tilde{z} = (z, z_{n+1}) \in X = R^{n+1} \times R^n \times R^1,$$

where $R^1 = \{\lambda \in R; \lambda < 0\}$, and regard $\tilde{F}(\tilde{z})$ as a distribution on $X$. We note that $\tilde{F}(\tilde{z})$ can be regarded as a distribution on $R^{n-1}$ and that $\text{supp} \tilde{F} \subset \{\tilde{z} \in R^{n+1}; z_n > 0\}$. In order to investigate the wave front set $WF(G)$ of $G(x, y)$ it suffices to study $WF(\tilde{F})$. Our main result is stated as follows:

**Theorem 1.1.** Assume that the condition $(A)$ is satisfied and that $\tilde{z}^0 \in R^{n+1}$. Then we have

$$t^{N/L} \{t^p \exp [-it\tilde{z} \cdot \tilde{\xi}^0] \tilde{F}(\tilde{z}) - \sum_{j=0}^{N} \tilde{F}_{\tilde{z}, j}(\tilde{z}) t^{-j/L}\} \to 0$$

as $t \to \infty$, in $\mathcal{D}'(X)$, $N = 0, 1, 2, \cdots$,

where $p_0$ is a rational number and $L$ is a positive integer. Moreover we have
\[
\bigcup_{\xi \in \mathbb{R}^{n+1} \setminus \{0\}} \bigcup_{j=0}^{\infty} \text{supp} \tilde{F}_{\xi,j}(x) \times \{\xi\} \subset WF(\tilde{F}(x)) \cap WF_{A}(\tilde{F}(x)) \cap \bigcup_{\xi \in \mathbb{R}^{n+1} \setminus \{0\}} K_{\xi}^{0} \times \{\xi\},
\]
(1.1)
\[
\text{ch}^{+}[\bigcup_{j=0}^{\infty} \text{supp} \tilde{F}_{\xi,j}(x)] \subset K_{\xi}^{0},
\]
(1.2)
where
\[
K_{\xi} = \{x \in X; \bar{z} \cdot \tilde{y} \geq 0 \text{ for all } \tilde{y} \in \Gamma_{\xi}\},
\]
\[
K_{\xi}^{0} = \{x \in X; \bar{z} \cdot \tilde{y} \geq 0 \text{ for all } \tilde{y} \in \Gamma_{\xi}^{0}\}
\]
and \(\Gamma_{\xi}\) and \(\Gamma_{\xi}^{0}\) are defined by (3.3) and (3.4), respectively.

**Remark.** The inclusion of (1.1) can be replaced by the equality except in certain exceptional cases (see Example 5.1 in [8]).

The remainder of this paper is organized as follows. In Section 2 we shall study some properties of symmetric functions of \(\lambda_{\xi}(\xi'), \cdots, \lambda_{\xi}(\xi')\). In Section 3 Theorem 1.1 will be proved. In Section 4 we shall give some remarks and examples.

\section{2. Algebraic Considerations}

In this section we assume without loss of generality that \(P(\xi)\) is irreducible. Let \(\xi''\) be fixed in \(\mathbb{R}^{n-1}\) and \(m_{\xi''}\), the multiplicity of \(\xi''\) relative to \(P_{m''}(\xi'')\). Let \(\xi''_{a} \in \mathbb{R}\) and write
\[
\nu^{n}P(\nu^{-1}\xi'' + \eta) = \sum_{j=m_{\xi''}}^{\infty} \nu^{j}Q_{\nu,j}(\eta), \quad Q_{\nu,m_{\xi''}}(\eta) \neq 0.
\]
It is easy to see that \(Q_{\nu,m_{\xi''}}(\eta) = P_{\nu}(\eta)\),
\[
Q_{\nu,j}(\eta) = \sum_{|a|+|\xi''| = \nu} \frac{1}{\nu!} \partial^{a}\partial^{\xi''}_{\alpha} P^{a}(\xi'') \cdot \eta^{a},
\]
where \(P(\xi) = P_{0}(\xi) + P_{1}(\xi) + \cdots + P_{m}(\xi)\) and \(P_{k}(\xi)\) is a homogeneous\(\dagger\)

\(\dagger\) \text{ch}[M] denotes the closed convex hull of \(M\) in \(X\).
polynomial of degree $m-k$. We can write

$$Q_{\eta, \xi} (\eta) = \sum_{k=0}^{r_j} q_{\eta, \xi_k} (\eta) \eta_k^k, \quad q_{\eta, \xi} (\eta) \neq 0 \quad \text{if} \quad Q_{\eta, \xi} (\eta) \neq 0,$$

where $r_j = r_j (\xi^0)$ depends on $\xi^0$. It follows that $r_{m_1 + m_2} = m'$ and $r_j < m'$ if $j < m' + m_1'$. We put

$$j_i = j_i (\xi^0) = m' + m_1' + i,$$

$$l_k = l_k (\xi^0) = \min \{ (r_{j_k} - r_j) / (j_k - j) ; m_2 \leq j < j_k \},$$

$$j_{k+1} = j_{k+1} (\xi^0) = \min \{ j ; m_2 \leq j < j_{k+1} \} \quad \text{and} \quad (r_{j_k} - r_j) / (j_k - j) = l_k,$$

and obtain the sequence $\{ j_k, l_k \}_{k=0, \ldots, s+1}$ so that

$$j_0 = m > j_1 = m' + m_1' > j_2 > \cdots > j_s > j_{s+1} = m_2,$$

$$l_0 = 0 < l_1 < l_2 < \cdots < l_s < l_{s+1} = \infty,$$

where $s = s (\xi^0)$ depends on $\xi^0$. For $\rho > 0$ we define the modified localization $P_{\rho, \xi_0} (\eta; \lambda)$ of $P$ at $\xi^0$ by

$$v^m P (v^{-1} \xi^0 + \eta', v^{-1} \xi^0 + \eta + \eta_n) = v^m \xi^0 (P_{\rho, \xi_0} (\eta; \lambda) + o(1))$$

as $\rho \downarrow 0$, $P_{\rho, \xi_0} (\eta; \lambda) \neq 0$ in $(\eta, \lambda)$.

Then we have

(2.1) \hspace{1cm} P_{\rho, \xi_0} (\eta; \lambda) = q_{\rho, \xi_{j_k}} (\eta') \chi^{j_k},

$$m_{\rho_2} (\rho) = j_k - r_{j_k} / \rho,$$

if $l_k > \rho > l_{k-1}$, $1 \leq k \leq s + 1$, and we have

(2.2) \hspace{1cm} P_{\rho, \xi_0} (\eta; \lambda) = [q_{\rho, \xi_{j_k}} (\eta') \chi^{j_k} - r_{j_{k+1}}]

$$\cdots + q_{\rho, \xi_{j_{k+1}} + r_{j_{k+1}}} (\eta') \chi^{j_{k+1}},$$

$$m_{\rho_2} (\rho) = j_k - r_{j_k} / \rho = j_{k+1} - r_{j_{k+1}} / \rho,$$

if $\rho = l_k$, $1 \leq k \leq s$. Moreover we have

$$P_{\rho, \xi_0} (\eta; \lambda) = P_{\xi_0} (\eta', \lambda + \eta_n), \quad m_\lambda (\rho) = m_{\xi_0},$$

if $\rho = l_{s+1} = \infty$. We note that $j_k (\xi^0)$ and $l_{k-1} (\xi^0)$ are independent of $\xi^0$ if $l_{k-1} < 1$. In fact, we have

$$P_{\rho, \xi_0} (\eta; \lambda) = P_{\rho, \xi_0} (\eta; \lambda) \quad \text{if} \quad l_{k-1} < \rho < \min (1, l_k).$$
Now we define the modified principal part $p_\phi^\theta(\eta; \lambda)$ and modified degree \( \deg_\phi \) for a polynomial \( p(\eta; \lambda) \) by

\[
p(t\eta; t^{\phi-t\eta}) = t^\theta (p_\phi^\theta(\eta; \lambda) + o(1)) \quad \text{as} \quad t \to \infty ,
\]

\[
p_\phi^\theta(\eta; \lambda) \not= 0 \quad \text{in} \quad (\eta, \lambda).
\]

**Lemma 2.1.** Let $\rho > 0$ and put $P_\rho^{\phi, \theta}(\eta; \lambda) = (P_\rho)^{\phi, \theta}(\eta; \lambda)$. Then we have

\[
P_\rho^{\phi, \theta}(\eta; \lambda) = (P_\rho)^{\phi, \theta}(\eta; \lambda), \quad \deg_\phi P_\rho^{\phi, \theta} = m_\theta(\rho).
\]

**Proof.**

\[
\nu_\rho^\phi (\nu^{-1} \xi_\rho^\phi + \eta', \nu^{-1} \xi_\rho^\phi + \nu^{-1} \lambda + \eta_n)
\]

\[
= \nu_\rho^\phi ((P_\rho)^{\phi, \theta}(\eta; \lambda) + Q(\eta, \lambda; \psi)),
\]

where $Q(\eta, \lambda; \psi)$ is a polynomial in $(\eta, \lambda)$, continuous in $(\eta, \lambda, \psi)$ and $Q(\eta, \lambda; 0) = 0$. Therefore we have

\[
\nu_\rho^\phi \partial^{[\psi]} / \partial \eta^\phi P_\rho (\nu^{-1} \xi_\rho^\phi + \eta', \nu^{-1} \xi_\rho^\phi + \nu^{-1} \lambda + \eta_n)
\]

\[
= \nu_\rho^\phi ((P_\rho)^{\phi, \theta}(\eta; \lambda) + \partial^{[\psi]} / \partial \eta^\phi Q(\eta, \lambda; \psi)).
\]

From this it follows that

\[
\nu_\rho^\phi \bar{p}(\eta, \lambda) = \sum_1 \partial^{[\psi]} / \partial \eta^\phi p(\eta; \lambda) ^\psi. \quad \text{Hyperbolicity of} \ P \ \text{implies that}
\]

\[
|P(\nu^{-1} \xi_\rho^\phi + \eta', \nu^{-1} \xi_\rho^\phi + \nu^{-1} \lambda + \eta_n) |
\]

\[
\leq \text{const.} \times \bar{p}^\phi (\nu^{-1} \xi_\rho^\phi + \eta', \nu^{-1} \xi_\rho^\phi + \nu^{-1} \lambda + \eta_n), \ \lambda \in \mathbf{R}, \ \eta \in \mathbf{R}^n
\]

(see [5]). Since there exists $(\eta^n, \lambda_0) \in \mathbf{R}^{n+1}$ such that $P_\rho^{\phi, \theta}(\eta^n; \lambda_0) \neq 0$, it follows that $\sigma_0 \leq m_\theta(\rho)$. Put

\[
\nu_\rho^\phi P_\rho (\nu^{-1} \xi_\rho^\phi + \eta', \nu^{-1} \xi_\rho^\phi + \nu^{-1} \lambda + \eta_n)
\]

\[
= \nu_\rho^\phi ((P_\rho)^{\phi, \theta}(\eta; \lambda) + o(1)) \quad \text{as} \ \nu \downarrow 0.
\]

Then we have $\deg_\phi (P_\rho)^{\phi, \theta} = \sigma_0 - \delta$ and $(P_\rho)^{\phi, \theta} = (P_\rho)^{\phi, \theta}$. Therefore it follows that $\sigma_0 = m_\theta(\rho)$. This proves the lemma. Q.E.D.
Lemma 2.2. Let $\rho > 0, \rho \neq 1$ and $\lambda_0 \in \mathbb{R} \setminus \{0\}$. Then $P_{p,\lambda}(\eta; \lambda_0)$ is a hyperbolic polynomial with respect to $\theta$. Moreover we have

$$P_{p,\lambda}(\eta; \lambda_0) \neq 0 \quad \text{for} \quad \eta \in \begin{cases} \mathbb{R}^n - i\Gamma \theta - i\Gamma(P_{\lambda_0,1}(\eta)) & \text{if} \quad 1 > \rho > 0, \\ \mathbb{R}^n - i\Gamma \theta - i\Gamma(P_{\lambda_0,1}(\eta)) & \text{if} \quad \infty > \rho > 1, \\ \mathbb{R}^n - i\Gamma \theta - i\Gamma & \text{if} \quad \rho = \lambda_{+1} = \infty. \end{cases}$$

In particular,

$$\Gamma(P_{p,\lambda}(\eta; \lambda_0)) \supset \begin{cases} \Gamma((P_{\lambda_0,1}(\eta)) & \text{if} \quad 1 > \rho > 0, \\ \Gamma_+ & \text{if} \quad \rho = \lambda_{+1} = \infty, \end{cases}$$

and

$$P_{p,\lambda}(\eta; \lambda_0) = \begin{cases} (P_{\lambda_0,1}(\eta))^{\lambda_0} & \text{if} \quad l_{1} \geq \rho > 0, \\ (P_{\lambda_0,1}(\eta)) \lambda_0^{l_{+1}} & \text{if} \quad \infty > \rho \geq l_{1}, \\ P_{\lambda_0}(\eta) & \text{if} \quad \rho = \lambda_{+1} = \infty, \end{cases}$$

where $(P_{p,\lambda})(\eta; \lambda_0)$ denotes the principal part of a polynomial $P_{p,\lambda}(\eta; \lambda_0)$ in $\eta$.

Remark. We note that $\Gamma \subset \Gamma((P_{\lambda_0,1}(\eta))$ and that $(P_{\lambda_0,1})(\eta)$ is independent of $\xi_\eta$.

Proof. Since $\rho \neq 1$, it follows that $P_{p,\lambda}(\eta; \lambda_0) \neq 0$ in $\eta$. In fact, from Lemma 2.1 we have

$$\deg q_{p,\eta_0,\eta_0}^{\lambda_0}(\eta') = j_k - j_{k-1}.$$ 

Thus

$$P_{p,\lambda}(\eta; \lambda_0) = \begin{cases} (q_{p,\eta_0,\eta_0})^{\lambda_0} & \text{if} \quad l_{1} \geq \rho \geq l_{k-1}, \\ \text{and} \quad 1 > \rho > 0, \\ (q_{p,\eta_0,\eta_0})^{\lambda_0} & \text{if} \quad l_{k} > \rho \geq l_{k-1}, \\ \text{and} \quad \rho > 1, \\ P_{\lambda_0}(\eta) & \text{if} \quad \rho = \lambda_{+1} = \infty. \end{cases}$$

Now let us assume that there exists $\eta \in \mathbb{R}^n - i\Gamma - i\Gamma$ such that $P_{p,\lambda}(\eta; \lambda_0) = 0$. Then there exist positive numbers $\varepsilon, \delta$ and $\xi_\eta \in C^\eta$ such that

$$|P_{p,\lambda}(\eta + \mu \xi_\eta; \lambda_0)| > \varepsilon > 0 \quad \text{for} \quad |\mu| = \delta > 0,$$

$$\eta + \mu \xi_\eta \in \mathbb{R}^n - i\Gamma - i\Gamma \quad \text{for} \quad |\mu| \leq \delta.$$
Therefore from Rouché's theorem it follows that there exists a positive number \( \nu_0 \) such that \( P(z^{1-\xi} + \xi' + \mu \xi + \nu^{-1} \eta_0 + \nu^{-1} \zeta_0 + \nu^{-1} \eta_0 + \nu^{-1} \zeta_0) \) has zeros within \( |z| < \delta \) if \( 0 < \nu \leq \nu_0 \), which is a contradiction to \( P(\xi) \neq 0 \) for \( \xi \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma \). So we have

\[
P_{p,\varepsilon} (\eta; \lambda_0) \neq 0 \quad \text{for} \quad \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma.
\]

This implies that \( P_{p,\varepsilon} (\eta; \lambda_0) \) is a hyperbolic polynomial with respect to \( \delta \) and that \( \Gamma (P_{p,\varepsilon} (\eta; \lambda_0)) \supset \Gamma \). Next let us prove (2.4). We note that (2.3) follows from (2.4) (see [1], [3]). One can easily verify (2.5).

Therefore (2.4) holds when \( \infty > \rho \geq l_k \) or \( l_k > \rho > 0 \). Let us prove (2.4) when \( 1 > \rho > 0 \). For we can prove (2.4) in the same manner when \( \rho > 1 \). Now assume that \( \Gamma (P_{p,\varepsilon} (\eta; \lambda_0)) \supset \Gamma ((P_{0,1})_\varepsilon) \) when \( 1 > l_k > \rho > 0 \). Then by (2.1) we have

\[
\Gamma (q_{\varepsilon, j_k+1} (\eta')) \supset \Gamma ((P_{0,1})_\varepsilon).
\]

Thus from (2.6) it follows that

\[
(2.7) \quad P_{\varepsilon, \varepsilon} (\eta; \lambda_0) \neq 0 \quad \text{for} \quad \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma ((P_{0,1})_\varepsilon).
\]

Assume that

\[
q_{\varepsilon, j_k+1} (\eta') = 0 \quad \text{for some} \quad \eta' \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma ((P_{0,1})_\varepsilon).
\]

From (2.2) we have

\[
l \rightarrow P_{\varepsilon, \varepsilon} (\eta; \lambda) \rightarrow q_{\varepsilon, j_k+1} (\eta') \lambda_{j_k+1} \neq 0
\]

(locally uniform), which leads us to a contradiction, using Rouché’s theorem. Therefore,

\[
(2.8) \quad P_{p,\varepsilon} (\eta; \lambda_0) = q_{\varepsilon, j_k+1} (\eta') \lambda_{j_k+1} \neq 0
\]

when \( l_{k+1} > \rho > l_k \) and \( \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma ((P_{0,1})_\varepsilon) \). From (2.7) and (2.8) it follows that

\[
\Gamma (P_{p,\varepsilon} (\eta; \lambda_0)) \supset \Gamma ((P_{0,1})_\varepsilon) \quad \text{when} \quad l_{k+1} > \rho > 0.
\]

Q.E.D.

We define \( q = q (\xi^\varepsilon) \) by

\[
(2.9) \quad \partial^k / \partial \xi^\varepsilon \partial_k P^\varepsilon (\xi^\varepsilon, \lambda) \equiv 0, \quad 0 \leq k \leq q - 1, \quad \partial^k / \partial \xi^\varepsilon \partial_k P^\varepsilon (\xi^\varepsilon, \lambda) \neq 0 \quad \text{in} \quad \lambda.
\]

Put
and define \( r = r(\xi^0) \) by

\[
(2.10) \quad \frac{\partial^{k+1}}{\partial \xi^0 \partial \xi^{k+1}} P(\xi) = 0, \quad 0 \leq k \leq r-1,
\]

\[
(2.11) \quad \frac{\partial^{r+1}}{\partial \xi^0 \partial \xi^{r+1}} P(\xi) \neq 0.
\]

Then we have the following

**Lemma 2.3.**

\[
(2.12) \quad q < m_{e'} \leq m - m', \quad p \leq \min(m', m - q),
\]

\[
q \leq m_{e'} \leq q + r \leq q + p \leq m' + m_{e'} \leq m,
\]

\[
p = m', \quad m_{e'} = m - m' \text{ if } q = m - m'.
\]

Moreover

\[
(2.13) \quad r_j \leq j - q \text{ for } m_{e'} \leq j \leq m,
\]

\[
(2.14) \quad r_j < j - q \text{ if } m_{e'} \leq j < q + r \text{ or } q + p < j \leq m,
\]

\[
(2.15) \quad r_j < m' \text{ if } m_{e'} \leq j < m' + m_{e'},
\]

\[
(2.16) \quad r_{q+r} = r, \quad r_{q+p} = p \text{ and } r_{m' + m_{e'}} = m'.
\]

**Remark.** This lemma yields us the following Newton polygon (Fig. 1).

\[
\begin{align*}
\alpha \quad \frac{\partial}{\partial \xi^0} P(\xi) + \lambda = 0 & \quad \text{in } \lambda. \\
\frac{\partial^{i+1}}{\partial \xi^0 \partial \xi^{i+1}} P(\xi) & = 0 \quad \text{in } \lambda.
\end{align*}
\]

**Proof.** If \(|\alpha| + k < q,\)

\[
(2.17) \quad \frac{\partial^{i+1}}{\partial \xi^0 \partial \xi^{i+1}} P(\xi) = 0 \quad \text{in } \lambda.
\]

In fact, for each \( \lambda \in \mathbb{R} \)
\[ v^m P(\nu^{-1} \xi^\alpha + \eta', \nu^{-1} \lambda_0 + \eta_n) \]
\[ = \sum_{f=0}^{m} v^f \sum_{|\alpha|+k\leq q} \frac{1}{\alpha!} \partial^{[\alpha]} / \partial \xi^a P^k(\xi^\alpha, \lambda_0) \eta^a. \]

If \( \partial^{[\alpha]} / \partial \xi^a P^k(\xi^\alpha, \lambda_0) \neq 0 \) for some \( \alpha \) and \( k \) with \( |\alpha| + k < q \), hyperbolicity of \( P \) implies that there exists a non-negative integer \( h \) such that \( h \leq |\alpha| + k < q \) and \( \partial^h / \partial \xi^a P^k(\xi^\alpha, \lambda_0) = h! P^k_{\{\theta \}}(\theta) \neq 0 \), which is a contradiction to (2.9). (2.13) easily follows from (2.17). (2.12), (2.15) and (2.16) are obvious. Now assume that

\[ p' = \max \{ \deg \partial^{[\alpha]} / \partial \xi^a P^k(\xi^\alpha, \lambda) ; |\alpha'| + k = q \} > p. \]

Then we have

\[ (2.18) \quad P^e_{\alpha, (\eta', 0)}(\theta; \lambda) = 0 \quad \text{for} \quad 1 > \rho > (m' - p) / (m' + 1 - p), \]

which is a contradiction to hyperbolicity of \( P^e_{\alpha, (\eta', 0)}(\eta; \lambda_0), \lambda_0 \in R \setminus \{0\} \). In fact, we have \( r_\rho + p = p' \) and \( r_\rho < j - q \) for \( q + p' < j \leq m \). Therefore, \( j - r_\rho / \rho > q + p' - p' / \rho \) when \( 1 > \rho > (m' - p') / (m' + 1 - p') \) and \( j \neq q + p' \). For it is obvious that \( j - r_\rho / \rho \geq j / (1 - 1 / \rho) + q / \rho > q + p' - p' / \rho \) if \( j < q + p' \). If \( j > q + p' \), then

\[ j - r_\rho / \rho = j - r_\rho + (1 - 1 / \rho) r_\rho \geq q + 1 + (1 - 1 / \rho) m' > q + p' - p' / \rho. \]

Thus we have \( P^e_{\alpha, (\eta', 0)}(\eta; \lambda) = a^e_{(\eta', 0), q + p' / \rho} (\eta') \lambda^p \). Since \( a^e_{(\eta', 0), q + p' / \rho} (\eta') = (q' ! p')^{-1} \partial^{p'+q'} / \partial \xi^a P^k(\xi, 0) \) we obtain (2.18). Therefore we have \( p = \max \{ \deg \partial^{[\alpha]} / \partial \xi^a P^k(\xi^\alpha, \lambda) ; |\alpha'| + k = q \}. \)

This implies that \( r_\rho < j - q \) if \( q + p < j \leq m \). Next let us prove that

\[ (2.19) \quad \partial^{[\alpha'+1]} / \partial \xi^a \partial \xi^a P^k(\xi^\alpha) = 0 \quad \text{for} \quad |\alpha'| + k = q \quad \text{and} \quad 0 \leq h \leq r - 1. \]

Assume that

\[ r' = \min \{ h ; \partial^{[\alpha'+1]} / \partial \xi^a \partial \xi^a P^k(\xi^\alpha) \neq 0 \quad \text{for some} \quad \alpha' \quad \text{and} \quad k \quad \text{with} \quad |\alpha'| + k = q \} < r. \]

Then similarly we have

\[ P^e_{\alpha', \nu} (\theta; \lambda) = 0 \quad \text{for} \quad (q + r' - m_{\nu} + 1) / (q + r' - m_{\nu}) > \rho > 1, \]

which is a contradiction to hyperbolicity of \( P^e_{\alpha', \nu} (\eta; \lambda_0), \lambda_0 \in R \setminus \{0\} \). From (2.19) it follows that \( r_\rho < j - q \) if \( m_{\nu} \leq j < q + r'. \)

Q.E.D.
From Lemma 2.3 it follows that there exist positive integers \( t = t(\xi^0) \) and \( t' = t'(\xi^0) \) such that \( 1 \leq t \leq t' \leq s + 1 \), \( j_t = q + p \) and \( j_t' = q + r \). If \( r < p \), then \( t' = t + 1 \) and \( l_t = 1 \). If \( r = p \), then \( t = t' \), \( l_t > 1 \) and \( l_{t-1} = 1 \). Thus \( j_t(\xi^0) \) and \( l_{t-1}(\xi^0) \), \( 0 \leq k \leq t(\xi^0) \), are independent of \( \xi^0 \). Put
\[
\begin{align*}
P_{k,\eta}(\xi'; \lambda) &= P_{k,\eta}(\xi'; \lambda) \lambda^{r_{j_k-1}}, \\
P_{k,\eta}(\xi'; \lambda) &= q_{t', j'_{t'-1}}(\xi') \lambda^{r_{j_{t'-1}} - j_{t'-1}} + \cdots + q_{t, j_{t-1}} r_{j_{t-1}}(\xi').
\end{align*}
\]
By Lemma 2.2 we obtain the following

**Lemma 2.4.** For \( 1 \leq k < t \) \( P_{k,\eta}(\xi'; \lambda) \) has no real zeros when \( \xi' \in \mathbb{R}^{n-1} - i\gamma_0 \delta' - i\bar{\Gamma}'(\eta_{(0,1)}) \). For \( t' \leq k \leq s \) \( P_{k,\eta}(\xi'; \lambda) \) has no real zeros when \( \xi' \in \mathbb{R}^{n-1} - i\gamma_0 \delta' - i\bar{\Gamma}'(\eta_{(0,1)}) \).

Denote the roots of \( P_{k,\eta}(\xi'; \lambda) = 0 \) by \( \lambda_{k, i}(\xi') \), \( \cdots \), \( \lambda_{k, t'}(\xi') \), \( \lambda_{k, i+1}(\xi') \), \( \cdots \), \( \lambda_{k, t}(\xi') \) so that the \( \lambda_{k, i}(\xi') \) are continuous and that
\[
\text{Im} \lambda_{k, i}(\xi' - i\gamma \delta') \geq 0 \quad \text{for} \quad \gamma > \gamma_0 \quad \text{and} \quad \xi' \in \mathbb{R}^{n-1},
\]
when \( r_{m+1} \neq 0 \). Then we easily obtain the following

**Lemma 2.5.** Assume that \( r_{m+1} \neq 0 \). Then
\[
\{ \lambda_{k, i}(\xi') \} \leq i \cup \{ \lambda_{k, i}(\xi') \} \leq i \neq r_{m+1} - i \bar{\Gamma}' \nu.
\]

Put
\[
P_{\eta, \xi}(\xi'; \lambda) = \sum_{|\alpha| + |\beta| = q} \frac{1}{\alpha!} \partial_{\xi^0}^{|\alpha|} P^\xi(\xi'; \lambda) \eta^\alpha.
\]
We note that \( P_{\eta, \xi}(\xi'; \lambda) \) is independent of \( \eta_m \). From the proof of Lemma 2.3 it follows that \( \deg P_{\eta, \xi}(\xi'; \lambda) \leq p \) in \( \lambda \) for fixed \( \eta \). The coefficient of \( \lambda^p \) in \( P_{\eta, \xi}(\xi'; \lambda) \) is equal to \( q_{t, q+p}(\xi') \), where \( \xi^0 \in \mathbb{R} \). Since \( q + p = j_t \), \( p = r_{j_t} \) and \( l_{t-1} < 1 \), it follows from (2.1) and Lemma 2.2 that
\[
q_{t, q+p}(\xi') \neq 0 \quad \text{for} \quad \xi' \in \mathbb{R}^{n-1} - i\gamma_0 - i\bar{\Gamma}'(\eta_{(0,1)}) \nu.
\]
Therefore we have
deg \( P_{\nu,\eta}(\eta; \lambda) = \rho \) in \( \lambda \) for fixed \( \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma((P_{0,1})_{\nu,0}). \)

**Lemma 2.6.** Let \( \xi_n^0 \in \mathbb{R} \) and \( \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma((P_{0})_{0,1}). \) Then \( \lambda = \xi_n^0 \) is a root of \( \partial^q/\partial \xi^q P^0(\xi^0, \lambda) = 0 \) with multiplicity \( r \) if and only if \( \lambda = \xi_n^0 \) is a root of \( P_{\nu,\eta}(\eta; \lambda) = 0 \) with multiplicity \( r. \)

**Proof.** Now assume that \( P_{\nu,\eta}(\eta; \xi_n^0) = 0 \) for some \( \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma((P_{0})_{0,1}). \) Then we have \( \partial^q/\partial \xi^q P^0(\xi^0) = 0. \) In fact, if \( \partial^q/\partial \xi^q P^0(\xi^0) \neq 0, \) we have \( (P_{0})_{0,1}(\xi^0) = P_{\nu,\eta}(\xi^0; \xi_n^0) = 0 \) for \( \xi \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma((P_{0})_{0,1}), \) which is a contradiction to \( P_{\nu,\eta}(\eta; \xi_n^0) = 0. \) Next assume that there exists a non-negative integer \( k \) such that \( k \leq r - 1 \) and \( \partial^q/\partial \lambda^k P_{\nu,\eta}(\eta; \xi_n^0) \neq 0 \) in \( \eta. \) Then we have \( r_{q+k} = k, \) which is a contradiction to (2.14).

For \( l_{r} > \rho > 1 \) we have

\[
P_{\nu,\eta}(\eta; \lambda) = \frac{1}{r!} \lambda^r \partial^r/\partial \lambda^r P_{\nu,\eta}(\eta; \xi_n^0).
\]

From Lemma 2.2 and this it follows that

\[\partial^q/\partial \lambda^r P_{\nu,\eta}(\eta; \xi_n^0) \neq 0 \quad \text{for} \quad \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma((P_{0})_{0,1}).\]

This proves the lemma. Q.E.D.

Lemma 2.6 yields the following

**Lemma 2.7.** Let \( \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma((P_{0})_{0,1}) \cap \Gamma((P_{0})_{0,1}) \cap \Gamma((P_{0})_{\nu,0}) \). The real zeros of \( \partial^q/\partial \xi^q P^0(\xi^0, \lambda) \) agree with those of \( P_{\nu,\eta}(\eta; \lambda) \) (including multiplicities). Moreover the number of the roots with positive imaginary part of \( P_{\nu,\eta}(\eta; \lambda) = 0 \) is equal to that of the roots with positive imaginary part of \( \partial^q/\partial \xi^q P^0(\xi^0, \lambda) = 0. \)

**Remark.** The non-real zeros of \( \partial^q/\partial \xi^q P^0(\xi^0, \lambda) \) do not always agree with those of \( P_{\nu,\eta}(\eta; \lambda) \). In fact, for \( P(\xi) = P^0(\xi) = \xi_1^2 - 2\xi_1(\xi_2^2 + \xi_3^2) + (\xi_2^2 + \xi_3^2 + \xi_4^2/2) \xi_4^2 \) we have

\[r = r(\xi^0) \text{ is defined by (2.10) and (2.11). Lemma 2.6 implies that } \lambda = \xi_n^0 \text{ is a root of } P_{\nu,\eta}(\eta; \lambda) = 0 \text{ with multiplicity } r \text{ and that } \partial^q/\partial \xi^q P^0(\xi^0) = 0 \text{ if } P_{\nu,\eta}(\eta; \xi_n^0) = 0 \text{ for some } \eta \in \mathbb{R}^n - i\gamma_0 \delta - i\Gamma((P_{0})_{0,1}).\]
\[ \frac{\partial^j}{\partial \xi^j} P^a(0, 0, 1, \lambda) = -4(1 + \lambda^2), \]
\[ P_{a,0,1,1}(\eta; \lambda) = -2(1 + \lambda^2) \eta_1^3 + (1 + \lambda^2/2) \eta_1^2. \]
Moreover if \( \frac{\partial^j}{\partial \xi^j} P^a(\xi^0) \neq 0 \), then \( P_{a}(\eta) = (P_{a,0,1}) (\eta) \) and \( P_{a,0,1}^\alpha (-i\eta'; \xi^0) = P_{a}^\alpha (-i\eta') = 0 \) for \( \eta' \in \partial \Gamma(P_{a,0}) \)

Put
\[ \sigma^k(\xi') = \sum_{j=1}^l \lambda_j^k (\xi')^k, \quad 1 \leq k \leq l', \]
\[ \hat{\Gamma}_{v} = \bigcap_{\xi^0 \in \mathbb{R}} \hat{\Gamma}_{v} \cap \hat{\Gamma}((P_{a,0,1})_{v',0}). \]

**Lemma 2.8.** Let \( 1 \leq k \leq l \). For any compact set \( K \) in \( \mathbb{R}^{n-1} - i\overline{\tau}_0 \theta' - i\hat{\Gamma}' \), there exists \( \nu_K > 0 \) such that \( \sigma^k(\nu^{-1}\xi^0 + \eta') \) is well-defined for \( \eta' \in K \) and \( 0 < \nu \leq \nu_K \) and
\[ \nu^m \sigma^k(\nu^{-1}\xi^0 + \eta') = \sum_{j=0}^m \sigma_{v,0}^k (\eta') \nu^j L, \quad \sigma_{v,0}^k (\eta') \neq 0, \]
whose convergence is uniform in \( K \times \{ \nu; 0 \leq \nu \leq \nu_K \} \), where \( s_k \) is a rational number and \( L \) is a positive integer. Moreover the \( \sigma_{v,0}^k \) are holomorphic in \( \mathbb{R}^{n-1} - i\overline{\tau}_0 \theta' - i\hat{\Gamma}' \).

**Proof.** We can assume without loss of generality that \( K \) is small so that
\[ \{ \lambda_{v,0}(\eta'); 1 \leq j \leq \ell \} \quad \text{and} \quad 1 \leq j \leq \ell' \]
\[ \text{and} \quad \eta' \in K \} = \emptyset \quad \text{if} \quad \xi^0 \in \mathbb{R} \quad \text{and} \quad \tau_{m+1} \neq 0 \]
(see Lemma 2.5). Let \( \xi^0 \in \mathbb{R} \) and \( \hat{\Gamma}_{v,1} (1 \leq j \leq \ell (\xi^0), \ell' (\xi^0) \leq j \leq s (\xi^0)) \) be simple closed curves enclosing only the roots with positive imaginary part of \( P_{v,0}(\eta'; \lambda) = 0 \) for \( \eta' \in K \) (see Lemma 2.4). Let \( \hat{\Gamma}_{v,0} \) be a simple closed curve enclosing only the roots \( \lambda_{v,0}(\eta'), 1 \leq j \leq \ell', \) of \( P_{v}(\eta', \lambda) = 0 \) for \( \eta' \in K \) if \( \tau_{m+1} \neq 0 \) and \( \hat{\Gamma}' \) a simple closed curve enclosing only the roots with positive imaginary part of \( P_{v,0}(\eta; \lambda) = 0 \) for \( \eta' \in K \) (see Lemma 2.7). From the relations between the roots of \( P(\nu^{-1}\xi^0 + \eta', \lambda) = 0 \) and

---

\[ \delta M \] denotes the boundary of \( M \).

\[ \lambda_j(\xi^0) \] are continuous and \( \text{Im } \lambda_j(\xi^0 - i\tau \theta') > 0 \) for \( \theta' \in \mathbb{R}^{n-1} \) and \( \tau > \tau' \).
the roots of $P_{f,v'}(y'; \lambda) = 0$, $P_{v}(y', \lambda) = 0$ and $P^{v-\epsilon}_{v}(y'; \lambda) = 0$ there exists $\nu'_{K}(>0)$ such that \( \{ \lambda \gamma (y^{-1} \xi^{v} + y') \}_{1 \leq \lambda \leq \nu'} \cap \{ \lambda \gamma (y^{-1} \xi^{v} + y') \}_{1 \leq \lambda \leq \nu'} = \emptyset \) for $y' \in K$, $0 < \nu' \leq \nu'_{K}$. So we can take $\gamma_{v'}$ to be a simple closed curve enclosing only the roots $\gamma (y^{-1} \xi^{v} + y')$, $1 \leq \lambda \leq \nu'_{K}$, of $P(y^{-1} \xi^{v} + y', \lambda) = 0$ for $y' \in K$, $0 < \nu' \leq \nu'_{K}$. For $y' \in K$ and $0 < \nu' \leq \nu'_{K}$ we have

\[
(2.20) \quad \sigma^{v}(y^{-1} \xi^{v} + y') = (2\pi i)^{-1} \int_{\gamma_{v'}} \lambda^{k} \theta / \partial \bar{\xi} \cdot P(y^{-1} \xi^{v} + y', \lambda) \times P^{v-\epsilon}_{v}(y^{-1} \xi^{v} + y', \lambda)^{-1} d\lambda
\]

\[
= \sum_{k=0}^{n-1} \frac{(2\pi i)^{-1}}{k!} \int_{\gamma_{v'}} \lambda^{k} \theta / \partial \bar{\xi} \cdot P(y^{-1} \xi^{v} + y', \lambda) \times (P_{y^{1},(v^{1})}(y', \lambda) + o(1))^{-1} \mu_{m-1/2^{l}-m_{v}(\hat{v})}^{1/2^{l}} d\lambda
\]

\[
+ (2\pi i)^{-1} \int_{\gamma_{v'}} \lambda^{k} \theta / \partial \bar{\xi} \cdot P(y^{-1} \xi^{v} + y', \lambda) \times (P_{v}(y', \lambda) + o(1))^{-1} \mu_{m-1/2^{l}-m_{v}(\hat{v})}^{1/2^{l}} d\lambda
\]

\[
+ \sum_{\nu'_{K} \in R, \theta(\nu')^{k} \nu^{v} \theta^{v} \theta = 0} \left[ \sum_{j=1}^{r} (2\pi i)^{-1}ight]
\]

\[
\times \int_{\gamma_{v'}} (y^{-1} \xi^{v} + \mu_{1/2^{l}}) \mu_{m-1/2^{l}-m_{v}(\hat{v})}^{1/2^{l}} d\lambda
\]

\[
\times (P_{y^{1},(v^{1})}(y', \lambda) + o(1))^{-1} \mu_{m-1/2^{l}-m_{v}(\hat{v})}^{1/2^{l}} d\lambda
\]

\[
+ (1 - \delta_{m_{v^{1}}}^{v^{1}}) \times (2\pi i)^{-1} \int_{\gamma_{v'}} (y^{-1} \xi^{v} + \lambda) \times (P_{v}(y', \lambda) + o(1))^{-1} \mu_{m-1/2^{l}-m_{v}(\hat{v})}^{1/2^{l}} d\lambda
\]

\[
\times \mu_{m-1/2^{l}-m_{v}(\hat{v})}^{1/2^{l}} d\lambda,
\]

where each $o(1)$ is a polynomial of $y'$, $\lambda$ and $\nu^{v}L$ and vanishes for $\nu = 0$ and $L$ is a positive integer. So there exists $\nu_{K}(>0)$ such that each integrand in (2.20) can be expanded in a power series of $\nu^{v}L$, which converges uniformly in $y' \in K$ and $0 < \nu' \leq \nu'_{K}$. From this the lemma easily follows.

Q.E.D.

**Lemma 2.9.** Let $1 \leq k \leq l$. For any compact set $K$ in $R^{n-1} - i\hat{v}^{1}$, there exist $\nu_{K}$ and $r_{K}(>0)$ such that $\sigma^{v}(y^{-1} \xi^{v} + r_{v})$ is well-defined
when \( r_\alpha \gamma' \in \mathbb{R}^{n-1} - i\gamma_0 \partial - i\Gamma_{\gamma''}, \alpha \gamma' \in K \) for some \( \alpha \in C \) \( (|\alpha| = 1), 0 < \nu \leq \nu_0 \) and \( r \geq r_0 \). We have

\[
(\nu r^{-1})^k \sigma^k r_\nu^{\mathbb{Z}_0} + r \gamma' = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} r^{q_k l} \sigma^{l_{j,j}}, (\gamma') \nu^{l_{j,j}} r^{-l},
\]

\[
\sigma^{l_{j,j}}, (\gamma') \neq 0 \quad \text{if} \quad \sigma^{l_{j,j}}, (\gamma') \neq 0,
\]

whose convergence is uniform in \( \{(\gamma', \nu, r) ; r_\gamma \gamma' \in \mathbb{R}^{n-1} - i\gamma_0 \partial - i\Gamma_{\gamma''}, \alpha \gamma' \in K \} \) for some \( \alpha \in C \) \( (|\alpha| = 1), 0 < \nu \leq \nu_0 \) and \( r \geq r_0 \), where the \( q_{kl} \) are rational numbers. Moreover the \( \sigma^{l_{j,j}} (\gamma') \) are holomorphic in \( \mathbb{R}^{n-1} - i\Gamma_{\gamma''} \) and homogeneous and

\[
\sigma^{l_{j,j}} (r \gamma') = r^{q_k l_{j,j}} \sum_{l=0}^{\infty} \sigma^{l_{j,j}}, (\gamma') r^{-l},
\]

whose convergence is uniform in \( \{(\gamma', r) ; r_\gamma \gamma' \in \mathbb{R}^{n-1} - i\gamma_0 \partial - i\Gamma_{\gamma''}, \alpha \gamma' \in K \} \) for some \( \alpha \in C \) \( (|\alpha| = 1) \) and \( r \geq r_0 \).

**Proof.** Modifying the curves \( \mathcal{C}^+, \mathcal{C}_{(\gamma', 0), j}, \mathcal{C}_{\pi}^{\nu}, \mathcal{C}_{\pi, j}^{\nu} \) and \( \mathcal{C}_{\pi, j}^{\nu} \) in the proof of Lemma 2.8, we have

\[
(2.21) \quad \sigma^k (r^{-1} r_\nu^{\mathbb{Z}_0} + r \gamma') = (2\pi i)^{-1} \int_{\mathcal{C}_{\pi}^{\nu}} \lambda^k \partial / \partial \xi_n P (r^{-1} r_\nu^{\mathbb{Z}_0} + r \gamma', \lambda) \times P (r^{-1} r_\nu^{\mathbb{Z}_0} + r \gamma', \lambda)^{-1} d\lambda
\]

\[
= \left[ \sum_{j=1}^{l_{j,j}} (2\pi i)^{-1} \right] \int_{\mathcal{C}_{(\gamma', 0), j}} \lambda^k \partial / \partial \xi_n P (r^{-1} r_\nu^{\mathbb{Z}_0} + r \gamma', \nu^{1/2} r \lambda) \times (P_{\pi, j}^{\nu, \nu} (\gamma', 0; \lambda) + o(1))^{-1} \nu^{m-(k+1)/j - m(\xi_n, \nu, \gamma)} d\lambda
\]

\[
+ (2\pi i)^{-1} \int_{\mathcal{C}_{\pi}^{\nu}} \lambda^k \partial / \partial \xi_n P (r^{-1} r_\nu^{\mathbb{Z}_0} + r \gamma', \nu^{1/2} r \lambda) \times (P_{\pi, j}^{\nu, \nu} (\gamma', 0; \lambda) + o(1))^{-1} \nu^{m-k-1} d\lambda
\]

\[
+ \sum_{l \in \mathbb{Z}_0} \left\{ \sum_{j=1}^{l_{j,j}} (2\pi i)^{-1} \right\} \int_{\mathcal{C}_{(\gamma', 0), j}} \lambda^{r^{-1} - 1/2} \xi_n^k \times \theta / \partial \xi_n P (r^{-1} r_\nu^{\mathbb{Z}_0} + r \gamma', \nu^{1/2} r \lambda) \times (P_{\pi, j}^{\nu, \nu} (\gamma', 0; \lambda) + o(1))^{-1} \nu^{m-1/2 - m(\xi_n, \nu, \gamma)} d\lambda
\]
\[ + (1 - \delta_{\nu r^+t}(\nu r^+t)) \times (2\pi i)^{-1} \int_{\mathbb{C}} \frac{1}{\lambda^k} \left( \nu^{-1} \xi^0 + \lambda \right)^k \]
\[ \times \frac{\partial}{\partial \xi^0} P(\nu^{-1} r^+ \xi^0, \nu^{-1} \xi^0 + r \lambda) \left( P^0_{\nu r^+t}(\nu^+ \lambda) + o(1) \right)^{-1} \]
\[ \times \nu^{m-m^+t} d\lambda \] \times r^{-m+k+1},

where each \( o(1) \) is a polynomial of \( \nu^+, \lambda^+ \) and \( r^{-1} \) and vanishes for \( \nu=0, r^{-1}=0 \). In fact, for example, we have
\[
(yr - 1) m P(\nu^{-1} r^+ \xi^0, \nu^{-1} \xi^0 + \nu^{-1/2} r \lambda) = \nu^{m+1/2} \sum_{n_k+m^+t(\nu, \lambda)} P_{\nu r^+t,n_k}(\nu^+ \lambda, 0, r \lambda),
\]
de \( \deg_{\nu r^+t,n_k}(\nu^+ \lambda, 0, \lambda) \leq n_k \).

Therefore we have
\[
(yr - 1) m P(\nu^{-1} r^+ \xi^0, \nu^{-1} \xi^0 + \nu^{-1/2} r \lambda) = \nu^{m+1/2} \sum_{n_k+m^+t(\nu, \lambda)} P_{\nu r^+t,n_k}(\nu^+ \lambda, 0, \lambda) + o(1) \quad \text{as} \quad \nu, r^{-1} \to 0.
\]

So there exist \( \nu_K \) and \( r_K > 0 \) such that each integrand in (2.21) can be expanded in a power series of \( \nu^+ \) and \( r^{-1} \), which converges uniformly in \( \{ \nu^+, \lambda^+ \}; \nu^+ \in K, \lambda^+ \in C, |\nu^+| \geq r_K \) and \( 0 \leq |\lambda^+| \leq \nu_K \}. \) We note that
\[
\sigma_{\nu^+, \lambda^+}(\nu^+ \lambda^+) = \nu^{\nu^+ \lambda^+} \sigma_{\nu^+, \lambda^+}(\nu^+ \lambda^+)
\]
when \( \nu^+, \lambda^+ \in \mathbb{R}^{n-1} - i\hat{I}_{\nu^+, \lambda^+} \), where \( 1^{\nu^+ \lambda^+} = 1 \). This completes the proof.
Q.E.D.

Let us consider \( \hat{I}_{\nu^+} \). Although \( \Gamma(P_{\nu^+, \lambda^+}) = \Gamma(P_{\nu^+, \lambda^+} \nu^+, \lambda^+) \) does not always hold, we can prove the inner semi-continuity of \( (\nu^+, \lambda^+) \).

**Lemma 2.10.** Let \( \xi^\nu = \mathbb{R}^{n-1} \) and assume that \( 0 < \rho < l_1(\xi^\nu, 0) \). Then for any compact set \( \bar{M} \) in \( \Gamma((P_{\nu^+, \lambda^+} \nu^+, \lambda^+)) \) there exist a neighborhood \( U \) of \( \xi^\nu \) and positive numbers \( r_0, t_0 \) such that
\[
P(r \xi^\nu - i\nu \eta^\nu - i\nu \eta^\nu, r \nu^+ \lambda - i\nu \eta^\nu) \neq 0
\]
when \( \gamma \in \bar{M}, \xi^\nu \in U, \lambda \in \mathbb{R}, |\lambda| \geq 1, r \geq r_0 \) and \( 0 < t \leq t_0 \).

**Proof.** Put
where \( 0 < \nu \leq \nu_0, \quad r \geq r_0, \quad \zeta' \in \mathbb{R}^{n-1}, \quad |\zeta'| \leq \varepsilon, \quad \text{Re } s \geq 0, \quad \text{Re } t \geq 0, \quad |s| \leq s_0, \quad |t| \leq t_0, \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 1 \quad \text{and } \eta \in \tilde{M}. \) Then we have

\[
(vr^{-1})^m f(v, r, \zeta', \lambda, s, t, \eta)
\]

\[
= (vr^{-1})^m \left( \nu \left( \zeta' - iv \eta' - i(s + \gamma_0) \theta' \right) \right) + O(1) \text{ as } v, r^{-1} \to 0,
\]

i.e. for any positive number \( \delta \) there exist \( r_0, \nu_0 \) \((>0)\) such that

\[
|vr^{-1} - \nu| < \delta, \quad |\zeta'| < \varepsilon, \quad |s| \leq s_0, \quad |\lambda| \geq 1, \quad |t| \leq t_0 \quad \text{and } \eta \in \tilde{M}.
\]

So we can apply the same argument as in Lemma 3.7 in [7] to \( f(v, r, \zeta', \lambda, s, t, \eta) \) and we obtain the lemma.

Q.E.D.

**Lemma 2.11.** Let \( \xi^{\nu} \in \mathbb{R}^{n-1} \) and \( M \) be a compact set in \( \dot{\Gamma}_{\xi^{\nu}} \). There exists a neighborhood \( U \) of \( \xi^{\nu} \) such that

\[
M \subset \dot{\Gamma}_{\xi^{\nu}} \quad \text{for } \xi^{\nu} \in U.
\]

**Remark.** From the proof of Lemma 2.11 it follows that

\[
\bigcup_{\xi^{(\nu)} \in M} \dot{\Gamma}(P_{\xi^{(\nu)}, \ell_0}) \supset \dot{\Gamma}(\ell_{0,11}) \cup \xi^{(\nu)}, 0).
\]
Proof. Assume that there exists a sequence \{\xi_j, \eta_j\}_{j=1,2,\ldots} such that
\[|\xi_j - \xi'| < 1/j, \quad \xi_j, \eta_j \in \mathbb{R}, \quad \eta_j \in M \text{ and } P_{\eta_j}(-i\eta_j, 0) = 0.\]
Then from the inner semi-continuity of \(\hat{I}'_\epsilon\) (or \(I'_\epsilon\)) it follows that \(|\xi_j'\| \to \infty\) as \(j \to \infty\). Let \(\tilde{M}\) be a compact set in \(\Gamma((P_{\alpha,0})(\omega', \omega))\) such that the interior of \(\tilde{M}\) includes \(M \times \{0\}\). Lemma 2.10 implies that there exist a neighborhood \(U\) of \(\xi'\) and \(\lambda_0, \epsilon_0 > 0\) such that
\[P^\epsilon(\xi' - i\eta', \lambda - it\eta_0) \neq 0\]
when \(\eta \in \tilde{M}, \quad \xi' \in U, \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq \lambda_0\) and \(0 < t \leq \epsilon_0\), which leads us to a contradiction, using Rouché's theorem. Q.E.D.

§ 3. Proof of Theorem 1.1

Let \(P(\xi)\) be written in the form
\[P(\xi) = \prod_{j=1}^{q} P_j(\xi),\]
where the \(P_j(\xi)\) are irreducible polynomials. We assume that \(\prod_{j=1}^{q} P_j(\xi', \lambda) = 0\) has roots \(\lambda_i(\xi'), \ldots, \lambda_q(\xi')\) when \(\xi' \in \mathbb{R}^{n-1} - i\gamma_0 \gamma' - i\Gamma_\omega\), i.e.
\[\prod_{j=1}^{q} P_j(\xi', \lambda) = 0\]
does not have roots with positive imaginary part when \(\xi' \in \mathbb{R}^{n-1} - i\gamma_0 \gamma' - i\Gamma_\omega\). Then put
\[\hat{I}'_\epsilon = \bigcap_{\epsilon' \in \tilde{M}} \hat{I}'(\{(P_j)_{\epsilon'} \cap \hat{I}'((P_j)_{\alpha,0})(\omega', \omega))\}.\]
We note that
\[(\{(P_j)_{\alpha,0})(\omega', \omega)) = \{(P_j)_{\alpha,0})(\omega', \epsilon_0, \gamma)\} \text{ for all } \xi_0 \in \mathbb{R}.\]
The following lemma is obvious.

Lemma 3.1. \[\int B_j(\xi', \lambda) \lambda^{-1} P_\pm(\xi', \lambda)^{-1} d\lambda\] is a polynomial of \(\xi'\) and \(\sigma^k(\xi')\), \(1 \leq k \leq l\), when \(P_\pm(\xi', \lambda)\) is well-defined.

From Lemma 2.8 we have the following

Lemma 3.2. Let \(\xi'' \in \mathbb{R}^{n-1}. \) For any compact set \(K\) in \(\mathbb{R}^{n-1} - i\gamma_0 \gamma' - i\Gamma_\omega\) there exists \(\nu_K(>0)\) such that \(R(\nu^{-1}\xi'' + \eta')\) is well-defined for \(\eta' \in K\) and \(0 < \nu \leq \nu_K\) and
\[ \psi^{jv} R^{(y^{-1} \xi^{(v)} + \eta')} = \sum_{j=0}^{\infty} \psi^{j/L} R^{jv}_{\psi'}, \]

\[ R^{jv,0}(\eta') \equiv R^{jv}_\psi(\eta') \equiv 0, \]

whose convergence is uniform in \((\eta', v) \in K \times \{0 \leq v \leq v_0\}\), where \(h_{\psi}\) is a rational number and \(L\) is a positive integer. Moreover the \(R^{jv,1}(\eta')\) are holomorphic in \(R^{n-1} - i\gamma_0 \delta' - i\hat{\Gamma} \psi'.\)

**Remark.** \(R^{jv,0}(\eta') = R^{jv}_\psi(\eta')\) is the localization of \(R(\xi')\) at \(\xi'.\) Moreover this lemma for \(\xi' = 0\) implies that \(R(\xi')\) is holomorphic in \(R^{n-1} - i\gamma_0 \delta' - i\hat{\Gamma} '\) (see [3]).

The following lemma is also obtained by Lemma 2.9.

**Lemma 3.3.** Let \(\xi' \in R^{n-1}.\) For any compact set \(K\) in \(R^{n-1} - i\hat{\Gamma} \psi',\) there exist \(v_K\) and \(r_K > 0\) such that \(R^{(v^{-1} \xi^{(v)} + r \eta')}\) is well-defined when \(r \xi \eta' \in R^{n-1} - i\gamma_0 \delta' - i\hat{\Gamma} \psi',\) \(\alpha \eta' \in K\) for some \(\alpha \in C\) \((|\alpha| = 1),\) \(0 < v \leq v_K\) and \(r \geq r_K.\) We have

\[ (\mu^{v^{-1}})^{h_{\psi}} R^{(v^{-1} \xi^{(v)} + r \eta')} = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} r^{h_j^{(v)}} - i\psi^{j/L} R^{jv,1}_{\psi'}(\eta'), \]

\[ R^{0,0}_{\psi'}(\eta') \equiv 0 \text{ if } R^{0,1}_{\psi'}(\eta') \equiv 0, \]

whose convergence is uniform in \((\eta', \nu, \nu) \in K \times \{0 \leq \nu \leq \nu_0\}\) with obvious modifications.

**Remark.** The principal part \((R_{\psi'})^0(\eta')\) of \(R_{\psi'}(\eta')\) is equal to \(R^{0,0}_{\psi'}(\eta').\) Moreover this lemma for \(\xi' = 0\) implies Lemma 3.2 in [3].

In the above two lemmas we can replace \(R(\xi')\) by \(R_{\psi}(\xi')\) or \(P(\xi', \lambda)\) with obvious modifications.
Lemma 3.4. Let $\xi^0 \in \mathbb{R}^n$. There exist the localizations $P_{z^\nu}(\hat{\xi})$ and $(P_{z^\nu})^\nu(\hat{\xi})$ of $P_z(\hat{\xi})$ and $P^0_z(\hat{\xi})$ at $\xi^0$, respectively, and

\begin{equation}
(3.1) \quad P^0_{z^\nu}(\hat{\xi}) = (P_{z^\nu})^\nu(\hat{\xi}) = (P_{z^\nu})^\nu(\hat{\xi}).
\end{equation}

Proof. The existence of $P_{z^\nu}(\hat{\xi})$, $(P_{z^\nu})^\nu(\hat{\xi})$ and $(P_{z^\nu})^\nu(\hat{\xi})$ follows from Lemmas 3.2 and 3.3 and the above remark. It easily follows that

\begin{equation*}
P_{z^\nu}(\hat{\xi}) = P_{z^\nu}(\hat{\xi}) P_{-z^\nu}(\hat{\xi}), \quad (P_{z^\nu})^\nu(\hat{\xi}) = (P^0_{z^\nu})(\hat{\xi}) = (P_{z^\nu})^\nu(\hat{\xi})(P^0_{z^\nu})(\hat{\xi})
\end{equation*}

and $\deg^+(P_{z^\nu})^\nu(\xi) \leq \deg^+(P_{z^\nu})^\nu(\hat{\xi})$. This implies (3.1). Q.E.D.

Let us denote by $\Gamma(R_{z^\nu})$ the component of the set $\{\eta \in \hat{l}_{z^\nu}; (R_{z^\nu})^0(-i\eta') \neq 0\}$ which contains $\partial''\cdot$. Then we have the following

Lemma 3.5 ([8]). Let $\xi^0 \in \mathbb{R}^{n-1}$. $\Gamma(R_{z^\nu})$ is an open convex cone and

\begin{equation*}
R_{z^\nu}(\eta') \neq 0 \quad \text{for} \quad \xi' \in \mathbb{R}^{n-1} - i\eta'; \partial' - i\Gamma(R_{z^\nu}),
\end{equation*}

\begin{equation*}
(R_{z^\nu})^\nu(\xi') \neq 0 \quad \text{for} \quad \xi' \in \mathbb{R}^{n-1} - i\Gamma(R_{z^\nu}).
\end{equation*}

Let us denote by $\Gamma(P_{z^\nu})$ the component of the set $\{\eta \in \hat{l}_{z^\nu} \times \mathbb{R}; P^0_{z^\nu}(-i\eta) \neq 0\}$ which contains $\partial$. Then we have also the following

Lemma 3.6. Let $\xi^0 \in \mathbb{R}^n$. $\Gamma(P_{z^\nu})$ is an open convex cone and

\begin{equation*}
P_{z^\nu}(\xi') \neq 0 \quad \text{for} \quad \xi' \in \mathbb{R}^n - i\eta'; \partial' = i\Gamma(P_{z^\nu}),
\end{equation*}

\begin{equation*}
P^0_{z^\nu}(\xi') \neq 0 \quad \text{for} \quad \xi' \in \mathbb{R}^n - i\Gamma(P_{z^\nu}),
\end{equation*}

\begin{equation*}
\Gamma(P_{z^\nu}) \supset \Gamma_{z^\nu}.
\end{equation*}

In our case we can prove Lemma 3.2 in [8].

Lemma 3.7. Let $\xi^0 \in \mathbb{R}^{n-1}$ and let $M$ be a compact set in $\hat{l}_{z^\nu}$. Then there exist a conic neighborhood $A$ of $\xi^0$ and positive numbers $C$, $t_0$ such that $P_{+}(\zeta', \lambda)$ is holomorphic in $\langle \zeta', \lambda \rangle \in A \times \mathbb{R}$, where

\begin{equation*}
A = \{\zeta' = \xi' - it|\xi'|^2, \eta' - i\eta' \partial'; \eta', \xi' \in A, |\xi'| \geq C, \eta', \xi' \in M\}
\end{equation*}

and $0 < t \leq t_0$.

---

\textsuperscript{1} \deg \rho^\nu(\xi) denotes the degree of homogeneity of $\rho^\nu$.

\textsuperscript{2} $(R_{z^\nu})^\nu(-i\partial') \neq 0$ was shown in [7].
Therefore $R(\zeta')$ and $R_P(\zeta')$ are also holomorphic in $\Lambda$.

**Proof.** The lemma is trivial for $\xi'^0 = 0$. So we assume that $\xi'^0 \in \mathbb{R}^{n-1} \setminus \{0\}$. Let $\lambda(\zeta')$ be a root of $p_1(\zeta', \lambda) = 0$. We can assume that $\lambda(\zeta')$ is continuous in $\Lambda$ when $C$ and $t_0$ are suitably chosen. In fact, there exist a conic neighborhood $\mathcal{A}$ of $\xi'^0$ and $C$, $t_0 (> 0)$ such that

$$p_{j_0}(\xi' - it|\xi'|\eta' - i\gamma' \partial') \neq 0 \text{ if } \xi' \in \mathcal{A}, \ |\xi'| \geq C, \ \eta' \in M \text{ and } 0 < t \leq t_0.$$  

For $p_{j_0}(\xi')$ is independent of $\xi'$ and $M \subset \hat{\mathcal{I}}^{\nu} \subset \hat{\mathcal{I}}(p_{j_0}(\xi', \mu))$. The argument in Section 2 shows that $\lim_{r \to 0} \nu \lambda(\nu^{-1} \xi'^0 - i\eta' - i\gamma' \partial') = \mu_0$ exists if $\lim_{r \to 0} |\nu \lambda(\nu^{-1} \xi'^0 - i\eta' - i\gamma' \partial')| = 0$, where $\eta' \in M$. Moreover from Lemmas 2.4 and 2.7 it follows that $\mu_0$ is a real root of $\partial^2 p_{j_0}(\xi', \lambda) = 0$. Now let us assume that $\mu_0$ is a real multiple root of $\partial^2 p_{j_0}(\xi', \lambda) = 0$. We can assume without loss of generality that $M$ is small so that $\{(\eta', \eta_n); \eta' \in M \text{ and } \eta_n \leq \eta_n \leq \eta^2_n\} \subset \hat{\mathcal{I}}(p_{j_0}(\xi', \mu))$ for some $\eta^1_n, \eta^2_n \in \mathbb{R}$. Then it follows that there exist a conic neighborhood $\mathcal{A}$ of $(\xi'^0, \mu_0)$ and $C$, $t_0 (> 0)$ such that

$$p_j(\xi' - \mu \xi' |\xi'| \eta' - i\gamma' \partial', \lambda - \mu \xi' |\eta|) \neq 0$$

if $(\xi', \lambda) \in \mathcal{A}, \ |\xi'| \geq C, \ \eta' \in M, \ \eta_1 \leq \eta_n \leq \eta^2_n$ and $0 < t \leq t_0$.

This implies that

$$\text{Im} \lambda(\xi' - \mu \xi' |\xi'| \eta' - i\gamma' \partial') \notin [-it|\xi'|\eta^2_n, -it|\xi'|\eta^1_n]$$

for $\xi' \in \mathcal{A}, \ |\xi'| \geq C, \ \eta' \in M$ and $0 < t \leq t_0$, modifying $\mathcal{A}$, $C$ and $t_0$, if necessary (see Lemma 3.2 in [8]). If $\partial' \notin M$, we choose a continuous curve $\gamma'(\theta)$ in $\hat{\mathcal{I}}^{\nu}$ such that $\gamma'(0) = \partial'$ and $\gamma'(1) \in M$ and we repeat the above argument for each small neighborhood of $\gamma'(\theta)$, $0 \leq \theta \leq 1$. This proves the lemma (see Lemma 3.2 in [8]). Q.E.D.

Put

$$t_0 = t_j(\xi'^0) = h_{\xi'} + h_j(\xi'^0),$$

where $h_{\xi'}$ and $h_j(\xi'^0)$ are defined in Lemma 3.3. Then it is easy to see
that \( t_j(\xi^\nu) \) is an integer and that \( t_j(\xi^\nu) \leq t_0(0) \). Put

\[
t = t(\xi^\nu) = \max t_j(\xi^\nu), \quad \omega = \omega(\xi^\nu) = \min \{ j; t(\xi^\nu) = t_j(\xi^\nu) \}.
\]

It easily follows that \( t(\xi^\nu) = t_0(0) \). From Lemma 3.3 we have the following

**Lemma 3.8.** Let \( \xi^\nu \in \mathbb{R}^{n-1} \). The principal part \( R_0(\xi') \) of \( R(\xi') \) is well-defined and there exists the localization \( (R^0)_{\xi^\nu}(\gamma') \) of \( R_0(\xi') \) at \( \xi^\nu \). Moreover for any compact set \( K \) in \( \mathbb{R}^{n-1} - i\hat{I}_{\xi^\nu} \) there exists \( \gamma_K(>0) \) such that

\[
(3.2) \quad \nu^{\xi_{\xi^\nu} - t_{\xi^\nu}(0) - \omega(\xi^\nu)/2} R^0(\xi^\nu + \nu\gamma') = \sum_{\ell \in \ell(\xi^\nu) \neq \nu(0)} \nu^{\ell - \omega(\xi^\nu)/2} R^{\ell, \gamma'}(\gamma'),
\]

whose convergence is uniform in \( \{(\gamma', \nu); \gamma' \in \mathbb{R}^{n-1} - i\hat{I}_{\xi^\nu}, \alpha\gamma' \in K \text{ for some } \alpha \in C \ (|\alpha| = 1) \text{ and } 0 \leq \nu \leq \gamma_K \} \), and

\[
(R^0)_{\xi^\nu}(\gamma') = R^{\gamma}_{\xi_{\xi^\nu}, \omega(\xi^\nu)}(\gamma').
\]

Let \( \Gamma((R^0)_{\xi^\nu}) \) be the component of the set \( \{ \gamma' \in i\hat{I}_{\xi^\nu}; (R^0)_{\xi^\nu}(-i\gamma') \neq 0 \} \) which contains \( \gamma'' \).

**Lemma 3.9 ([8]).** Let \( \xi^\nu \in \mathbb{R}^{n-1} \). \( \Gamma((R^0)_{\xi^\nu}) \) is an open convex cone and

\[
(R^0)_{\xi^\nu}(\xi') \neq 0 \text{ for } \xi' \in \mathbb{R}^{n-1} - i\Gamma((R^0)_{\xi^\nu}).
\]

**Lemma 3.10 ([8]).** Let \( \xi'' \in \mathbb{R}^{n-1} \). For any compact set \( M \) in \( \Gamma((R^0)_{\xi''}) \), there exist a conic neighborhood \( \Delta, (\subset \mathbb{R}^{n-1}) \) of \( \xi'' \) and positive numbers \( C, t_0 \) such that

\[
R(\xi' - it|\xi'' - i\gamma'; \theta') \neq 0 \text{ if } \gamma' \in M, \xi' \in \Delta, \ |\xi'| \geq C \text{ and } 0 < t \leq t_0.
\]

Let \( \xi^\delta \in \mathbb{R}^{n+1} \) and put

\[
\Gamma(\gamma') = R^{\gamma}_{\xi_{\xi^\nu} + \gamma(0)} \sum_{\ell \in \ell(\xi^\nu)} R^{\ell, \gamma}(\gamma').
\]

\( (R^0)_{\xi^\nu}(-i\theta') \neq 0 \) was shown in [8].
Propagation of Singularities of Hyperbolic Mixed Problem

(3.3) \[ \Gamma_t^* = \{ \xi \in \mathbb{R}^{n+1}; (\eta', \eta_{n+1}) \in \Gamma(P_{q_{n+1}r^{n+1}}) \} \]
\[ \cap (\Gamma(P_{\xi \eta}) \times R) \cap (\Gamma(R_{q_{n+1}}) \times R^n). \]

(3.4) \[ \Gamma_t^0 = \{ \xi \in \mathbb{R}^{n+1}; (\eta', \eta_{n+1}) \in \Gamma(P_{q_{n+1}r^{n+1}}) \} \]
\[ \cap (\Gamma(P_{\xi \eta}) \times R) \cap (\Gamma((R^{0}_t)^{n+1}) \times R^n). \]

Then Theorem 1.1 can be proved by the same arguments as in [7], [8].

(1.2) follows from Lemma 4.1.

§ 4. Some Remarks and Examples

Lemma 4.1 ([8]). Let \( \xi^0 \in \mathbb{R}^{n-1} \). \( \Gamma((R^{0}_t)^{n+1}) \subset \Gamma(R_{q_{n+1}}). \)

Let us prove the inner semi-continuity of \( \Gamma((R^{0}_t)^{n+1}) \) and, therefore, \( \Gamma_t^0 \).

Lemma 4.2. Let \( \xi^0 \in \mathbb{R}^{n-1} \) and let \( M \) be a compact set in \( \Gamma((R^{0}_t)^{n+1}) \). Then there exist a neighborhood \( U \) of \( \xi^0 \) and positive number \( t_0 \) such that \( R^0(\xi') \) is holomorphic in \( U - iD \) and \( \Gamma(\xi') \neq 0 \) for \( \xi' \in U - iD \), where \( D = \{ t\eta'; \eta' \in M \text{ and } 0 < t \leq t_0 \} \).

Proof: We can assume without loss of generality that \( P(\xi) \) is irreducible. Since \( M \subset \Gamma((P_{(a,1)}(q_{1+a}))) \), it follows that there exist a neighborhood \( U \) of \( \xi^0 \) and \( t_0, \nu_0 \) (\( > 0 \)) such that
\[ P_{(a,1)}(v^{-1}\xi) = P_{m'}(v^{-1}\xi') \neq 0 \]
if \( \xi' \in U - iD - iv_0\eta \), \( 0 < v \leq \nu_0 \). Let \( K \) be a compact set in \( U - iD \). Then there exists \( \nu_K \) (\( > 0 \)) such that \( \nu_K \leq \nu_0 \) and \( K \subset U - iD - iv_K\eta \). Let \( \lambda^{\frac{a}{b}}(\xi'; v) \) be a root of \( P(v^{-1}\xi', v^{-1}\lambda) = 0 \) such that \( \lambda^{\frac{a}{b}}(\xi'; v) = v\lambda^{\frac{a}{b}}(v^{-1}\xi') \) when \( \xi' \in K \) and \( 0 < |v| \leq \nu_K \). In fact, since \( P_{m'}(v^{-1}\xi') \neq 0 \) for \( \xi' \in K \) and \( 0 < |v| \leq \nu_K \), modifying \( \nu_K \) if necessary, the above statement is meaningful. Moreover we can assume that \( \lambda^{\frac{a}{b}}(\xi'; v) \) is continuous when \( \xi' \in K \) and \( 0 \leq |v| \leq \nu_K \). Since \( \lambda^{\frac{a}{b}}(\xi'; 0) \) is a root of \( P(\xi', \lambda) = 0 \), the same argument as in Lemma 3.7 gives

\[ M \text{ denotes the interior of } M. \]
\{\lambda_j^\pm (\xi'; 0) \} \cap \{\lambda_j^\pm (\xi'; 0) \} = \emptyset \quad \text{for} \quad \xi' \in K,

modifying \( U \) and \( t_0 \) if necessary. Therefore it follows from continuity of \( \lambda_j^\pm (\xi'; \nu) \) that

\[
(4.1) \quad \{\lambda_j^\pm (\xi'; \nu) \} \cap \{\lambda_j^\pm (\xi'; \nu) \} = \emptyset \quad \text{for} \quad \xi' \in K \quad \text{and} \quad |\nu| \leq \nu_K,
\]

modifying \( \nu_K \) if necessary. Put

\[
P_+ (\xi', \lambda; \nu) = \prod_{j=1}^i (\lambda - \lambda_j^+ (\xi'; \nu)) = \lambda^1 + b_*^+ (\xi'; \nu) \lambda^{i-1} + \cdots + b_*^+ (\xi'; \nu),
\]

\[
P_- (\xi', \lambda) = \prod_{j=1}^i (\lambda - \lambda_j^- (\xi')) = \lambda^1 + a_*^+ (\xi') \lambda^{i-1} + \cdots + a_*^+ (\xi') .
\]

(4.1) implies that the \( b_*^j (\xi'; \nu) \) are holomorphic in \( \{(\xi', \nu); \xi' \in K \quad \text{and} \quad |\nu| \leq \nu_K \} \). Moreover we have \( a_*^j (\nu^{-1} \xi') = \nu^{-1} b_*^j (\xi'; \nu) \). Therefore we have

\[
b_*^j (\xi'; \nu) = a_*^j (\xi') + \nu a_*^j (\xi') + \nu^2 a_*^j (\xi') + \cdots ,
\]

whose convergence is uniform in \( \{(\xi', \mu); \xi' \in K \quad \text{and} \quad |\nu| \leq \nu_K \} \). \( a_*^j (\xi') \) is holomorphic in \( U-iD \) and homogeneous of degree \( j-k \). So \( R^0 (\xi') \) is well-defined and holomorphic in \( U-iD \). (3.2) and the above result yields us

\[
R^0 (\xi') \neq 0 \quad \text{for} \quad \xi' \in U-iD ,
\]

using the same argument as in the proof of Lemma 3.7 in [8].

Q.E.D.

**Theorem 4.3.** Let \( \xi^{0^0} \in \mathbb{R}^{n-1} \) and let \( M \) be a compact set in \( \Gamma ((R^0)_{\nu'}) \). There exists a neighborhood \( U \) of \( \xi^{0^0} \) such that

\[
M \subset \Gamma ((R^0)_{\nu'}) \quad \text{for} \quad \xi' \in U .
\]

**Proof.** It is obvious that \( M \subset \Gamma_{\nu^*} \) for \( \xi' \in U \), shrinking \( U \). Now assume that there exist \( \xi^{0^0} \in U \) and \( \eta^{0^0} \in M \) such that \( (R^0)_{\nu^*} (-i \eta^{0^0}) = 0 \), where \( U \) is sufficiently small. Since \( (R^0)_{\nu^*} (-i \eta^{0^0}) \neq 0 \), there exists \( \xi^{0^0} \in C^{n-1} \) such that \( \xi^{0^0} - i (\eta^{0^0} + \mu \xi^{0^0}) \in U-iM \) for \( |\mu| \leq 1 \) and \( (R^0)_{\nu^*} (-i (\eta^{0^0} + \xi^{0^0})) \neq 0 \). Therefore it follows that there exist \( \varepsilon, \delta \) \((>0)\) such that

\[
| (R^0)_{\nu^*} (-i (\eta^{0^0} + \mu \xi^{0^0})) | \geq 2\varepsilon \quad \text{for} \quad |\mu| = \delta.
\]
On the other hand from (3.2) we have
\[
|t^{n^* - t_1(0) - \omega(t_1)t^{*L}} R^0 (\xi' - it (\eta'' + \mu \xi''')) - (R^0)_{t_1} (i(\eta'' + \mu \xi'''))| < \varepsilon \quad \text{for} \quad |\mu| = \delta \quad \text{and} \quad 0 < t \leq t_1 \quad (\leq t_2),
\]
where \( t_0 \) and \( t_1 \) are suitably chosen. Rouche’s theorem implies that \( R^0 (\xi' - it (\eta'' + \mu \xi''')) \) has zeros within \(|\mu| < \delta\) for \( 0 < t \leq t_1 \), which is a contradiction to Lemma 4.2. Q.E.D.

Theorem 4.3 yields us the following

**Theorem 4.4.** \( \bigcup_{\xi \in \mathbb{R}^{n-1}[0]} K^\sharp \times \{\tilde{\xi}\} \) is closed in \( T^*X \setminus 0 \).

In Section 2 the developments of \( \sigma^p (\nu^{-1} \xi'' + \eta') \) and \( \sigma^p (\nu^{-1} r \xi'' + \eta') \) was given. However we can similarly obtain the developments \( f (\nu^{-1} \xi'' + \eta') \) and \( f (\nu^{-1} r \xi'' + \eta') \), where
\[
f (\xi') = (2\pi i)^{-1} \int_{\gamma} g (\xi', \lambda) P (\xi', \lambda)^{-1} d\lambda
\]
and \( g (\xi', \lambda) \) is a polynomial of \( (\xi', \lambda) \) and \( \gamma^p \) encloses only the roots \( \lambda^i (\xi') \) \( \cdots \), \( \lambda^i (\xi') \) of \( P (\xi', \lambda) = 0 \). This will be useful for hyperbolic systems.

Next let us consider some examples.

**Example 4.5.** Put \( n = 4 \) and
\[
P (\tilde{\xi}) = (\xi^1 - \xi^2 - \xi^3 - \xi^4 + a \xi^4) (\xi^1 - \xi^4), \quad a > 0,
\]
\[
B_1 (\tilde{\xi}) = 1, \quad B_2 (\tilde{\xi}) = (-\tilde{\xi}_1 - i \tilde{\xi}_2) \tilde{\xi}_4 - \tilde{\xi}_4.
\]
Then we have \( R (\tilde{\xi}) = i \tilde{\xi}_2 + \sqrt{\tilde{\xi}_1^2 - \tilde{\xi}_2^2 - \tilde{\xi}_3^2 + a \tilde{\xi}_3^2} \). It is obvious that \( \{P, B_1, B_2\} \) satisfies the condition (A). We can show that \( \bigcup \bigcup_{\xi \in \mathbb{R}^{n-1}[0]} K^\sharp \times \{\tilde{\xi}\} \) is not closed in \( T^*X \setminus 0 \) and that
\[
\bigcup_{\xi \in \mathbb{R}^{n-1}[0]} \bigcup_{j=0}^{m} \supp \tilde{F}_{\xi,j} \times \{\tilde{\xi}\} = \bigcup_{\xi \in \mathbb{R}^{n-1}[0]} K^\sharp \times \{\tilde{\xi}\}
\]
\( \subset WF (\tilde{F}) \subset WF_A (\tilde{F}) \subset \bigcup_{\xi \in \mathbb{R}^{n-1}[0]} K^\sharp \times \{\tilde{\xi}\} \)
(see [9]). Moreover we have
\[
\overline{ch} [ WF (\tilde{F}) ]_{\xi^4} = \overline{ch} [ WF_A (\tilde{F}) ]_{\xi^4} = K^\sharp_{\xi^4}, \quad \text{for} \quad \xi^4 \neq 0.
\]
Example 4.6. Put $n=3$ and

$$P(\xi) = ((\xi_1 - \xi_2)^2 - \xi_3^2 + a) \cdot ((2\xi_1 - \xi_2)^2 - \xi_3^2),$$

$$B_1(\xi) = 1, \quad B_2(\xi) = \xi_3.$$

Then $R(\xi') = -1$ and $\{P, B_1, B_2\}$ satisfies the condition (A). We note that $(\xi_1 - \xi_2)^2 - \xi_3^2 + a$ is irreducible when $a \neq 0$. It is easy to see that

$$WF(\overline{F}) |_{\alpha_1, \alpha_2, \alpha_3} = \{z \in X; z = \alpha(2, -1, 0, -1) + \beta(2, -1, 1, 0)$$

$$+ \gamma(1, -1, 0, 0); \alpha, \beta > 0 \text{ and } \gamma \geq 0 \text{ when } a \neq 0,$$

$$WF(\overline{F}) |_{\alpha_1, \alpha_2, \alpha_3} = \{z \in X; z = \alpha(2, -1, 0, -1) + \beta(2, -1, 1, 0)$$

$$\text{ and } \alpha, \beta > 0 \text{ when } a = 0.$$

This shows that so called lateral wave appears when $a \neq 0$.

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References


