On Lie Algebras of Vector Fields on Smooth Orbifolds

By

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§ 0. Introduction

In the previous paper [1], we studied Pursell–Shanks type theorem for orbit spaces of \(G\)-manifolds. The purpose of this paper is to prove that this still holds for smooth orbifolds and for fibrations over smooth orbifolds with compact generic fibre.

Let \(B\) and \(B'\) be connected smooth orbifolds. Let \(\mathfrak{X}(B)\) (resp. \(\mathfrak{X}(B)\)) be the Lie algebra of all smooth vector fields (resp. strata preserving smooth vector fields) on \(B\) with compact support.

**Theorem 0.1.** The following statements are equivalent.

1. There exists a Lie algebra isomorphism \(\Phi : \mathfrak{X}(B) \to \mathfrak{X}(B')\).
2. There exists a Lie algebra isomorphism \(\Phi : \mathfrak{X}(B) \to \mathfrak{X}(B')\).
3. There exists a diffeomorphism \(\sigma : B \to B'\).

\(E\) admits a natural orbifold structure. Let \(\rho : E \to B\) (resp. \(\rho' : E' \to B'\)) be a fibration over \(B\) (resp. \(B'\)) with generic fibre \(F\) (resp. \(F'\)), a connected closed smooth manifold. Let \(\mathfrak{D}(E; \rho)\) (resp. \(\mathfrak{X}(E; \rho)\)) be the subalgebra of \(\mathfrak{D}(E)\) (resp. \(\mathfrak{X}(E)\)) consisting of fibration preserving vector fields (see § 1). Using Theorem 0.1, we prove the following.

**Theorem 0.2.** The following statements are equivalent.

1. There exists a Lie algebra isomorphism \(\Phi : \mathfrak{D}(E; \rho) \to \mathfrak{D}(E'; \rho')\).
2. There exists a Lie algebra isomorphism \(\Phi : \mathfrak{X}(E; \rho) \to \mathfrak{X}(E'; \rho')\).
3. There exists a fibration preserving diffeomorphism \(\sigma : E \to E'\).

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Theorem 0.1 is a special case of Theorem 0.2, but we prove Theorem 0.2 by using Theorem 0.1. Theorem 0.2 was proved by Omori [6] for the case that $E$ and $E'$ are smooth fibre bundles.

The paper is organized as follows. In §1 we define smooth vector fields on smooth orbifolds and on fibrations over smooth orbifolds. §2 is devoted to preliminaries. In §3 and §4 we determine the maximal ideals of $\mathfrak{X}(B), \mathfrak{X}(E; p)$ and some subalgebras of $\mathfrak{X}(E; p)$. In §5 we prove that Theorem 0.2 (2) implies (3). In §6 and §7 we determine the maximal ideals of $\mathfrak{D}(B)$ and $\mathfrak{D}(E; p)$, and prove Theorem 0.1 and Theorem 0.2.

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§1. Definitions

In this section we shall define smooth vector fields on smooth orbifolds and fibrations over smooth orbifolds.

Definition 1.1 (See Satake [7] and Thurston [10])

A paracompact Hausdorff space $M$ is called a smooth orbifold if there exists an open covering $\{U_i\}$ of $M$, closed under finite intersections, satisfying the following conditions.

1. For each $U_i$, there are a finite group $\Gamma_i$, a smooth action of $\Gamma_i$ on an open set $\bar{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\phi_i : U_i \to \bar{U}_i/\Gamma_i$.

2. Whenever $U_i \subset U_j$, there is a smooth embedding $\phi_{ij} : \bar{U}_i \to \bar{U}_j$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\bar{U}_i & \xrightarrow{\phi_{ij}} & \bar{U}_j \\
\downarrow \phi_i^{-1} & & \downarrow \phi_j^{-1} \\
\bar{U}_i/\Gamma_i & \xrightarrow{\phi_{ij}^{-1}} & \bar{U}_j/\Gamma_j
\end{array}
\]

Remark. We can choose the finite group $\Gamma_i$ such that the action $\Gamma_i \times \bar{U}_i \to \bar{U}_i$ is effective, and then there is a unique injective group homomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ such that $\phi_{ij}$ is equivariant with respect
Two coverings give rise to the same orbifold structure if they can be combined consistently to give a larger cover still satisfying the conditions.

Let $F$ be a smooth manifold. A paracompact Hausdorff space $E$ with a continuous map $p: E \to M$ is called a smooth fibrations over a smooth orbifold $M$ with generic fibre $F$ if the following conditions are satisfied.

(1) For each $U_i$, there exists a smooth $\Gamma_i$-action on $F$ such that $p^{-1}(U_i) = U_i \times F/\Gamma_i$ and the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{U}_i \times F & \longrightarrow & p^{-1}(U_i) \\
\downarrow & & \downarrow \\
\tilde{U}_i & \longrightarrow & U_i
\end{array}
$$

(2) Whenever $U_i \subset U_j$, there exists a smooth embedding $\phi_{ij}: \tilde{U}_i \times F \to \tilde{U}_j \times F$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{U}_i \times F & \xrightarrow{\phi_{ij}} & \tilde{U}_j \times F \\
\downarrow & & \downarrow \\
\tilde{U}_i \times F/\Gamma_i & \cong & \tilde{U}_j \times F/\Gamma_j
\end{array}
$$

Definition 1.2 (See [1], §1) Let $M$ be a smooth orbifold given in Definition 1.1. A function $f: M \to R$ is said to be smooth if the composition $\tilde{U}_i \to \tilde{U}_i/\Gamma_i = U_i \to R$ is smooth for any $U_i$. Let $C^\infty(M)$ denote the algebra of all smooth functions on $M$. As in [1], §1 for any $b \in M$ we can define a tangent space $\tau_b(M)$. We define a smooth vector field on $M$ to be a real linear derivation on $C^\infty(M)$. Let $\mathfrak{D}(M)$ be the Lie algebra of all smooth vector fields on $M$ with compact support. A smooth vector field $X$ on $M$ is called strata preserving if $X$ preserves the ideals in $C^\infty(M)$ of smooth functions which vanish on the strata of $U_i = \tilde{U}_i/\Gamma_i$ for any $U_i$ (cf. [10], §2). Let $\mathfrak{X}_s(M)$ denote the Lie algebra of strata preserving vector fields on $M$ with compact support.

Let $p: E \to M$ be a smooth fibration over a smooth orbifold $M$ with generic fibre $F$. Then $E$ has a natural smooth orbifold structure. Let $X \in \mathfrak{X}(E)$. $X$ is said to be fibration preserving if $(dp)_{e_1}(X_{e_1}) = (dp)_{e_2}(X_{e_2})$ for any $b \in M$ and $e_1, e_2 \in p^{-1}(b)$. Let $\mathfrak{X}(E; p) = \{X \in \mathfrak{X}(E) ; X$
is fibration preserving. Let \( p': E' \to M' \) be another smooth fibration over a smooth orbifold \( M' \) with generic fibre \( F' \). We say a map \( \sigma: E \to E' \) is smooth if \( f \circ \sigma \in C^\infty(E) \) for any \( f \in C^\infty(E') \), and \( \sigma \) is diffeomorphic if \( \sigma \) and \( \sigma^{-1} \) are smooth. For a diffeomorphism \( \sigma: E \to E' \), let \( \sigma_*: \mathfrak{X}(E) \to \mathfrak{X}(E') \) be a Lie algebra isomorphism defined by \( \sigma_*(X)(f)(e) = X(f \circ \sigma)(\sigma^{-1}(e)) \) for \( X \in \mathfrak{X}(E), f \in C^\infty(E'), e \in E' \).

\section{Preliminaries}

Let \( \Gamma \) be a finite group and let \( \Gamma \times \tilde{U} \to \tilde{U} \) be a smooth action on an open set \( \tilde{U} \) of \( \mathbb{R}^n \). Let \( U = \tilde{U} / \Gamma \) be the orbit space of \( \tilde{U} \) and \( \pi: \tilde{U} \to U \) the natural projection. In this section, we shall consider that \( p: E \to U \) is a smooth fibration with generic fibre \( F \), a closed connected smooth \( m \)-manifold, and \( E = \tilde{U} \times F / \Gamma \). Let \( \mathfrak{X}(U) \) be the Lie algebra of all \( \Gamma \)-invariant smooth vector fields on \( \tilde{U} \) with compact support. Let \( \tilde{E} = \tilde{U} \times F \). Let \( \pi: \tilde{E} \to E \) and \( \tilde{p}: \tilde{E} \to \tilde{U} \) be the natural projections.

**Lemma 2.1.** The induced map \( \tilde{\pi}_*: \mathfrak{X}(\tilde{U}) \to \mathfrak{X}(U) \) is a Lie algebra isomorphism.

**Proof.** By Bierstone \cite{2} and Schwarz \cite{8}, \( \tilde{\pi}_* \) is epimorphic. For \( a \in \tilde{U} \), the isotropy subgroup \( \Gamma_a \) at \( a \) acts on the tangent space \( \tau_a(\tilde{U}) \). Let \( \tau_a(\tilde{U})^{\Gamma_a} \) denote the set of \( \Gamma_a \)-invariant vectors of \( \tau_a(\tilde{U}) \). Let \( X \in \mathfrak{X}(\tilde{U}) \) such that \( \tilde{\pi}_*(X) = 0 \). Then \( X_a \in \tau_a(\tilde{U})^{\Gamma_a} \) for any \( a \in \tilde{U} \). Since \( (d\pi)_\cdot: \tau_a(\tilde{U}) \to \tau_a(\tilde{U}) \) is monomorphic on \( \tau_a(\tilde{U})^{\Gamma_a} \), we have \( X_a = 0 \), and this completes the proof of Lemma 2.1.

Let \( \mathfrak{X}(\tilde{E}; \tilde{p}) = \{ X \in \mathfrak{X}(\tilde{E}) : X \text{ is fibration preserving} \} \).

**Lemma 2.2.** The induced map \( \tilde{\pi}_*: \mathfrak{X}(\tilde{E}; \tilde{p}) \to \mathfrak{X}(E; p) \) is a Lie algebra isomorphism.

**Proof.** By Lemma 2.1, the map \( \tilde{\pi}_* \) is monomorphic. Let \( X \in \mathfrak{X}(E; p) \). By Lemma 2.1, there exists a vector field \( Y \in \mathfrak{X}(\tilde{E}) \) such that \( \tilde{\pi}_*(Y) = X \). We shall prove that \( Y \) is fibration preserving. Let \( \tilde{e}_i, \tilde{e}_2 \in \tilde{E} \) such that \( \tilde{p}(\tilde{e}_i) = \tilde{p}(\tilde{e}_2) = \tilde{b} \in \tilde{U} \). Let \( b = \pi(\tilde{b}) \) and \( e_i = \pi(\tilde{e}_i) \) for \( i = 1, 2 \). Then \( (d\pi)_\cdot((d\tilde{p})_{\tilde{e}_i}(Y_{\tilde{e}_i})) = (dp)_{e_i}(X_{e_i}) \) for \( i = 1, 2 \). Since \( X \) is
fibration preserving, \((d\pi)_s ((d\bar{p})_{\bar{t}}(Y_{\bar{t}})) = (d\pi)_s ((d\bar{p})_{\bar{t}}(Y_{\bar{t}}))\). Note that, if \(\Gamma_s\) is a principal isotopy group, then \((d\pi)_s : \pi_s : (\bar{U})_s \rightarrow \pi_s (U)\) is isomorphic, and \((d\bar{p})_{\bar{t}}(Y_{\bar{t}}) = (d\bar{p})_{\bar{t}}(Y_{\bar{t}})\). Since the set of principal orbits of \(\bar{U}\) is open dense in \(\bar{U}\), we see that \(Y\) is fibration preserving. This completes the proof of Lemma 2.2.

For \(\bar{\varepsilon} \in \bar{E}\), let \((V ; x_1, \ldots, x_n, y_1, \ldots, y_m)\) be a local coordinate at \(\bar{\varepsilon}\) such that \((x_1, \ldots, x_n)\) is a local coordinate of \(R^n\) and \((y_1, \ldots, y_m)\) is a local coordinate of the fibre \(F\). We can assume that \(V\) is a \(\Gamma_s\)-invariant neighborhood of \(\bar{\varepsilon}\). Then we see that each \(X \in \mathcal{X}_r (\bar{E} ; \bar{p})\) is described on \(V\) as follows:

\[
X = \sum_{i=1}^{n} a_i (x_1, \ldots, x_n) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} b_j (x_1, \ldots, x_n, y_1, \ldots, y_m) \frac{\partial}{\partial y_j},
\]

where \(a_i\) and \(b_j\) are smooth functions on \(V\).

**Lemma 2.3.** If \(X \in \mathcal{X}_r (\bar{E} ; \bar{p})\) satisfies \((d\bar{p})_{\bar{t}} X = 0\), then there exists a local coordinate \((W ; x_1, \ldots, x_n, y_1, \ldots, y_m)\) at \(\bar{\varepsilon}\) such that

1. \((x_1, \ldots, x_n)\) is a local coordinate of \(\bar{U}\) satisfying \(x_i (\bar{e}_1) = x_i (\bar{e}_2)\) for \(\bar{b} \in \bar{U}\) and \(\bar{e}_1, \bar{e}_2 \in p^{-1}(\bar{b}) \cap W\).
2. \(x_1\) is a \(\Gamma_s\)-invariant smooth function on \(W\) such that \(X = \frac{\partial}{\partial x_1}\).
3. \((y_1, \ldots, y_m)\) is a local coordinate of the fibre \(F\).

**Proof.** We can prove by easy computations.

**Lemma 2.4.** Let \(X \in \mathcal{X}_r (\bar{E} ; \bar{p})\) satisfying \((d\bar{p})_{\bar{t}} X = 0\). Then for any \(Y \in \mathcal{X}_r (\bar{E} ; \bar{p})\) with \(\text{supp}(Y) \subseteq W\), there are a neighborhood \(W_1 \subset W\) of \(\bar{\varepsilon}\) and a vector field \(Z \in \mathcal{X}_r (\bar{E} ; \bar{p})\) such that \([X, Z] = Y\) on \(W_1\), where \(W\) is a neighborhood of \(\bar{\varepsilon}\) as in Lemma 2.3.

**Proof.** Let \((W ; x_1, \ldots, x_n, y_1, \ldots, y_m)\) be a local coordinate at \(\bar{\varepsilon}\) as Lemma 2.3. Let \(Y = \sum_{i=1}^{n} a_i (x_1, \ldots, x_n) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} b_j (x_1, \ldots, x_n, y_1, \ldots, y_m) \frac{\partial}{\partial y_j}\) on \(W\). Put \(Z_1 = \sum_{i=1}^{n} \left( \int_{-\infty}^{1} a_i (x_1, \ldots, x_n) \, dx_i \right) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \left( \int_{-\infty}^{1} b_j (x_1, \ldots, x_n, y_1, \ldots, y_m) \, dx_i \right) \frac{\partial}{\partial y_j}\). Then \(Z_1\) is a fibration preserving \(\Gamma_s\)-invariant smooth
vector field on $W$. By using a $\Gamma$-invariant partition of unity on $\bar{E}$, we can extend $Z_1$ to a $\Gamma$-invariant smooth vector field on $\bar{E}$ with compact support such that $Z=Z_1$ on a neighborhood $W_1 \subset W$ of $\bar{e}$. Obviously, $[X, Z] = Y$ on $W_1$, and this completes the proof of Lemma 2.4.

**Lemma 2.5.** (cf. Omori [6], 10.7.1) If $X$ satisfies $X_\gamma \neq 0$ and $(dp)_\gamma X_\gamma = 0$, then there exists a local coordinate $(W; x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that $(y_1, \ldots, y_m)$ is a local coordinate of the fibre $F$ and

$$X = \sum_{i=1}^n a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_1}$$

with $a_i(0, \ldots, 0) = 0$,

where the origin of the coordinate corresponds to the point $\bar{e}$. Moreover, the coordinate can be chosen such that $y_1$ is $\Gamma_\gamma$-invariant.

**Proof.** We can show by easy computations.

§ 3. Maximal Ideals of $\mathfrak{X}(B)$ and $\mathfrak{X}(E; p)$

Let $B$ be a connected smooth orbifold. Let $p: E \to B$ be a fibration over $B$ with generic fibre $F$, a connected closed manifold. In this section we shall determine the maximal ideals of $\mathfrak{X}(E; p)$.

Let $\mathfrak{G}_b(B) = \{X \in \mathfrak{X}(B) : X = 0 \text{ on a neighborhood of } b \text{ for } b \in B\}$.

**Lemma 3.1** (cf. Koriyama, Maeda, Omori [4], Lemma 4.3) Let $M$ be a maximal ideal of $\mathfrak{X}(B)$ such that $M \supset [\mathfrak{X}(B), \mathfrak{X}(B)]$. Then there exists a unique point $b \in B$ such that $M \supset \mathfrak{G}_b(B)$.

**Proof.** We give a metric on $B$. Let $\mathfrak{G}_b(\varepsilon) = \{X \in \mathfrak{X}(B) : X = 0 \text{ on an } \varepsilon\text{-neighborhood of } b\}$. Assume that $M + \mathfrak{G}_b(\varepsilon) = \mathfrak{X}(B)$ for any $b \in B$. As in the proof of Koriyama, Maeda and Omori [4], Lemma 4.1, we can prove that $M \supset [\mathfrak{X}(B), \mathfrak{X}(B)]$, contradicting the assumption for $M$. Thus there exists a point $b \in B$ with $M \supset \mathfrak{G}_b(\varepsilon)$.

Let $A_\varepsilon = \{b \in B; \mathfrak{G}_b(\varepsilon) \subset M\}$. It is easy to see that, if $\varepsilon > \delta$, then $A_\varepsilon \supset cl(A_\delta)$ (the closure of $A$). Then there exists $b \in \bigcap_{\varepsilon > \delta} A_\varepsilon$, and we see that $M \supset \mathfrak{G}_b(B)$. If $b \neq b'$, then we can prove that $\mathfrak{G}_b(B) + \mathfrak{G}_{b'}(B) = \mathfrak{X}(B)$. Therefore such a point $b$ must be unique and this completes the proof of Lemma 3.1.
Let $B_0 = \{ b \in B ; X_b = 0 \text{ for any } X \in \mathfrak{X}(B) \}$ and let $B_1 = B - B_0$. From [1], Lemma 3.9 and Proposition 3.10 we have the following.

**Proposition 3.2.** (1) For any point $b \in B_1$, there exists a unique maximal ideal $\mathfrak{N}(b)$ such that $\mathfrak{N}(b) \supseteq \mathfrak{G}_b(B)$, $\mathfrak{N}(b) \supseteq [\mathfrak{X}(B), \mathfrak{X}(B)]$ and $\text{codim } \mathfrak{N}(b) = \infty$.

(2) Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{X}(B)$ such that $\mathfrak{M} \supseteq \mathfrak{G}_b(B)$ for $b \in B_0$. Then $\text{codim } \mathfrak{M} < \infty$.

Let $B^*$ be the space of all maximal ideals $\mathfrak{M}$ of $\mathfrak{X}(B)$ such that $\mathfrak{M} \supseteq [\mathfrak{X}(B), \mathfrak{X}(B)]$, with Stone topology (see [1], Definition 4.1). Using Lemma 3.1, we define a map $\tau : B^* \to B$ such that $\tau(\mathfrak{M}) = b$ if $\mathfrak{M} \supseteq \mathfrak{G}_b(B)$. Let $B^*_b = \{ \mathfrak{M} \in B^* ; \text{codim } \mathfrak{M} = \infty \}$.

From [1], Lemma 4.2 we have the following.

**Proposition 3.3.** $\tau : B^*_b \to B_1$ is homeomorphic.

Let $\mathfrak{G}_e(E; p) = \{ X \in \mathfrak{X}(E; p) ; X = 0 \text{ on a neighborhood of } e \text{ for } e \in E \}$. Then we have the following useful lemma.

**Lemma 3.4.** Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{X}(E; p)$ such that $fX \in \mathfrak{A}$ for any $f \in C^\infty(E)$ and $X \in \mathfrak{A} \cap (\text{Ker } p_*)$. Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{A}$ such that $\mathfrak{M} \supseteq [\mathfrak{A}, \mathfrak{A}]$ and $p_*(\mathfrak{M}) = p_*(\mathfrak{A})$. Then there exists a unique point $e \in E$ such that $\mathfrak{M} \supseteq \mathfrak{G}_e(\mathfrak{A})$, where $\mathfrak{G}_e(\mathfrak{A}) = \mathfrak{G}_e(E; p) \cap \mathfrak{A}$.

**Proof.** Suppose that $\mathfrak{A} = \mathfrak{M} + \mathfrak{G}_e(\varepsilon; \mathfrak{A})$ for any $e \in E$, where $\mathfrak{G}_e(\varepsilon; \mathfrak{A}) = \{ X \in \mathfrak{A} ; X = 0 \text{ on an } \varepsilon\text{-neighborhood of } e \}$. For any vector field $X \in \mathfrak{A}$, there exists a vector field $Y \in \mathfrak{M}$ such that $Z = X - Y \in \text{Ker } p_*$. For each positive number $\varepsilon > 0$, $Z$ can be written as finite sums $Z = \sum Z_i$ such that $Z_i \in \mathfrak{A} \cap (\text{Ker } p_*)$ and $\text{diam } (\text{supp } Z_i) < \varepsilon$. Then as in the proof of Koriyama, Maede and Omori [4], Lemma 4.1, $[\mathfrak{A} \cap (\text{Ker } p_*)$, $\mathfrak{A} \cap (\text{Ker } p_*)]$ is contained in $\mathfrak{M}$. Then $[\mathfrak{A}, \mathfrak{A}]$ is contained in $\mathfrak{M}$, contradicting the assumption for $\mathfrak{M}$. Therefore there exists a point $e \in E$ such that $\mathfrak{M} \supseteq \mathfrak{G}_e(\varepsilon; \mathfrak{A})$. As in the proof of Lemma 3.1, there exists a unique point $e \in E$ such that $\mathfrak{M} \supseteq \mathfrak{G}_e(\varepsilon; \mathfrak{A})$, and Lemma 3.4 follows.
Let \( F_b = p^{-1}(b) \) for \( b \in B_0 \). Put \( F_{b,0} = \{ e \in F_b \mid X_e = 0 \text{ for any } X \in \mathfrak{X}(E;\rho) \} \) and put \( F_{b,1} = F_b - F_{b,0} \). Let \( \mathfrak{X}_F(E;\rho) = \{ X \in \mathfrak{X}(E;\rho) \mid p_*(X) = 0 \} \). For \( b \in B_0 \), \( e \in F_{b,0} \) put \( \mathcal{F}(e) = \{ X \in \mathfrak{X}(E;\rho) \mid (ad Y_1) \cdots (ad Y_k) X_e = 0 \text{ for any } Y_i \in \mathfrak{X}_F(E;\rho) \text{ and any integer } k \geq 0 \} \), where \( (ad Y)(Z) = [Y, Z] \) for \( Y, Z \in \mathfrak{X}(E;\rho) \).

**Lemma 3.5.** If \( b \in B_0 \) and \( e \in F_{b,1} \), then \( \mathcal{F}(e) \) is an infinite codimensional maximal ideal of \( \mathfrak{X}(E;\rho) \). Moreover \( \mathcal{F}(e) \) is a unique maximal ideal containing \( \mathfrak{O}_e(E;\rho) \), and \( \mathcal{F}(e) \supseteq [\mathfrak{X}_F(E;\rho), \mathfrak{X}_F(E;\rho)] \).

**Proof.** There exist a finite group \( \Gamma \), a smooth action of \( \Gamma \) on an open set \( \tilde{U} \) of \( \mathbb{R}^n \) such that \( \tilde{U}/\Gamma = U \) is a neighborhood of \( b \), and there exists a smooth \( \Gamma \)-action on the fibre \( F \) such that \( p^{-1}(U) = \tilde{U} \times F/\Gamma \). Put \( E_0 = \tilde{U} \times F/\Gamma \) and \( E_0 = \tilde{U} \times F \). Let \( \pi : \tilde{E}_0 \to E_0 \) and \( \tilde{\pi} : \tilde{E}_0 \to \tilde{U} \) be the projections. By Lemma 2.2 \( \pi_* : \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \to \mathfrak{X}(E_0;\rho) \) is a Lie algebra isomorphism. Let \( \tilde{e} \) be a point of \( \tilde{E}_0 \) such that \( \pi(\tilde{e}) = e \), and let \( \tilde{\mathcal{F}}(\tilde{e}) = \{ X \in \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \mid (ad Y_1) \cdots (ad Y_k) X_{\tilde{e}} = 0 \text{ for any } Y_i \in \mathfrak{X}_\Gamma, (\tilde{E}_0;\rho) \text{ and any integer } k \geq 0 \} \). Note that \( \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \) can be identified with \( \mathfrak{X}_\Gamma(\tilde{U}) \times \mathfrak{X}_\Gamma, F_0(\tilde{E}_0;\rho) \) as an \( \mathbb{R} \)-algebra, where \( \mathfrak{X}_\Gamma, F_0(\tilde{E}_0;\rho) = \{ X \in \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \mid p_*(X) = 0 \} \). By easy computations we can see that \( \tilde{\mathcal{F}}(\tilde{e}) \) is an ideal of \( \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \). Since \( \pi_*^{-1}(\mathcal{F}(e) \cap \mathfrak{X}(E_0;\rho)) = \tilde{\mathcal{F}}(\tilde{e}) \), \( \mathcal{F}(e) \) is an ideal of \( \mathfrak{X}(E;\rho) \).

Let \( \mathfrak{M} \) be a maximal ideal containing \( \mathfrak{O}_e(E;\rho) \). Assume that there exists a vector field \( X \in \mathfrak{M} \) with \( X_e \neq 0 \). There exists \( \tilde{X} \in \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \) such that \( \pi_*(\tilde{X}) = X \) on a neighborhood \( V \) of \( e \). Since \( b \in B_0 \), \( (d \rho)_e \tilde{X}_e = 0 \). By Lemma 2.5, there exists a local coordinate \( (W; x_1, \ldots, x_n, y_1, \ldots, y_m) \) around \( e \) such that \( (y_1, \ldots, y_m) \) is a local coordinate for a fibre \( F \) and \( \tilde{X} = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_1} \) with \( a_i(0, \ldots, 0) = 0 \), where the origin of the coordinate corresponds to the point \( \tilde{e} \). There is a vector fields \( \tilde{X}_1 \in \mathfrak{X}_\Gamma, F(\tilde{E}_0;\rho) \) such that \( \tilde{X}_1 = y_1 \frac{\partial}{\partial y_1} \) on a neighborhood \( W_1 \subseteq W \) or \( \tilde{e} \). Note that \( \tilde{X}_2 = [\tilde{X}, \tilde{X}_1] = \frac{\partial}{\partial y_1} \) on \( W_1 \). Put \( X_2 = [X, \pi_* (\tilde{X}_1)] \). Then \( X_2 \in \mathfrak{M} \). Let \( Y \in \mathfrak{X}(E_0;\rho) \) such that \( \text{supp}(Y) \subset V \). Let \( \tilde{Y} \in \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \) such that \( \pi_*(\tilde{Y}) = Y \). As in the proof of Lemma 2.4, we can prove that there are a neighborhood \( W_1 \) of \( e \) and a vector field \( \tilde{Z} \in \mathfrak{X}_\Gamma(\tilde{E}_0;\rho) \) such that \( [\tilde{X}_2, \tilde{Z}] = \tilde{Y} \) on \( W_1 \subseteq W \). Put \( Z = \pi_*(\tilde{Z}) \).
Then \([X_2, Z] \in \mathcal{M}\) and \([X_2, Z] = Y\) on a neighborhood of \(e\). Since \(\mathcal{M}\) contains \(\mathfrak{g}_e(E; p)\), \(\mathcal{M} = \mathfrak{X}(E; p)\). This is a contradiction. Thus \(X_e = 0\) for any \(X \in \mathcal{M}\), and we have \(\mathcal{M} = \mathfrak{I}(e)\). Therefore \(\mathfrak{I}(e)\) is a unique maximal ideal containing \(\mathfrak{g}_e(E; p)\). There are vector fields \(\tilde{Y}_i \in \mathfrak{X}_f(\tilde{E}_0; \tilde{p})\) \((i = 0, 1, 2, \ldots)\) such that \(\tilde{Y}_i = -\frac{\partial}{\partial y_1}\) on a neighborhood of \(\tilde{e}\). Put \(Y_i = \pi_\ast(\tilde{Y}_i)\). Then \(Y_i \not\in \mathfrak{I}(e)\), and \(\mathfrak{I}(e)\) is infinite codimensional. Since \([Y_0, Y_i] \not\in \mathfrak{I}(e)\), \(\mathfrak{I}(e) \supset [\mathfrak{X}_f(E; p), \mathfrak{X}_f(E; p)]\). This completes the proof of Lemma 3.5.

Let \(b \in B_0\) and \(e \in F_{b_0}\). Let \(U = \tilde{U}/\Gamma\) be a neighborhood of \(b\) as in \(\S 2\). For \(X \in \mathfrak{X}(E; p)\), there is \(Y \in \mathfrak{X}_f(\tilde{U} \times F; \tilde{p})\) such that \(\pi_\ast(Y) = X\) on a neighborhood \(W\) of \(e\), where \(\pi : \tilde{U} \times F \to \tilde{U} \times F / \Gamma\) is the natural projection.

Choose a point \(\tilde{e} \in \tilde{U} \times F\) with \(\pi(\tilde{e}) = e\). Let \(j_1^i(Y)\) denote the 1-jet of \(Y\) at \(\tilde{e}\). Note that \(j_1^i(Y)\) defines an element of \(\mathfrak{gl}_f(\tau_e(\tilde{U} \times F))\), where \(\mathfrak{gl}_f(\tau_e(\tilde{U} \times F))\) denotes the set of all \(\Gamma_{\text{f}}\)-invariant endomorphism of the tangent space \(\tau_e(\tilde{U} \times F)\) at \(\tilde{e}\). Let \(j_0^i(E; p) = \{A \in \mathfrak{gl}_f(\tau_e(\tilde{U} \times F)); (d\tilde{p})_*(A(v)) = 0\text{ for any }v \in \tau_e(\tilde{U} \times F)\text{ with } (d\tilde{p})_*(v) = 0\}, \) where \(\tilde{p} : \tilde{U} \times F \to \tilde{U}\) is the natural projection. Since \(Y\) is fibration preserving, \(j_0^i(Y) \subseteq j_0^i(E; p)\). Let \(j_1^i(\mathfrak{X}(E; p) \to j_0^i(E; p)\) be a map defined by \(j_1^i(X) = j_1^i(Y)\). Then it is easy to see the following.

**Lemma 3.6.** \(j_1^i : \mathfrak{X}(E; p) \to j_0^i(E; p)\) is an onto Lie algebra homomorphism.

**Lemma 3.7.** Let \(\mathcal{M}\) be a maximal ideal of \(\mathfrak{X}(E; p)\) such that \(p_\ast(\mathcal{M}) = \mathfrak{X}(B)\) and \(\mathcal{M} \supset \mathfrak{g}_e(E; p)\) for some point \(e \in F_{b_0}\) with \(b \in B_0\). Then \(\mathcal{M} = (j_0^i)^{-1}(\mathcal{L})\) for some maximal ideal \(\mathcal{L}\) of \(j_0^i(E; p)\), and \(\mathcal{M}\) is finite codimensional.

**Proof.** Assume that \(j_1^i(\mathcal{M}) = j_0^i(E; p)\). We use the same notations as in the proof of Lemma 3.8. Take a vector field \(X \in \mathcal{M}\) such that \(j_1^i(X)\) is a unit matrix. There exists \(Y \in \mathfrak{X}_f(\tilde{U} \times F; \tilde{p})\) such that \(\pi_\ast(Y) = X\) on a neighborhood of \(e\). By Sternberg's linearization theorem [9], there exists a local coordinate \((W; z_1, \ldots, z_{n+m})\) at \(e\) such that \(Y = \sum_{i=1}^{n+m} z_i \frac{\partial}{\partial z_i}\) on \(W\).
Let \( Z \in \mathfrak{g}(E; p) \) with \( j^1(Z) = 0 \). There exists \( \tilde{Z} \in \mathfrak{g}_F(\bar{U} \times F; \bar{p}) \) such that \( \pi_\ast(\tilde{Z}) = Z \) on a neighborhood of \( e \). Then \( j^1(\tilde{Z}) = 0 \). Let \( \tilde{Z} = \sum_0^\infty \Ad(\exp tY) \tilde{Z} dt \), where \( (\Ad(\exp tY) \tilde{Z})_x = (d\phi_t^{-1}(x)) \tilde{Z} \phi_t^{-1}(x) \), \( \phi_t = \exp tY \) (see Koriyama, Maeda and Omori [5], § 1). \( \tilde{Z} \) can be written in the form \( \tilde{Z} = \sum a_i(z_1, \ldots, z_{n+m}) \frac{\partial}{\partial z_i} \) on \( W \) with \( \frac{\partial a_i}{\partial z_j}(0, \ldots, 0) = 0 \) for \( i, j = 1, \ldots, n+m \). Then \( \Ad(\exp tY) \tilde{Z} = \sum a_i(e^{-t}z_1, \ldots, e^{-t}z_{n+m}) e^t \frac{\partial}{\partial z_i} \) on \( W \). Since \( \frac{\partial a_i}{\partial z_j}(0, \ldots, 0) = 0 \), it is clear that \( \int_0^\infty a_i(e^{-t}z_1, \ldots, e^{-t}z_{n+m}) e^t dt \) exists. Hence \( \tilde{Z} \in \mathfrak{g}(\bar{U} \times F) \). We can see that \( \exp s(\Ad(\exp tY) \tilde{Z}) = \phi_s \circ \phi_t^{-1} \), where \( \phi_s = \exp s \tilde{Z} \). Thus \( \tilde{Z} \) is fiber preserving, and \( \tilde{Z} \in \mathfrak{g}_F(\bar{U} \times F; \bar{p}) \). Note that \([Y, \tilde{Z}] = \tilde{Z} \), and \([X, \pi_\ast(\tilde{Z})] = Z \) on a neighborhood of \( e \). Since \( \mathfrak{m} \) contains \( \mathfrak{g}_s(E; p) \), \( \mathfrak{m} \) contains the ideal \( \Ker j^1_\ast \) of \( \mathfrak{g}(E; p) \). But \( j^1_\ast(\mathfrak{m}) = j^1_\ast(E; p) \), hence \( \mathfrak{m} = \mathfrak{g}(E; p) \). This is a contradiction. Thus \( j^1_\ast(\mathfrak{m}) \) must be a proper ideal \( \mathfrak{g} \) of \( j^1_\ast(E; p) \), and \( \mathfrak{m} = (j^1_\ast)^{-1}(\mathfrak{g}) \). Since \( j^1_\ast(E; p) \) is a finite dimensional Lie algebra, \( \mathfrak{m} \) is finite codimensional. This completes the proof of Lemma 3.7.

Let \( E^\ast \) be the space of all maximal ideals \( \mathfrak{m} \) of \( \mathfrak{g}(E; p) \) such that \( \mathfrak{m} \supseteq [\mathfrak{g}(E; p), \mathfrak{g}(E; p)] \), with Stone topology. Let \( E^\ast_1 = \{ \mathfrak{m} \in E^\ast; \text{ codim } \mathfrak{m} = \infty \} \). Combining Proposition 3.2, Lemmas 3.4, 3.5 and 3.7, we have the following.

**Proposition 3.8.** If \( \mathfrak{m} \in E^\ast_1 \), then
(1) \( \mathfrak{m} = \phi^1_\ast(\mathfrak{m}') \) for some \( \mathfrak{m}' \in B_1^\ast \), or
(2) \( \mathfrak{m} = \mathfrak{g}(e) \) for a point \( e \in F_{b,1} \) with \( b \in B_0 \).

Let \( E_i = B_1 \cup ( \bigcup_{b \in B_0} F_{b,1} ) \) be a subspace of the disjoint union \( B \cup E \).

Using Proposition 3.8 we can define a map \( \tau: E^\ast_1 \rightarrow E_i \) as follows.
(1) If \( \mathfrak{m} = \phi^1_\ast(\mathfrak{m}') \) for some \( \mathfrak{m}' \in B_1^\ast \), then \( \tau(\mathfrak{m}) = \tau(\mathfrak{m}') \).
(2) If \( \mathfrak{m} = \mathfrak{g}(e) \) for \( e \in F_{b,1} \) with \( b \in B_0 \), then \( \tau(\mathfrak{m}) = e \).

**Proposition 3.9.** \( \tau: E^\ast_1 \rightarrow E_i \) is homeomorphic.

**Proof.** By Proposition 3.8, \( \tau \) is a bijection. It is enough to show that \( \tau(CL(S)) = cl(\tau(S)) \) for any subset \( S \) of \( E^\ast_1 \). Here \( CL \) and \( cl \) are closure operators of \( E^\ast_1 \) and \( E_i \), respectively. Assume that
\[ \tau(\mathcal{M}) \in \text{cl}(\tau(S)) \] for some \( \mathcal{M} \in \text{CL}(S) \). In the case that \( \tau(\mathcal{M}) = b \in B_0 \), we can find \( X \in \mathcal{X}(E;p) \) such that \( p_*(X)_b \neq 0 \) and \( X = 0 \) on a neighborhood of \( \text{cl}(\tau(S)) \). Then we see that \( X \in \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} \subset \mathcal{M} \) (cf. [1], § 4). Since \( p_*(\mathcal{M}) = \mathcal{J}(b) \), \( p_*(X)_b = 0 \). This is a contradiction. In the case that \( \tau(\mathcal{M}) = e \in F_{0,1}(b \in B_0) \), we can find \( X \in \mathcal{X}(E;p) \) such that \( p_*(X)_e = 0 \), \( X_e \neq 0 \) and \( X = 0 \) on a neighborhood of \( \tau(S) \). Then \( X \in \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} = \mathcal{J}(e) \), and \( X_e = 0 \). This is a contradiction. Therefore \( \tau(\text{CL}(S)) \) is contained in \( \text{cl}(\tau(S)) \).

If \( X \in \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} \), then \( p_*(X)_e = 0 \) on \( \tau(S) \cap B_1 \), so \( p_*(X)_e = 0 \) on \( \text{cl}(\tau(S) \cap B_1) \). Hence, for any \( X \in \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} \), \( p_*(X)_e = 0 \) on \( \text{cl}(\tau(S) \cap B_1) \). Then \( \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} \) is an ideal of \( \mathcal{X}(E;p) \) contained in \( p_*(\mathcal{J}(b)) \) for any \( b \in \text{cl}(\tau(S) \cap B_1) \). Similarly, \( \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} \) is contained in \( \mathcal{J}(e) \) for any \( e \in \text{cl}(\tau(S) \cap p^{-1}(B_0)) \). Thus \( \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} \) is contained in \( \text{cl}(\tau(S)) \), and \( \tau(\text{CL}(S)) \) contains \( \text{cl}(\tau(S)) \). This completes the proof of Proposition 3.9.

§ 4. Maximal Ideals of Some Subalgebras of \( \mathcal{X}(E;p) \)

Let \( \mathcal{X}(B)_b = \{ X \in \mathcal{X}(B) ; X_b = 0 \} \) for \( b \in B \), and \( \mathfrak{A}_b = \bigcap_{\mathcal{M} \in \mathcal{S}} \mathcal{M} \) be a subalgebra of \( \mathcal{X}(E;p) \). For \( b, b' \in B \), let \( \mathcal{F}^b_{b,b'} = \{ e \in F_e ; X_e = 0 \} \) for \( X \in \mathfrak{A}_b \) and let \( \mathcal{F}^b_{b,b'} = F_{b'} - \mathcal{F}^b_{b,b'} \). Let \( \mathcal{J}(e)_b = \mathfrak{A}_b \cap \mathcal{J}(e) \) and let \( \mathfrak{G}_e(\mathfrak{A}_b) = \mathfrak{A}_b \cap \mathfrak{G}_e(E;\mathfrak{A}_b) \) for \( e \in E \). As in the proofs of Lemmas 3.5 and 3.7, we prove the following lemmas respectively.

Lemma 4.1. If \( e \in \mathcal{F}^b_{b,b'} \) for \( b' \in B_0 \) or \( b' = b \), \( \mathcal{J}(e)_b \) is an infinite codimensional maximal ideal of \( \mathfrak{A}_b \). Moreover \( \mathcal{J}(e)_b \) is a unique maximal ideal containing \( \mathfrak{G}_e(\mathfrak{A}_b) \).

Lemma 4.2. Let \( \mathcal{M} \) be a maximal ideal of \( \mathfrak{A}_b \) such that \( p_*(\mathcal{M}) = \mathcal{X}(B)_b \) and \( \mathcal{M} \supset \mathfrak{G}_e(\mathfrak{A}_b) \) for some \( e \in \mathcal{F}^b_{b,b'} \) with \( b' \in B_0 \) or \( b' = b \). Then \( \mathcal{M} \) is finite codimensional.

Let \( A_b^* \) be the set of all maximal ideals \( \mathcal{M} \) of \( \mathfrak{A}_b \) such that \( \mathcal{M} \supset [\mathfrak{A}_b, \mathfrak{A}_b] \) and \( p_*(\mathcal{M}) = \mathcal{X}(B)_b \). Let \( A^* = \{ \mathcal{M} ; \mathcal{M} \in A_b^* \} \) for some \( b \in B \).

We give the Stone topology on \( A^* \). Combining Lemmas 3.4, 4.1 and 4.2 we have the following.

Proposition 4.3. If \( \mathcal{M} \in A_b^* \), then we have the following cases.
(1) There exists a point \( e \in p^{-1}(B_0) \) such that \( \mathcal{M} \supseteq \mathcal{G}_B(\mathcal{A}_B) \), and \( \mathcal{M} = \hat{\mathcal{M}} \cap \mathcal{A}_B \) for some \( \hat{\mathcal{M}} \in E^* \).

(2) There exists a point \( e \in F_b (b \in B_1) \) such that \( \mathcal{M} \supseteq \mathcal{G}_B(\mathcal{A}_B) \), and \( \mathcal{M} \subseteq \hat{\mathcal{M}} \) for any \( \hat{\mathcal{M}} \in E^* \).

Using Lemma 3.4, we have a map \( t : A^* \rightarrow E \) such that \( \mathcal{M} \supseteq \mathcal{G}_B(\mathcal{A}_B) \) for \( \mathcal{M} \in \mathcal{A}_B \). Let \( \hat{A}_B = \{ \mathcal{M} \in A^* ; \text{codim} \mathcal{M} = \infty \} \) and \( \hat{\mathcal{M}} \subseteq \hat{\mathcal{M}} \) for any \( \hat{\mathcal{M}} \in E^* \) for \( b \in B_1 \) and \( \hat{A}_B = \{ \mathcal{M} \in A^* ; \text{codim} \mathcal{M} = \infty \} \) for \( b \in B_0 \). Let \( \hat{\mathcal{M}} = \bigcup_{b \in B} \hat{A}_B \). Setting \( \hat{E}_1 = \bigcup_{b \in B} \hat{F}_b \), we have a bijective map \( t : \hat{A}^* \rightarrow \hat{E}_1 \) by Lemma 4.1 and Proposition 4.3.

**Proposition 4.4.** The map \( t : \hat{A}^* \rightarrow \hat{E}_1 \) is homeomorphic.

**Proof.** It is enough to show that \( t(\text{cl}(S)) = \text{cl}(t(S)) \) for any subset \( S \) of \( \hat{A}^* \), where \( \text{CL} \) and \( \text{cl} \) are closure operators of \( \hat{A}^* \) and \( \hat{E}_1 \) respectively. As in the proof of Proposition 3.9, we see that \( t(\text{CL}(S)) \) is contained in \( \text{cl}(t(S)) \).

Note that, for \( X \in \mathcal{X}(E; p) \) and \( e \in \hat{E}_1 \), \( X \in \mathfrak{F}(\mathcal{E}_B) \) if \(((ad Y_1) \ldots (ad Y_k) X)_{e} = 0 \) for any \( Y_i \in \mathfrak{X}(E; p) \) and any integer \( k \geq 0 \). Then we see that \( \bigcap_{e \in \hat{E}_1} \mathfrak{F}(\mathcal{E}_B) = \bigcap_{e \in \text{CL}(S)} \mathfrak{F}(\mathcal{E}_B) \). Thus \( \text{cl}(t(S)) \) is contained in \( t(\text{CL}(S)) \). This completes the proof of Proposition 4.4.

**§ 5. Theorem 0.2 (2) Implies (3)**

Let \( B \) and \( B' \) be connected smooth orbifolds without boundary. Let \( p : E \rightarrow B \) (resp. \( p' : E' \rightarrow B' \)) be a fibration over \( B \) (resp. \( B' \)) with generic fibre \( F \) (resp. \( F' \)), a connected closed manifold. Suppose that there exists a Lie algebra isomorphism \( \Phi : \mathcal{X}(E; p) \rightarrow \mathcal{X}(E'; p') \).

**Lemma 5.1.** \( \Phi(\mathcal{X}_p(E; p)) = \mathcal{X}_{p'}(E'; p') \).

**Proof.** From Proposition 3.9, there exists a homeomorphism \( \phi : B_1 \cup \bigcup_{b \in B_0} F_{b,1} \xrightarrow{\tau^{-1}} E^* \xrightarrow{\phi} E' \xrightarrow{\tau'} B_1 \cup \bigcup_{b' \in B_0} F_{b',1} \). Note that \( B_1, B_{b,1}, F_{b,1}, F_{b',1} \) are connected smooth orbifolds. Assume that \( \phi(B_1) = F_{b',1} \) for some \( b' \in B_0 \). Put \( \mathfrak{A}_{b',0} = \{ X \in \mathcal{X}(E'; p') ; X = 0 \} \) on \( p'^{-1}(b') \). We see that \( \bigcap_{b \in B_1} \tau^{-1}(b) = \mathfrak{X}_p(E; p) \) and \( \bigcap_{e' \in F_{b',1}} \tau'(e') = \mathfrak{A}_{b',0} \). Let \( E_{b,0} = \{ e \in E ; X_e \neq 0 \} \) for some \( X \in \mathfrak{X}_p(E; p) \). Using the method of the proof of
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[1], Proposition 3.2, we see that $E_1 \neq E_2$ is dense in $E$. As in the proof of Lemma 3.5, $\mathcal{F}(e) \cap \mathcal{X}_F(E; p)$ is an infinite codimensional maximal ideal of $\mathcal{X}_F(E; p)$ and we see that $\mathcal{F}(e) \cap \mathcal{X}_F(E; p) \supset \{ \mathcal{X}_F(E; p), \mathcal{X}_F(E; p) \}$. Hence the intersection of all maximal ideals $\mathcal{M}$ of $\mathcal{X}_F(E; p)$ with $\mathcal{M} \supset \{ \mathcal{X}_F(E; p), \mathcal{X}_F(E; p) \}$ is zero.

Let $\mathcal{M}'$ be a maximal ideal of $\mathcal{H}_{\mathcal{X}_F}$ with $\mathcal{M} \supset \{ \mathcal{X}_F(E; p), \mathcal{X}_F(E; p) \}$. If $p'_* (\mathcal{M}')$ is a proper ideal of $\mathcal{X}(B')$, then $\mathcal{M}' = (p'_*)^{-1}(\mathcal{M}')$ for some $\mathcal{M}' \subset B'$. If $p'_* (\mathcal{M}') = \mathcal{X}(B')$, then, by Lemma 3.4, there exists a unique point $e' \in E'$ such that $\mathcal{M}' \supset \mathcal{O}_{p'}(\mathcal{H}_{\mathcal{X}_F})$ for $e' \in B'$. Then $\phi' = \phi' \circ \phi = \phi$. Choose a point $e' \in E'$ such that $\phi'(e') \in B'$. There exists $X' \in \mathcal{H}_{\mathcal{X}_F}$ such that $X'_{e'} = 0$ and $X' = 0$ on a neighborhood of $p'^{-1}(B')$. Then $X'$ is contained in any maximal ideal of $\mathcal{H}_{\mathcal{X}_F}$. But, for each maximal ideal $\mathcal{M}'$ of $\mathcal{H}_{\mathcal{X}_F}$, there exists a maximal ideal $\mathcal{M}$ of $\mathcal{X}_F(E; p)$ with $\mathcal{M} \supset \{ \mathcal{X}_F(E; p), \mathcal{X}(E; p) \}$ such that $\phi(\mathcal{M}) = \mathcal{M}'$. Thus the intersection of these maximal ideals $\mathcal{M}'$ of $\mathcal{H}_{\mathcal{X}_F}$ must be zero, and this is a contradiction. Therefore $\phi(B_1) = B_1$ and we have $\phi(\mathcal{X}_F(E; p)) = \mathcal{X}_F(E'; p')$. This completes the proof of Lemma 5.1.

**Proposition 5.2.** If there exists a Lie algebra isomorphism $\Phi : \mathcal{X}(E; p) \to \mathcal{X}(E'; p')$, then there exists a Lie algebra isomorphism $\Phi : \mathcal{X}(B) \to \mathcal{X}(B')$ and a diffeomorphism $\sigma : B \to B'$ such that $p'_* \circ \Phi = \tilde{\Phi} \circ p_*$ and $\Phi = \sigma_*$.\[\]

**Proof.** By Lemma 5.1, the Lie algebra isomorphism $\Phi$ induces a Lie algebra isomorphism $\Phi : \mathcal{X}(B) \to \mathcal{X}(B')$. Using a partition of unity, we can reduce the proof of Proposition 5.2 to the special case when the orbifolds $B$ and $B'$ are orbit spaces of representation spaces of finite groups. Thus Proposition 5.2 follows from Theorem of [1].\[\]

We shall use the notations given in § 4. From Proposition 5.2, we have a Lie algebra isomorphism $\Phi : \mathcal{H}_b \to \mathcal{H}_b$ for each $b \in B$. By Proposition 4.4, there exists a homeomorphism $\sigma_1 : \hat{E}_1 \to \hat{E}_1$ such that $p'_* \circ \sigma = \tilde{\Phi} \circ p_*$.\[\]

**Proposition 5.3.** The map $\sigma_1 : \hat{E}_1 \to \hat{E}_1$ is extended to a homeomorphism $\sigma : E \to E'$ such that $p'_* \circ \sigma = \tilde{\Phi} \circ p_*$.\[\]

Let $\hat{A}_b^* = \{ \mathcal{M} \in \mathcal{A}_b^* : \mathcal{M} \subset \mathcal{M} \text{ for any } \mathcal{M} \in \mathcal{E}^* \}$ for $b \in B_1$ and $\hat{A}_b^* = \{ \mathcal{M} \in \mathcal{E}^* \}$ for $b \in B_2$. The map $\sigma_1 : \hat{E}_1 \to \hat{E}_1$ is extended to a homeomorphism $\sigma : E \to E'$ such that $p'_* \circ \sigma = \tilde{\Phi} \circ p_*$.\[\]
$E=A^*; \rho(t(\mathcal{M}))=b$ for $b \in B$. Let $\hat{A}^* = \{ \mathcal{M} : \mathcal{M} \in \hat{A}^*_b \text{ for some } b \in B \}$. Let $CL$ and $cl$ be the closure operators of $\hat{A}^*$ and $E$, respectively. To prove Proposition 5.3, we need the following.

**Lemma 5.4.** $t(CL(S)) = cl(t(S))$ for any subset $S$ of $\hat{A}^*$.

**Proof.** As in the proof of Proposition 3.9, we see that $t(CL(S))$ is contained in $cl(t(S))$. Let $e \in cl(t(S))$. By the similar argument to Proposition 4.4, we see that $\mathcal{M}(e)_{p(e)}$ for $e \in cl(t(S)) \cap \hat{E}_1$. Thus $cl(t(S)) \cap \hat{E}_1$ is contained in $t(CL(S))$.

Now assume that $f = t(e)$ for some $e \in cl(t(S)) \cap \hat{E}_1$, where $\mathcal{M} = \bigcap_{n \in S} \mathcal{M}_n$, $b = p(e)$. As in the proof of Lemma 3.7, we can prove that, for any $X \in \mathcal{A}_b \cap (ker p)$, there exists $Y \in \mathcal{M}$ such that $X = Y$ on an open neighborhood $W$ of $e$. There exists $\mathcal{M} \in S$ with $t(\mathcal{M}) \in W$. Since $\mathcal{M}$ contains $\mathcal{G}_r(\mathcal{A}_b)$, we have that $\mathcal{M} = \mathcal{A}_b$, where $e' = t(\mathcal{M})$ and $b' = p(e')$. This is a contradiction. Therefore $f = t(e)$ for any $e \in cl(t(S)) \cap \hat{E}_1$. Hence, for any $e \in cl(t(S)) \cap \hat{E}_1$, there exists a maximal ideal $\mathcal{M} \in CL(S)$ such that $t(\mathcal{M}) = e$. This completes the proof of Lemma 5.4.

**Proof of Proposition 5.3.** Note that the map $t : \hat{A}^* \to E$ is surjective. We define $\sigma : E \to E'$ by $\sigma(t(\mathcal{M})) = t'(\Phi(\mathcal{M}))$. First we shall prove that $\sigma$ is well defined. Note that $\sigma = \sigma_1$ on $E_1$ by the definition, and $\rho'(\sigma(t(\mathcal{M}))) = \sigma(t(\mathcal{M}))$ for any $\mathcal{M} \in A^*$. By the same argument as the proof of [1], Proposition 3.2, we see that $\hat{K}_b(\mathcal{M})$ is discrete in $F_b$ for each $b \in B$. Thus, for any $e \in E - \hat{E}_1$, there exists a closed neighborhood $D$ of $e$ in $F_b$ such that $D \cap \hat{K}_b(\mathcal{M}) = \{ e \}$, where $b = p(e)$. Let $D_0 = D - \{ e \}$ and let $D^*_b = t^{-1}(D_0)$, $D^*_b = t^{-1}(D)$. Since $\bigcap_{\mathcal{M} \in \mathcal{G}_r(\mathcal{A}_b)} cl(t'(\Phi(\mathcal{M}))) = D^*_b$.

Together with Lemma 5.4, we have that $\rho'(\sigma(t(D_0))) = \sigma(t(D^*_b))) = cl(t'(\Phi(D^*_b))) = cl(t'(\Phi(D))) = cl(t'(\Phi(D^*_b))) = cl(t'(\Phi(D^*_b))) = t'(\Phi(D^*_b))).$ Since $\sigma_1$ is homeomorphic, $\rho'(\sigma(D_0)) \cap \hat{K}_b(\mathcal{M})$ consist of a point $e'$. Thus $t'(\Phi(D^*_b)) \cap \hat{K}_b(\mathcal{M}) = \{ e' \}$. Therefore $\sigma$ is a well defined bijective map. It is clear by the definition that $\rho' \circ \sigma = \sigma \circ p$. By Lemma 5.4, $\sigma$ is homeomorphic, and this completes the proof of Proposition 5.3.

**Lemma 5.5.** (Cf. Koriyama, Maeda and Omori [4], Lemma 5.1).
Let $X \in \mathfrak{X}(E; p)$ and $e \in E_1$, $b = p(e)$. Then $X_e \neq 0$ if and only if $[p_*(X), \mathfrak{X}(B)] + \mathfrak{g}(b) = \mathfrak{X}(B)$ or $[X, \mathfrak{g}_b] + \mathfrak{g}(e)_b = \mathfrak{g}_b$.

**Proof.** From [1], Lemma 5.2, $p_*(X)_e \neq 0$ if and only if $[p_*(X), \mathfrak{X}(B)] + \mathfrak{g}(b) = \mathfrak{X}(B)$. We assume that $p_*(X)_e = 0$. By the same argument as the proof of Lemma 3.5, we see that if $X_e = 0$, then $[X, \mathfrak{g}_b] + \mathfrak{g}(e)_b = \mathfrak{g}_b$. Now assume that $[X, \mathfrak{g}_b] + \mathfrak{g}(e)_b = \mathfrak{g}_b$ but $X_e = 0$. Let $\mathfrak{g}(e)_b = \{Z \in \mathfrak{g}_b; (Ad Y) Z = 0 \text{ for } Y \in \mathfrak{X}_p(E; p)\}$ be a subalgebra of $\mathfrak{g}_b$. It is easy to see that $[X, Z] \in \mathfrak{g}(e)_b$ for any $Z \in \mathfrak{g}(e)_b$. Note that $\mathfrak{g}(e)_b$ is finite dimensional in $\mathfrak{g}_b$.

Let $ad X: \mathfrak{g}_b \rightarrow \mathfrak{g}_b$ be a map defined by $(ad X)(Z) = [X, Z]$ for $Z \in \mathfrak{g}_b$. Then the map $ad X$ induces a map $A(X)_*: \mathfrak{g}_b/\mathfrak{g}(e)_b \rightarrow \mathfrak{g}_b/\mathfrak{g}(e)_b$. Since $[X, \mathfrak{g}_b] + \mathfrak{g}(e)_b = \mathfrak{g}_b$, $A(X)_*$ is epimorphic. Since $A(X)_*$ is an endomorphism of the finite dimensional vector space $\mathfrak{g}_b/\mathfrak{g}(e)_b$, $A(X)_*$ is isomorphic. Let $\{X\}$ be the equivalence class of $X$ in $\mathfrak{g}_b/\mathfrak{g}(e)_b$. Then $A(X)_*[X] = 0$, hence $X \in \mathfrak{g}(e)_b$. Therefore $[X, \mathfrak{g}_b] \subset \mathfrak{g}(e)_b$, where $\mathfrak{g}_b = \{Y \in \mathfrak{g}_b; Y_e = 0\}$.

This completes the proof of Lemma 5.5.

Let $\mathfrak{X}(E; p)_e = \{X \in \mathfrak{X}(E; p); X_e = 0\}$ for each $e \in E$. From Propositions 5.2, 5.3 and Lemma 5.5, we have the following.

**Proposition 5.6.** The Lie algebra isomorphism $\Phi: \mathfrak{X}(E; p) \rightarrow \mathfrak{X}(E'; p')$ induces a Lie algebra isomorphism $\Phi: \mathfrak{X}(E; p)_e \rightarrow \mathfrak{X}(E'; p')_{\phi(e)}$ for each $e \in E$.

Let $V$ be a product $\Gamma$-module $V_1 \times V_2$ of a finite group $\Gamma$ such that $V^\Gamma = \{0\}$, and $\rho_1: V \rightarrow V_1$ the natural projection. Let $\{\eta_1, \ldots, \eta_s\}$ be a minimal set of homogeneous generators for $R[V_2]^\Gamma$. Here $R[V_2]^\Gamma$ is the algebra of $\Gamma$-invariant polynomials which vanish at 0. Let $\{y_1, \ldots, y_m\}$ be a canonical coordinate of $V_2$ such that $\Gamma$ acts orthogonally on this coordinate. We can assume $\eta_1 = y_1 + \cdots + y_m$. Let $C^\gamma(V)$ denote the set of all $\Gamma$-invariant smooth functions on $M$.

**Lemma 5.7.** Let $g$ be a $\Gamma$-invariant continuous function on $V$ such that $\eta_ig \in C^\gamma(V)$ for $i = 1, \ldots, s$. Then $g$ is a $\Gamma$-invariant smooth function.
Proof. If $s=1$, by using Taylor expansion, it is easy to prove that $g$ is a smooth function. We consider the case that $s \geq 2$. Put $h_i = \eta_i g$. Then we see that $h_i \eta_2 = h_2 \eta_i$. Put $G = h_2 \eta_1$. Let $T_a G$ be a formal Taylor expansion at $a \in V$. Then $T_a G = \eta_1 T_\eta(h_i) = \eta_i T_\eta(h_i)$. Since $\{\eta_1, \ldots, \eta_s\}$ is a minimal set of homogeneous generators for $R[V_2]_0^s$, there exists a smooth function $h'_i$ on $V$ such that $h_1 = \eta_i h'_i$. Then $g$ is a smooth function.

**Theorem 5.8.** There exists a Lie algebra isomorphism $\Phi: \mathfrak{X}(E'; \rho) \rightarrow \mathfrak{X}(E'; \rho')$ if and only if there exists a fibration preserving diffeomorphism $\sigma: E \rightarrow E'$ such that $\Phi = \sigma_*$.  

**Proof.** Using Proposition 5.6 and Lemma 5.7, we can prove Theorem 5.8 by the same argument as in [1], §5.

**§ 6. Reflection Groups**

Let $V$ be an $n$-dimensional product $\Gamma$-module $V_1 \times V_2$ of a finite group $\Gamma$. Let $\Gamma_v$ denote the isotropy subgroup of $\Gamma$ at $v \in V$. Let $V^{(i)} = \{v \in V; \Gamma_v \text{ is a cyclic group of order two generated by a reflection of } V\}$. Let $\Gamma_1$ be the subgroup of $\Gamma$ generated by reflections $\{\gamma \in \Gamma_v; v \in V^{(i)}\}$. $\Gamma_1$ is a normal subgroup of $\Gamma$ which is a reflection group. Let $\Gamma_1'(i=1, 2)$ be the subgroup of $\Gamma_1$ generated by reflections $\{\gamma \in \Gamma_v; v \in V^{(i)} \cap V_i\}$. Then $\Gamma_1' = \Gamma_1' \times \Gamma_1'$ since $\Gamma_1'$ is a reflection subgroup, there is a homogeneous minimal set of generators $\{\theta_1, \ldots, \theta_n\}$ for $R[V_1]_0^{\Gamma_1'}$ (see Bourbaki [3], Chapitre $V$, §5, Théorème 3). Let $n_i = \dim V_i$ for $i=1, 2$. We can assume that $\{\theta_1, \ldots, \theta_{n_1}\}$ and $\{\theta_{n_1+1}, \ldots, \theta_n\}$ are homogeneous minimal sets of generators for $R[V_1]_0^{\Gamma_1}$ and $R[V_2]_0^{\Gamma_1}$ respectively. Let $\theta^1 = (\theta_1, \ldots, \theta_{n_1}): V_1 \rightarrow R^{n_1}$, $\theta^2 = (\theta_{n_1+1}, \ldots, \theta_n): V_2 \rightarrow R^{n_2}$ and $\theta = (\theta^1, \theta^2): V \rightarrow R^n$ be polynomial maps. Let $\tilde{\theta}^1: V_1/\Gamma_1' \rightarrow R^{n_1}$, $\tilde{\theta}^2: V_2/\Gamma_1' \rightarrow R^{n_2}$ and $\tilde{\theta}: V/\Gamma_1 \rightarrow R^n$ be the induced orbit maps which are embeddings. Since $\Gamma_1$ is a normal subgroup of $\Gamma$, the $\Gamma$-action on $V$ induces an action $\Psi_0: \overline{\Gamma} \times V/\Gamma_1 \rightarrow V/\Gamma_1$, where $\overline{\Gamma}$ is a factor group $\Gamma/\Gamma_1$.

**Lemma 6.1.** There exists a linear action $\Psi: \overline{\Gamma} \times R^n \rightarrow R^n$ such that
(1) \( \psi(\gamma, \bar{\theta}(X)) = \bar{\theta}(\psi_0(\gamma, x)) \) for \( \gamma \in \tilde{\Gamma} \), \( x \in V_i/\Gamma_i \), and \( \bar{\theta}: V_i/\Gamma_i \rightarrow R^n \) is a \( \tilde{\Gamma} \)-equivariant embedding.
(2) \( \psi(\gamma, \bar{\theta}(x)) = \bar{\theta}(\psi_0(\gamma, x)) \) for \( \gamma \in \tilde{\Gamma} \), \( x \in V_i/\Gamma_i^i (i = 1, 2) \), and \( \bar{\theta}^i: V_i/\Gamma_i^i \rightarrow R^{n_i} \) is a \( \tilde{\Gamma}^i \)-equivariant embedding.

**Proof.** Since \( \Gamma_i \) is a normal subgroup of \( \Gamma \), \( \tilde{\Gamma} \) acts on \( R[V_i]^{\tilde{\Gamma}_i} \) by \( (\gamma \cdot f)(v) = f(\gamma^{-1} \cdot v) \) for \( f \in R[V_i]^{\tilde{\Gamma}_i} \), \( \gamma \in \tilde{\Gamma} \), \( v \in V \). We can choose a homogeneous minimal set of generators for \( R[V_i]^{\tilde{\Gamma}_i} \) such that \( \gamma \cdot \theta_i = 2 \deg \gamma_i - \deg j a_{ij}(\gamma) \theta_j \) with \((a_{ij}(\gamma)) \in GL(n, R)\) for \( \gamma \in \tilde{\Gamma} \) (cf. Schwarz [8], Lemma 8.1). We can assume that \( a_{ij} = 0 \) if \( 1 \leq i \leq n_1 \), \( n_1 + 1 \leq j \leq n \) or \( 1 \leq j \leq n_3 \), \( n_1 + 1 \leq i \leq n \). Let \( \{e_1, \ldots, e_n\} \) be a canonical basis of \( R^n \). We define a linear action \( \psi:\tilde{\Gamma} \times R^n \rightarrow R^n \) by \( \psi(\gamma, \Sigma_j x_j e_j) = \Sigma_j \gamma^{-1} a_{ij}(\gamma^{-1}) x_j e_i \). Then we can check that the action satisfies (1) and (2). This completes the proof of Lemma 6.1.

Let \( \rho: V_i/\Gamma_i \rightarrow V_1/\Gamma_1 \) and \( \rho: R^n/\tilde{\Gamma} \rightarrow R^n/\tilde{\Gamma} \) be maps induced from the natural projections \( V \rightarrow V_1 \) and \( R^n \rightarrow R^n/\tilde{\Gamma} \) respectively. We can regard \( \rho \) and \( \tilde{\rho} \) as fibrations over smooth orbifolds. Since \( \bar{\theta}: V_i/\Gamma_i \rightarrow R^n \) is a \( \tilde{\Gamma} \)-equivariant embedding by Lemma 6.1, we have an induced embedding \( \bar{\theta}: V_i/\Gamma_i \rightarrow R^n/\tilde{\Gamma} \). It is easy to see that the induced map \( \bar{\theta}^* : C^\infty(V_i/\Gamma_i) \rightarrow C^\infty(V_i/\Gamma) \) is epimorphic. We define \( \bar{\theta}^* : \mathbb{D}(R^n/\tilde{\Gamma}) \rightarrow \mathbb{D}(V_i/\Gamma) \) by \( \bar{\theta}^*(X)(\bar{\theta}^*(f)) = X(f) \circ \bar{\theta} \) for \( X \in \mathbb{D}(R^n/\tilde{\Gamma}) \), \( f \in C^\infty(R^n/\tilde{\Gamma}) \). Since \( \bar{\theta}(V_i/\Gamma) \) is the closure of an open set in \( R^n/\tilde{\Gamma} \), we see that \( \bar{\theta}^* \) is a well defined Lie algebra homomorphism. Clearly \( \bar{\theta}^* \) induces a Lie algebra homomorphism \( \bar{\theta}^* : \mathbb{D}(R^n/\tilde{\Gamma}; \rho) \rightarrow \mathbb{D}(V_i/\Gamma; \bar{\theta}) \).

Let \( \{\gamma_i, \ldots, \gamma_k\} \) be a minimal set of generators for \( R[R^n]_0 \) such that \( \{\gamma_i, \ldots, \gamma_k\} (k_1 \leq k) \) is a minimal set of generators for \( R[R^n]_0 \). Let \( \eta: R^n/\tilde{\Gamma} \rightarrow R^k \) be the orbit map of a polynomial map \( \eta = (\gamma_1, \ldots, \gamma_k): R^n \rightarrow R^k \). Let \( \nu = (\nu_1, \ldots, \nu_k): V \rightarrow R^k \), where \( \nu_i = \gamma_i \circ \bar{\theta} \). Since \( \bar{\theta} \) and \( \eta \) are smooth embeddings, \( \nu(V) = \eta(\bar{\theta}(V_i/\Gamma)) \) is diffeomorphic to \( V_i/\Gamma \). Let \( r_i \in R[\gamma_1, \ldots, \gamma_k] \), \( i = 1, \ldots, l \), be the generators of the ideal \( I \) of algebraic relations among \( \gamma_1, \ldots, \gamma_k \). Let \( S \) be a subset of \( R^n \) and \( I^\infty(S) \) denote the ideal in \( C^\infty(R^n) \) which vanish on \( S \). For a smooth vector field \( X \) on \( R^n \), we say that \( X \) is tangent to \( S \) if \( X \) preserves the ideal \( I^\infty(S) \).
Lemma 6.2. Let \( X \) be a polynomial vector field on \( \mathbb{R}^k \). Then

1. \( X \) is tangent to \( \eta(\mathbb{R}^n) \) if and only if \( X(r_i) \in I, \ i = 1, \ldots, l \).
2. \( X \) is tangent to \( \eta(\mathbb{R}^n) \) if and only if \( X \) is tangent to \( \nu(V) \).

Proof. Since the ring of polynomial functions on \( \eta(\mathbb{R}^n) \) can be identified with \( R[y_1, \ldots, y_r]/I \), we see (1). Since \( \nu(V) \) contains an open set in \( \mathbb{R}^n \), we can prove that \( I \) is also the ideal of algebraic relations among \( \nu_1, \ldots, \nu_n \), and (2) follows.

Let \( D(\eta(\mathbb{R}^n)) \) be the Lie algebra of polynomial vector fields on \( \mathbb{R}^k \) which tangent to \( \eta(\mathbb{R}^n) \). By the same argument as in Schwarz [8, Proposition 6.14], we see the following.

Proposition 6.3. (1) \( \mathfrak{D}(\eta(\mathbb{R}^n)) = C^\infty(\eta(\mathbb{R}^n)) \cdot D(\eta(\mathbb{R}^n)) \).
(2) \( \mathfrak{D}(\nu(V)) = C^\infty(\nu(V)) \cdot D(\nu(V)) \).

Proposition 6.4. The Lie algebra homomorphism \( \bar{\theta}^*: \mathfrak{D}(\mathbb{R}^n/\bar{\Gamma}) \rightarrow \mathfrak{D}(V/\Gamma) \) is epimorphic.

Proof. By Lemma 6.2 \( D(\eta(\mathbb{R}^n)) = D(\nu(V)) \). Let \( j: \nu(V) \subset \eta(\mathbb{R}^n) \) be the inclusion. Then the induced map \( j^*: C^\infty(\eta(\mathbb{R}^n)) \rightarrow C^\infty(\nu(V)) \) is epimorphic. Since \( \bar{\theta}^* \) is identified with a homomorphism \( \mathfrak{D}(\mathbb{R}^n/\bar{\Gamma}) = \mathfrak{D}(\eta(\mathbb{R}^n)) \rightarrow \mathfrak{D}(\nu(V)) = \mathfrak{D}(V/\Gamma) \), \( \bar{\theta}^* \) is epimorphic from Proposition 6.3. This completes the proof of Proposition 6.4.

Note that the maps \( \rho: V/\Gamma \rightarrow V_1/\Gamma \) and \( \rho: R^n/\bar{\Gamma} \rightarrow R^n/\bar{\Gamma} \) induce Lie algebra homomorphisms \( p_*: \mathfrak{D}(V/\Gamma; \rho) \rightarrow \mathfrak{D}(V_1/\Gamma) \) and \( \rho_*: \mathfrak{D}(R^n/\bar{\Gamma}; \rho) \rightarrow \mathfrak{D}(R^n/\bar{\Gamma}) \). We set \( \mathfrak{D}_F(V/\Gamma; \rho) = \ker \rho_* \) and \( \mathfrak{D}_F(R^n/\bar{\Gamma}; \rho) = \ker \rho_* \). Then \( \mathfrak{D}_F(V/\Gamma; \rho) \) is a \( C^\infty(V/\Gamma) \)-module and \( \mathfrak{D}_F(R^n/\bar{\Gamma}; \rho) \) is a \( C^\infty(R^n/\bar{\Gamma}) \)-module respectively. Let \( q: R^k \rightarrow R^k \) be the natural projection \( q: \mathbb{R} \rightarrow \mathbb{R} \) be the natural projection \( q: R^k \rightarrow R^k \) be the natural projection \( q: \mathbb{R} \rightarrow \mathbb{R} \) be the natural projection \( q: \mathbb{R} \rightarrow \mathbb{R} \) be the natural projection. Let \( q_*: \mathfrak{D}(R^k; q) \rightarrow \mathfrak{D}(\mathbb{R}^k) \). Let \( D_F(\eta(\mathbb{R}^n); q) = \ker q_* \cap D(\eta(\mathbb{R}^n)) \) and \( D_F(\nu(V); q) = \ker q_* \cap D(\nu(V)) \). By the same argument as in Proposition 6.3 and Proposition 6.4, we have the following.

Proposition 6.5. (1) \( \mathfrak{D}_F(\eta(\mathbb{R}^n); q) = C^\infty(\eta(\mathbb{R}^n)) \cdot D_F(\eta(\mathbb{R}^n); q) \).
(2) \( \mathfrak{D}_F(\nu(V); q) = C^\infty(\nu(V)) \cdot D_F(\nu(V); q) \).
The Lie algebra homomorphism $\tilde{\theta}^*: D_\mathbb{F}(\mathbb{R}^n/\Gamma; \rho) \to D_\mathbb{F}(V/\Gamma; \rho)$ is epimorphic.

Combining Propositions 6.4 and 6.5, we have.

**Corollary 6.6.** The Lie algebra homomorphism $\tilde{\theta}^*: D_\mathbb{F}(\mathbb{R}^n/\Gamma; \rho) \to D(V/\Gamma; \rho)$ is epimorphic.

**Lemma 6.7** If $V^{(1)}$ is not empty, then there exists a smooth vector field $X \in D(V/\Gamma; \rho)$ such that $X_{\pi(0)} \neq 0$, where $\pi: V \to V/\Gamma$ is the natural projection.

**Proof.** $V$ is an orthogonal representation with respect to a suitable basis of $V$. Let $W$ be the orthogonal complement of $V^\Gamma$. Then $W$ is a $\Gamma$-module. Since $V^{(1)}$ is not empty, $\dim W > 0$. Let $\{x_1, \ldots, x_p\}$ be a canonical coordinate of $W$. If $\dim W^\Gamma > 0$, we see that there exists a $\Gamma$-invariant smooth vector field $Y$ on $V$ such that $X = \pi^*(Y) \in D(V/\Gamma; \rho)$ and $X_{\pi(0)} \neq 0$. If $\dim W^\Gamma = 0$, then we can choose a homogeneous minimal set of generators $\{\theta_1, \ldots, \theta_n\}$ for $R[V]_{\pi(0)}$ such that $\theta_i = x_1^2 + \cdots + x_i^2$ for some $i$. Since $\gamma \cdot \theta_i = \theta_i$ for any $\gamma \in \Gamma$, we see that $\dim (R^n)^\Gamma > 0$. Then we have a $\Gamma$-invariant smooth vector field $Z$ on $R^n$ such that $X = \pi^*(Z) \in D(R^n/\Gamma; \rho)$ and $X_{\pi(0)} \neq 0$, where $\pi': R^n \to R^n/\Gamma$ is the natural projection. Since $(d\theta)_{\pi(0)}: \tau_{\pi(0)}(R^n/\Gamma) \to \tau_{\pi(0)}(R^n/\Gamma)$ is isomorphic, $\theta^*(X)_{\pi(0)} \neq 0$. This completes the proof of Lemma 6.7.

Let $D(V/\Gamma; \rho)(x) = \{Y_x; Y \in D(V/\Gamma; \rho)\}$ for $x \in V/\Gamma$.

**Corollary 6.8.** $D(V/\Gamma; \rho)(\pi(0)) = \{0\}$ if and only if $\dim V^\Gamma = 0$ and $V^{(1)} = \phi$.

**Proof.** If $V^{(1)} = \phi$, then $D(V/\Gamma; \rho) = \mathcal{X}(V/\Gamma; \rho)$ by Schwarz [8, Chapter I, Proposition 3.5]. By Lemma 2.2 $\mathcal{X}(V/\Gamma; \rho) = \mathcal{X}_1(V/\rho)$, where $\beta: V \to V_1$ is the natural projection. Thus if $V^{(1)} = \phi$ and $\dim V^\Gamma = 0$, $D(V/\Gamma; \rho)(\pi(0)) = \{0\}$ from Lemmas 2.3 and 2.5. If $V^{(1)} \neq \phi$, then $D(V/\Gamma; \rho)(\pi(0)) \neq \{0\}$ by Lemma 6.7, and Corollary 6.8 follows.
§ 7. Proof of Theorem 0.1 and Theorem 0.2

Let \( p: E \to B \) be a fibration over a connected smooth orbifold \( B \) with generic fibre \( F \), a connected smooth manifold. We set \( B_0 = \{ b \in B ; X_b = 0 \text{ for any } X \in \mathcal{D}(B) \} \), \( B_1 = B - B_0 \) and \( \mathcal{D}(B)_3 = \{ X \in \mathcal{D}(B) ; X_b = 0 \} \).

**Proposition 7.1.** For \( b \in B_1 \), there exists a unique infinite codimensional maximal ideal \( \mathfrak{S}_b(B) \) of \( \mathcal{D}(B) \) which is contained in \( \mathcal{D}(B)_3 \).

**Proof.** For \( b \in B \), there exist a finite group \( \Gamma \) and a linear action \( \Gamma \times V \to V \) on an \( n \)-dimensional vector space \( V \) such that \( V/\Gamma \) is diffeomorphic to an open neighborhood \( U \) of \( b \). By Lemma 6.1 and Proposition 6.4, we have a linear action \( \bar{\Gamma} \times R^n \to R^n \) such that there exists an embedding \( \bar{\theta}: V/\bar{\Gamma} \to \mathbb{R}^n/\bar{\Gamma} \) and \( \bar{\theta}^*: \mathcal{D}(\mathbb{R}^n/\bar{\Gamma}) \to \mathcal{D}(V/\Gamma) \) is a Lie algebra epimorphism. Using the property of the reflection group \( \Gamma_1 \), the natural group homomorphism \( \Gamma \to \Gamma_1 \) has a right inverse. Then it is easy to see that \( \bar{\Gamma} \) has no reflection subgroups. Then \( \mathcal{D}(\mathbb{R}^n/\bar{\Gamma}) \) is an infinite codimensional maximal ideal of \( \mathcal{D}(V/\Gamma) \). By Corollary 6.6 and Lemma 6.7, there exists a smooth vector field \( \bar{X} \in \mathcal{D}(\mathbb{R}^n/\bar{\Gamma}) \) such that \( \bar{X}(0) = 0 \).

From [1], Lemma 3.9, there exists a unique infinite codimensional maximal ideal \( \mathfrak{R}_1 \) of \( \mathcal{X}(\mathbb{R}^n/\bar{\Gamma}) \) which is contained in \( \mathcal{X}(\mathbb{R}^n/\bar{\Gamma})_3 \). Then \( \mathfrak{R}_2 = \bar{\theta}^*(\mathfrak{R}_1) \) is an infinite codimensional maximal ideal of \( \mathcal{D}(V/\Gamma) \) which is contained in \( \mathcal{D}(V/\Gamma)_3 \). \( \mathfrak{S}_b(B) = \{ X \in \mathcal{D}(B) ; X = Y \text{ on a neighborhood of } b \text{ in } B \} \). In the proof of [1], Proposition 3.8, we can prove that \( \mathfrak{S}_b(B) \) is an infinite codimensional maximal ideal of \( \mathcal{D}(B) \). This completes the proof of Proposition 7.1.

**Lemma 7.2.** Let \( b \in B_1 \). Then, for \( X \in \mathcal{D}(B) \), \( X(b) \neq 0 \) if and only if \( [X, \mathcal{D}(B)] + \mathfrak{S}_b(B) = \mathcal{D}(B) \).

**Proof.** We use the same notation as in the proof of Proposition 7.1. To prove Lemma 7.2, it is sufficient to prove that, for \( X \in \mathcal{D}(V/\Gamma) \), \( X_{\pi(0)} \neq 0 \) if and only if \( [X, \mathcal{D}(V/\Gamma)] + \mathfrak{R}_2 = \mathcal{D}(V/\Gamma) \) where \( \pi: V \to V/\Gamma \) is the natural projection. Note that \( \mathcal{D}(\mathbb{R}^n/\bar{\Gamma}) = \mathcal{X}(\mathbb{R}^n/\bar{\Gamma}) \). It follows from [1], Lemma 5.2 that, for \( Y \in \mathcal{D}(\mathbb{R}^n/\bar{\Gamma}) \), \( Y_{\rho(0)} \neq 0 \) if and only if \( [Y, \mathcal{D}(\mathbb{R}^n/\bar{\Gamma})] + \mathfrak{R}_3 = \mathcal{D}(\mathbb{R}^n/\bar{\Gamma}) \), where \( \rho: \mathbb{R}^n \to \mathbb{R}^n/\bar{\Gamma} \) is the
natural projection. Let $X \in \mathcal{D}(V/\Gamma)$ with $X_{x(0)} \neq 0$. There exists $Y \in \mathcal{D}(R^*/\Gamma)$ such that $\tilde{\theta}^*(Y) = X$. Then $Y_{x(0)} \neq 0$ and hence $[Y, \mathcal{D}(R^*/\Gamma)] + \mathfrak{g}_1 = \mathcal{D}(R^*/\Gamma)$. Since $\tilde{\theta}^*$ is epimorphic, $[X, \mathcal{D}(V/\Gamma)] + \mathfrak{g}_2 = \mathcal{D}(V/\Gamma)$. Conversely, suppose that $[X, \mathcal{D}(V/\Gamma)] + \mathfrak{g}_2 = \mathcal{D}(V/\Gamma)$. Let $Y \in \mathcal{D}(R^*/\Gamma)$ such that $\tilde{\theta}^*(Y) = X$. It is easy to see that $\mathcal{D}(R^*/\Gamma) = [Y, \mathcal{D}(R^*/\Gamma)] + \mathfrak{g}_1$. Then $Y_{x(0)} \neq 0$. Hence $X_{x(0)} \neq 0$, and this completes the proof of Lemma 7.2.

**Proof of Theorem 0.1.** Using the result of [1], we see that (2) implies (3). By Schwarz [8], Corollary 1.7, if $\sigma: B \to B'$ is diffeomorphic, then $\sigma$ is strata preserving. Then we see that (3) implies (2). Assume that there exists a Lie algebra isomorphism $\Phi: \mathcal{D}(B) \to \mathcal{D}(B')$. As in [1], § 5, using Proposition 7.1, and Lemma 7.2, we can prove that there exists a diffeomorphism $\sigma: B \to B'$ such that $\Phi = \sigma_*$. This completes the proof of Theorem 0.1.

**Proof of Theorem 0.2.** By Theorem 5.8, (2) implies (3). By Schwarz [8], Corollary 1.7, (3) implies (2). Assume that there exists a Lie algebra isomorphism $\Phi: \mathcal{D}(E; p) \to \mathcal{D}(E'; p')$. As the proof of Theorem 5.7, using Corollary 6.6 and Theorem 0.1, we can prove that there exists a fibration preserving diffeomorphism $\sigma: E \to E'$ such that $\Phi = \sigma_*$. This complete the proof of Theorem 0.2.

**References**


