The Theory of Vector Valued Fourier Hyperfunctions of Mixed Type. II

By

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Abstract

The soft resolution \( \mathcal{Q}_{k,1} \) of the sheaf \( \mathcal{O}_{k,1} \) of rapidly decreasing holomorphic functions of \((k,1)\) type is constructed. Using the above resolution, we prove \( H^k_\mathbb{C}(V,^{p,\bar{p}}\mathcal{O}_{k,1}) \) \( \cong L(\mathcal{O}_{k,1}(K), E) \).

§ 1. Introduction

In the first part of the present paper (S. Nagamachi [4]), which will be referred to as [I], we defined the mixed type Fourier hyperfunctions which take values in a Fréchet space \( E \). The purpose of this second part is to prove that the space \( H^k_\mathbb{C}(V,^{p,\bar{p}}\mathcal{O}_{k,1}) \) of \( E\)-valued Fourier hyperfunctions with support contained in a compact set \( K \) is isomorphic to the space \( L(\mathcal{O}_{k,1}(K), E) \) of continuous linear mappings of \( \mathcal{O}_{k,1}(K) \) into \( E \). We proved this theorem in [I] only for \( E = \mathcal{C} \) (Theorem 5.13 of [I]).

In Section 2, we study the Fourier transformation for slowly increasing \( C^\infty \) functions and rapidly decreasing distributions. In Section 3, we prepare the theory of cohomology with bounds in an appropriate form.

In Section 4, we construct a soft resolution of the sheaf \( \mathcal{Q}_{k,1} \) of rapidly decreasing holomorphic functions (Theorem 4.9),

\[ 0 \rightarrow \mathcal{Q}_{k,1} \rightarrow \mathcal{Q}'_{(0,0)} \rightarrow \cdots \rightarrow \mathcal{Q}'_{(0,p)} \rightarrow 0, \]

where \( \mathcal{Q}'_{(0,p)} \) is the sheaf subordinate to the presheaf \( \{ \mathcal{Q}'_{(0,p)}(\mathcal{O}) \} \) of \((0,p)\)-forms whose coefficients are rapidly decreasing distributions in \( \mathcal{O} \).

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(Definition 4.1). To do this, we use the method similar to that developed in 7.6 of L. Hörmander [1], that is, the duality arguments, using the property of the Fourier transformation (Propositions in § 2) and the estimate of the solutions of certain system of linear equations with polynomial coefficients (Proposition 3.5, which is an extension of Theorem 7.6.11 of L. Hörmander [1]).

Using this method, we construct also the following resolution of $\mathcal{O}_{k,1}$, on $Q^{k,1}$:
\[
0 \rightarrow \mathcal{O}_{k,1} \rightarrow \mathcal{F} (\alpha, 0) \rightarrow \cdots \rightarrow \mathcal{F} (\alpha, n) \rightarrow 0,
\]
which is an extension of Theorem 4.11 of [I], where the resolution has been obtained only on the open subset $\mathcal{O}$ of $Q^{k,1}$ satisfying a certain condition.

In Section 5, we prove $H^2_{t} (V, \mathcal{O}, K) = L (\mathcal{O}_{k,1} (K), E)$ (Theorem 5.7) using the Serre-Komatsu duality theorem and properties of tensor products of $E$ with nuclear Fréchet spaces.

We continue to use the same notions and notations as those in [I].

§ 2. Function Spaces

In this section we study the Fourier transformation for slowly increasing $C^\infty$ functions and rapidly decreasing distributions.

**Definition 2.1.** Let $K$ be the closure of $\bigcap_{i=1}^{l} (\Gamma_i \times B_i)$ in $Q^{k,1}$, where $\Gamma_i$ is the strictly convex closed cone in $R^{2k_i + l_i}$ whose vertex is at the origin and $B_i$ is the closed ball in $R^{l_i}$ whose center is at the origin.

In this section we always denote by $K$ the compact set defined in Definition 2.1.

We identify $C^\infty$ with $R^{2n}$ and denote by $\langle x, \eta \rangle$ the inner product in $R^{2n}$, i.e., $\langle x, \eta \rangle = \sum_{i=1}^{2n} x_i \eta_i$.

**Definition 2.2.** Let $h_{K, \delta} (\eta) = \sup_{x \in K \cap R^{2n}} (- \langle x, \eta \rangle + \varepsilon |x|)$. Define $K^0 = \{ \eta \in R^{2n}; h_{K, \delta} (\eta) < \infty \}$ and $K_0 = \bigcap_{i=1}^{l} (\Gamma_i \cap R^{k_i})$, where $\Gamma_i =$
\{\eta \in \mathbb{R}^{m+1}; \langle x, \eta \rangle > 0 \text{ for all } 0 \neq x \in \Gamma_i\}.

**Proposition 2.3.** Let \( K^\circ \) be the interior of \( K^\circ \). If \( \eta = (\alpha_i, \beta_i) \in K^\circ \) for \( i = 1, \ldots, j \) and arbitrary real \( s_i, 1 \leq i \leq j \).

**Proof.** Let \( \eta_i = (\alpha_i, \beta_i) \in \Gamma_i^\circ \times \mathbb{R}^i \). Then

\[ h_{K^\circ}(\eta) = \sup_{x \in K^\circ \cap \mathbb{R}^m} \left( -\langle x, \eta \rangle + \varepsilon |x| \right) \]

\[ = \sum_{i=1}^j \sup_{x_i \in \Gamma_i} \left( -\langle x_i, \alpha_i \rangle + \varepsilon |x_i| \right) + \sum_{i=1}^j \sup_{y_i \in \mathbb{R}^i} \left( -\langle y_i, \beta_i \rangle + \varepsilon |y_i| \right) \]

\[ = \sum_{i=1}^j h_{\Gamma_i^\circ}(\alpha_i) + \sum_{i=1}^j h_{\mathbb{R}^i}(\beta_i) \]

\( h_{K^\circ}(\eta) < \infty \) implies that \( h_{\Gamma_i^\circ}(\alpha_i) < \infty \) for all \( i \) and this shows that \( \langle x_i, \alpha_i \rangle > 0 \) for \( 0 \neq x_i \in \Gamma_i \) because if \( \langle x_i, \alpha_i \rangle \leq 0 \) for some \( 0 \neq x_i \in \Gamma_i \), then \( -\langle t x_i, \alpha_i \rangle + \varepsilon |t x_i| \) tends to infinity as \( t \to \infty \), this is a contradiction. Thus we have \( K^\circ \supset K^\circ_\varepsilon \) and \( K^\circ \supset \cup_{\varepsilon>0} K^\circ_\varepsilon \). Conversely if \( \eta \in K^\circ \), then let \( \inf_{x \in \mathbb{R}^m} \langle x_i, \alpha_i \rangle = \delta_i > 0 \) and choose \( \varepsilon > 0 \) satisfying \( \varepsilon < \delta_i \) for all \( i \). Then we have \( -\langle t x_i, \alpha_i \rangle + \varepsilon |t x_i| \leq 0 \) for \( x_i \in \Gamma_i, |x_i|=1 \) and \( t \geq 0 \), consequently \( h_{\Gamma_i^\circ}(\alpha_i) \leq 0 \) for all \( i \). Since \( h_{\Gamma_i^\circ}(\beta_i) < \infty \) for all \( i \), \( h_{K^\circ}(\eta) = \sum_{i=1}^j h_{\Gamma_i^\circ}(\alpha_i) + \sum_{i=1}^j h_{\mathbb{R}^i}(\beta_i) < \infty \). Thus we have \( K^\circ \subset \cup_{\varepsilon>0} K^\circ_\varepsilon \).

**Proposition 2.4.** If \( \eta = ((\alpha_i, \beta_i), \ldots, (\alpha_j, \beta_j)) \in K^\circ \), then \( ((t_i \alpha_i, s_i \beta_i), \ldots, (t_j \alpha_j, s_j \beta_j)) \in K^\circ_\varepsilon \) for \( t \geq 1 \) and arbitrary real \( s_i, 1 \leq i \leq j \).

**Proof.** \( \eta \in K^\circ \) is equivalent to \( h_{\Gamma_i^\circ}(\alpha_i) < \infty \) for \( 1 \leq i \leq j \). Since \( h_{\Gamma_i^\circ}(\alpha_i) = \sup_{x \in \Gamma_i^\circ, |x|=1, t \geq 0} (-\langle x, \alpha_i \rangle + \varepsilon) s_i \), \( h_{\Gamma_i^\circ}(\alpha_i) < \infty \) is equivalent to \( \inf_{x \in \mathbb{R}^m} \langle x_i, \alpha_i \rangle \geq \varepsilon \) and \( \inf_{x \in \mathbb{R}^m} \langle x_i, t_i \alpha_i \rangle \geq \varepsilon \) implies \( \inf_{x \in \mathbb{R}^m} \langle x_i, t_i \alpha_i \rangle \geq \varepsilon \) for \( t_i \geq 1 \). Thus we have \( ((t_i \alpha_i, s_i \beta_i), \ldots, (t_j \alpha_j, s_j \beta_j)) \in K^\circ \).

**Corollary 2.5.** Let \( \text{Int } K^\circ \) be the interior of \( K^\circ \). If \( \eta = ((\alpha_i, \beta_i), \ldots, (\alpha_j, \beta_j)) \in \text{Int } K^\circ \), then for \( t \geq 1 \) and arbitrary real \( s_i, 1 \leq i \leq j \), \( \eta(t, s) = ((t_i \alpha_i, s_i \beta_i), \ldots, (t_j \alpha_j, s_j \beta_j)) \in \text{Int } K^\circ \).

**Proof.** If \( \eta \in \text{Int } K^\circ \), then there exists a neighbourhood \( V \) of zero such that \( \eta + V \subset K^\circ \). By Proposition 2.4 we have \( \eta(t, s) + V(t, 1) \subset K^\circ \), where \( V(t, 1) = \{v(t, 1); v \in V\} \) is a neighbourhood of zero. Thus we
Proposition 2.6. Let $0<\delta<\varepsilon$. Then $K^\circ_\varepsilon$ is strictly contained in $K^\circ_\delta$, that is, the distance between $K^\circ_\varepsilon$ and the complement $(K^\circ_\varepsilon)^c$ of $K^\circ_\varepsilon$ is positive. Therefore $K^\circ_\varepsilon \subset \text{Int } K^\circ_\delta$.

Proof. Let $\eta \in K^\circ_\varepsilon$ and $e \in C^\omega$ with $|e|<\varepsilon-\delta$. Since $\eta \in K^\circ_\varepsilon$ is equivalent to $\inf_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, \alpha_i \rangle \geq \varepsilon$ for $i=1, \ldots, j$,

$$\inf_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, \alpha_i + e_i \rangle \geq \inf_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, \alpha_i \rangle - \sup_{x_i \in \Gamma_i, |x_i|=1} \langle x_i, e_i \rangle \geq \varepsilon - (\varepsilon - \delta) = \delta.$$ 

Thus we have shown that $\eta + e \in K^\circ_\delta$ for all $\eta \in K^\circ_\varepsilon$ and $|e|<\varepsilon-\delta$. This shows that $K^\circ_\varepsilon$ is strictly contained in $K^\circ_\delta$.

Proposition 2.7. Let $f$ be a $C^N$ function with support contained in $K \cap R^{2n}$. Suppose there exist positive constants $\delta$ and $C$ such that $|D^\alpha f(x)| \leq Ce^{|\alpha|}$ for all $|\alpha| \leq N$. Define

$$\tilde{f}(\zeta) = (2\pi)^{-n} \int_{R^{2n}} e^{ic(x, \zeta)} f(x) dx.$$ 

Then $\tilde{f}(\zeta)$ is an analytic function defined in $\{\zeta \in C^{2n}, \text{Im } \zeta \in \text{Int } K^\circ_\varepsilon\}$ for any $\varepsilon>\delta$, and satisfies

$$(2.1) \quad |\tilde{f}(\zeta)| \leq C\varepsilon e^{h_\varepsilon (\text{Im } \zeta)} / (1 + |\zeta|)^N$$

for some constant $C_\varepsilon>0$ and $\text{Im } \zeta \in K^\circ_\varepsilon$.

Proof. Let $\text{Im } \zeta \in K^\circ_\varepsilon$. The inequalities

$$|\zeta^\alpha \tilde{f}(\zeta)| \leq (2\pi)^{-n} \int_{R^{2n}} e^{-(\langle x, \text{Im } \zeta \rangle + \varepsilon |x|)} e^{-\varepsilon |x|} |D^\alpha f(x)| dx$$

$$\leq C\varepsilon e^{h_\varepsilon (\text{Im } \zeta)}$$

imply that $\tilde{f}(\zeta)$ is analytic in $\text{Im } \zeta \in \text{Int } K^\circ_\varepsilon$ and satisfies (2.1) for $\text{Im } \zeta \in K^\circ_\varepsilon$.

Corollary 2.8. Let $f \in \mathcal{T}_\varepsilon(K)$ (Definition 2.14 of [1]), then $\tilde{f}(\zeta)$ is an analytic function defined in $\{\zeta \in C^{2n}; \text{Im } \zeta \in K^\circ_\varepsilon\}$ and satisfies
in $\Im \zeta \in K^\circ_\varepsilon$ for any $\varepsilon>0$ and $N>0$, where $C_{N,\varepsilon}$ is a positive number independent of $\zeta$.

**Proof.** The corollary follows from Propositions 2.3, 2.6 and 2.7.

**Proposition 2.9.** Let $K$ be the set defined in Definition 2.1. For any $0<\varepsilon \leq 1$, there exists an $\eta_\varepsilon \in \text{Int } K^\circ_\varepsilon$ satisfying $|\eta_\varepsilon| \leq A\varepsilon$ for some positive constant $A$ not depending on $\varepsilon$.

**Proof.** Let $\eta \in K^\circ_\varepsilon$ and $A = |\eta|$. Define $\eta_\varepsilon = \varepsilon \eta$ for $0<\varepsilon \leq 1$, then $|\eta_\varepsilon| = A\varepsilon$ and

$$h_{K,2\varepsilon}(\eta_\varepsilon) = \sup_{x \in K^\circ_\varepsilon \cap \mathbb{R}^n} \left( -\langle x, \varepsilon \eta \rangle + 2\varepsilon |x| \right) = \varepsilon h_{K,\varepsilon}(\eta) < \infty.$$  

This shows that $\eta_\varepsilon \in K^\circ_{2\varepsilon} \subset \text{Int } K^\circ_\varepsilon$.

**Proposition 2.10.** Let $N \geq 3n$, and let $g(\zeta)$ be an analytic function in $\{ \zeta \in C^n; \Im \zeta \in \text{Int } K^\circ_\varepsilon \}$ which satisfies

$$|g(\zeta)| \leq C \frac{1}{(1 + |\zeta|)^N} e^{h_{K,\varepsilon}(\Im \zeta)}$$

for $\Im \zeta \in K^\circ_\varepsilon$. If we define

$$(2.2) \quad \hat{g}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \hat{g}(\xi) d\xi \quad \text{for } \eta \in \text{Int } K^\circ_\varepsilon,$$

$\hat{g}(x)$ is a $C^\infty$ function with support contained in $K \cap \mathbb{R}^n$, satisfying $|D^a\hat{g}(x)| < M\varepsilon^{|a|}$ for some constant $M$ and $\varepsilon = A\varepsilon$, where $A$ is the constant appeared in Proposition 2.9.

**Proof.** The inequalities

$$|\hat{g}(x)| = |(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} e^{\varepsilon \zeta \cdot v} g(\xi + i\eta) d\xi|$$

$$\leq C e^{\varepsilon \zeta \cdot v} \int |g(\xi + i\eta)| d\xi$$

$$\leq C e^{\varepsilon \zeta \cdot v} e^{h_{K,\varepsilon}(\zeta)}$$
hold for \( \eta \in \text{Int} K^o_{\varepsilon} \). \( x \notin K \) implies \( x_i \notin \Gamma_i \) or \( y_k \notin B_k \) for some \( l, k \). Hence there exists \( \alpha_i \in \Gamma^o \) such that \( \langle x_i, \alpha_i \rangle < 0 \) or \( y_k \) satisfies \( \langle y_k, s y_k \rangle > h_{B_k, \varepsilon} (s y_k) \) for large \( s > 0 \). Since (2.2) is independent of \( \eta \in \text{Int} K^o_{\varepsilon} \) by the Cauchy-Poincaré theorem, we have, for large \( s > 0 \),

\[
(2.3) \quad |\tilde{g}(x)| \leq C \exp \left( \langle x_i, t \alpha_i \rangle + \langle y_k, -s y_k \rangle + h_{B_k, \varepsilon} (-s y_k) \right) \\
+ \sum_{i \in \Omega} \langle x_i, \alpha_i \rangle + \sum_{i \in \Omega} \langle y_i, \beta_i \rangle + \sum_{i \in \Omega} h_{B_i, \varepsilon} (\beta_i),
\]

where we have used the facts that \( h_{r \xi, \varepsilon} (\alpha_i) \leq 0 \) and \( \eta = ((\alpha_i, \beta_i), \cdots, (t \alpha_i, \beta_i), \cdots, (\alpha_s, s y_s), \cdots, (\alpha_j, \beta_j)) \in \text{Int} K^o_{\varepsilon} \) for large \( t, s \) (Proposition 2.4). The right hand side of (2.3) vanishes as \( t \) or \( s \) tends to infinity. Thus we have \( g(x) = 0 \) if \( x \notin K \).

Let \( |\alpha| \leq N - 3n \). The inequalities

\[
(2.4) \quad |D^s \tilde{g}(x)| = (2\pi)^{-n} \left| \int_{R^{1n}} e^{-i(x, \xi)} e^{i(x, \xi)} (-i \xi + \eta) \alpha (\xi + i \eta) d\xi \right| \\
\leq C e^{(x, \xi)} e^{h_{K, \varepsilon}(\xi)} \\
\leq C e^{(x, \xi)} e^{h_{K, \varepsilon}(\xi)}
\]

hold for \( \eta = \eta_\varepsilon \in \text{Int} K^o_{\varepsilon} \) such that \( |\eta_\varepsilon| \leq \delta = A \varepsilon \) by Proposition 2.9. Hence

\[
(2.5) \quad |D^s \tilde{g}(x)| \leq M e^{(x, \xi)}
\]

holds for some constant \( M > 0 \).

**Proposition 2.11.** Let \( f \) be a \( C^r \) function satisfying the conditions in Proposition 2.7, then \( \tilde{f} = f \).

**Proof.** Let \( \eta \in \text{Int} K^o_{\varepsilon} \), then \( e^{-(\eta, \xi)} = f(y) \) is rapidly decreasing. Therefore we have

\[
\tilde{f}(x) = (2\pi)^{-n} \int_{R^{1n}} e^{-i(x, \xi)} \left( \int_{R^{1n}} e^{i(y, \xi)} f(y) dy \right) d\xi \\
= (2\pi)^{-n} \int_{R^{1n}} e^{-i(x, \xi)} \left( \int_{R^{1n}} e^{i(y, \xi)} e^{-(\eta, \xi)} f(y) dy \right) d\xi \\
= f(x).
\]

**Proposition 2.12.** Let \( g(\xi) \) be an analytic function satisfying the condition in Proposition 2.10. Then \( \tilde{g} = g \).
Proof. Let $\zeta = \xi + i\eta$ and $\eta \in \text{Int } K^0_{\delta}$, then $g(x + i\eta)$ is integrable with respect to $x$. Therefore we have

$$\tilde{g}(\zeta) = (2\pi)^{-2n} \int_{\mathbb{R}^n} e^{i(x, \xi)} \left( \int_{\mathbb{R}^n + i\eta} e^{-(y, \eta)}g(x) \, dx \right) \, du$$

$$= (2\pi)^{-2n} \int_{\mathbb{R}^n} e^{-(y, \eta)} e^{i(x, \xi)} \left( \int_{\mathbb{R}^n} e^{-(y, \eta)} e^{i(x, \xi)} g(x + i\eta) \, dx \right) \, du$$

$$= g(\xi + i\eta) = g(\zeta).$$

Proposition 2.13. Let $f \in \mathcal{D}(K)$, we define

$$(2.6) \quad \|f\|_{\delta, \zeta} = \int_{\mathbb{R}^n - iK_{\delta}} |\tilde{f}(\zeta)|^2 e^{-2|\zeta|, \zeta \in \text{Im } \zeta} \left( 1 + |\zeta|^{\delta} \right) \, d\lambda$$

then there exists a seminorm $\|f\|_{\delta, \zeta} = \sup_{x \in \mathbb{R}^n, |x| \leq M} |e^{-\zeta|x|} D^{\alpha} f(x)|$ of $\mathcal{D}(K)$ such that $\|f\|_{\delta, \zeta} \leq C \|f\|_{\delta, \zeta}$.

Proof. The inequalities

$$|e^{-\zeta, \zeta \in \text{Im } \zeta} |\tilde{f}(\zeta)|$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\zeta, \zeta \in \text{Im } \zeta} e^{i(x, \xi)} D^\alpha f(x) \, dx$$

$$\leq \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} e^{(x, \zeta) - \zeta, \zeta \in \text{Im } \zeta} e^{i(x, \xi) \zeta} D^\alpha f(x) \, dx \right|$$

$$\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| e^{-\zeta|x|} D^\alpha f(x) \right| \, dx$$

$$\leq C_{\delta} \|f\|_{\delta, \zeta},$$

for $0 < \delta < \varepsilon$ and $\text{Im } \zeta \in K^0_{\delta}$, show that

$$e^{-2|\zeta|, \zeta \in \text{Im } \zeta} \left( 1 + |\zeta|^{\delta} \right) |\tilde{f}(\zeta)| \leq C' \|f\|_{\delta, \zeta}.$$

Then we have

$$\|f\|_{\delta - 2n, \zeta} = \int_{\mathbb{R}^n + iK_{\delta}} e^{-2|\zeta|, \zeta \in \text{Im } \zeta} \left( 1 + |\zeta|^{\delta} \right) |\tilde{f}(\zeta)|^2 \, d\lambda$$

$$\leq C \|f\|_{\delta, \zeta}.$$

Thus we have, for $M = N + 3n$ and $\delta = \varepsilon/2$

$$\|f\|_{\delta, \zeta} \leq C \|f\|_{\delta, \zeta}.$$
Proposition 2.14. \( \text{Int } K_\epsilon^0 = \bigcup_{\epsilon > \epsilon} K_\epsilon^0 \).

Proof. \( \text{Int } K_\epsilon^0 \supset \bigcup_{\epsilon > \epsilon} K_\epsilon^0 \) is clear from Proposition 2.6. Let \( \eta \in \text{Int } K_\epsilon^0 \), then there exists a positive number \( \gamma \) such that for every \( \epsilon \in C^n \) with \( |\epsilon| \leq \gamma \), \( \eta + \epsilon \in K_\epsilon^0 \). Thus we have

\[
0 \geq \sup_{x_i \in R_\epsilon, |x_i| = 1, |\epsilon| \leq \gamma} (-\langle x_i, \alpha_i + \epsilon_i \rangle + \epsilon) = \sup_{x_i \in R_\epsilon, |x_i| = 1} (-\langle x_i, \alpha_i \rangle + \epsilon + \gamma).
\]

This shows that \( \eta \in K_{\epsilon + \gamma}^0 \) and \( \text{Int } K_\epsilon^0 \subset \bigcup_{\epsilon > \epsilon} K_\epsilon^0 \).

Proposition 2.15. \( \text{Int } K_\epsilon^0 \) is a convex set and \( h_{K,\epsilon}(\eta) \) is a convex function in \( \text{Int } K_\epsilon^0 \).

Proof. Let \( \xi, \eta \in \text{Int } K_\epsilon^0 \), then there exist \( \delta > \epsilon \) such that \( \xi, \eta \in K_{\epsilon + \delta}^0 \). For \( \lambda, \mu \geq 0, \lambda + \mu = 1 \), we have

\[
(2.7) \quad h_{K,\delta}(\lambda \xi + \mu \eta) = \sup_{x \in K \cap R^m} (-\langle x, \lambda \xi + \mu \eta \rangle + \delta |x|) = \sup_{x \in K \cap R^m} (-\lambda \langle x, \xi \rangle - \mu \langle x, \eta \rangle + \delta (\lambda + \mu) |x|) \\
\leq \lambda h_{K,\delta}(\xi) + \mu h_{K,\delta}(\eta) < \infty.
\]

This shows that \( \lambda \xi + \mu \eta \in K_{\epsilon + \delta}^0 \subset \text{Int } K_\epsilon^0 \). Hence \( \text{Int } K_\epsilon^0 \) is convex. The equation (2.7) shows that \( h_{K,\delta}(\eta) \) is a convex function defined in \( K_{\epsilon + \delta}^0 \), hence \( h_{K,\epsilon}(\eta) \) is convex in \( \text{Int } K_\epsilon^0 \).

Proposition 2.16. \( h_{K,\epsilon}(\eta) \) is Lipschitz continuous in \( K_\epsilon^0 \), that is,

\[
|h_{K,\epsilon}(\eta) - h_{K,\epsilon}(\eta')| \leq C|\eta - \eta'|
\]

for some constant \( C \).

Proof. Let \( h_{R,\epsilon}(\alpha_i) = \sup_{x_i \in R_\epsilon} (-\langle x_i, \alpha_i \rangle + \epsilon |x_i|) \) and \( h_{B,\epsilon}(\beta_i) = \sup_{y_i \in B_\epsilon} (-\langle y_i, \beta_i \rangle + \epsilon |y_i|) \). Then

\[
h_{K,\epsilon}(\eta) = \sum_{i=1}^j h_{R,\epsilon}(\alpha_i) + \sum_{i=1}^j h_{B,\epsilon}(\beta_i)
\]

where \( \eta = (\alpha_i, \beta_i, \ldots, (\alpha_j, \beta_j)) \) and \( x = ((x_i, y_i), \ldots, (x_j, y_j)) \). Let \(|B_\epsilon|\)
be the diameter of the ball $B_t$. We have
\[ |h_{B_t,\epsilon}(\beta_t) - h_{B_t,\epsilon}(\beta'_t)| \leq \sup_{x_t \in B_t} |\langle x_t, \beta_t - \beta'_t \rangle| \leq |B_t| |\beta_t - \beta'_t| . \]
Since $\eta \in K_0$ implies that $h_{F_t,\epsilon}(\alpha_t) = 0$ for all $i$, we have
\[ |h_{K,\epsilon}(\eta) - h_{K,\epsilon}(\eta')| \leq C|\eta - \eta'| \]
where $C = \max_i (|B_t|)$.

§ 3. Cohomology with Bounds

For the later use, we develop the theory of cohomology with bounds on the pseudoconvex domain $\Omega$ in $\mathbb{C}^n$, which is an extension of what is developed in 7.6 of L. Hörmander [1], where the case $\Omega = \mathbb{C}^n$ is treaded.

Here we use the same notation that is used in 7.6 of L. Hörmander [1]. We denote by $U^{(n)}$ the covering of $\mathbb{C}^n$ which consists of the cubes $U^{(n)}_v$ with side equal to $2 \cdot 3^{-n}$ and center at $g \cdot 3^{-n}$, where $g$ runs through the set $I$ of points in $\mathbb{C}^n$ with integral coordinates. For every $v$ and $g$ we can find precisely one $g'$ such that $U^{(n)}_v$ contains the cube with the same center as $U^{(n)}_v$ but twice the side; we set $p_v, v = g'$. More generally if $v < u$, we define
\[ \rho_{u,v} = \rho_{u-1,v} \rho_{u-1,v} \cdots \rho_{u-1,v} . \]

Let $\Omega$ be an open subset of $\mathbb{C}^n$, then $U^{(n)} \cap \Omega = \{ U^{(n)}_v \cap \Omega; g \in I \}$ is an open covering of $\Omega$. We also define
\[ \Omega^{n,v} = \bigcup \{ U^{(n)}_v; U^{(n)}_v \subset \Omega \} \quad \text{for} \quad g' = \rho_{u,v} \]
and
\[ \Omega_\varepsilon = \{ z \in \Omega; \text{dist} (z, \Omega^c) > \varepsilon \} \]
where dist $(z, \Omega^c)$ is the distance between the point $z$ and the complement $\Omega^c$ of $\Omega$. We use the abbreviation $\Omega^{n,v}$ for $(\Omega^{n,v})^{\omega}$.

Let $P = (P_{j,k}), j = 1, \ldots, p, k = 1, \ldots, q$ be the matrix with polynomial entries, and consider the sheaf homomorphism
\[ (3.1) \quad P: \mathcal{O}^q \rightarrow \mathcal{O}^p \]
defined by the mapping $(f_1, \ldots, f_q) \in \mathcal{O}^q$ to $\{ \sum P_{j,k} f_k \}^q_{j=1}$. Let $R_P$ be the
kernel of the sheaf homomorphism (3.1). It is known that \( R_p \) is a coherent analytic sheaf and finitely generated by the germs of \( q \)-tuples \( Q = (Q_1, \cdots, Q_p) \) with polynomial components such that

\[
\sum_{k=1}^{p} P_{j,k} Q_k = 0, \quad j = 1, \cdots, p.
\]

(See Lemma 7.6.3 of L. Hörmander [1].)

If \( \phi \) is a continuous function, we define \( C^{q} (U \cap Q, R_p, \phi) \) as the set of alternating cochains \( c = \{ c_s \} \in I^{r-1} \) where \( c_s \in \Gamma (U \cap Q, R_p) \), and

\[
\| c \| = \sum \int_{U \cap Q} | c_s |^2 e^{-\phi} d\lambda < \infty.
\]

We define \( \rho_{s, \phi} : C^{q} (U \cap Q, R_p, \phi) \to C^{q} (U \cap Q, R_p, \phi) \) by setting \( (\rho_{s, \phi} c) \), equal to the restriction of \( c_{s+1} (\cdot, \cdots, \cdot, \cdot, \cdot) \) to \( U \).

**Proposition 3.1.** Let \( \phi \) be a plurisubharmonic function in an open set \( V \) in \( C^* \), and \( Q \) be a pseudoconvex domain contained in \( V \). For every cochain \( c \in C^{q} (U \cap V, \mathcal{O}, \phi) \) with \( \delta c = 0 \), one can find a cochain \( c' \in C^{q-1} (U^\alpha \setminus O, \mathcal{O}, \phi) \) so that \( \delta c' = \rho_{s, \phi} c \)

\[
(3.2) \quad \| c' \|_\phi \leq K \| c \|_\phi.
\]

Here \( K \) is a constant independent of \( \phi \) and \( c \), and \( \psi \) is defined by \( \psi (z) = \phi (z) + 2 \log (1 + |z|^2) \).

We prove this in a way similar to Proposition 7.6.1 of L. Hörmander [1], so that we need the following lemma.

**Lemma 3.2.** Let \( Q \) be a pseudoconvex domain and let \( Q' \) be a relatively compact subset of \( Q \). For every plurisubharmonic function \( \phi \) in \( Q \) and every \( f \in L^r_{(0, q+1)} (Q, \phi) \) with \( \partial f = 0 \), one can find \( u \in L^r_{(0, q)} (Q, \text{loc}) \) with \( \partial u = f \) and

\[
\int_{Q'} |u|^2 e^{-\phi} d\lambda \leq K \int_{Q} |f|^2 e^{-\phi} d\lambda
\]

where \( K \) is independent of \( u \) and \( \phi \).

**Proof.** See Lemma 7.6.2 of L. Hörmander [1].
Proof of Proposition 3.1. We introduce the space $C^p(\mathcal{U}^{(\omega)} \cap V, \mathcal{L}_q, \phi)$ of all alternating cochains $c = \{c_s\}, s \in I^{p+1}$, where $c_s \in L^2(\mathcal{U}^{(\omega)} \cap V, \phi)$, $\partial c_s = 0$ and

$$\|c\|_p^2 = \sum_{|s| = p+1} \int_{\mathcal{U}^{(\omega)} \cap V} |c_s|^2 e^{-\phi} d\lambda < \infty.$$  

We wish to prove that if $\partial c = 0$, then one can find $c' \in C^{p-1}(\mathcal{U}^{(\omega)} \cap Q^{p-1}, \mathcal{L}_q, \phi)$ so that $\partial c' = \rho_{p+p-1}^* c$ and (3.2) hold.

For $q=0$, this assertion is precisely Proposition 3.1. We shall prove it assuming, if $p>1$, that it has already been proved for smaller values of $p$ and all $q$.

Choose a non-negative function $\chi \in C^\infty_c(\mathcal{U}^{(\omega)})$ such that $\sum \chi(z - g) = 1$. Now set $b_s = \sum \chi(z - g) c_{p,s}, s \in I^p$, then we have $\partial b = c$ and

$$|b_s|^2 \leq \sum \chi(z - g) |c_{p,s}|^2,$$

hence

$$\|b\|_p \leq \|c\|_p.$$  

Let $\bar{\partial} b$ be the cochain belonging to $C^{p-1}(\mathcal{U}^{(\omega)} \cap V, \mathcal{L}_q, \phi)$ defined by $$(\bar{\partial} b)_s = \bar{\partial} b_s = \sum \bar{\partial} \chi(z - g) / \chi.$$  

Then we obtain with a constant $K$

$$\|\bar{\partial} b\|_p \leq K \|c\|_p.$$  

Now $\partial \bar{\partial} b = \bar{\partial} b = \partial c = 0$. If $p>1$, we can by the inductive hypothesis find a cochain $b' \in C^{p-2}(\mathcal{U}^{(\omega)} \cap Q^{p-2}, \mathcal{L}_q, \phi)$ such that $\partial b' = \rho_{p+p-1}^* \bar{\partial} b$ and for some constant $K_1$

$$\|b'\|_p \leq K_1 \|\bar{\partial} b\|_p \leq K K_1 \|c\|_p.$$  

Since $\bar{\partial} b' = 0$ and $\psi$ is plurisubharmonic, by Lemma 3.2 we can choose $b'' \in L^2(\mathcal{U}^{(\omega)} \cap Q^{p+1}, \psi)$ for every $s \in I^{p-1}$ satisfying $U^{(\omega)}_s \subset Q^{p+1}$, $s' = \rho_{p-2,p+p-1}^* s$ so that $\bar{\partial} b'' = b''$ in $U^{(\omega)}_s$ and with a constant $K_2$,

$$\int_{U^{(\omega)}_s} |b''|^2 e^{-\phi} d\lambda \leq K_2 \int_{U^{(\omega)}_s} |b''|^2 e^{-\phi} d\lambda.$$  

Now set

$$c' = \rho_{p+p-1}^* b - \bar{\partial} b''.$$  

Then $\partial c' = \rho_{p+p-1}^* \partial b = \rho_{p+p-1}^* c$, and

$$\bar{\partial} c' = \rho_{p+p-1}^* \bar{\partial} b - \partial \bar{\partial} b'' = \rho_{p+p-1}^* \bar{\partial} b - \partial \rho_{p+p-1}^* \partial b \partial b''.$$
Summing up the estimates for $b$, $b'$ and $b''$ given above, we obtain $c' \in C^{p-1}(\mathcal{U}^{(\sigma+\nu)}, \mathcal{Q}^{\nu+\sigma-1}, \mathcal{L}_q, \psi)$ and the estimate (3.2).

It remains to consider the case $p=1$. The fact that $\overline{\partial}b = 0$ then means that $\overline{\partial}b$ defines uniquely a form $f$ of type $(0, q+1)$ in $\mathcal{V}$ with $\overline{\partial}f = 0$ and

$$\int \left| f \right|^2 e^{-\mu}d\lambda \leq \left\| \overline{\partial}b \right\|_2^2 \leq K^2 \left\| c \right\|_2^2.$$ 

By Theorem 4.4.2 of L. Hörmander [1], we can find a form $u \in L^{2b}(\mathcal{Q}, \psi)$ so that $\overline{\partial}u = f$ and

$$\int \left| u \right|^2 e^{-\mu}d\lambda \leq \int \left| f \right|^2 e^{-\mu}d\lambda.$$

Setting $c' = b - u$, we obtain $c' \in C^q(\mathcal{U}^{(\sigma)}, \mathcal{Q}, \mathcal{L}_q, \psi)$ and the estimate (3.2).

**Proposition 3.3.** Let $P$ be a matrix with polynomial entries and $\mathcal{Q}$ be a neighbourhood of 0. Then there exists a neighbourhood $\mathcal{Q}'$ of 0 such that for every $u \in \mathcal{O}(\mathcal{Q} + z)^q$ one can find $v \in \mathcal{O}(\mathcal{Q} + z)^q$ satisfying $Pv = Pu$, and

$$\sup_{\mathcal{A} + \mathbf{z}} |v| \leq C(1 + |z|)^x \sup_{\mathcal{A} + \mathbf{z}} |Pu|,$$

where the constants $C$ and $N$ are independent of $u$ and $z \in \mathcal{C}^q$.

**Proof.** See Proposition 7.6.5 of L. Hörmander [1].

**Proposition 3.4.** Let a matrix $P$ and an integer $v$ be given. Then there exist integers $\mu$ and $N$ such that, if $\phi$ is plurisubharmonic in a pseudoconvex domain $\mathcal{Q}$ and for some constant $C > 0$

$$|\phi(z) - \phi(z')| < C, \quad |z - z'| < 1,$$

then for every $c \in C^q(\mathcal{U}^{(\sigma)} \cap \mathcal{Q}^{\nu+\phi}, \mathcal{R}_p, \psi)$ with $\delta c = 0$, $\sigma > 0$, $\lambda \leq \nu$, one can find $c' \in C^{q-1}(\mathcal{U}^{(\sigma)} \cap \mathcal{Q}^{\nu+\phi}_{(\delta-\nu)}, \mathcal{R}_p, \phi_y)$ so that $\delta c' = \rho^\phi c$ and for some constant $K$

$$\|c'\|_{\phi} \leq K\|c\|_{\phi}.$$
Here \( \phi_N(z) = \phi(z) + N \log(1 + |z|^2) \), \( \tau = 2^n \) and \( \varepsilon \geq \sqrt{2n} 3^{1-1} \).

**Proof.** We can also prove the proposition in a way similar to the proof of Theorem 7.6.10 of L. Hörmander [1]. We shall prove it by induction for decreasing \( \sigma \), noting that it is valid when \( \sigma > 2^n \), since there are no non-zero \( c \in C^s(Q^{(\omega)} \cap Q^{(\mu)}, \mathcal{R}, \phi) \). Thus assume that the theorem has been proved for all \( P \) when \( \sigma \) is replaced by \( \sigma + 1 \). By Lemma 7.6.4 of L. Hörmander [1], we have \( c = Qd_\psi \) for \( d \in C(Q^{(\omega)} \cap Q^{(\mu)}, \mathcal{O}, \phi_N) \). By Proposition 3.3 and the condition (3.4), if \( \mu \) is large we can choose \( d'_\psi \in \mathcal{O}(P^{(\sigma)}) \) so that \( Qd'_\psi = Qd_\psi = c_\psi \) in \( P^{(\sigma)} \) and

\[
\left| \int_{U^{(\sigma)}} |d'_\psi|^2 (1 + |z|^2)^{-\sigma} e^{-\phi} d_\lambda \right| \leq C \int_{U^{(\sigma)}} |c_\psi|^2 e^{-\phi} d_\lambda
\]

for \( s' = \rho_{\mu, s} \) and \( U^{(\sigma)} \subset Q^{(\mu)} \). Thus we have \( d'_\psi \in C^s(Q^{(\omega)} \cap Q^{(\mu)}, \mathcal{O}, \phi_N) \), \( \rho_{\mu, s} c = Qd_\psi \) and

\[
\|d'_\psi\|_{s'} \leq C_1 \|c\|_{\phi}.
\]

Since \( \delta c = 0 \), it follows that \( \delta Qd'_\psi = Q\delta d'_\psi = 0 \). Thus \( \delta d' = d'' \in C^{s+1}(Q^{(\omega)} \cap Q^{(\mu)}, \mathcal{R}, \phi_N) \), and since \( \delta d'' = 0 \) and \( \phi_N \) is plurisubharmonic, it follows by the inductive hypothesis that for suitable \( N' \) and \( \mu' > \mu \) we can find \( d'' \in C^s(Q^{(\omega)} \cap Q^{(\mu'}, \mathcal{R}, \phi_{N'}) \) so that \( \delta d'' = \rho_{\mu_{\mu'}, s} d'' \) and

\[
\|d''\|_{s_{\mu'}} \leq C_2 \|d''\|_{s_{\mu}}.
\]

Setting \( r = \rho_{\mu_{\mu'}, s} d'' \), we have \( \delta r = \rho_{\mu_{\mu'}, s} d'' \) and

\[
\|r\|_{s_{\mu'}} \leq C_3 \|r\|_{s_{\mu}} \leq C_4 \|c\|_{\phi}.
\]

Hence Proposition 3.1 shows that for some \( \mu'' > \mu' \) and \( N'' > N' \) one can find \( r' \in C^{s-1}(Q^{(\omega)} \cap Q^{(\mu'' \times \mu_{\mu'} \times \mu_{\mu''}), \mathcal{O}, \phi_{N''}) \) so that \( \rho_{\mu_{\mu'}, r, \mu''} = \delta r' \) and

\[
\|r'\|_{s_{\mu''}} \leq C_5 \|r\|_{s_{\mu'}} \leq C_6 \|c\|_{\phi}.
\]

Here we used the fact that \( Q^{(\mu'') \times \mu_{\mu'} \times \mu_{\mu''}) \) is a pseudoconvex domain contained in \( Q^{(\mu', \mu'')} \) as \( \varepsilon \geq \sqrt{2n} 3^{1-1} \). If we set \( c' = Qr' \), it follows that

\[
\partial c' = Q\partial r' = Q\rho_{\mu_{\mu'}, r, \mu''} = \rho_{\mu_{\mu'}, s} \rho_{\mu_{\mu'}, r, \mu''} d' - \rho_{\mu_{\mu'}, s} Qd'' = \rho_{\mu_{\mu'}, s} c.
\]

Since (3.6) implies (3.5) for suitable \( \mu \) and \( N \), the proposition is
Proposition 3.5. Let \( \mathcal{Q}' \) be an open set which is strictly contained in a pseudoconvex domain \( \mathcal{Q} \) of \( \mathbb{C}^n \) (dist \((\mathcal{Q}', \mathcal{Q}) \geq \delta > 0\)). Given the system \( P \) there is a constant \( N \) such that, if \( \phi \) is a plurisubharmonic function satisfying (3.4), then for all \( u \in \mathcal{O}(\mathcal{Q})^q \) one can find \( v \in \mathcal{O}(\mathcal{Q}')^q \) with \( Pv = Pu \) and

\[
(3.7) \quad \int_{\mathcal{Q}'} |v|^2 e^{-\phi} (1 + |z|^2)^{-N} d\lambda \leq C \int_{\mathcal{Q}} |Pu|^2 e^{-\phi} d\lambda
\]

where \( C \) is a constant independent of \( u \).

Proof. First, choose \( \nu \) so that \( \delta > \tau z = 2^m \sqrt{2n} 3^{1-\nu} \). By Proposition 3.3 we can choose \( \nu < \mu \) so that there exists an element \( u \in \mathcal{O}(U_{\nu}^{(q)})^q \) such that \( Pu = Pu \) in \( U_{\nu}^{(q)} \subset U_{\nu}^{(q)} \subset \mathcal{Q} \), and for some constants \( C \) and \( N \) independent of \( u \) and \( g \in I \)

\[
(3.8) \quad \int_{U_{\nu}^{(q)}} |u|^2 e^{-\phi} (1 + |z|^2)^{-N} d\lambda \leq C \int_{U_{\nu}^{(q)}} |Pu|^2 e^{-\phi} d\lambda
\]

where \( g' = \rho_{m,\nu} g \). Let \( c_{\nu,\mu} = u_{\nu} - u_{\mu} \). This defines a cocycle \( c \in C^\infty(U_{\nu}^{(q)} \cap \mathcal{Q}^q, R_p, \phi_{\nu}) \) and by (3.8) we obtain

\[
(3.9) \quad \|c\|_{g,\nu}^2 \leq C' \int_{\mathcal{Q}} |Pu|^2 e^{-\phi} d\lambda.
\]

Proposition 3.4 asserts that for some \( \lambda > \mu \) and \( N' > N \) there exists a cochain \( c' \in C^\infty(U_{\lambda}^{(q)} \cap \mathcal{Q}', R_p, \phi_{\lambda'}) \) such that \( \delta c' = \rho_{\lambda,\mu} c |\mathcal{Q}' \) and

\[
(3.10) \quad \|c'\|_{g,\nu}^2 \leq C'' \|c\|_{g,\nu}.
\]

Here we used the fact that \( \mathcal{Q}' \) is contained in \( \Omega_{\nu}^{(i)} \) as \( \delta > \tau z \). This means that if we set \( v = u_{\nu,\mu} + c' \in U_{\nu}^{(q)} \cap \mathcal{Q}' \), we define uniquely an element \( v \in \mathcal{O}(\mathcal{Q}')^q \). Since \( Pu' = 0 \), it follows that \( Pv = Pu' \), and from the estimates (3.8), (3.9) and (3.10) we obtain (3.7) with \( N \) replaced by \( N' \).

§ 4. Soft Resolution of \( \mathcal{Q}_{k,i} \)

In this section, we define the space \( \mathcal{Q}(\mathcal{Q}) \) of rapidly decreasing distributions, and using this space we make a resolution of \( \mathcal{Q}^{(i)} \), that is,
Definition 4.1. Let $\mathcal{Q}$ be an open set in $\mathbb{Q}^{k,1}$. We denote by $\mathcal{G}(\mathcal{Q})$ the inductive limit $\lim_{K \subset 2 \mathcal{F}_e(K)}$ of $\mathcal{F}_e(K)$, where $K$ is a compact set in $\mathcal{Q}$. We denote by $\mathcal{G}'(\mathcal{Q})$ the dual space of $\mathcal{G}(\mathcal{Q})$.

Since the injection of $\mathcal{G}(\mathcal{Q})$ into $\mathcal{F}(\mathcal{Q})$ (Definition 2.13 of [I]) is continuous and of dense range, $\mathcal{F}'(\mathcal{Q})$ is a linear subspace of $\mathcal{G}'(\mathcal{Q})$. Moreover, we have the following proposition.

Proposition 4.2. An element of $\mathcal{G}'(\mathcal{Q})$ belongs to $\mathcal{F}'(\mathcal{Q})$ if and only if it has a compact support.

Proof. Let $T \in \mathcal{F}'(\mathcal{Q})$. By the definition of the topology of $\mathcal{F}(\mathcal{Q})$ (see Definition 2.13 of [I]), there are a compact set $K$ in $\mathcal{Q}$, an integer $m \geq 0$, and a constant $C > 0$ such that for all $f \in \mathcal{F}(\mathcal{Q})$,

$$|\langle T, f \rangle| \leq C \sup_{|x| \leq m, x \in K \cap \mathbb{C}^n} |D^m f(x)| e^{-|x|/(m+1)}.$$

This implies immediately that $\langle T, f \rangle = 0$ whenever the support of $f$ is contained in the complement of $K$, which means that $\text{supp} T \subset K$.

Conversely if $T$ is an element of $\mathcal{G}'(\mathcal{Q})$ with the compact support $K$. Let $\alpha(x) \in \mathcal{F}_e(\mathcal{Q})$ be equal to one in some neighbourhood of $K$. Then $\langle T, f \rangle = \langle T, \alpha f \rangle$ and if $\phi$, converges to zero in $\mathcal{F}(\mathcal{Q})$, $\alpha \phi$, converges to zero in $\mathcal{G}(\mathcal{Q})$. Therefore $\mathcal{F}(\mathcal{Q}) \ni \phi \to \langle T, \phi \rangle$ is continuous, hence $T \in \mathcal{F}'(\mathcal{Q})$.

Proposition 4.3. If $\mathcal{Q}$ is a bounded open set in $\mathcal{C}^n$ then $\mathcal{G}'(\mathcal{Q}) = \mathcal{D}'(\mathcal{Q})$.

Proof. It is obvious, since $\mathcal{G}(\mathcal{Q}) = \mathcal{D}(\mathcal{Q})$.

Proposition 4.4. Let $K$ be a compact subset of $\mathbb{Q}^{k,1}$ defined in Definition 2.1, and $\mathcal{Q}$ be a neighbourhood of $K$. For $f \in \mathcal{F}'(\mathcal{Q})$, define

$$\hat{f}(\xi) = \langle f, e^{-i\xi \cdot \cdot \cdot} \rangle / (2\pi)^n.$$
then \( \hat{f}(\zeta) \) is analytic in \( \{ \zeta \in \mathbb{C}^n; |\text{Im } \zeta| < \varepsilon \} \) for some \( \varepsilon > 0 \) and there exists an \( N \) satisfying \( |\hat{f}(\zeta)| \leq C(1 + |\zeta|)^N \) for \( |\text{Im } \zeta| < \varepsilon \). The equality

\[
(4.2) \quad \langle f, v \rangle = \int_{\mathbb{R}^{2n}} \hat{f}(\xi + i\eta) \overline{v}(\xi + i\eta) \, d\xi
\]

holds for \( v \in \mathcal{D}(K) \) and \( \eta \in K^0 \) with \( |\eta| < \varepsilon \).

Proof. By the definition of the topology of \( \mathcal{D}(\mathcal{O}) \), there exists a seminorm \( \| \cdot \|_{L, N, \varepsilon} \) satisfying \( \| f, v \|_{L, N, \varepsilon} \leq C \| v \|_{L, N, \varepsilon} \) for some constant \( C \), where \( \| v \|_{L, N, \varepsilon} = \sup_{x \in L \cap \mathbb{R}^{2n}, |a| \leq N} |D^\alpha f(x) e^{-\varepsilon|x|}| \) for the compact set \( L \) in \( \mathcal{O} \) and \( \varepsilon > 0, N > 0 \). If \( |\text{Im } \zeta| < \varepsilon \), then

\[
\| e^{-i(x, \zeta)} \|_{L, N, \varepsilon} = \sup_{x \in L \cap \mathbb{R}^{2n}, |a| \leq N} |e^{-i(x, \zeta)}| e^{-\varepsilon|x|} \leq \sup_{|a| \leq N} \{ |\zeta| \} \leq (1 + |\zeta|)^N < \infty .
\]

Hence \( \hat{f}(\zeta) = \langle f, e^{-i(x, \zeta)} \rangle / (2\pi)^n \) is analytic in \( |\text{Im } \zeta| < \varepsilon \) and satisfies \( |\hat{f}(\zeta)| \leq C(1 + |\zeta|)^N \). Since

\[
v(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-i(x, \xi + i\eta)} \overline{v}(\xi + i\eta) \, d\xi
\]

by Proposition 2.11, and the Riemann sum converges with respect to the seminorm \( \| \cdot \|_{L, N, \varepsilon} \), then

\[
\langle f, v \rangle = \langle f, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-i(x, \xi + i\eta)} \overline{v}(\xi + i\eta) \, d\xi \rangle = \int_{\mathbb{R}^{2n}} \hat{f}(\xi + i\eta) \overline{v}(\xi + i\eta) \, d\xi .
\]

Remark 4.5. The equality (4.2) holds when \( v \) satisfies \( |D^\alpha v(x)| \leq C e^{\varepsilon|x|} \) for \( |\alpha| \leq N + 3n \) and \( \delta > 0 \) such that \( K^\delta \) has an element \( \gamma \) satisfying \( |\gamma| < \varepsilon \).

Let \( \overline{\partial} \) be the Cauchy-Riemann operator defined by

\[
\overline{\partial}: u = \sum_{i_1, \ldots, i_p < i_p} u_{i_1, \ldots, i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \rightarrow
\]

\[
\overline{\partial} u = \sum_{i_1, \ldots, i_p < i_p} \left( \partial u_{i_1, \ldots, i_p} / \partial \bar{z}_j \right) d\bar{z}_j \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} .
\]
If we identify forms $u$ and $w$ with vector functions $\tilde{u}$ and $\tilde{w}$ having $\binom{n}{p}$ and $\binom{n}{p+1}$ components respectively, $\delta_p$ can be represented by $P_p(D)$ where $P_p(\zeta)$ is a $\binom{n}{p} - \binom{n}{p+1}$ matrix with polynomial entries, and $D = i\partial/\partial x$. It is known as the Koszul resolution that the following sequence is exact:

$$0 \rightarrow A^{tP_{p-1}(\zeta)} \rightarrow A^n \rightarrow \cdots \rightarrow A^{tP_p(\zeta)} \rightarrow \cdots$$

$$\cdots \rightarrow tP_p(\zeta) \rightarrow A \rightarrow \text{Coker } tP_p(\zeta) \rightarrow 0,$$

where $A$ is the polynomial ring of the variable $\zeta = (\zeta_1, \cdots, \zeta_n)$ and $tP_p(\zeta)$ is the transpose of $P_p(-\zeta)$ (see Example 4 in §7 of Chapter VII of V. P. Palamodov [5]). It is known that $R_{tP_p}$ is generated by the germs of the lows of the matrix $P_{p-1}(\zeta)$ (see Lemma 7.6.3 of L. Hörmander [1]). Since $R_{tP_p}$ is a coherent analytic sheaf, we have the following proposition.

**Proposition 4.6.** Let $\Omega$ be a pseudoconvex domain. If $f \in \mathcal{O}^s(\Omega)$ satisfies the equation $tP_p(\zeta)f(\zeta) = 0$, then there exists a $g \in \mathcal{O}^s(\Omega)$ satisfying $f(\zeta) = tP_{p+1}(\zeta)g(\zeta)$, where $r = \binom{n}{p+1}$ and $s = \binom{n}{p+2}$.

**Proof.** See Theorem 7.2.9 of L. Hörmander [1].

**Definition 4.7.** (The sheaf of rapidly decreasing distributions.) We denote by $\mathcal{G}'$ the sheaf determined by a presheaf $\{\mathcal{G}'(\mathcal{O})\}$, where $\Omega$ is an open set in $Q^{k,1}$.

For any locally finite covering $\{U_a\}$ of $\Omega$, there exists a partition of unity $\{\phi_a\}$ subordinate to the covering $\{U_a \cap \mathcal{C}^n\}$ such that all derivatives of $\phi_a$ are bounded. Then $\mathcal{G}'(\mathcal{O})$ is the section module of the sheaf $\mathcal{G}'$ and $\mathcal{G}'$ is a soft sheaf.

**Theorem 4.8.** Let $\Omega$ be a neighbourhood of a point $z_\infty$ at infinity in $Q^{k,1}$. If $f \in \mathcal{G}'_{(0,p)}(\Omega)$ satisfies $\bar{\partial}f = 0$, then there exists a neighbourhood $\omega$ of $z_\infty$ with $\omega \subseteq \Omega$ and $u \in \mathcal{G}'_{(0,p-1)}(\omega)$ such that $\bar{\partial}u = f$ in $\omega$. 
Proof. First we choose a neighbourhood \( \omega \) of \( z_0 \) having the form \( \omega = a + \text{Int } K \), where \( K \) is the compact set in \( Q^{k,i} \) defined in Definition 2.1 and \( a \in \mathbb{R}^n \).

Let \( L \) be a compact set in \( \varrho \) containing \( \omega \). Then \( f \in \mathcal{F}'_\varepsilon (L)^i \) and satisfies, for some \( m > 0, \varepsilon > 0, |\langle f, \psi \rangle| \leq C \| \psi \|_{m, \varepsilon} \) for all \( \psi \in \mathcal{F}_\varepsilon (L)^i \), where \( J = \left\{ \frac{n}{p} \right\} \). Hence

\[
(4.3) \quad |\langle f, \phi \rangle| \leq C \| \phi \|_{m, \varepsilon} \quad \text{for } \phi \in \varrho (\omega)^i,
\]

where \( \| \phi \|_{m, \varepsilon} = \sum_{j=1}^J \sup_{x \in \mathbb{R}^n, |\omega| \leq \varepsilon} |D^p \phi_j(x)| e^{-\varepsilon^{|x|}} \). If we can show that there exist \( M > 0 \) and \( \varepsilon > 0 \) satisfying

\[
(4.4) \quad |\langle f, \nu \rangle| \leq C \| \vartheta \nu \|_{m, \varepsilon} \quad \text{for all } \nu \in \varrho (\omega)(n) \omega,
\]

by the Hahn-Banach theorem there exists a \( \nu \in \mathcal{F}'_\varepsilon (\omega)^i \) satisfying \( \langle f, \nu \rangle = \langle u, \vartheta \nu \rangle \), that is, \( \vartheta u = f \) in \( \omega \), where \( \vartheta \) is the dual operator of \( \bar{\vartheta} \). Let \( \nu \in \varrho (\omega)^i \), then \( \text{supp } \nu \subset a + K \). By the coordinate transformation (translation) we may assume \( \text{supp } \nu \subset K \). Then, by Corollary 2.8, \( \vartheta (\zeta) \) is analytic for \( \text{Im } \zeta \in K^\circ \) and satisfies, for any \( \varepsilon > 0 \) and \( \nu > 0 \),

\[
|\vartheta (\zeta)| \leq C_{\varepsilon, \nu} \frac{1}{(1 + |\zeta|)} e^{h_{K, \varepsilon} (\text{Im } \zeta)} \quad \text{for } \text{Im } \zeta \in K^\circ.
\]

Let \( \vartheta_p \) be represented by \( P_p (D) \), then by Proposition 3.5 there exists an \( N \) such that for any \( \nu \) there exists a function \( V (\zeta) \) analytic for \( \text{Im } \zeta \in K^\circ_\varepsilon \) and satisfying

\[
\int_{\mathbb{R}^n + i \text{Int } K^\circ_\varepsilon} |V (\zeta)|^2 e^{-2h_{K, \varepsilon} (\text{Im } \zeta)} (1 + |\zeta|) e^{-N} d\lambda \leq \int_{\mathbb{R}^n + i K^\circ_\varepsilon} |V_p (\zeta) \vartheta (\zeta)|^2 e^{-2h_{K, \varepsilon} (\text{Im } \zeta)} (1 + |\zeta|) e^{-N} d\lambda < \infty,
\]

where we have used the fact that \( h_{K, \varepsilon} (\text{Im } \zeta) \) is a convex (hence plurisubharmonic) function satisfying the condition (3.4) and \( \mathbb{R}^n + i \text{Int } K^\circ_\varepsilon \) is a pseudoconvex domain strictly contained in \( \mathbb{R}^n + i \text{Int } K^\circ_\varepsilon \) (see Propositions 2.15 and 2.16). From the above inequality, we have

\[
|V (\zeta)| \leq C \frac{1}{(1 + |\zeta|) e^{h_{K, \varepsilon} (\text{Im } \zeta)}} \quad \text{for } \text{Im } \zeta \in K^\circ_\varepsilon.
\]
Propositions 2.10, 2.12 and the above inequality imply that $V(\zeta) = \vartheta_1(\zeta)$ for a $C^{r-N_{-n}}$ function $v_1$ with support contained in $K$ satisfying $\|v_1\|_{r-N_{-n}, \epsilon \lambda} < \infty$. From Propositions 3.5 and 4.6, there exists a function $\Phi(\zeta)$ analytic in $\{\zeta \in \mathcal{C}^n; \text{Im}\zeta \in \text{Int} K_{\mu}^0\}$ and satisfying $V(\zeta) - \vartheta(\zeta) = \epsilon P_{\mu}(\zeta) \Phi(\zeta)$ and

$$\int_{R^{2n} \times fK_{\mu}^0} |\Phi(\zeta)|^2 e^{-2hK_{\mu} \epsilon(\text{Im}\zeta)} (1 + |\zeta|^2)^{-N_{-n}} d\lambda < \infty,$$

for some constant $\epsilon$ depending only on $P_{\mu}(\zeta)$ and $P_{\mu-1}(\zeta)$. This implies that there exists a $C^{r-N_{-n}}$ function $\Phi$ with support contained in $K$, satisfying $\Phi(\zeta) = \tilde{\Phi}(\zeta)$ and $\|\tilde{\Phi}\|_{r-N_{-n}, \epsilon \lambda} < \infty$.

Considering the inequality (4.3), if we take sufficiently large $\nu>0$ and small $\epsilon>0$, we have

$$\langle f, v_1 \rangle - \langle f, v \rangle = \langle f, \epsilon P_{\mu}(D) \phi \rangle = \langle P_{\mu}(D)f, \phi \rangle = \langle \tilde{\phi}, \phi \rangle = 0.$$

Let $\alpha \in \mathcal{F}_{e}(L)$ with $\alpha(x) = 1$ on a neighbourhood of $\omega \cap R^m$. Define $f_0 = \alpha f$, then $f_0 \in \mathcal{F}'(D)$ by Proposition 4.2, and $\langle f, v \rangle = \langle f_0, v \rangle$ for any $C^\infty$ function $v$ with support contained in $\omega$ and satisfying $\|v\|_{m, \epsilon} < \infty$.

By Remark 4.5 if we take sufficiently large $\nu>0$ and small $\epsilon>0$, we have

$$|\langle f, v \rangle|^2 = |\langle f, v_1 \rangle|^2 = |\langle f_0, v_1 \rangle|^2 \leq \left( \int_{R^{2n}} \left| \tilde{f}_0(\xi + i\eta) \vartheta_1(\xi + i\eta) |d\xi|^2 \right|^2$$

$$\leq \int_{R^{2n}} |\tilde{f}_0(\xi + i\eta)|^2 (1 + |\xi|^2)^{r-N_{-n}} d\xi$$

$$\times \int_{R^{2n}} |\tilde{V}(\xi + i\eta)|^2 (1 + |\xi|^2)^{r-N_{-n}} d\xi$$

$$\leq C_1 \int_{R^{2n} \times fK_{\mu}^0} |\tilde{V}(\zeta)|^2 e^{-2hK_{\mu} \epsilon(\text{Im}\zeta)} (1 + |\zeta|^2)^{-N_{-n}} d\lambda$$

$$\leq C \int_{R^{2n} \times fK_{\mu}^0} \left| \epsilon P_{\mu-1}(\zeta) \tilde{\vartheta}(\zeta) \right|^2 e^{-2hK_{\mu} \epsilon(\text{Im}\zeta)} (1 + |\zeta|^2)^{-N_{-n}} d\lambda$$

$$\leq C \|\epsilon P_{\mu-1}(D)v\|_{N_{-\epsilon} \lambda} = C\|\tilde{\theta} v\|_{N_{-\epsilon} \lambda},$$

The last inequality follows from Proposition 2.13. Thus we have shown (4.4), and completed the proof.
Theorem 4.9. We have the following soft resolution of the sheaf \( \mathcal{O}_{k,1} \):

\[
0 \to \mathcal{O}_{k,1} \to \mathcal{O}_{(0,0)}' \to \mathcal{O}_{(0,1)}' \to \cdots \to \mathcal{O}_{(0,n)}' \to 0.
\]

Proof. Since the restriction of \( \mathcal{O}_{k,1} \) or \( \mathcal{O}' \) to \( C^n \) is \( \mathcal{O} \) or \( \mathcal{D}' \), respectively, and it is well known that the following sequence is exact:

\[
0 \to \mathcal{O} \to \mathcal{D}'_{(0,0)} \to \cdots \to \mathcal{D}'_{(0,n)} \to 0.
\]

In order to obtain the resolution (4.5), we have only to make it at points at infinity. It is done in Theorem 4.8.

Definition 4.10. Let \( K \) be the compact set in \( Q^{k,1} \) defined in Definition 2.1. Define \( I_{K,\varepsilon}(\gamma) = \sup_{x \in K \cap R^n} (|x, \gamma| - \varepsilon|x|) \) and \( K_{\varepsilon} = \{ \gamma \in R^{2n}; I_{K,\varepsilon}(\gamma) < \infty \} \).

Proposition 4.11. Let \( \Omega \) be an open set in \( Q^{k,1} \) containing \( K \). If \( f \in \mathcal{F}'(\Omega) \) satisfies the inequality \( \langle f, v \rangle \leq C \|v\|_{K,\varepsilon} \) for all \( v \in \mathcal{F}(\Omega) \), where \( \|v\|_{K,\varepsilon} = \sup_{x \in K \cap R^n, |\alpha| \leq N} |D^\alpha f(x)| e^{-\varepsilon|x|} \), then \( \hat{f}(\zeta) = \langle f, e^{-i(x,\zeta)} \rangle / (2\pi)^n \) is analytic in \( \{ \zeta \in C^{2n}; \text{Im} \zeta \in \text{Int} K_{\varepsilon} \} \) and satisfies, for some constant \( C \geq 0 \),

\[
|\hat{f}(\zeta)| \leq C (1 + |\zeta|)^N e^{iK,\varepsilon(\text{Im} \zeta)} \quad \text{for Im} \zeta \in K_{\varepsilon}.
\]

Proof. Let \( \zeta = x + i\eta \) and \( \gamma \in K_{\varepsilon} \). Then we have

\[
\|e^{-i(x,\zeta)}\|_{K,\varepsilon} = \sup_{z \in K \cap R^n, |\alpha| \leq N} |\zeta^\alpha e^{i(x,\eta)}| e^{-\varepsilon|x|} \leq (1 + |\zeta|)^N e^{iK,\varepsilon(\text{Im} \zeta)}.
\]

Since \( (e^{-i(x,\zeta)} - e^{-i(x,\zeta)}) / h \) converges to \( -ixe^{-i(x,\zeta)} \) as \( h \to 0 \) with respect to \( \| \cdot \|_{K,\varepsilon} \) for \( \text{Im} \zeta \in \text{Int} K_{\varepsilon} \), \( \hat{f}(\zeta) \) is analytic.

Proposition 4.12. Let \( F(\zeta) \) be an analytic function in \( \{ \zeta \in C^{2n}; \text{Im} \zeta \in \text{Int} K_{\varepsilon} \} \) satisfying the inequality (4.6). Then \( F(\zeta) \) defines an element \( f \in \mathcal{F}'(Q^{k,1}) \) with support contained in \( K \) satisfying

\[
\langle f, \phi \rangle = \int_{R^{2n} + i\eta} F(\zeta) \overline{\phi}(\zeta) d\zeta \quad \text{for} \quad \phi \in C_0^\infty(R^{2n}).
\]
Proof. If $\phi \in C^\omega_c (\mathbb{R}^n)$, then $\tilde{\phi} (\zeta)$ is an entire function satisfying for any $\nu > 0$

$$|\tilde{\phi} (\zeta)| \leq C e^{\alpha \nu (\text{Im} \zeta) / (1 + |\zeta|)^{\nu}} ,$$

where $B$ is the support of $\phi$ and $h_B (\eta) = \sup_{x \in B} (-<x, \eta>)$. Hence the linear form

$$\int_{\mathbb{R}^n} F (\zeta) \tilde{\phi} (\zeta) d\zeta = \langle f, \phi \rangle$$

defines a distribution $f$. Let $B$ be convex and $B \cap K = \phi$, then there exists a vector $\eta \in (-K^0) \subset K^0_{\delta}$ such that for some $\delta > 0$

$$\sup_{x \in K \cap \mathbb{R}^n} \langle x, \eta \rangle \leq \langle y, \eta \rangle - \delta |\eta|$$

for all $y \in B$, hence $I_{K, \varepsilon} (\eta) + h_B (\eta) \leq -\delta |\eta|$. Thus we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} F (\zeta) \tilde{\phi} (\zeta) d\zeta$$

$$\leq \lim_{\varepsilon \to 0} C e^{\alpha \varepsilon (\text{Im} \zeta) + h_B (\zeta)} \leq \lim_{\varepsilon \to 0} C e^{-\delta |\zeta|} = 0 .$$

Hence the support of $f$ is contained in $K$. Let $L$ be a neighbourhood of $K$ having the form of Definition 2.1. If $\psi \in \mathcal{F}_c (L)$ then $\tilde{\phi} (\zeta)$ is analytic in \{ $\zeta \in \mathbb{C}^n; \text{Im} \zeta \in L^0$ \} and satisfies for any $\nu > 0$ and $\varepsilon > 0$

$$|\tilde{\phi} (\zeta)| \leq C e^{\alpha \varepsilon (\text{Im} \zeta) / (1 + |\zeta|)^{\nu}}$$

for $\text{Im} \zeta \in L^0_\varepsilon$. Hence it follows from the formula

$$\int_{\mathbb{R}^n} F (\zeta) \tilde{\phi} (\zeta) d\zeta$$

that the distribution $f$ can be extended to $\mathcal{F}_c (L)$. Let $\alpha \in \mathcal{F}_c (L)$ such that $\alpha (x) = 1$ in a neighbourhood of $K$, then $\alpha \psi \in \mathcal{F}_c (L)$ for $\psi \in \mathcal{F}(\mathbb{R}^n_i)$. Since the support of $f$ is contained in $K$, we have $\langle f, \psi \rangle = \langle f, \alpha \psi \rangle$. This shows that $f \in \mathcal{F}'(\mathbb{R}^n_i)$.

Let $\mathcal{Q}$ be an open set in $\mathbb{Q}^{k \cdot i}$ which has the form $a + \text{Int} K$, where $K$ is the convex set defined in Definition 2.1 and $a \in \mathbb{C}^n$.

**Theorem 4.13.** If $\tilde{\partial}_p \nu = 0$ for $\nu \in \mathcal{F}_{(0, p)} (\mathcal{Q})$, then there exists $u \in \mathcal{F}_{(0, p - 1)} (\mathcal{Q})$ satisfying $\tilde{\partial}_{p - 1} u = \nu$. 
Proof. We represent $\delta_\nu$ by $P_\nu(D)$. Since all the spaces of the sequence

$$
\mathbb{T}(\mathcal{Q})^q \xrightarrow{P_{\nu-1}(D)} \mathbb{T}(\mathcal{Q})^r \xrightarrow{P_\nu(D)} \mathbb{T}(\mathcal{Q})^s
$$

are $FS$ spaces (see Remark 2.27 in [I]), we have only to show that the dual sequence

$$
\mathbb{T}'(\mathcal{Q})^q \xleftarrow{^tP_{\nu-1}(D)} \mathbb{T}'(\mathcal{Q})^r \xleftarrow{^tP_\nu(D)} \mathbb{T}'(\mathcal{Q})^s
$$

is exact and the range of $^tP_{\nu-1}(D)$ is closed.

Let $g \in \mathbb{T}'(\mathcal{Q})^r$, then there exist a convex set of the form $b + L$ contained in $\mathcal{Q}$ and constants $N > 0$, $\varepsilon > 0$ such that the estimate

$$
|\langle g, v \rangle| \leq C\|v\|_{b + L, N, \varepsilon}
$$

holds for all $v \in \mathbb{T}(\mathcal{Q})^r$. We may assume that $L$ is also a convex set of the type in Definition 2.1. By coordinate transformation (translation) we may also assume $b = 0$. Then, by Proposition 4.11, $\tilde{g}(\zeta)$ is analytic in $\{\zeta \in C^n; \text{Im } \zeta \in \text{Int } L^0_\nu\}$ and satisfies

$$
|\tilde{g}(\zeta)| \leq C(1 + |\zeta|)^N e^{tL_{1/2}(|\text{Im } \zeta|)} \quad \text{for } \text{Im } \zeta \in L_\nu^0.
$$

The equation $^tP_{\nu-1}(D)g = 0$ implies $^tP_{\nu-1}(-\zeta)\tilde{g}(\zeta) = 0$ in $\{\zeta \in C^n; \text{Im } \zeta \in \text{Int } L^0_\nu\}$. Then by Propositions 3.5 and 4.6, there exists an analytic function $F(\zeta)$ such that $^tP_{\nu}(-\zeta)F(\zeta) = \tilde{g}(\zeta)$ for $\text{Im } \zeta \in \text{Int } L^0_{\nu/2}$ and satisfying for some $\nu > 0$

$$
|F(\zeta)| \leq C(1 + |\zeta|)^N e^{tL_{1/2}(|\text{Im } \zeta|)} \quad \text{for } \text{Im } \zeta \in L^0_{\nu/2}.
$$

Here we used the fact that $I_{L, \varepsilon}(\eta)$ is convex and Lipschitz continuous, and $L^0_{\nu/2}$ is a convex set contained strictly in $L^0_\nu$. This shows that there exists $f \in \mathbb{T}'(\mathcal{Q})^s$ such that

$$
\langle f, P_\nu(D)v \rangle = \int_{\mathbb{R}^n + t\eta} F(\zeta) P_\nu(\zeta) \overline{v}(\zeta) \, d\zeta
$$

$$
= \int_{\mathbb{R}^n + t\eta} ^tP_\nu(-\zeta) F(\zeta) \overline{v}(\zeta) \, d\zeta
$$

$$
= \int_{\mathbb{R}^n + t\eta} \tilde{g}(\zeta) \overline{v}(\zeta) \, d\zeta = \langle g, v \rangle
$$

for all $v \in \mathbb{T}(K)$, that is, $^tP_\nu(D)f = g$.

Next we prove the closedness of the range of $^tP_\nu(D)$. Assume
$F_j \to F$ in $\mathcal{E}'(\Omega)$ with $F_j = \iota^* P_0(D) G_j$ for $G_j \in \mathcal{E}'(\Omega)^n$. Since the sequence \{\(F_j\)\} is a bounded set in the DFS space $\mathcal{E}'(\Omega)$, there exist a compact set $L$ in $\Omega$ (we may assume that $L$ is a convex set of the type in Definition 2.1) and constants $C>0$, $\varepsilon>0$ satisfying
\[
\left| \tilde{F}_j(\zeta) \right| \leq C (1 + |\zeta|)^N e^{\varepsilon \Re(z)} \quad \text{for} \quad \Im \zeta \in L_{\varepsilon}. \]

By Proposition 3.5 we can choose $\Psi_j(\zeta)$ satisfying
\[
(4.8) \quad |\Psi_j(\zeta)| \leq C' (1 + |\zeta|)^N e^{\varepsilon \Re(z)} \quad \text{for} \quad \Im \zeta \in L_{\varepsilon}. \]

Since \{\(\Psi_j(\zeta)\)\} forms a normal family, there exists a subsequence which converges to $\Psi(\zeta)$ which also satisfies (4.8). Thus there exists $G \in \mathcal{E}'(\Omega)^n$ satisfying
\[
\langle G, P_0(D) v \rangle = \int_{\mathbb{R}^n + z \in L} \Psi(\zeta) P_0(\zeta) \overline{v}(\zeta) d\zeta
= \lim_{k \to \infty} \int_{\mathbb{R}^n + z \in L} i P_0(-\zeta) \Psi_j(\zeta) \overline{v}(\zeta) d\zeta
= \lim_{k \to \infty} \langle F_j, v \rangle = \langle F, v \rangle.
\]

This shows that $F = \iota^* P_0(D) G$, that is, the range of $\iota^* P_0(D)$ is closed.

At the end of this section, we give an extension of Theorem 4.11 of [1].

**Theorem 4.14.** We have the following soft resolution of the sheaf $\mathcal{O}_{k,1}$ on $\mathbb{Q}^{k+1}$:
\[
(4.7) \quad 0 \to \mathcal{O}_{k,1} \xrightarrow{\partial} \mathcal{E}(\mathbb{Q},0) \to \cdots \to \mathcal{E}(\mathbb{Q},n) \to 0. \]

**Proof.** Since the restriction of $\partial \mathcal{O}_{k,1}$ or $\mathcal{E}$ to $\mathbb{C}^n$ is $\mathcal{O}$ or $\mathcal{E}$ respectively, and it is well known that the following sequence is exact:
\[
0 \to \mathcal{O} \xrightarrow{\partial} \mathcal{E}(\mathbb{Q},0) \to \cdots \to \mathcal{E}(\mathbb{Q},n) \to 0. \]

In order to obtain the resolution (4.7) of $\partial \mathcal{O}_{k,1}$, we have only to make the resolution at points at infinity. Since the point $z_\infty$ at infinity has a fundamental system of neighbourhoods whose member has the form $a + \text{Int } K$, Theorem 4.13 gives the resolution at points at infinity.
Remark 4.15. In the above theorem the resolution is obtained on the whole $Q^{k,1}$, while in Theorem 4.11 of [I], it is obtained on the open subset $\mathcal{O}$ which satisfies the condition (i) of Definition 4.5 of [I].

§ 5. Fourier Hyperfunctions with Compact Supports

In this section, we show that the space $\mathcal{H}^*_K(V, \mathcal{O}_{k,1})$ of $E$-valued Fourier hyperfunctions is isomorphic to the space $L(Q_{k,i}(K), E)$ of continuous linear mappings from $Q_{k,i}(K)$ to a Fréchet space $E$.

Let $K$ be a compact set in $\bigcap_{i=1}^j D^{\alpha_i}$ and $V$ be an $\mathcal{O}_{k,1}$-pseudoconvex neighbourhood of $K$ in $Q^{k,1}$. From Theorem 5.8 and Corollary 5.10 of [I], we have $H^p_0(V, Q_{k,1}) = 0$ for $0 \leq p \leq n - 1$ and $H^p(K, Q_{k,1}) = 0$ for $p \geq 1$. Therefore from the long exact sequence of cohomology groups with compact supports,

$$0 \rightarrow H^0(V-K, Q_{k,1}) \rightarrow H^*_0(V, Q_{k,1}) \rightarrow H^0(K, Q_{k,1})$$

follows that $\delta : H^0(K, Q_{k,1}) \cong H^*_0(V-K, Q_{k,1})$ and $H^*_0(V-K, Q_{k,1}) = 0$, for $n \geq 2$.

Since by Theorem 4.9 we have the soft resolution

$$0 \rightarrow Q_{k,1} \rightarrow G_0^1 \rightarrow G_0^2 \rightarrow \cdots \rightarrow G_0^n \rightarrow 0,$$

$H^*_0(V-K, Q_{k,1})$ can be represented by the first cohomology group of the complex $(Q_{\alpha,1}, (V-K), \delta)$. Then $\delta$ can be represented as the following continuous mapping. Let $U$ be an open neighbourhood of $K$ and $\alpha \in \mathcal{F}_e(U)$ such that $\alpha = 1$ in $W \cap R^m$, where $W$ is some neighbourhood of $K$ in $U$. The map

$$\partial_{U,\alpha} : H^0(U, Q_{k,1}) \rightarrow \{u \in \mathcal{F}^1_{\alpha,1}(V-K) ; \bar{\delta}u = 0\}$$

defined by $\partial_{U,\alpha}(f) = \bar{\delta}(\alpha f)$ is continuous and induces a continuous map of $H^0(U, Q_{k,1})$ into $H^*_0(V-K, Q_{k,1})$. These maps define the map $\delta$ on the inductive limit $H^0(K, Q_{k,1}) = \lim_{\overrightarrow{U \supset K}} H^0(U, Q_{k,1})$ of $H^0(U, Q_{k,1})$ and therefore $\delta$ is continuous. Moreover we can show that $\delta$ is an open mapping.
Proposition 5.1. Let $n \geq 2$. Consider the dual complex,

$$
\begin{align*}
&\mathcal{F}(V-K, \mathcal{O}_{n-1}) \xrightarrow{\delta_{n-1}} \mathcal{F}(V-K, \mathcal{O}_{n-2}) \\
&\mathcal{F}'(V-K, \mathcal{O}_{n-1}) \xrightarrow{-\delta_{n-1}} \mathcal{F}'(V-K, \mathcal{O}_{n-2}) \\
&\mathcal{F}(V-K, \mathcal{O}_{n-1}) \xrightarrow{\delta_{n-1}} \mathcal{F}(V-K, \mathcal{O}_{n-2}) \\
&\mathcal{F}'(V-K, \mathcal{O}_{n-1}) \xrightarrow{-\delta_{n-1}} \mathcal{F}'(V-K, \mathcal{O}_{n-2})
\end{align*}
$$

Then the ranges of the operators are all closed.

Proof. $H^2_e(V-K, \mathcal{O}_{k,1}) = 0$ shows that the range of $-\delta_1$ is closed, and from Theorem 5.11 of [I], it follows that the range of $-\delta_{n-1}$ is closed. The closedness of ranges of other operators is a consequence of the so-called Serre-Komatsu duality theorem (see Theorem 4.7 of [I]).

Proposition 5.2. Let $n \geq 2$, then $H^0(K, \mathcal{O}_{k,1})$ and $H^1_e(V-K, \mathcal{O}_{k,1})$ are DFS spaces.

Proof. Proposition 2.7 of [I] shows that $H^0(K, \mathcal{O}_{k,1}) = \mathcal{O}_{k,1}(K)$ is a DFS space. $\mathcal{F}(V-K)$ is a DFS space as the dual space of an FS space $\mathcal{F}(V-K)$ (see Remark 2.27 of [I]). Since a closed subspace and a quotient space (by its closed subspace) of a DFS space are also DFS spaces, it follows from the fact that the range of $-\delta_0$ is closed, that $H^1_e(V-K, \mathcal{O}_{k,1})$ is a DFS space.

Theorem 5.3. Let $E$ be a fully complete space, and let $F$ be a barreled space. Let $f$ be a linear mapping of a subspace $E_0 \subset E$ onto $F$. Suppose that the graph of $f$ is closed in $E \times F$. Then $f$ is open.

Proof. See Theorem 4.10 of V. Pták [6].

Proposition 5.4. Let $n \geq 2$, then $\delta : H^0(K, \mathcal{O}_{k,1}) \to H^1_e(V-K, \mathcal{O}_{k,1})$ is a homeomorphism.

Proof. It is known that DFS spaces are fully complete and barreled spaces (see Theorems 4.3.28 and 4.3.40 of H. Komatsu [3]). Since $\delta$ is a one-to-one onto continuous mapping, it follows from Theorem
5.3 that $\delta$ is a homeomorphism.

**Proposition 5.5.** Let $n \geq 2$, then $H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,i}) \cong [\mathcal{O}_{k,i}(K)]'$.

*Proof.* $H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,i})$ is represented by the $(n-1)$-th cohomology group of the complex $(\mathcal{F}_{(0,n-1)}(V-K), \delta)$. It follows from Proposition 5.1 and the so-called Serre-Komatsu duality theorem (Theorem 4.7 of [I]) that

$$H^{n-1}(V-K, \tilde{\mathcal{O}}_{k,i}) \cong [H^i_k(V-K, \mathcal{O}_{k,i})]' \cong [\mathcal{O}_{k,i}(K)]'.$$

Let $E$ be a Fréchet space. From the exact sequence,

$$(5.2) \quad \cdots \to H^p_k(V, \mathcal{O}_{k,i}) \to H^p(V, \mathcal{O}_{k,i}) \to H^{p+1}_k(V, \mathcal{O}_{k,i}) \to \cdots$$

and the fact that if $V$ is $\mathcal{O}_{k,i}$-pseudoconvex, $H^p(V, \mathcal{O}_{k,i}) = 0$ for $p > 0$ (see Theorem 6.6 of [I]), it follows that $H^p_k(V, \mathcal{O}_{k,i}) \cong H^{n-1}(V-K, \mathcal{O}_{k,i})$, for $n \geq 2$.

**Proposition 5.6.** Let $n \geq 2$, then $H^{n-1}(V-K, \mathcal{O}_{k,i}) \cong H^{n-1}(V-K, \mathcal{O}_{k,i}) \hat{\boxtimes} E$ for a Fréchet space $E$.

*Proof.* We represent $H^{n-1}(V-K, \mathcal{O}_{k,i})$ by the $(n-1)$-th cohomology group of the complex,

$$\cdots \to \mathcal{F}_{(0,n-1)}(V-K) \xrightarrow{\delta_{n-2}} \mathcal{F}_{(0,n-2)}(V-K) \xrightarrow{\delta_{n-3}} \cdots$$

Since the range of $\delta_{n-2}$ is closed by Proposition 5.1 and $\mathcal{F}_{(0,n-1)}(V-K)$ is a Fréchet nuclear space, we have the exact sequence

$$(5.3) \quad 0 \to \text{im} \delta_{n-2} \to \text{ker} \delta_{n-1} \to \text{ker} \delta_{n-1}/\text{im} \delta_{n-2} \to 0$$

where all the spaces are Fréchet nuclear spaces. Since the tensoring by $\hat{\boxtimes} E$ is an exact functor (see Theorem 6.5 of [I]), we have the following exact sequence:

$$(5.4) \quad 0 \to (\text{im} \delta_{n-2}) \hat{\boxtimes} E \to (\text{ker} \delta_{n-1}) \hat{\boxtimes} E \to H^{n-1}(V-K, \mathcal{O}_{k,i}) \hat{\boxtimes} E \to 0.$$
\[ \ker(\tilde{\sigma}_{n-1} \otimes 1_E) = [f \otimes v \in \mathcal{F}_{(n-1)}(V-K) \hat{\otimes} E; \tilde{\sigma}_{n-1}f = 0] \\
= (\ker(\tilde{\sigma}_{n-1}) \hat{\otimes} E). \]

By Proposition 43.9 of F. Treves [7], we also have \( \text{im} (\tilde{\sigma}_{n-2} \otimes 1_E) = (\text{im} \tilde{\sigma}_{n-2}) \hat{\otimes} E \). Since \( H^{n-1}(V-K, \mathcal{E}_{k,1}) \) can be represented by the \((n-1)\)-th cohomology group of the complex \( (\mathcal{F}_{\omega^2}, (V-K, E), \mathcal{E}) \) and \( \mathcal{F}_{\omega^2}, (V-K, E) \cong \mathcal{F}_{\omega^2}, (V-K) \hat{\otimes} E \) and \( \mathcal{E} = \tilde{\sigma} \hat{\otimes} 1_E \), we have \( H^{n-1}(V-K, \mathcal{E}_{k,1}) \cong H^{n-1}(V-K, \mathcal{E}_{k,1}) \hat{\otimes} E \).

**Theorem 5.7.** Let \( E \) be a Fréchet space and \( K \) be a compact set in \( \mathfrak{M}' \). Then \( H^n_{\mathfrak{K}}(V, \mathcal{E}_{k,1}) \cong L(\mathcal{Q}_{k,1}(K), E) \).

**Proof.** By Proposition 50.5 of F. Treves [7], we have \( L(\mathcal{Q}_{k,1}(K), E) \cong \left[ \mathcal{Q}_{k,1}(K) \right]' \hat{\otimes} E \). Propositions 5.5 and 5.6 show that \( \left[ \mathcal{Q}_{k,1}(K) \right]' \hat{\otimes} E \cong H^{n-1}(V-K, \mathcal{E}_{k,1}) \), for \( n \geq 2 \). Thus we have \( H^n_{\mathfrak{K}}(V, \mathcal{E}_{k,1}) \cong L(\mathcal{Q}_{k,1}(K), E) \), for \( n \geq 2 \).

If \( n = 1 \), \( H^1(W, \mathcal{E}_{k,1}) = 0 \) for any open set \( W \) in \( Q^{k,1} \) satisfying the condition (i) of Definition 5.1 of [I] (Theorem 5.11 of [I]). Consider the dual complex,

\[
0 \rightarrow \mathcal{F}_{(n,0)}(W) \xrightarrow{\tilde{\delta}} \mathcal{F}_{(n,1)}(W) \rightarrow 0 \\
\downarrow \quad \downarrow \\
0 \leftarrow \mathcal{F}_{(n,0)}'(W) \leftarrow \mathcal{F}_{(n,0)}(W) \leftarrow 0.
\]

Then the range of \( \tilde{\delta} (= \mathcal{F}_{(n,0)}'(W)) \) is closed, therefore the range of \(-\tilde{\delta}\) is closed and

\[
\mathcal{E}_{k,1}(W) \cong \left[ H^1_k(W, \mathcal{Q}_{k,1}) \right]'.
\]

The mapping \( \rho \) of the exact sequence

\[
0 \rightarrow H^k(K, \mathcal{E}_{k,1}) \xrightarrow{\delta} H^1_k(V-K, \mathcal{Q}_{k,1}) \xrightarrow{\rho} H^1_k(V, \mathcal{Q}_{k,1}) \rightarrow 0
\]

is continuous since it is induced by the continuous injection of \( \mathcal{F}'(V-K) \) into \( \mathcal{F}'(V) \). Therefore the dual sequence

\[
0 \rightarrow \mathcal{E}_{k,1}(V) \xrightarrow{\delta} \mathcal{E}_{k,1}(V-K) \xrightarrow{\delta^*} \mathcal{Q}_{k,1}(K) \rightarrow 0
\]

is exact. Since all the spaces of the above sequence are Fréchet nuclear, we have the exact sequence.
where we used the fact that \( \mathcal{O}_{\mathcal{K},i}(W, E) \cong \mathcal{O}_{\mathcal{K},i}(W) \otimes E \) for an open set \( W \) in \( Q^{k,i} \) \((6.6) \) of \([I]\)) and the tensoring \( \otimes E \) is an exact functor (Theorem 6.5 of \([I]\)). Thus we have

\[
H^p_k(V, \mathcal{O}_{\mathcal{K},i}) \cong \mathcal{O}_{\mathcal{K},i}(V - K, E) \otimes [\mathcal{O}_{\mathcal{K},i}(K)]' \otimes E
\]

for \( n = 1 \).

**Corollary 5.8.** Let \( \Omega \) be an open set in \( \prod_{i=1}^{j} D^{v_i} \). Then \( E^* \mathcal{R}_{\mathcal{K},i}(\Omega) \cong L(\mathcal{O}_{\mathcal{K},i}(\Omega), E) / L(\mathcal{O}_{\mathcal{K},i}(\partial \Omega), E) \).

**Proof.** The corollary follows from Proposition 6.10 of \([I]\) and Theorem 5.7.

Without changing the proof of Theorem 5.7, we can prove the following theorem, which corresponds to Theorem 5.12 of \([I]\) in the scalar valued case.

**Theorem 5.9.** Let \( K \) be a compact set in \( Q^{k,i} \), and \( V \) be an \( \mathcal{O}_{\mathcal{K},i} \)-pseudoconvex domain containing \( K \). Suppose \( H^p(K, \mathcal{O}_{\mathcal{K},i}) = 0 \) for \( p \geq 1 \). Then we have

\[
H^p_k(V, \mathcal{O}_{\mathcal{K},i}) \cong L(\mathcal{O}_{\mathcal{K},i}(K), E).
\]

**Remark 5.10.** We can also prove \( H^p_k(V, \mathcal{O}_{\mathcal{K},i}) = 0 \) for \( p \neq n \), for a compact set \( K \) satisfying the condition of the above theorem, in the same way as Theorem 6.8 of \([I]\).

**References**


