Hodge Spectral Sequence on Compact Kähler Spaces

By

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Introduction

Let \( X \) be a complex manifold of dimension \( n \) and \( E^p_q \) the Hodge spectral sequence on \( X \). The following is fundamental in the study of algebraic varieties.

**Theorem (W.V.D. Hodge [8])** If \( X \) is a compact Kähler manifold, then

\[
\begin{align*}
(E) & \quad E^p_q = \sum_{i=0}^p E^p_i \\
& \quad E^p_q \cong E^p_q
\end{align*}

for any \( p \) and \( q \).

In 1972, P. Deligne [3] succeeded in generalizing it for an arbitrary quasi-projective variety by analyzing a different spectral sequence. His so-called mixed Hodge theory explains how the singular cohomology is composed of the analytic cohomology attached to the variety.

On the other hand, Grauert-Riemenschneider [7] and Fujiki [5] tried to understand the Hodge spectral sequence itself on pseudoconvex manifolds. Inspired by these works, the author [11] could show that (H) is valid for the range \( p+q \geq n+r \) on any “very strongly \( r \)-convex” Kähler manifold of dimension \( n \). The crucial point was to establish an isomorphism between the ordinary cohomology and the \( L^2 \) cohomology with respect to a certain complete Kähler metric on pseudoconvex domains.

Since it has long been known that for any projective variety over \( \mathbb{C} \) the complement of the singular locus admits a complete Kähler metric (Grauert [6]), it is natural to ask for a reasonable extension of [11] in such a case.

The purpose of the present paper is to show the following in this spirit.

**Theorem 1** Let \( X \) be a compact Kähler space of pure dimension \( n \) whose...
singular points are isolated, and let $X^*$ be the complement of the singular points. Then (H) holds on $X^*$ for the range $p+q<n-1$.

Note that the range is optimal since $\dim H^{0,n-1}(X^*) = \infty$ if $X^* \neq X$, where $H^{0,n-1}$ denotes the Dolbeault cohomology of type $(n-1, 0)$.

We shall also give a partial answer to a question of Cheeger-Goreski-MacPherson [2] by showing the following:

**Theorem 2** Under the situation of Theorem 1,
\[
\begin{cases}
H^r(X^*) \cong H_{(2)}^r(X^*) & \text{if } r < n-1 \\
H^{p,q}(X^*) \cong H_{(2)}^{p,q}(X^*) & \text{if } p+q < n-1
\end{cases}
\]

and
\[
\begin{aligned}
H_0^r(X^*) & \cong H_{(2)}^r(X^*) & \text{if } r > n+1 \\
H_{0,q}^r(X^*) & \cong H_{(2)}^{p,q}(X^*) & \text{if } p+q > n+1.
\end{aligned}
\]

Here, $H$, $H_0$, and $H_{(2)}$ denote respectively the ordinary cohomology, the cohomology with compact support, and the $L^2$ cohomology.

Note that the duality between $H_{(2)}^r$ and $H_{(2)}^{2n-r}$ is not obvious since the metric on $X^*$ is not complete as long as $X^* \neq X$.

Since the intersection cohomology $IH^r(X)$ is isomorphic to $H^r(X^*)$ if $r < n$ and isomorphic to $H_0^r(X^*)$ if $r > n$, Theorem 2 implies the following.

**Corollary** $IH^r(X) \cong H_{(2)}^r(X^*)$ if $r \neq n, n \pm 1$.

Cheeger-Goreski-MacPherson conjectured that the above isomorphism is valid for any degree, and in some special cases it has been verified (cf. [2], [10] and [12]).

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### §1. Preliminaries

Let $(M, ds^2)$ be a complete Hermitian manifold of dimension $n$. We put
\[
L_{(2)} (= L_{(2)}^0(M)) : = \{ \text{square integrable complex differential forms on } M \}.
\]
\[
L_r^{(2)} (= L_r^{(2)}(M)) : = \{ f \in L_{(2)}; \deg f = r \}.
\]
\[ L^2_{(2)} ( = L^2_{f}(M) ) : = \{ f \in L_{(2)} ; f \text{ is of type } (p, q) \} . \]

The norms and the inner products in \( L_{(2)} \) shall be denoted by \( \| \| (=\| | \|_M) \) and \( (, ) (=\langle , \rangle_M) \), respectively. The exterior differentiations \( d, \bar{\partial} \) and \( \partial \) are regarded as densely defined closed linear operators on \( L_{(2)} \) whose domains of definition are given by

\[
\text{Dom } d : = \{ f \in L_{(2)} ; \langle f, L^2_{(2)} \rangle \}
\]

Here the differentiation is in distribution sense.

**Definition**

\[
H_{(2)} ( = H_{(2)}(M) ) : = \text{Ker } d/\text{Im } d .
\]

\[
H_{(2)}^{*} ( = H_{(2)}^{*}(M) ) : = \text{Ker } d \cap L_{(2)}/\text{Im } d \cap L_{(2)} .
\]

\[
H_{(2)}^{*} ( = H_{(2)}^{*}(M) ) : = \text{Ker } \bar{\partial} \cap L_{(2)}^{*}/\text{Im } \bar{\partial} \cap L_{(2)}^{*} .
\]

We denote by \( d^* \) and \( \bar{\partial}^* \) the adjoints of \( d \) and \( \bar{\partial} \), respectively. Note that \( H_{(2)} \cong \text{Ker } d \cap \text{Ker } d^* \) (resp. \( H_{(2)}^{*} \cong \text{Ker } \bar{\partial} \cap \text{Ker } \bar{\partial}^* \cap L_{(2)}^{*} \)) if and only if \( \text{Im } d \) is closed (resp. \( \text{Im } \bar{\partial} \cap L_{(2)}^{*} \) is closed). \( H_{(2)} \) are called \( L^2 \) cohomologies of \( M \). For any family of supports \( \Phi \), \( L^2 \) cohomologies with supports in \( \Phi \) are also defined similarly as above.

The following is first due to H. Donnelly and C. Fefferman, but the proof below is different from theirs.

**Theorem 1.1 (cf. [4])** Suppose that there exists a \( C^\infty \) real valued function \( F \) on \( M \) such that the fundamental form of \( ds^2 \) is \( i \partial \bar{\partial} F \) and that \( |\partial F|_{\infty} (:=\text{sup } |\partial F|) < \infty \). Then, for any \( u \in \text{Ker } d \cap L_{(2)} \) with \( r \neq n \), there exists a \( v \in \text{Dom } d \cap L_{(2)}^{r} \) such that \( dv = u \) and \( \|v\| \leq 2|\partial F|_{\infty}\|u\| \). Similarly, if \( p+q = n \), then for any \( u \in \text{Ker } \bar{\partial} \cap L_{(2)}^{p+q} \) there exists a \( v \in \text{Dom } \bar{\partial} \cap L_{(2)}^{p+q} \) such that \( \bar{\partial}v = u \) and \( \|v\| \leq (1+\sqrt{2})|\partial F|_{\infty}\|u\| \). In particular,

\[
\begin{cases}
H_{(2)} = 0 & \text{if } r \neq n , \\
H_{(2)}^{*} = 0 & \text{if } p+q \neq n .
\end{cases}
\]

**Proof.** The assertions are equivalent to that

\[
\|u\| \leq 2|\partial F|_{\infty}|d^*u| , \quad \text{for any } u \in \text{Ker } d \cap \text{Dom } d^* \cap L_{(2)}
\]

and

\[
\|u\| \leq (1+\sqrt{2})|\partial F|_{\infty}|\bar{\partial}^*u| , \quad \text{for any } u \in \text{Ker } \bar{\partial} \cap \text{Dom } \bar{\partial}^* \cap L_{(2)}^{p+q} ,
\]

respectively (cf. [9]). They are proved as follows:
For any differential form $\theta$, let $e(\theta)$ be the multiplication by $\theta$ from the left. Then we have the following formula.

$$[\bar{\partial}, e(\bar{\partial}F)^{*}] + [\partial^{*}, e(\partial F)] = [e(i\partial\bar{\partial}F), e(i\partial\bar{\partial}F)^{*}] .$$

Here $[\ , \ ]$ denotes the commutator with weight (i.e., $[S, T] := S^o T - (-1)^{degS degT} T^o S$) and $*$ denotes the adjoint.

In fact, with respect to the operator $A := e(i\partial\bar{\partial}F)^{*}$ we have $[\bar{\partial}, A] = i\partial^{*}$ and $[e(\partial F), A] = i e(\bar{\partial}F)^{*}$ (cf. [12]). Therefore

$$[\partial^{*}, e(\partial F)] = -i e(\partial F) [\bar{\partial}, A] - i [e(\partial F), A] \bar{\partial}$$

$$= [e(i\partial\bar{\partial}F), A] + i e(\partial F) [\bar{\partial}, A] + i [\partial^{*}, A] e(\partial F)$$

$$= [e(i\partial\bar{\partial}F), A] - [e(\partial F), \partial^{*}] .$$

Hence, for any compactly supported $C^\infty r$-form $u$,

$$(|e(i\partial\bar{\partial}F), A| u, u) \leq |\partial F| \omega(\|\partial u\|^2 + \|\partial^{*} u\|^2 + \|\partial u\|^2 + \|\partial^{*} u\|) .$$

Since the metric is Kählerian, we have

$$\|\partial u\|^2 + \|\partial^{*} u\|^2 = \|\partial u\|^2 + \|\partial^{*} u\|^2$$

$$= \frac{1}{2} (\|du\|^2 + \|d^{*} u\|^2) \quad (\text{cf. } [14]) .$$

On the other hand, $[e(i\partial\bar{\partial}F), A] u = (r-n)u$. Thus we obtain

$$\|u\| \leq 2 |\partial F| \omega(\|du\|^2 + \|d^{*} u\|^2)^{1/2}$$

and

$$\|u\| \leq (1 + \sqrt{2}) |\partial F| \omega(\|\partial u\|^2 + \|\partial^{*} u\|) ,$$

if $r \neq n$.

Since the metric $ds^2$ is complete, the required estimate follows from the above (cf. [13]).

§2. A Poincaré-Dolbeault Lemma

Let $X$ be a complex analytic space of pure dimension $n$. In what follows the nonsingular part of $X$ will be denoted by $X^*$. Suppose that $o$ is an isolated singular point of $X$. Then we have a holomorphic embedding of the
germ \((X, 0) \to (\mathbb{C}^N, O)\). We fix in the followings a holomorphic coordinate \(z = (z_1, \cdots, z_N)\) of \(\mathbb{C}^N\) and the euclidean norm \(|z|\) of \(z\). We put \(B^*_c := \{z; 0 < |z| < c\}\) and \(X^*_c := X \cap B^*_c\) (\(c\) sufficiently small). As a candidate of the potential \(F\) in Theorem 1.1, we put

\[
F_c(z) := - \log \log \left( \frac{c}{|z|} \right).
\]

Proposition 2.1 The length of \(\partial(F_c|X^*_c)\) with respect to the metric \(2\partial\bar{\partial}(F_c|X^*_c)\) is bounded.

Proof. On \(B^*_c\) we have

\[
\partial F_c = - \frac{\partial \log |z|}{\log(c/|z|)}
\]

and

\[
\partial\bar{\partial} F_c \geq \frac{\partial \log |z| \bar{\partial} \log |z|}{\log^2(c/|z|)}.
\]

Hence \(|\partial(F_c|X^*_c)| \leq 1\).

In what follows we fix \(c\) and regard \(X^*_c\) for \(b \leq c\) as a Kähler manifold with metric \(2\partial\bar{\partial}(F_b|X^*_c)\). Moreover \(c\) is fixed so that \(\partial\bar{\partial}X^*_c\) is compact for all \(b \leq c\). It is clear from (3) that \(X^*_c\) are then complete Kähler manifolds.

Combining (1) in Theorem 1.1 and Proposition 2.1 we obtain the following:

Proposition 2.2 For any \(b \leq c\),

\[
\begin{cases}
H^{(r)}(X^*_c) = 0 & \text{if } r \neq n, \\
H^{(r)}(X^*_c) = 0 & \text{if } p + q \neq n.
\end{cases}
\]

The following observation was already made in [10], but we shall repeat the proof because of the completeness.

Lemma 2.3 Let \(r > n\) and \(u \in L^2(\Omega^{p,q}(X^*_c)).\) Then, \(u|X^*_c \in L^2(\Omega^{p,q}(X^*_c)),\) for any \(b \leq c\).

Proof. Since

\[
\partial\bar{\partial} F_b = \frac{\partial\bar{\partial} \log |z| + \partial \log |z| \bar{\partial} \log |z|}{\log(b/|z|) \log^2(b/|z|)},
\]

for any \(b\), the eigenvalues \(\lambda_1 \leq \cdots \leq \lambda_N\) of \(\partial\bar{\partial} F_b\) measured by \(\partial\bar{\partial} F_b\) are given by

\[
\lambda_j = \frac{\log(c/|z|)}{\log(b/|z|)}, \quad \text{for } 1 \leq j \leq N - 1
\]
and

$$\lambda_k = \frac{\log(c/|z|)}{\log(b/|z|)}.$$  

Let $\mu_1 \leq \cdots \leq \mu_n$ be the eigenvalues of $\partial \bar{\partial}(F \mid X^k)$ measured by $\partial \bar{\partial}(F \mid X^k)$. Then, by Courant's minimax principle,

$$\mu_j = \frac{\log(c/|z|)}{\log(b/|z|)}, \quad \text{for} \quad 1 \leq j \leq n-1$$

and

$$\frac{\log(c/|z|)}{\log(b/|z|)} \leq \mu_n \leq \frac{\log(c/|z|)}{\log(b/|z|)}.$$  

Now it is easy to see that $||u||_{X^*} \leq ||u||_{X^*}$, for any $u \in L^r(X^k)$ with $r > n$.

On the opposite side $r < n$ we have the following, which will be used to prove Theorem 2.

Lemma 2.4 Let $b < c$ and $u \in L^r_{(2)}(X^k)$ with $r < n$. Let $\tilde{u}$ be a form in $L^r_{(2)}(X^k)$ defined by $\tilde{u} := u$ on $X^k$ and $\tilde{u} := 0$ on $X^k \setminus X^k$. Then $\tilde{u} \in L^r_{(2)}(X^k)$. Moreover if $r < n - 1$, then $\tilde{u} \in \text{Dom} \ d$ (resp. $\tilde{u} \in \text{Dom} \ \bar{\partial}$) if $u \in \text{Dom} \ d$ (resp. $u \in \text{Dom} \ \bar{\partial}$).

Proof. The first part is proved similarly as in the proof of Lemma 2.3. The latter part follows from the first part and (3). In fact, let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that $\chi \equiv 1$ on $(-\infty, -2)$ and $\chi \equiv 0$ on $(-1, \infty)$. Then, if $u \in \text{Dom} \ d$ (resp. \text{Dom} $\bar{\partial}$), the sequence $\{\chi(k(|z| - b)) u_k\}_{k=1}^\infty$ is convergent on $X^k$ with respect to the graph norm of $d$ (resp. the graph norm of $\bar{\partial}$), if $r < n - 1$.

Let $\Phi$ be a family of closed subsets of $X^k$ defined by $\Phi := \{K \subset X^k; K \cap X_{c/2}^k \text{ is compact}\}$.

Then the following is an immediate consequence of Proposition 2.2 and Lemma 2.4.

Theorem 2.5 The images of the following natural homomorphisms are zero.

$$H^r_0(X^k) \to H^r_0(X^k), \quad \text{for} \quad r < n,$$

$$H^{p,q}_0(X^k) \to H^{p,q}_0(X^k), \quad \text{for} \quad p + q < n.$$

Here $H_\phi$ denotes the cohomology with supports in $\Phi$. 
Theorem 2.5 is not used to prove Theorem 1 and Theorem 2, but it may have some application in the theory of isolated singularities.

§3. Proof of Theorem 1

Let $X$ be a complex space of dimension $n$. For any isolated singular point $q \in X$ we shall freely use the notations $X_q$, $F_q$, etc. in §2. $X$ is called a Kähler space if there exist an open covering $U = \{U_j\}_{j \in J}$ of $X$ and a system of $C^\infty$ strictly plurisubharmonic functions $\varphi_j$, each $\varphi_j$ being defined on $U_j \in U$, such that $\varphi_j - \varphi_k$ is pluriharmonic on $U_j \cap U_k$. Given such a system of functions, $\{\partial \bar{\partial} \varphi_j\}_{j \in J}$ defines a Kähler metric on $X^*$. Clearly, this metric is locally quasi-isometric to those induced from the euclidean one by embedding $X$ locally into $\mathbb{C}^n$.

Proposition 3.1 Let $X$ be a compact Kähler space with isolated singularities. Then $X^*$ admits a complete Kähler metric which is quasi-isometric to $\partial \bar{\partial} F_{q,e}$ on $X^*_{q,e}$ for each singular point $q$.

Proof. Let $\varphi_v$ ($v = 1, \cdots, m$) be the singular points of $X$. We choose $e$ so that $X^*_v,e$ are mutually disjoint regarded as subsets of $X$. Let $\rho_v$ be $C^\infty$ functions on $X^*$ such that $\rho_v = 1$ on $X^*_{v,e}$ and $\rho_v = 0$ on $X^* \setminus X^*_v,e$. Then, for $A > 0$,

$$\sum_{v=1}^{m} \partial \bar{\partial} \left( \rho_v F_{v,e} \right) + A \partial \bar{\partial} \varphi_j$$

gives a complete Kähler metric with the required property.

Proof of Theorem 1 Once for all we regard $X^*$ as a complete Kähler manifold with a metric such as in Proposition 3.1. Since $H^{r,q}_2(\mathbb{C}^n) = \bigoplus_{p+q=r} H^{p,q}_2(\mathbb{C}^n)$ and $H^{p,q}_2(\mathbb{C}^n) = H^{p,q}(\mathbb{C}^n)$, it suffices to show that

$$\begin{cases} H^{r}_2(X^*) \cong H^{r}(X^*) & \text{if } r < n - 1, \\ H^{p,q}_2(X^*) \cong H^{p,q}(\mathbb{C}^n) & \text{if } p+q < n - 1. \end{cases}$$

(4)

Since $\dim H^r(\mathbb{C}^n)$ and $\dim H^{p,q}(\mathbb{C}^n)$ are finite on the above ranges (cf. [1]), by Serre’s duality (4) is equivalent to that

$$\begin{cases} H^{r}_2(X^*) \cong H^{r}_0(X^*) & \text{if } r > n + 1, \\ H^{p,q}_2(X^*) \cong H^{p,q}_0(\mathbb{C}^n) & \text{if } p+q > n + 1. \end{cases}$$

(5)

But (5) is immediate from Proposition 2.2 and Lemma 2.3. In fact, to show that the natural homomorphism from $H^r_0(X^*)$ to $H^{r}_2(X^*)$, say $\alpha$, is surjec-
tive, one has only to know that square integrable forms on $X^*$ are in $L_{(2)}^r(X^*_e)$ around each singular point, is already assured for $r > n$ by Lemma 2.3. To show that $\alpha$ is injective, let $u$ be in $L_{(2)}^r(X^*)$ and compactly supported, such that there exists a $v \in L_{(2)}^r(X^*)$ with $dv = u$. Since $dv = 0$ near the singularity, by the same reason as above one can replace $v$ by a compactly supported form in $L_{(2)}^{r-1}(X^*)$. The other isomorphism is proved similarly.

Remark. It is also easy to prove (4) directly from Proposition 2.2 by using Lemma 2.4 instead of Lemma 2.3.

§4. Proof of Theorem 2

Let $X$ be a compact Kähler space of pure dimension $n$. Now we need to distinguish two metrics on $X^*$, i.e. the original Kähler metric and a complete Kähler metric given in Proposition 3.1. Let us denote the original metric by $ds^2$ and make the distinction by $H_{(2)}^r(X^*_e)ds^2$, etc.

While Theorem 1 was a consequence of Proposition 2.2, the proof of Theorem 2 is clearly reduced to the following local cohomology vanishing.

**Proposition 4.1** For each singular point $o \in X$,

$$\lim_{c \to 0} H_{(2)}^r(X^*_e)ds^2 = 0 \quad \text{if} \quad r > n$$

and

$$\lim_{c \to 0} H_{(2)}^p(X^*_e)ds^2 = 0 \quad \text{if} \quad p + q > n$$

and

$$\lim_{c \to 0} H_{(2)}^r(X^*_e)\Phi ds^2 = 0 \quad \text{if} \quad r < n$$

and

$$\lim_{c \to 0} H_{(2)}^p(X^*_e)\Phi ds^2 = 0 \quad \text{if} \quad p + q < n.$$ 

Here $H_{(2)}^*(X^*_e)\Phi ds^2$ denote the $L^2$ cohomologies with supports in $\Phi$ and the limits are taken by letting $c \to 0$.

**Proof.** We put $F_\varepsilon(z) := -\log((c^2 - |z|^2)\log(c/|z|))$ for any $\varepsilon \geq 0$. Then $\partial \bar{\partial} F_\varepsilon > 0$ on $X^*_e$ and $\partial \bar{\partial} F_\varepsilon$ converges to $-\partial \bar{\partial} \log(c^2 - |z|^2)$ on the compact subsets of $X^*_e$. We have

$$\partial \bar{\partial} F_\varepsilon = \frac{\partial \bar{\partial} |z|^2}{c^2 - |z|^2} + \frac{\partial |z|^2 \partial |z|^2}{(c^2 - |z|^2)^2}$$

$$+ \varepsilon \left( \frac{\partial \bar{\partial} \log |z|}{\log(c/|z|)} + \frac{\partial \log |z| \bar{\partial} \log |z|}{\log^2(c/|z|)} \right)$$
\[ \geq \partial \log (c^2 - |z|^2) \bar{\partial} \log (c^2 - |z|^2) \]
\[ + \varepsilon^{-1} \partial \log \log^2(c/|z|) \bar{\partial} \log \log^2(c/|z|) . \]

From the above inequality it is clear that \( \partial \bar{\partial} c^2_{\varepsilon} | X_\varepsilon \) is a complete Kähler metric on \( X_\varepsilon \) and \( | \partial \bar{\partial} c^2_{\varepsilon} | \leq 2 \) if \( 0 \leq \varepsilon < 1 \). Here \( | \cdot | \) denotes the length with respect to \( \partial \bar{\partial} F_\varepsilon \).

From the above, the eigenvalues \( \xi_1, \ldots, \xi_N \) of \( \partial \bar{\partial} c^2 \) measured by the euclidean metric \( \partial \bar{\partial} |z|^2 \) are given by

\[ \xi_j = \frac{1}{c^2 - |z|^2} + \frac{\varepsilon}{|z|^2 \log(c/|z|)} , \quad 1 \leq j \leq N-1 , \]
\[ \xi_N = \frac{c^2}{(c^2 - |z|^2)^2} + \frac{\varepsilon}{|z|^2 \log^2(c/|z|)} . \]

Thus, similarly as in Lemma 2.3, one can find a constant \( A \) such that

\[ ||u||_e \leq A ||u||_{ds^2} \quad \text{for any} \quad u \in L^2 (X_\varepsilon^*, ds^2) , \]

if \( 0 \leq \varepsilon < 1 \). Here \( || \cdot || \) denotes the \( L^2 \)-norm with respect to \( \partial \bar{\partial} F_\varepsilon | X_\varepsilon \). If \( r > n \) and \( du=0 \), then by Theorem 1.1, there exist \( v_\varepsilon \in L^2 (X_\varepsilon^*) \) such that \( dv_\varepsilon = u \) and \( ||v_\varepsilon||_e \leq 4A ||u||_{ds^2} \) if \( 0 < \varepsilon < 1 \). Let \( \{ v_{\varepsilon, j} \}_{j=1}^\infty \) be a subsequence of \( \{ v_{\varepsilon, j} \}_{j=1}^\infty \) which converges weakly on each compact subset of \( X_\varepsilon^* \), and let \( v \) be the limit on \( X_\varepsilon^* \). Then \( ||v||_e \leq 4A ||u||_{ds^2} \) and \( dv = u \). Since \( \partial \bar{\partial} F_\varepsilon \) is quasi-isometric near \( q \) to \( ds^2 \), this proves that \( \lim_{\varepsilon \to 0} H^r (\varepsilon, X_\varepsilon^*) = 0 \) for \( r > n \). The proofs of the other vanishings are similar except that for the vanishing with supports in \( \emptyset \) one should use Lemma 2.4. This is a slight change and we shall not repeat the whole argument. The detail is left to the reader.

References


