Microhyperbolic Operators in Gevrey Classes

By

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§ 1. Introduction

Kashiwara and Kawai [16] defined microhyperbolicity and proved that the microlocal Cauchy problem for microhyperbolic pseudodifferential operators is well-posed in the framework of microfunctions, which is a microlocalization of the results obtained by Bony and Schapira [3]. In the microlocal studies of pseudodifferential operators, the concept of microhyperbolicity is very useful. From their results one can obtain results on propagation of analytic singularities (propagation of micro-analyticities) of solutions for microhyperbolic operators (see [28]). On the other hand, Bronshtein [5] proved that the hyperbolic Cauchy problem is well-posed in some Gevrey classes which are intermediate spaces between the space of real analytic functions and $C^\infty$ (see, also, [14], [15]). So we can generalize the definition of microhyperbolicity in the framework of some Gevrey classes, to say the least of it. In doing so, we expect to get a clue to a generalization of microhyperbolicity and microlocal studies of microhyperbolic operators in the framework of $C^\infty$.

In this paper we shall consider microhyperbolic operators in Gevrey classes and prove microlocal well-posedness of the microlocal Cauchy problem and theorems on propagation of singularities for microhyperbolic operators. Our aims are to show how one can obtain microlocal results (microlocal well-posedness and, therefore, a microlocal version of Holmgren's uniqueness theorem) from methods to prove well-posedness of the Cauchy problem and to show that theorems on propagation of singularities are immediate consequences of a microlocal version of Holmgren's uniqueness theorem, using

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generalized Hamilton flows. We shall prove microlocal well-posedness, reducing the problems to those in $L^2$. From this point of view one may assert that consideration in $L^2$ (or $C^\infty$) are much more important than in Gevrey classes. However, in the framework of Gevrey classes one can easily solve some problems, which seem difficult to be solved in the framework of $C^\infty$, and obtain some conjectures on the problems in the framework of $C^\infty$. We should note that Uchikoshi [27] investigated a related problem.

Let $K$ be a regular compact set in $\mathbb{R}^n$, and let $\kappa>1$ and $h>0$. We denote by $\mathscr{E}^{(\kappa),h}(K)$ the space of all $f \in C^\infty(K)$ which satisfies, with some constant $C\geq 0$,

$$|D^\alpha f(x)| \leq Ch^{|\alpha|}|\alpha|!^\kappa$$

for $x \in K$ and $|\alpha| = 0, 1, 2, \ldots$, where $x=(x_1, \ldots, x_n) \in \mathbb{R}^n$, $D=i^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $\alpha=(\alpha_1, \ldots, \alpha_n)$ is a multi-index and $|\alpha| = \sum_{j=1}^n \alpha_j$. We also denote by $\mathscr{D}_K^{(\kappa),h}$ the space of all $f \in C^\infty(\mathbb{R}^n)$ with support in $K$ satisfying (1.1). $\mathscr{E}^{(\kappa),h}(K)$ and $\mathscr{D}_K^{(\kappa),h}$ are Banach spaces under the norm defined by

$$\|f\|_{\mathscr{E}^{(\kappa),h}(K)} = \sup_{x \in K, \alpha} |D^\alpha f(x)|/(h^{|\alpha|}|\alpha|!^\kappa).$$

Let $\Omega$ be an open set in $\mathbb{R}^n$. We introduce the following locally convex spaces (Gevrey classes):

$$\mathscr{E}^{(\kappa)}(\Omega) = \lim_{K \subset \Omega} \mathscr{E}^{(\kappa)}(K), \quad \mathscr{E}^{(\kappa)}(K) = \lim_{h \to 0} \mathscr{E}^{(\kappa),h}(K),$$

$$\mathscr{D}^{(\kappa)}(\Omega) = \lim_{K \subset \Omega} \mathscr{D}_K^{(\kappa)}, \quad \mathscr{D}_K^{(\kappa)} = \lim_{h \to 0} \mathscr{D}_K^{(\kappa),h},$$

where $A \subset B$ means that the closure $\overline{A}$ of $A$ is compact and included in the interior $\mathring{B}$ of $B$. We denote by $\mathscr{D}^*(\Omega)$ and $\mathscr{E}^*(\Omega)$ the strong dual spaces of $\mathscr{D}^*(\Omega)$ and $\mathscr{E}^*(\Omega)$, respectively, where $*$ denotes $(\kappa)$ or $\{\kappa\}$. We also write $\mathscr{E}^*, \ldots$, instead of $\mathscr{E}^*(\mathbb{R}^n), \ldots$ (see, e.g., [18]). Let us define symbol classes $S_{\kappa}^m$, where $m \in \mathbb{R}$. We say that a symbol $p(x, \xi)$ belongs to $S_{\kappa}^m$ (resp. $S_{\{\kappa\}}^m$) if $p(x, \xi) \in C^\infty(\mathbb{T}^*\mathbb{R}^n)$ and for any compact subset $K$ of $\mathbb{R}^n$ and any $A>0$ there is $C= C_{K,A} > 0$ (resp. for any compact subset $K$ of $\mathbb{R}^n$ there are $A \equiv A_{K} > 0$ and $C \equiv C_{K} > 0$) such that
for $x \in K$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and any multi-indeces $\alpha$ and $\beta$, where $T^*\mathbb{R}^n$ is identified with $\mathbb{R}^n \times \mathbb{R}^n$, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ and $p^{(\alpha)}(x, \xi) = \partial_\xi^\alpha D_\xi^\beta p(x, \xi)$. We impose the following conditions:

(A-1) $p(x, \xi) \in S^m_{\text{a}},$ where $*1$ denotes $(\kappa_1)$ or $\{\kappa_1\}$, and $\kappa_1 > 1$ and $m \in \mathbb{R}$. And $p(x, D)$ is properly supported.

(A-2) There is a symbol $p_m(x, \xi)$, which is positively homogeneous of degree $m$ in $\xi$, such that $p(x, \xi) - \sigma(\xi) p_m(x, \xi) \in S^{m-1}_{\text{a}}$, $\sigma(\xi) \in \mathfrak{S}(\kappa_1)$ and $\sigma(\xi) = 1$ for $|\xi| \geq 1$ and $\sigma(\xi) = 0$ for $|\xi| \leq 1/2$.

Definition 1.1. Let $z^0 = (x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0$ and $\theta \in T^*_\rho(T^*\mathbb{R}^n) \simeq \mathbb{R}^{2n}$. We say that $p(x, \xi)$ (or $p_m(x, \xi)$) is microhyperbolic with respect to $\theta$ at $z^0$ if there are a neighborhood $\mathcal{U}$ of $z^0$ in $T^*\mathbb{R}^n \setminus 0$, $l \in \mathbb{N} \cup \{0\}$ and positive constants $c$ and $t_0$ such that

$$\left| \sum_{l=0}^l (-it\theta)^j p_m(x, \xi) / j! \right| \geq ct^l \quad \text{for } (x, \xi) \in \mathcal{U} \text{ and } 0 \leq t \leq t_0$$

where $\theta = (\theta_x, \theta_\xi)$ is regarded as a vector field $\theta = \theta_x \cdot (\partial / \partial x) + \theta_\xi \cdot (\partial / \partial \xi)$.

Remark. (i) The above definition coincides with the definition given in [33]. (ii) When $p_m(x, \xi)$ is real analytic, the above definition coincides with the definition of partially microhyperbolicity given by Kashiwara and Kawai [16].

Let $\Omega$ be an open conic set in $T^*\mathbb{R}^n \setminus 0$. We assume that

(A-3) $p_m(x, \xi)$ is microhyperbolic at each point in $\Omega$.

For $z^0 \in T^*\mathbb{R}^n \setminus 0$ we can write

$$p_m(z^0 + s\delta z) = s^\mu (p_m^\delta(\delta z) + o(1)) \quad \text{as } s \to 0,$$

where $p_m^\delta(\delta z) \neq 0$ in $\delta z \in T^*_\rho(T^*\mathbb{R}^n)$, if there are multi-indeces $\alpha$ and $\beta$ such that $p_m^{(\alpha)}(z^0) \neq 0$. $p_m^\delta(\delta z)$ is called the localization polynomial of $p_m(z)$ at $z^0$ and $\mu \equiv \mu(z^0)$ is called the multiplicity of $p_m(z)$ at $z^0$. If $p_m(z)$ is microhyperbolic with respect to $\theta$ at $z^0$, then $p_m^\delta(\delta z)$ is hyperbolic with respect to $\theta$, i.e.,

$$p_m^\delta(\delta z - is\theta) \neq 0 \quad \text{for } \delta z \in T^*_\rho(T^*\mathbb{R}^n) \text{ and } s > 0$$

(see, e.g., [11]). Therefore, we can define $\Gamma(p_m^\delta, \theta)$ as the connected
component of the set \( \{ \delta z \in T^*_\mathbb{R}(T^*\mathbb{R}^n); p_{m\delta}(\delta z) \neq 0 \} \) which contains \( \mathcal{O} \), when \( p_m(z) \) is microhyperbolic with respect to \( \mathcal{O} \) at \( z^0 \). For some properties of hyperbolic polynomials and \( \Gamma'(p_{m\delta}, \mathcal{O}) \) we refer to Atiyah, Bott and Gårding [2].

**Definition 1.2.**

(i) \( t(x, \xi) \in C^1(\mathcal{O}) \) is called a time function for \( p_m \) in \( \mathcal{O} \) if \( t(x, \xi) \) is real-valued and positively homogeneous of degree 0 in \( \xi \), and if \( p_m(z) \) is microhyperbolic with respect to \( -H_t(z) \) at every \( z \in \mathcal{O} \), where \( H_t(z) = \sum_{\gamma=1}^{\gamma(T, \mathbb{R}^n)} \left( \frac{\partial}{\partial \xi_\gamma} \right)(z) \left( \frac{\partial}{\partial x_\gamma} \right)(z) \left( \frac{\partial}{\partial \xi_\gamma} \right) \). (ii) Let \( t(x, \xi) \in C^1(\mathcal{O}) \) be a time function for \( p_m \) in \( \mathcal{O} \), and let \( z \in \mathcal{O} \). We define the generalized Hamilton flows \( K^\pm(z; \mathcal{O}; t) \) by

\[
K^\pm(z; \mathcal{O}; t) = \{ z(s) \in \mathcal{O}; \pm s \geq 0, \text{ and } \{ z(s) \} \text{ is a Lipschitz continuous curve in } \mathcal{O} \text{ satisfying } (d/ds)z(s) \in H_t(z(s)) \}
\]

where \( H_t(z) = \{ (\delta x, \delta \xi) \in T^*_\mathbb{R}(T^*\mathbb{R}^n); \sigma((\delta y, \delta \eta), (\delta x, \delta \xi)) = \delta x \cdot \delta \eta - \delta y \cdot \delta \xi \} \geq 0 \) for any \( (\delta y, \delta \eta) \in \Gamma' \) for \( z \in T^*\mathbb{R}^n \setminus 0 \) and \( \Gamma \subset T_z(T^*\mathbb{R}^n) \).

**Remark.** We should note that Leray [21] and Lascar [20] defined flows similar to \( K^\pm(z; \mathcal{O}; t) \).

**Definition 1.3.** Let \( \kappa > \kappa_i \) and \( f \in \mathcal{D}^{(\kappa)} \). \( WF(\mathcal{O}) \) (resp. \( WF(\kappa) \) (\( f \)) is defined as the complement in \( T^*\mathbb{R}^n \setminus 0 \) of the collection of all \( (x^0, \xi^0) \) in \( T^*\mathbb{R}^n \setminus 0 \) such that there are a neighborhood \( U \) of \( x^0 \) and a conic neighborhood \( \Gamma \) of \( \xi^0 \) such that for every \( \varphi \in \mathcal{D}^{(\kappa)}(U) \) and every \( A > 0 \) there is a positive constant \( C \) (resp. for every \( \varphi \in \mathcal{D}^{(\kappa)}(U) \) there are positive constants \( A \) and \( C \) satisfying

\[
|\mathcal{F}[\varphi f](\xi)| \leq C \exp[-A|\xi|^\kappa] \text{ for } \xi \in \Gamma,
\]

where \( \mathcal{F}[f](\xi) \equiv \hat{f}(\xi) \) denotes the Fourier transform of \( f \) (see [10], [28]).

Moreover, we assume that

\[
(A-4) \quad \mu(\mathcal{O}) = \sup_{x \in \mathcal{O}} \mu(z) < +\infty, \text{ and } \kappa \leq \mu(\mathcal{O}) = \min \{2, \mu(\mathcal{O}) / (\mu(\mathcal{O}) - 1)\} \text{ if } *1 = (\kappa_i), \text{ and } \kappa_i < \kappa(\mathcal{O}) \text{ if } *1 = \{\kappa_i\}.
\]

**Theorem 1.4.** Assume that \( (A-1)-(A-4) \) are valid, and let \( \mathcal{O} \ni z \)}}
\( \Theta(z) \in T_\gamma(\Omega) \) be a continuous vector field such that \( p_m(z) \) is microhyperbolic with respect to \( \Theta(z) \) at each \( z \in \Omega \). We denote by \( \star(\kappa) \) or \( \{ \kappa \} \), and assume that \( \kappa_1 \leq \kappa \leq \kappa(\Omega) \) and \( \star = (\kappa) \) when \( \star 1 = (\kappa_1) \) and that \( \kappa_1 \leq \kappa < \kappa(\Omega) \) and \( \star = \{ \kappa \} \) when \( \star 1 = \{ \kappa_1 \} \). If \( \mu \in \mathcal{D}^{*1'} \), \( z^0 \in \text{WF}_*(\mu) \cap \Omega \) and \( \text{WF}_*(\mu u) \cap \Omega = \emptyset \), then there are \( a \in (-\infty, 0) \cup \{ -\infty \} \) and a Lipschitz continuous function \( z(t) \) defined on \( (a, 0] \) with values in \( \Omega \) such that \( z(t) \in \text{WF}_*(\mu) \) for \( t \in (a, 0] \), \( (d/dt)z(t) \in \Gamma(p_m(z(t)), \Theta(z(t))) \cap \{ \partial z; |\partial z| = 1 \} \) for a.e. \( t \in (a, 0] \), and \( z(0) = z^0 \), and \( \lim_{t \to 0} z(t) \in \partial \Omega \) if \( a > -\infty \), where \( \partial \Omega \) denotes the boundary of \( \Omega \) in \( T_*^\gamma R^n \).

**Theorem 1.5.** Assume that \((A-1)-(A-4)\) are valid and that \( t(z) \in C^1(\Omega) \) is a time function for \( p_m \) in \( \Omega \). Moreover, assume that \( \kappa_1 \leq \kappa \leq \kappa(\Omega) \) and \( \star = (\kappa) \) when \( \star 1 = (\kappa_1) \) and that \( \kappa_1 \leq \kappa < \kappa(\Omega) \) and \( \star = \{ \kappa \} \) when \( \star 1 = \{ \kappa_1 \} \). (i) Let \( z^0 \in \Omega \) and \( t_0 \in \mathbb{R} \) satisfy \( t_0 \leq t(z^0) \), and assume that \( K^- (z^0; \Omega; t) \cap \{ z \in \Omega; t(z) \geq t_0 \} \subset \Omega \). Then \( z^0 \in \text{WF}_*(\mu) \) if \( \mu \in \mathcal{D}^{*1'} \), \( \text{WF}_*(\mu u) \cap K^- (z^0; \Omega; t) \cap \{ z \in \Omega; t(z) \geq t_0 \} = \emptyset \) and \( \text{WF}_*(\mu u) \cap K^- (z^0; \Omega; t) \cap \{ z \in \Omega; t(z) = t_0 \} = \emptyset \). (ii) Furthermore, assume that \( K^- (z; \Omega; t) \cap \{ z \in \Omega; t(z) \geq t_0 \} \subset \Omega \) for every \( z \in \Omega \). Then

\[
\text{WF}_*(\mu u) \cap \{ z \in \Omega; t(z) \geq t_0 \} \subset \{ z \in \Omega; z \in K^-(u; \Omega; t) \text{ for some } u \in (\text{WF}_*(\mu) \cap \{ z \in \Omega; t(z) \geq t_0 \}) \cup (\text{WF}_*(\mu) \cap \{ z \in \Omega; t(z) = t_0 \}) \} \text{ for } u \in \mathcal{D}^{*1'}.
\]

**Remark.** Theorem 1.5 is an immediate consequence of Theorem 1.4. We note that there do not always exist time functions for \( p_m \) even locally (see Proposition 5.1).

The remainder of this paper is organized as follows. In \( \S 2 \) we shall give preliminary lemmas on calculus of pseudo-differential operators. In \( \S 3 \) we shall investigate hypoellipticity in Gevrey classes for operators which satisfy the so-called \((H)\)-condition (see [9]). The microlocal Cauchy problem will be studied and microlocal parametrics will be constructed in \( \S 4 \). We shall give the proof of Theorem 1.4 and some remarks in \( \S 5 \).

**\( \S 2. \) Calculus of Pseudo-Differential Operators**

Using pseudo-differential operators of infinite order, we can reduce
the problem in Gevrey classes to the problem in the Sobolev spaces and prove Theorem 1.4. In doing so, we must establish calculus of pseudo-differential operators of infinite order. By results in this section (Proposition 2.13 below) we can calculate the symbols of the reduced operators. Throughout this paper we denote by $C_{a,b,\ldots}(A,B,\ldots)$ a constant depending on $a,b,\ldots$ and $A,B,\ldots$ which is locally bounded in $A,B,\ldots$. Let $\varepsilon>1$ and $\varepsilon\in\mathbb{R}$, and define

$$\mathcal{H}_{\varepsilon,A} = \{v(\xi) \in C^m(\mathbb{R}^n) ; \exp[\varepsilon\langle\xi\rangle^{1/\varepsilon}] v(\xi) \in \mathcal{S}\}.$$ 

We say that $v_j \to v$ in $\mathcal{H}_{\varepsilon,A}$ as $j \to \infty$ if $\exp[\varepsilon\langle\xi\rangle^{1/\varepsilon}] v_j(\xi) \to \exp[\varepsilon\langle\xi\rangle^{1/\varepsilon}] v(\xi)$ in $\mathcal{S}$ as $j \to \infty$. Since $\mathcal{D}$ is dense in $\mathcal{H}_{\varepsilon,A}$, it is obvious that the dual space $\mathcal{H}_{\varepsilon,A}'$ of $\mathcal{H}_{\varepsilon,A}$ is identified with $\{\exp[\varepsilon\langle\xi\rangle^{1/\varepsilon}] v(\xi) \in \mathcal{D}' ; v \in \mathcal{D}\}$.

For $\varepsilon \geq 0$, we can define

$$\mathcal{S}_{\varepsilon,A} = \mathcal{F}^{-1}[\mathcal{H}_{\varepsilon,A}] \quad (= \mathcal{F} [\mathcal{H}_{\varepsilon,A}] = \{u \in \mathcal{S} ; \exp[\varepsilon\langle\xi\rangle^{1/\varepsilon}] u(\xi) \in \mathcal{S}\}).$$

We introduce the topology in $\mathcal{S}_{\varepsilon,A}$ so that $\mathcal{F} : \mathcal{S}_{\varepsilon,A} \to \mathcal{S}_{\varepsilon,A}$ is homeomorphic. Denote by $\mathcal{S}'_{\varepsilon,A}$ the dual space of $\mathcal{S}_{\varepsilon,A}$ for $\varepsilon \geq 0$. Then we can define the transposed operators $^t\mathcal{F}$ and $^t\mathcal{F}^{-1}$ of $\mathcal{F}$ and $\mathcal{F}^{-1}$ which map $\mathcal{S}'_{\varepsilon,A}$ and $\mathcal{S}_{\varepsilon,A}$ onto $\mathcal{S}'_{\varepsilon,A}$ and $\mathcal{S}_{\varepsilon,A}$, respectively. Since $\mathcal{S}_{\varepsilon,A} \subset \mathcal{S}'_{\varepsilon,A} \subset (\mathcal{D}')$ for $\varepsilon \geq 0$, we can define $\mathcal{S}_{\varepsilon,A} = ^t\mathcal{F}^{-1}[\mathcal{S}_{\varepsilon,A}]$ for $\varepsilon \geq 0$. It is easy to see that $\mathcal{S}'_{\varepsilon,A} = \mathcal{F}[\mathcal{S}_{\varepsilon,A}]$ is the dual space of $\mathcal{S}_{\varepsilon,A}$, $\mathcal{S}'_{\varepsilon,A} \subset \mathcal{S}'_{\varepsilon,A} \subset \mathcal{S}'_{\varepsilon,A} \subset \mathcal{S}_{\varepsilon,A}$ for $\varepsilon \geq 0$ and that $\mathcal{F} = ^t\mathcal{F}$ on $\mathcal{S}'$.

So we write $^t\mathcal{F}$ as $\mathcal{F}$. Define

$$H_{\varepsilon,A}^m = \{u \in \mathcal{S}_{\varepsilon,A} ; \langle \xi \rangle^m \exp[\varepsilon\langle\xi\rangle^{1/\varepsilon}] u(\xi) \in L^2\}, \quad L_{\varepsilon,A}^2 = H_{\varepsilon,A}^0,$$

where $m \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$.

**Lemma 2.1.** (i) $\mathcal{D}^{(\varepsilon)}$ is a dense subspace of $\mathcal{S}_{\varepsilon,A}$. (ii) $\mathcal{D}_{m}^{(\varepsilon)} \subset \bigcup_{\varepsilon>0} \mathcal{S}_{\varepsilon,A}$. (iii) $\mathcal{S}_{\varepsilon,A} \subset \bigcup_{\varepsilon > \varepsilon'} \mathcal{S}_{\varepsilon',A}$ and $\mathcal{D}_{m}^{(\varepsilon)} \subset \bigcap_{\varepsilon < \varepsilon'} \mathcal{S}_{\varepsilon,A}$. (iv) $\mathcal{D}_{m}^{(\varepsilon)} \subset \mathcal{S}_{\varepsilon,A} \subset H_{m}^{\varepsilon} \subset H_{m}^{\varepsilon'} \subset \mathcal{D}_{m}^{(\varepsilon')}$, where $\varepsilon \geq \varepsilon' > \varepsilon$, $m \geq m'$ and $m' \in \mathbb{R}$.

**Proof.** The assertions (ii) and (iii) can be proved by the Paley-Wiener theorem in Gevrey classes (see, e.g., [18]). We can also prove that $u_j(\xi) \equiv \chi(\xi/k) v(\xi) \to v(\xi)$ in $\mathcal{S}_{\varepsilon,A}$ as $k \to \infty$ and that $u_j(\xi) \equiv \int \chi(\eta) w(\xi - \eta/j) d\eta \to w(\xi)$ in $\mathcal{S}_{\varepsilon,A}$ as $j \to \infty$, where $\chi \in \mathcal{D}^{(\varepsilon)}$, $\chi(\xi) = 1$ if $|\xi| \leq 1$ and $\chi(\xi) = 0$ if $|\xi| \geq 2$, and $d\eta = (2\pi)^{-n}d\eta$. This proves the assertion (i). Q.E.D.
In this paper we shall frequently use the following facts without quoting.

**Lemma 2.2.**

(i) \( N! \leq c e^{-N} N^{N+1/2} \) for \( N \geq 1 \), where \( c \) is a positive constant.

(ii) For \( t \geq 1 \)

\[ \inf_{N=0,1,2,...} N! t^{-N} \leq c \inf_{N=1,2,...} N^{N+1/2} (et)^{-N} e^{t/2} e^{-t}. \]

(iii) \( |\alpha| \leq n |\alpha|! \) and \( \sum_{|\beta|=1, \beta \leq \alpha} (\frac{\alpha}{\beta}) = \left( \frac{|\alpha|}{t} \right) \), where \( (\frac{\alpha}{\beta}) = \alpha! / (\beta! (\alpha - \beta)!) \).

(iv) \( \sum_{|\alpha|=N} (\alpha! / N!) ^{1/2} \leq c_{\kappa}^{-1} \) if \( \kappa > 1 \), where \( c_{\kappa} \) is a constant depending only on \( \kappa \).

(v) \( \sum_{k=0}^{\infty} k!^{-t} = c_{\kappa}(t) \rightarrow \infty \) if \( t > 0 \) and \( \kappa > 1 \).

(vi) \( \langle \xi + \eta \rangle \leq \langle \xi \rangle + |\eta| \), where \( \langle \xi \rangle = (h^2 + |\xi|^2)^{1/2} \).

(vii) \( |\partial_{\xi}^k \langle \xi \rangle | \leq (1 + \sqrt{2})^{|\alpha|} (|\alpha| + |k|) ! \langle \xi \rangle ^{|\alpha|-|k|}/[k]! \), where \( k > 0 \) and \([k] \) denotes the largest integer \( \leq k \).

(viii) Let \( 1 \leq \kappa' < \kappa \) and \( N \in \mathbb{N} \cup \{0\} \), and assume that \( \chi(\xi) \in C^\infty (\mathbb{R}^n) \) satisfies

\[ |\chi^{(\alpha+\beta)}(\xi)| \leq CA^{|\alpha|} B^{|\beta|} N^{N-1} |\beta|^{-|\beta|} \text{ for } |\alpha| \leq N \text{ and any } \beta. \]

Then, for any \( c > 0 \), and \( d > 0 \) there is \( C_{A,B,C,d} > 0 \) such that

\[ |\partial_{\eta}^{\alpha+\beta} \partial_{\xi}^{|\beta|} \chi(\eta \langle \xi \rangle^{h_{\kappa}}) | \leq C_{A,B,C,d} A^{|\alpha|} d^{|\beta|+|\gamma|} N^{|\alpha|} (|\beta| + |\gamma|)!^k \langle \xi \rangle ^{(|\alpha|+|\beta|)-|\gamma|} \]

for \( |\eta| \langle \xi \rangle^{h_{\kappa}} \leq c \), \( h > 0 \), \( |\alpha| \leq N \) and any \( \beta \) and \( \gamma \).

**Proof.** The assertions (i)-(iii), (v) and (vi) are obvious. The assertion (vii) can be proved by induction on the dimension \( n \). The assertion (viii) can be proved by induction on \( |\alpha| \). We note that a similar estimate to (vii) can be also obtained by Cauchy's estimates. In order to prove (viii) it suffices to prove that

\[ |\partial_{\eta}^{\alpha+\beta} \partial_{\xi}^{|\beta|} \chi(\eta \langle \xi \rangle^{h_{\kappa}}) | \leq C_{A,B,C,d} A^{|\alpha|} d^{|\beta|+|\gamma|} N^{|\alpha|} (|\beta| + |\gamma|)!^k \langle \xi \rangle ^{(|\alpha|+|\beta|)-|\gamma|} \]

for \( |\eta| \langle \xi \rangle^{h_{\kappa}} \leq c \), \( |\alpha| \leq N \) and any \( \beta \) and \( \gamma \), which can be proved by induction on \( |\gamma| \). Here \( B_1 \) depends on \( A \), \( B \) and \( C \). Q. E. D.

Let \( p(\xi, y, \eta) \) be a symbol satisfying

\[ |\partial_{\eta}^{\alpha} \partial_{\xi}^{\beta} D_{\eta} D_{\xi} p(\xi, y, \eta) | \leq C_{A,B} A^{|\alpha|} |\eta|^{k} \exp[\delta_{1} \langle \xi \rangle^{1/\kappa} + \delta_{2} \langle \eta \rangle^{1/\kappa}] \]

for \( (\xi, y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) and any multi-indeces \( \alpha, \beta \) and \( \gamma \), where \( A > 0 \), \( \delta_1, \delta_2 \in \mathbb{R} \) and the positive constants \( C_{A,B} \) depend on \( \alpha \) and \( \beta \). Define

\[ p(D_x, y, D_y) u(x) = \mathcal{F}_{\xi}^{-1} \left[ e^{-i\gamma \xi} \left( \int e^{iy \eta} p(\xi, y, \eta) \hat{u}(\eta) d\eta \right) d\xi \right](x) \]
for $u \in \mathcal{D}(\kappa)$.

**Proposition 2.3.** $p(D_x, y, D_y)$ maps continuously $\mathcal{S}_{\kappa, \varepsilon_2}$ to $\mathcal{S}_{\kappa, \varepsilon_1}$ if $\delta_2 - \kappa(nA)^{-1/\kappa} < \varepsilon_2$, $\varepsilon_1 \leq \varepsilon_2 - \delta_1 - \delta_2$ and $\varepsilon_1 < \kappa(nA)^{-1/\kappa} - \delta_1$. In particular, $p(D_x, y, D_y)$ maps continuously $\mathcal{S}_{\kappa, \varepsilon}$ to $\mathcal{S}_{\kappa, \varepsilon - \delta_2}$ if $|\varepsilon - \delta_2| < \kappa(nA)^{-1/\kappa}$.

**Proof.** Let $u \in \mathcal{D}(\kappa)$ and write

$$<\xi|D_x^2[p(D_x, y, D_y) u(x)](\xi)> = \int F(\xi, \eta) d\eta,$$

where $F(\xi, \eta) = \sum_{\alpha}^{\alpha+1} a_\alpha f(\xi, \eta, \eta) = \sum_{\alpha}^{\alpha+1} a_\alpha (a!a^{2!})^{-1}$

$$<\xi|^2 f(\xi, \eta, \eta) > 2 <\xi, \eta> <\xi|^2 f(\xi, \eta, \eta) > 2N \exp[<\xi | \mathcal{E}(\varepsilon_2)^{1/\kappa}] F(\xi, \eta, \eta) \hat{u}(\eta) \text{ and } N = [(n + |\alpha|)/2]+1.$$

Then we have

$$|D_x^2 f(\xi, \eta, \eta)| \leq C_\alpha A^{|\alpha|} |\beta| |<\xi|^2 f(\xi, \eta, \eta) > 2N \exp[<\xi | \mathcal{E}(\varepsilon_2)^{1/\kappa}] F(\xi, \eta, \eta) \hat{u}(\eta) |.$$

Since $<\xi + \eta | \mathcal{E}(\varepsilon_2)^{1/\kappa}$, it follows from Lemma 2.2 that

$$|F(\xi, \eta)| \leq C_\alpha A^{|\alpha|} |\beta| |<\xi|^2 f(\xi, \eta, \eta) > 2N \exp[<\xi | \mathcal{E}(\varepsilon_2)^{1/\kappa}] F(\xi, \eta, \eta) \hat{u}(\eta) |.$$

where $A' > A$. Noting that $\pm <\eta | \mathcal{E}(\varepsilon_2)^{1/\kappa} < \pm <\xi | \mathcal{E}(\varepsilon_2)^{1/\kappa}$, we have

if $A' > A$, $\delta_2 - \varepsilon_2 < \kappa(nA)^{-1/\kappa}$ and $M > j + n$. This proves the proposition.

Q.E.D.

**Corollary.** $p(D_x, y, D_y)$ maps continuously $\mathcal{S}_{\kappa, \varepsilon_2}$ to $\mathcal{S}_{\kappa, \varepsilon_1}$ if $\delta_2 - \kappa(nA)^{-1/\kappa} < \varepsilon_2$, $\varepsilon_1 \leq \varepsilon_2 - \delta_1 - \delta_2$ and $\varepsilon_1 < \kappa(nA)^{-1/\kappa} - \delta_1$.

Let $\{\phi_{R}(\xi) \in \mathcal{S}(\alpha)\}$ satisfy the following conditions: $0 \leq \phi_{R}(\xi) \leq 1$, $\phi_{R}(\xi) = 1$ if $<\xi|^4 > 2R_j$, $\phi_{R}(\xi) = 0$ if $<\xi|^4 < R_j$, and $|\phi_{R(\alpha)}(\xi)| \leq C_\alpha^{|\alpha|} |\alpha|^{1/\kappa} <\xi|^{|\alpha|}$ for any $d > 0$, where $R > 0$, $j = 0, 1, 2, \ldots$, and $C_\alpha$ is a positive constant depending on $d$. For example, $\phi_{R}(\xi) \equiv 1$ and $\phi_{R}(\xi) \equiv \chi(\xi/R_j)$ (j = 1, 2, \ldots) satisfy the above conditions if $1 < \kappa R \leq \kappa$, $\chi \in \mathcal{S}(\kappa') (\mathcal{R}^d)$, $0 \leq \chi(t) \leq 1$, $\chi(t) = 1$ if $t \geq 2^\varepsilon$ and $\chi(t) = 0$ if
Lemma 2.4. Let $R_0 > 0$, $\kappa' > 0$ and $h > 0$. If

$$|q^{R,(\alpha)}_{(\beta)}(x, \xi)| \leq C_a A^{\beta} |B|^\beta |\xi|^{\kappa - |\alpha| - 1/\kappa} \exp[\delta(\xi)^{1/\kappa}]$$

for any $\alpha, \beta$, $j = 0, 1, 2, \ldots$ and $\langle \xi \rangle^{1/\kappa} \geq R_0 j$, then $q_R(x, \xi) = \sum_{\alpha, \beta} q_{(\alpha)}^{R,(\beta)}(x, \xi)$. $q_j(x, \xi)$ is well-defined and satisfies

$$|q_j^{R,(\alpha)}(x, \xi)| \leq C (\alpha, \beta) A^{\beta} |\xi|^{\kappa - |\alpha| - 1/\kappa} \exp[\delta(\xi)^{1/\kappa}]$$

for $R \geq \max(R_0, 2e^{-1}B^{\kappa^*})$. Moreover, if

$$|q_j^{R,(\alpha)}(x, \xi)| \leq C A^{\alpha + |\beta|} B^\beta (\alpha + |\beta|)^{1/\kappa} \langle \xi \rangle^{\kappa - |\alpha| - 1/\kappa} \exp[\delta(\xi)^{1/\kappa}]$$

for any $\alpha, \beta$, $j = 0, 1, 2, \ldots$ and $\langle \xi \rangle^{1/\kappa} \geq R_0 j$, then

$$|q_j^{R,(\alpha)}(x, \xi)| \leq C \langle \alpha, \beta \rangle A^{\beta} |\xi|^{\kappa - |\alpha| - 1/\kappa} \exp[\delta(\xi)^{1/\kappa}]$$

for $R \geq \max(R_0, 2e^{-1}B^{\kappa^*})$.

Let $h \leq 1$ and $m_1, m_2 \in R$, and let $p(x, \xi, y, \eta)$ be a symbol satisfying

$$p(x, D_x, y, D_y) u(x) = \int e^{ix \xi} \left( \int e^{iy \eta} p(x, \xi, y, \eta) u(\eta) \right) d\eta \right) d\xi$$

if $\delta \leq (nA_d)^{-1}$. Here we have applied the same argument as in the proof of Proposition 2.3. Put

$$q_j(x, \xi, y, \eta) = \sum_{\lambda, \mu = 0}^{\infty} \sum_{\nu = 0}^{\infty} I(\mu, \nu) \langle \xi \rangle^{-1} (\alpha + |\beta| + j)^{\nu} \langle \xi \rangle^{\kappa - |\alpha| - 1/\kappa} \exp[\delta(\xi)^{1/\kappa}]$$

Then we have

$$|q_j^{(\alpha)}(x, \xi)| \leq (\max_{\lambda, \mu, \nu} L_{\lambda, \mu}) (\alpha + |\beta| + j)^{\nu} \langle \xi \rangle^{\kappa - |\alpha| - 1/\kappa} \exp[\delta(\xi)^{1/\kappa}]$$

where $I(\mu, \nu) = \langle \alpha | \beta | \eta | \mu | \eta | \nu | j \rangle (\alpha - \mu)^{\mu} (\beta + j - \nu)^{\nu}$. It is easy to see that
Applying Lemma 2.2, we have

\[
I(\mu, \nu) \leq \begin{cases} 
|\alpha|^{-\kappa} (|\beta| - \nu)^{1-\kappa} & \text{if } \kappa' = \kappa, \\
|\beta|^{-\kappa} (\mu - |\nu|)^{1-\kappa} & \text{if } \kappa' = 1.
\end{cases}
\]

where

\[
(2.3) \quad \tilde{A} = \max(A, A_1) \quad \text{and} \quad \tilde{A} = A_2 \quad \text{if } \kappa' = \kappa,
\]
\[
\tilde{A} = \max(A, A_2) \quad \text{and} \quad \tilde{A} = A_1 \quad \text{if } \kappa' = 1,
\]

and \( B = 2^\epsilon nA_1A_2 \). By Lemma 2.4

\[
(2.4) \quad q^R(x, \xi) = \sum_{\gamma \in \mathcal{O}} q_\gamma^R(\xi) q_\gamma(x, \xi)
\]
can be defined for \( R \geq 4e^{-1}(nA_1A_2)^{1/\kappa} \) and satisfies

\[
(2.5) \quad |q_{(\gamma)}^{R}(x, \xi)| \leq C_{A^1}(|\alpha|, L_{|\alpha|}, \tilde{A}, A^{-1}, \tilde{A}/A) \tilde{A}^{1+|\beta|} \tilde{B}^{\kappa + |\beta|} \\
\times \frac{1}{A^{1/\kappa}} \exp[(\delta_1 + \delta_2) \langle \xi \rangle_{nA_1}^{1/\kappa}],
\]
if \( R \geq 4e^{-1}(nA_1A_2)^{1/\kappa} \), \( \tilde{A} > A \) and \( L_{|\alpha|x} A^{1+x} = C_n \), and

\[
(2.6) \quad |q_\gamma^{R}(x, \xi)| \leq C_{A^1}(C, \tilde{A}/A) \tilde{A}^{1+|\beta|} \tilde{B}^{\kappa + |\beta|} \\
\times \frac{1}{A^{1/\kappa}} \exp[(\delta_1 + \delta_2) \langle \xi \rangle_{nA_1}^{1/\kappa}],
\]
if \( R \geq 4e^{-1}(nA_1A_2)^{1/\kappa} \) and \( L_{|\alpha|x} = C \). Therefore, Proposition 2.3 shows

that \( q^R(x, D) \) maps continuously \( \mathcal{S}_{\kappa, \xi} \) to \( \mathcal{S}_{\kappa, \xi - \delta_1 - \delta_2} \) when \( |\xi - \delta_1 - \delta_2| < \kappa (nA)^{-1/\kappa} \).

**Lemma 2.5.** Let \( \chi(x) \) be a function in \( \mathcal{D}^{(\kappa)} \) such that \( 0 \leq \chi(x) \leq 1 \)

and \( \chi(x) = 1 \) near the origin. Then,

\[
\sigma(p(x, D_x, y, D_y))(x, \xi)
\]

\[
= \lim_{\rho \to 0} \int \frac{e^{-iy \cdot \rho} p(x, \xi + \eta, x + y, \xi) \chi(\eta/j) \chi(\eta/j) \chi(\eta/j + \eta) \chi(\eta/j) \chi(\eta/j + \eta) \chi(\eta/j + \eta) \chi(\eta/j + \eta)}{\rho^d} dy d\eta
\]

if \( \delta_1 < \kappa (nA^2)^{-1/\kappa} \) when \( \kappa' = \kappa \). Here \( \sigma(p(x, D_x, y, D_y))(x, \xi) \) denotes the simplified symbol of \( p \), that is, \( p(x, D_x, y, D_y) u(x) = \sigma(p(x, D_x, y, D_y))(x, D) u(x) \) for \( u \in \mathcal{D}^{(\infty)} \).

**Proof.** Assume that \( \delta_1 < \kappa (nA^2)^{-1/\kappa} \) when \( \kappa' = \kappa \). By the same argument as in the proof of Proposition 2.3, we have
where $G(x, \eta) = \int \left( \int e^{i\xi \cdot x} e^{i(x-y) \cdot \eta} \cdot \psi (x, y, \xi) \cdot \hat{u}(\xi) \, d\xi \right) \, dy$ is integrable in $\eta$. And we have also

$$G(x, \eta) = \lim_{r \to \infty} G(x, \eta),$$

where $G(x, \eta) = \int e^{i\xi \cdot x} e^{i(x-y) \cdot \eta} \chi((\eta - \xi) / j) \chi((y-x) / j) \, \psi (x, y, \xi) \times \hat{u}(\xi) \, d\xi \, dy$. Moreover, from the same argument as in the proof of Proposition 2.3, it follows that there is a function $F(x, \eta)$ integrable in $\eta$ satisfying $|G(x, \eta)| \leq F(x, \eta)$ $(j=1, 2, \ldots)$. Therefore, applying Lebesgue's theorem and Fubini's theorem, we have

$$p(x, D_x, y, D_y) \, u(x) = \lim_{r \to \infty} \int e^{i\xi \cdot x} \left( \int e^{-i\eta \cdot y} \right.$$

$$\times p(x, \xi + \eta, x + y, \xi) \chi(\eta / j) \chi(y / j) \, d\eta \, dy \hat{u}(\xi) \, d\xi.$$

Similarly, there is a function $F(x, \xi)$ integrable in $\xi$ such that

$$|\hat{u}(\xi) \left( \int e^{-i\eta \cdot y} \right. \times p(x, \xi + \eta, x + y, \xi) \chi(\eta / j) \chi(y / j) \, d\eta \, dy |$$

$$= |\hat{u}(\xi) \left( \int e^{-i\eta \cdot y} \chi((\eta - \xi) / j) \chi((y-x) / j) \, d\eta \right) \chi(\eta / j) \chi(y / j) \, d\eta |$$

where $M = [n/2] + 1$. So we can apply Lebesgue's theorem to (2.7), which proves the lemma.

Q. E. D.

Let $1 \leq \xi \leq \kappa$, and let $\{\phi_N\}_{N=0, 1, 2, \ldots}$ be a sequence in $B^{(\xi)}$ such that

$$\phi_N(\xi) = 1 \text{ if } |\xi| \leq 1/4, \quad \phi_N(\xi) = 0 \text{ if } |\xi| \geq 1/2, \text{ and}$$

$$|\phi_N^{|a| + \beta}|(\xi)| \leq C(A_3(N + 1) / 2)^{|a|} B^{|\beta|} |\beta|! \quad \text{for } |a| \leq N + 1,$$

where $A_3$, $B$ and $C$ are positive constants. By Lemma 2.2, for any $d > 0$ there is $C_d > 0$ such that

$$|\hat{u}(\xi) \left( \int e^{-i\eta \cdot y} \right. \times p(x, \xi + \eta, x + y, \xi) \chi(\eta / j) \chi(y / j) \, d\eta | \leq$$

$$C_d 2^N A_3^{(|a| + |\beta| + 1)} |\alpha|! |\beta|! |\gamma|! \chi(\xi) \times \chi(\eta) \leq$$

$$\leq C_d 2^N A_3^{(|a| + |\beta| + 1)} |\alpha|! |\beta|! |\gamma|! \chi(\xi) \times \chi(\eta) \leq$$

$$\leq C_d 2^N A_3^{(|a| + |\beta| + 1)} |\alpha|! |\beta|! |\gamma|! \chi(\xi) \times \chi(\eta) \leq$$

$$\leq C_d 2^N A_3^{(|a| + |\beta| + 1)} |\alpha|! |\beta|! |\gamma|! \chi(\xi) \times \chi(\eta) \leq$$

since $(N + 1)^{|a|} \leq (N + |\alpha|)! / N! \leq 2^{N + |\alpha|} |\alpha|!$. Define for $R \geq 4e^{-1}(nA_1A_2)^{1/\kappa}$

$$r^R(x, D) = p(x, D_x, y, D_y) - q^R(x, D),$$

$$r^R_{\xi}(x, \xi) = (\phi_N^R(\xi) - \phi_N^{R+i}(\xi)) \{Os - \int e^{-i\eta \cdot y} p(x, \xi + \eta, x + y, \xi) \times \chi(\xi) \chi(\eta) \, dy \, d\eta - \sum_{\xi} \hat{u}(\xi) \}$$. 


\[ r_{2N}^r(x, \xi) = (\psi_N^r(\xi) - \psi_{N+1}^r(\xi)) \{O \delta - \int e^{-i\gamma \cdot \eta} p(x, \xi + \eta, x + y, \xi) \times (1 - \phi_N(\eta/|\xi|)) d\eta \}. \]

Then it is obvious that
\[ r^R(x, \xi) = \sigma(r^R(x, D))(x, \xi) = \sum_{n=0}^\infty \{r_{0N}^r(x, \xi) + r_{2N}^r(x, \xi) \}. \]

First consider \( r_{0N}^R(x, \xi) \). We can write
\[ r_{0N}^R(x, \xi) = (\psi_N^r(\xi) - \psi_{N+1}^r(\xi)) \sum_{\gamma=1}^1 \sum_{|\gamma|=N+1} (N+1)! \gamma!^{-1} (1 - \theta)^N \times \left( \int e^{-i\gamma \cdot \eta} \gamma!^{-2M} r_{1N}^r(x, \xi, \theta, \eta) d\eta \right) d\theta, \]
where \( r_{1N}^r(x, \xi, \theta, \eta) = \langle D_\eta \rangle^{2M} \delta_{\eta} [\psi_N(\eta/|\xi|) (D_\xi p)(x, \xi + \eta, x + \theta y, \xi) \rangle \) and \( M = \lfloor n/2 \rfloor + 1 \).

**Lemma 2.6.** Put
\[ \delta = \begin{cases} (3/2)^{1/\delta_1} \delta_1 + \delta_2 & \text{if } \delta_1 \geq 0, \\ 2^{-1/\delta_1} \delta_1 + \delta_2 & \text{if } \delta_1 < 0, \end{cases} \]
\[ A' = \begin{cases} A & \text{if } \kappa' = \kappa, \\ \max(A, A_2) & \text{if } \kappa' = 1, \end{cases} \]
\[ A_3 = \begin{cases} A_1 & \text{if } \kappa' = \kappa, \\ \max(A_1, A_3/3) & \text{if } \kappa' = 1. \end{cases} \]

Then,
\[ |r_{1N}^R(x, \xi, \theta, \eta)| \leq C(|\alpha|, L_{1\alpha}, A, A_1, A_2/A, A_3/A_1) (2^A A')^{1/|\beta|} \times |\beta|^{e/|\xi|} \langle \xi \rangle_{1}^{m_1 + m_2 - |\alpha| + N + 1} \exp[(\delta - \kappa/(2R))|\xi|^{1/\kappa}] (N+1)^{e/2}\rho^{N+1}, \]
where \( \rho = 7 \cdot 2^e a A_1 A_2 R^{-e} \).

**Proof.** For \(|\gamma| = N + 1 \) we have
\[ |\partial_\xi D_N^L r_{1N}^r(x, \xi, \theta, \eta)| \leq 2^E C(|\alpha|, L_{1\alpha}, A, A_1, A_2/A, A_3/A_1) (2^A A')^{1/|\beta|} \times |\beta|^{e/|\xi|} \langle \xi \rangle_{1}^{m_1 + m_2 - |\alpha| - N-1} \exp[\delta|\xi|^{1/\kappa}] \times \sum_{|\mu|=0}^{1/|\beta|} \sum_{|\nu|=0}^{N+1} I(\mu, \nu) A^\mu (3A_{1})^{\nu} A_{2}^{N+1+1/|\beta| - |\mu|} A_{3}^{N+1+1-|\nu|}, \]
where \( I(\mu, \nu) = |\beta|^{1/\delta_1} \mu^{1/\kappa} (N+1 + |\beta| - \mu)^{1/\kappa} \mu! (|\beta| - \mu)! \nu! (N+1 + |\beta|)^{1/\kappa} \). Here we have used the facts that \((j+k)! \leq C_0(j) (1+\varepsilon)^k \times k! \) for \( \varepsilon > 0 \) and that \(|\langle \xi \rangle_{1/2} \leq \langle \xi + \eta \rangle_{1/2} \leq \langle \xi \rangle_{1/2} + |\eta| \leq 3|\xi|/2 \) if \( \phi_N(\eta/|\xi|) \neq 0 \). It is obvious that
Therefore, applying Lemma 2.2, we have
\[
I(\mu, \nu) \leq \begin{cases} 
( |\beta| - \mu ) |(N + 1 - \nu) |^{1 - \varepsilon} & \text{if } \kappa' = \kappa, \\
\left( \frac{|\beta|}{\mu} \right)^{1 - \varepsilon} & \text{if } \kappa' = 1.
\end{cases}
\]

This gives
\[
|r_{2N}^{(a)}(x, \xi)| \leq C'(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_4) \times (7A_4)^{N+1}(N + 1 + |\beta|)^{1+1+|\beta|} \times \exp[(\delta - \delta')^{1+1+|\beta|}] \\
\times \Phi_8(\xi),
\]
where \(\Phi_8(\xi)\) is the characteristic function of \(\{\xi \in \mathbb{R}^n; \ N \leq \langle \xi \rangle^{1/\kappa} \leq 2(N + 1)\}. \) From Lemma 2.2 it follows that
\[
(2.12) \quad |r_{1N}^{(a)}(x, \xi)| \leq C'(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_4) \times (2^sA')^{1+1+|\beta|} \times \exp[(\delta - \delta')^{1+1+|\beta|}]
\]
for \(\delta \leq 0.\) (2.12) with \(\delta = \kappa/(2R)\) shows (2.11). \(\quad Q. E. D.\)

Lemma 2.6. implies that
\[
|\sum_{N=0}^{R} r_{1N}^{(a)}(x, \xi)| \leq C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_4) (2^sA')^{1+1+|\beta|} \times \exp[(\delta - \kappa/(2R))^{1+1+|\beta|}]
\]
for \(R \geq 2^{1+1+|\beta|} (nA_1A_2)^{1/\kappa}.\) Next let us estimate \(r_{2N}(x, \xi).\) We can write
\[
r_{2N}(x, \xi) = \langle \varphi_N^R(\xi) - \varphi_{n+1}(\xi) \rangle \int \int e^{-i\xi y} r_{2N}(x, \xi, y, \eta) dy d\eta,
\]
where \(r_{2N}(x, \xi, y, \eta) = \langle y \rangle^{-2M} \langle D_y \rangle^{2M} \{p(x, \xi + y, \eta, x + y, \xi) (1 - \varphi_N(\eta/\langle \xi \rangle^\alpha))\} \) and \(M = [n/2] + 1.\)

Lemma 2.7. Let \(A'\) be defined in Lemma 2.6, and let \(B > 0\) if \(\kappa' = \kappa\) and \(B = A_2\) if \(\kappa' = 1.\) Then,
\[
|r_{2N}^{(a)}(x, \xi)| \leq C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_2/B, 1/B) \times (2^sA')^{1+1+|\beta|} \times \exp[(\delta - \kappa/(2R))^{1+1+|\beta|}]
\]
for \(|\delta| \leq 2^{-3}\kappa(nB)^{-1/\kappa}\) and \(\delta' = \delta_1 + 4^{-3-1/\kappa}\kappa(nB)^{-1/\kappa}.
\]

Proof. The same calculation as in the proof of Lemma 2.6 yields
where \( \Psi(\xi) \) is the characteristic function of \( \{ \xi \in \mathbb{R}^k : |\xi| \geq 1/4 \} \). Thus we have

\[
|\partial_\xi D_\xi^2 \int e^{-i\xi \cdot r_{2N}(x, \xi, \eta)} \, dy | \\
\leq C_{\mathcal{F}} \left( |\alpha|, L_{|\alpha|}, A, A_1, \frac{A_2}{A}, \frac{A_2}{B}, 1/B \right) (2^n A')^{|\beta| |\eta|^{n-1}} \\
\times \exp \left[ -2^{-1/k} (nB')^{-1/k} |\eta|^{1/k} + \frac{1}{2} \left( |\xi| + \frac{1}{2} |\xi|^{1/k} + \frac{1}{2} \left( |\xi|^{1/k} \right)^{k'} \right) \right] \\
\times \Psi(\eta/\langle \xi \rangle_{k}) < \frac{1}{k}^{2/n} \\
\leq C_{\mathcal{F}} \left( |\alpha|, L_{|\alpha|}, A, A_1, \frac{A_2}{A}, \frac{A_2}{B}, 1/B \right) (2^n A')^{|\beta| |\eta|^{n-1}} \\
\times \exp \left[ \beta \left( |\xi|^{1/k} \right)^{k'} \right] < \frac{1}{k}^{2/n} \\
\text{if } B' > B, \quad |\beta| \leq 2^{-1/k} (nB')^{-1/k} \quad \text{and } \beta = \beta_1 + \beta_2 + 4^{-1/k} (|\beta| - 2^{-1/k} (nB')^{-1/k}) \],
\]
which proves the lemma. \( \text{Q. E. D.} \)

Lemma 2.7. implies that

\[
|\sum_{\mathcal{F}} q_{R(\xi)}(\alpha, \xi, \eta) | \leq C_{\mathcal{F}} \left( |\alpha|, L_{|\alpha|}, A, \frac{A_2}{A}, \frac{A_2}{B}, 1/B, 1/R \right) \\
\times (2^n A')^{|\beta| |\eta|^{n-1}} \exp \left[ \beta \left( |\xi|^{1/k} \right)^{k'} \right] \\
\text{if } |\alpha| \leq 2^{-3/k} (nB')^{-1/k} \quad \text{so we have the following estimates}
\]

**Proposition 2.8.** Let \( \rho(x, \xi, \eta) \) satisfy (2.1). Then there are \( r_0 > 0 \) and \( \delta(1/\tilde{A}_1, 1/A_2) > 0 \) such that \( \delta(1/\tilde{A}_1, 1/A_2) = \delta(1/A_2) A_{1}^{-\kappa} \) if \( \kappa' = \kappa \), \( \delta(1/\tilde{A}_1, 1/A_2) = \delta(1/A_2) A_{1}^{-\kappa} \) if \( \kappa' = 1 \), and the following estimates hold if \( a \geq 1 \) and \( R = ar_0^{-1/\kappa} A_{1}^{-\kappa/2} \):

\[
|q_{R(\xi)}(\alpha, \xi, \eta) | \leq C_{\mathcal{F}} \left( |\alpha|, L_{|\alpha|}, \tilde{A}, \tilde{A}, A''/A, 1/A_2 \right) \tilde{A}^{1/\kappa} |\beta| \| |\eta|^{n-1} \\
\times \exp \left[ \beta \left( |\xi|^{1/k} \right)^{k'} \right] \\
\text{if } \tilde{A}'' > \tilde{A} \text{ and } L_{k, L'A} A^k \equiv C_{\mathcal{F}},
\]

\[
|q_{R(\xi)}(\alpha, \xi, \eta) | \leq C_{\mathcal{F}} \left( |\alpha|, \tilde{A}, A''/A, \tilde{A}, A_2/A, 1/A_2 \right) (2^n A')^{|\beta| |\eta|^{n-1}} \\
\times \exp \left[ \beta \left( |\xi|^{1/k} \right)^{k'} \right] \\
\text{if } |\alpha| \leq 2^1 (A_{1}/1, 1/A_2) \equiv C_{\mathcal{F}},
\]

\[
|r_{R(\xi)}(\alpha, \xi, \eta) | \leq C_{\mathcal{F}} \left( |\alpha|, L_{|\alpha|}, A, A''/A, 1/A_2, A_2/A, 1/A_2 \right) (2^n A')^{|\beta| |\eta|^{n-1}} \\
\times \exp \left[ \beta \left( |\xi|^{1/k} \right)^{k'} \right] \\
\text{if } |\alpha| \leq 2^1 (A_{1}/1, 1/A_2) \equiv C_{\mathcal{F}},
\]

where \( q_{R}(x, \xi) \) is the symbol defined by (2.2) and (2.4), \( r_{R}(x, \xi) = \sigma(p(x, D_x, y, D_y)) (x, \xi) - q_{R}(x, \xi) \), and \( \tilde{A} \) and \( \tilde{A} \) are defined by (2.3), (2.10) and (2.9), respectively.
Proof. If, for example, we choose \( r_0 = 2^{1+3/k} n^{1/k} \) and
\[
\delta(1/\hat{A}_1, 1/\hat{A}_2) = \begin{cases} 
2^{-3-3/k} \kappa (n \hat{A}_1 \hat{A}_2)^{-1/k} \text{ when } \kappa' = \kappa, \\
2^{-3-3/k} \kappa (n \hat{A}_2)^{-1/k} \min(\hat{A}_1^{-1/k}, 2^{-3/k}) \text{ when } \kappa' = 1,
\end{cases}
\]
then the proposition easily follows from (2.5), (2.6) and Lemmas 2.6 and 2.7. Q. E. D.

Let \( A(x, \xi) \) be a symbol satisfying
\[
|A^{(a)} (x, \xi)| \leq C_0 A_0^{|\alpha|+|\beta|} (|\alpha| + |\beta|)! \langle \xi \rangle_k^{k-1-|\alpha|},
\]
and set \( \omega_{\beta} (A; x, \xi) = e^{-A(x, \xi)} (e^{A(x, \xi)})^{(a)} \).

Lemma 2.9. If \( A_1 > A_0, \rho > 0 \) and \( A_1/A_2 + C_0 A_0 \rho^{-1} A_2^{-1} (1 - A_0/A_1)^{-1} \leq 1 \), then
\[
|\omega_{\beta}^{(a)} (A; x, \xi)| \leq A_1^{-1+|\beta|} A_2^{-|\alpha|+|\beta|} (|\alpha| + |\beta|)! \langle \xi \rangle_k^{k-1-|\alpha|} \times \langle \xi \rangle_k^{-|\alpha|+|\beta|} \sum_{\mu=0}^{|\alpha|+|\beta|} \rho^\mu \langle \xi \rangle_k^{\mu/k}/k!.
\]
In particular, we can take \( A_1 = (1 + (C_0/\rho)^{1/2}) A_0 \) and \( A_2 = (1 + (C_0/\rho)^{1/2})^2 A_0 \) for \( \rho > 0 \).

Proof. It is obvious that (2.14) holds for \( |\alpha| + |\beta| = 0 \). Assume that (2.14) holds for \( |\alpha| + |\beta| \leq N \). Let \( |\alpha| + |\beta| = N \) and \( |\varepsilon| + |\varepsilon'| = 1 \). Then
\[
|\omega_{\beta}^{(a)} (A; x, \xi)| = |\omega_{\beta}^{(a)} (A; x, \xi)| + |A^{(a)} (x, \xi) \omega_{\beta} (A; x, \xi)| \leq A_1^{-1+|\beta|} A_2^{-|\alpha|+|\beta|} (N + |\gamma| + |\delta| + 1)! \langle \xi \rangle_k^{k-1-|\alpha|} \times \sum_{\mu=0}^{|\alpha|+|\beta|} \rho^\mu \langle \xi \rangle_k^{\mu/k}/k! [A_1/A_2 + \sum_{\mu=0}^{|\alpha|+|\beta|} (|\gamma| + |\delta|)] \\
\times (N + |\gamma| + |\delta| + 1)^{-1} (N+1) C_0 A_0 \rho^{-1} A_2^{-1} (A_0/A_1)^{\rho},
\]
which proves the lemma. Q. E. D.

Corollary. For \( \rho > 0 \),
\[
|e^{A(x, \xi)}^{(a)} (x, \xi)| \leq [(1 + (C_0/\rho)^{1/2})^2 A_0]^{|\alpha|+|\beta|} (|\alpha| + |\beta|)! \times \langle \xi \rangle_k^{-|\alpha|+|\beta|} \text{exp}[\rho \langle \xi \rangle_k^{\mu/k} + \text{Re } A(x, \xi)].
\]

Lemma 2.10. Let \( p(x, \xi) \) be a symbol satisfying
\[
p^{(a)} (x, \xi) \leq L_{|a|+|\beta|} A_1^{|\alpha|+|\beta|} |\alpha|! |\beta|! |\xi|^{|\alpha|} \langle \xi \rangle_k^{m-|\alpha|} \text{exp}[\delta \langle \xi \rangle_k^{l/k}],
\]
where \( m, \delta \in \mathbb{R} \) and \( L_{k,A} A^{k|x|} \equiv C_k \) or \( L_{k,A} \equiv C \), and set \( \lambda_0 = \inf_{L>0} \sup_{x \in \mathbb{R}^n} |x| \). Then \( (e^A)(x, D) \) maps continuously to \( \mathscr{G}_{k, \varepsilon}^{\rho} \) if \( \rho > \lambda_0 + \delta \) and \( |\varepsilon - \delta| < \kappa (nA)^{-1/k} \). Moreover there are \( r(A_0) > 0 \) and \( \delta_{A_0} > 0 \) such that \( q^R(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} q_{\alpha}^R(x, \xi) \) is well-defined, and \( q^R(x, \xi) \) and \( r^R(x, \xi) \) satisfy the following estimates if \( a \geq 1, R = ar(A_0) A^\infty \) and \( \rho > \lambda_0 + C_0 + \delta \), where \( q_{\alpha}^R(x, \xi) = \sum_{|\alpha| = -} \alpha!^{-1} \omega^\alpha (x; x, \xi) p_{\alpha}(x, \xi) e^{i(\xi, \xi)} \):

\[
(2.16) \quad |q_{\alpha}^R(x, \xi)| \leq C_{\rho, A} (|\alpha|, L_{|\alpha|}, A, A^\infty, A_0, A) (2\pi A^\infty)^{|\beta|} \times |\beta|^{1/2} \exp (\rho^\beta \xi^\beta \xi^{-\beta}) \\
\quad \text{if } A > A \text{ and } L_{k,A} A^{k|x|} \equiv C_k,
\]

\[
(2.17) \quad |q_{\alpha}^R(x, \xi)| \leq C_{\rho, A} (C, A_0 / A) (2\pi A^\infty)^{|\alpha| + |\beta|} (|\alpha| + |\beta|)^{|\kappa|} \times \exp (\rho^\beta \xi^\beta \xi^{-\beta}) \text{ if } L_{k,A} \equiv C,
\]

\[
(2.18) \quad |r_{\alpha}^R(x, \xi)| \leq C_{\rho, A_0} (|\alpha|, L_{|\alpha|}, A^\infty)^{|\beta|} \times |\beta|^{1/2} \exp (|\rho - a^{-1} \delta_{A_0} A^{-1/\kappa}| \xi^\beta \xi^{-\beta}) \\
\quad \text{if } |\lambda_0 + C_0| < |\rho - a^{-1} \delta_{A_0} A^{-1/\kappa}|.
\]

**Remark.** (i) If \( |\varepsilon - \delta| < \kappa (nA)^{-1/\kappa} \) and \( |\varepsilon - \lambda_0 - C_0 - \delta| < 2^{-1/k} (nA)^{-1/\kappa} \), then \( (e^A)(x, D) \) maps continuously to \( \mathscr{G}_{k, \varepsilon}^{\rho} \). (ii) For example, one can take \( r(A_0) = 2^{1+3\kappa (nA)^{1/\kappa}} \) and \( \delta_{A_0} = 2^{-3-3/\kappa} \kappa^{-1/\kappa} \min (A^{-1/\kappa}, 2^{1/2}) \), where \( A_1 = \max \{8A_0, A_3/3\} \) and \( A_3 \) is the constant in (2.8).

**Proof.** From the corollary of Lemma 2.9 it follows that

\[
| (e^{i(x, D) A}(x, \eta))_{(\beta)}^{(\gamma)} | \leq C_{\rho} (A_0 / A) L_{|\gamma|, A} A^{|\beta| + |\gamma|} \\
\times (8A_0)^{|\alpha| + |\beta| + |\gamma|} |\alpha| ! |\beta| ! |\gamma| ! |\delta| ! |\rho^\beta \xi^\beta \xi^{-\beta} | \exp (\rho^\beta \xi^\beta \xi^{-\beta}) \\
\times \exp (|\rho - a^{-1} \delta_{A_0} A^{-1/\kappa}| \xi^\beta \xi^{-\beta})
\]

if \( \rho > \lambda_0 + C_0 \). Therefore, the lemma immediately follows from Propositions 2.3 and 2.8. Q.E.D.

**Lemma 2.11** ([6], [7], [17], [19]). Let \( 0 \leq \rho < 1 \) and \( m \in \mathbb{R} \). Then, for each \( s \in \mathbb{R} \) there are \( C_s > 0 \) and a non-negative integer \( \bar{N}_s \) such that

\[
|D|^a (x, D) u|_{L^2} \leq C_s M |D|^a u|_{L^2} \quad \text{for } u \in H^{s+\kappa}
\]

if \( (a)^{\beta}(x, \xi) \leq M (\xi^\beta \xi^{-\beta})^{(1/\beta) - 1/\kappa} \) for \( (x, \xi) \in T^* \mathbb{R}^n \), \( h \geq 1 \), \( |\alpha| \leq \bar{N}_s \) and \( |\beta| \leq \bar{N}_s \), where \( H^s \) denotes the Sobolev space of order \( s \).
Proof. Make a change of variables: \( y = h^p x \). Taking \( \lambda(\xi) = h^{p-1} \times \langle \xi \rangle^{1-p} \) as a basic weight function, Theorem 1.6 in Chapter 7 of [19] gives the lemma. Q. E. D.

**Proposition 2.12.** There is \( \varepsilon_0 > 0 \) such that \( p(x, D) \) maps continuously \( H^s_{\varepsilon} \) to \( H^s \) if \( p(x, \xi) \) satisfies (2.15) and \( |\varepsilon - \delta| < \varepsilon_0 A^{-1/K} \).

**Remark.** Proposition 2.12 was proved in [13] and [24] when \( \delta = 0 \).

**Proof.** It suffices to show that \( \exp \left[ (\varepsilon - \delta) \langle D \rangle^{1/\kappa} \right] p(x, D) \exp \left[ -\varepsilon \times \langle D \rangle^{1/\kappa} \right] \) maps continuously \( H^s \) to \( H^s \). By Lemma 2.10 and its remark we can write

\[
\exp \left[ (\varepsilon - \delta) \langle D \rangle^{1/\kappa} \right] p(x, D) = q(x, D) + r(x, D),
\]

where \( q(x, \xi) = \sum_{\alpha} \psi_{\alpha}(\xi) q(x, \xi), q(x, \xi) = \sum_{|\alpha|} \alpha^{-1} \omega_{\alpha}(\xi) p_{(\alpha)}(x, \xi) \exp[\varepsilon - \delta] \langle \xi \rangle^{1/\kappa} \), \( R = 2^{-2+3/\kappa} (n \mathcal{A})^{1/\kappa}, \mathcal{A} = \max(8A_0, A_0/3), A_0 = 1 + \sqrt{2} \) and \( \omega_{\alpha}(\xi) \equiv \omega_{\alpha}(\langle \varepsilon - \delta \rangle \langle \xi \rangle^{1/\kappa}; x, \xi) \). Moreover, we have

\[
|r^{(\beta)}(x, \xi)| \leq C_{p, \alpha}(c |\alpha|, L_{|\alpha|}, \langle 2eA \rangle^{\beta} |\beta|^{\varepsilon} \times \exp[\langle \rho - 2\varepsilon_0 A^{-1/K} \rangle ^{1/\kappa}] \]

if \( \rho > \varepsilon + |\varepsilon - \delta| \) and \( |\varepsilon - \delta| + \varepsilon - \delta < 2\varepsilon_0 A^{-1/K} \), where \( \varepsilon_0 = 2^{-2+3/\kappa} (n \mathcal{A})^{-1/K} \). Therefore, we have

\[
(2.19) \quad \left| (r(x, \xi) \exp[\varepsilon \langle \xi \rangle^{1/\kappa}] \right|^{(\beta)} \leq C_{\alpha}(c |\alpha|, |\beta|, L_{|\alpha|}) \langle \xi \rangle^{m-|\alpha|}
\]

if \( |\varepsilon - \delta| < \varepsilon_0 A^{-1/K} \). On the other hand, a simple calculation yields

\[
|q(x, \xi)\exp[\varepsilon \langle \xi \rangle^{1/\kappa}]^{(\beta)}| \leq C(\epsilon |\alpha|, |\beta|, L_{|\alpha|}, A) (n\mathcal{A}_0)^{-1/\kappa} \langle \xi \rangle^{m-1/\kappa} \langle \xi \rangle^{1/\kappa} \]

\[
\times \sum_{\alpha} C_0 \langle \xi \rangle^{\beta/\kappa} \leq C(\epsilon |\alpha|, |\beta|, L_{|\alpha|}, A) \langle \xi \rangle^{m-1/\kappa} \langle \xi \rangle^{1/\kappa} \langle \xi \rangle^{1/\kappa} \]

\[
\times \{e^{1-\kappa n\mathcal{A}_0 R^{-1}} \text{max}(C_0, R^{-1})\} \text{ if } \langle \xi \rangle^{1/\kappa} \geq R_j,
\]

where \( C_0 = |\varepsilon - \delta| \) and \( \mathcal{A}_0 = 8A_0 \). Since

\[
e^{1-\kappa n\mathcal{A}_0 R^{-1}} \text{max}(C_0, R^{-1}) < 1 \text{ when } C_0 < \varepsilon_0 A^{-1/K},
\]

we have

\[
(2.20) \quad \left| (q(x, \xi)\exp[-\varepsilon \langle \xi \rangle^{1/\kappa}] \right|^{(\beta)} \leq C(\epsilon |\alpha|, |\beta|, L_{|\alpha|}, A) \langle \xi \rangle^{m-1/\kappa} \langle \xi \rangle^{1/\kappa} \]

if \( |\varepsilon - \delta| < \varepsilon_0 A^{-1/K} \). Thus (2.19), (2.20) and Lemma 2.11 show that
Proposition 2.13. Assume that $\Lambda(x, \xi)$ satisfies (2.13) and that $p(x, \xi)$ is a symbol satisfying (2.15) with $L_{k, A} \equiv C$. Then $(e^A)(x, D) p(x, D) R(e^{-A})(x, D)$ maps continuously $S_{k, e}$ to $S_{k, e-p}$ if $\rho > \lambda_{0} + \lambda_1 + \delta$ and $|\varepsilon - \lambda_1 - \delta| < \kappa(nA)^{-1/\kappa}$, and $H_{k, e}$ to $H_{k, e-p}$ if $\rho > \lambda_{0} + \lambda_1 + \delta$, $|\varepsilon - \lambda_1 - \delta| < \epsilon_{0} A^{-1/\kappa}$ and $s \in \mathbb{R}$, where $\epsilon_{0}$ and $\lambda_{0}$ are the constants defined in Proposition 2.12 and Lemma 2.10, respectively, and $\lambda_{1} = \inf_{k \gg 0} \sup_{x \in \mathbb{R}^n} \Re \Lambda(x, \xi) \langle \xi \rangle^{-1/\kappa}$. Here $R(e^{-A})(x, D)$ denotes the transposed operator of $(e^{-A}) (x, -D)$. Moreover there is $\epsilon_{A_0} > 0$ such that there are symbols $p_{\Lambda}(x, \xi)$ and $r_{\Lambda}(x, \xi)$ satisfying the following properties if $C_{0} < \epsilon_{A_0} A^{-1/\kappa}$ and $|\delta| < \epsilon_{A_0} A^{-1/\kappa}$:

\[(e^{A})(x, D) p(x, D) R(e^{-A})(x, D) = p_{\Lambda}(x, D) + r_{\Lambda}(x, D),\]

\[(2.21) \quad |f_{A}(x, \xi) - \sum_{|\alpha| + |\beta| < N} (\alpha! \beta!)^{-1} (p_{A}(x, \xi) \omega^{\alpha} (A; x, \xi) \times \omega^{\beta} (-A; x, \xi)) (\alpha) \beta | \leq C_{A_0} (C, N) (2^{A} A)^{|\alpha| + |\beta|} \times \exp[(\delta - e_{A_0} A^{-1/\kappa}) \langle \xi \rangle^{1/\kappa}],\]

\[N = 0, 1, 2, \ldots,\]

\[(2.22) \quad |f_{A}(x, \xi) | \leq C_{A_0} (|\alpha|, C) (2^{A} A)^{|\beta|} \langle \beta \rangle^{1/\kappa} \times \exp[(\delta - e_{A_0} A^{-1/\kappa}) \langle \xi \rangle^{1/\kappa}],\]

\[(2.23) \quad r_{\Lambda} : S_{k, e} \rightarrow S_{k, e-p} \text{ continuously if } \rho = \delta - e_{A_0} A^{-1/\kappa} \text{ and } |\varepsilon - \delta| < 2^{-t} \kappa(nA)^{-1/\kappa},\]

\[(2.24) \quad r_{\Lambda} : H_{k, e} \rightarrow H_{k, e-p} \text{ continuously if } \rho = \delta - e_{A_0} A^{-1/\kappa},\]

\[s, s' \in \mathbb{R} \quad \text{and} \quad |\varepsilon - \delta| < 2^{-t} \epsilon_{A_0} A^{-1/\kappa},\]

Proof. We set $q(x, \xi) = \sum_{j=0}^{\infty} \omega_{j}^{\beta}(\xi)(q_{j}(x, \xi))$, and $r(x, \xi) = \sigma(e^{A})(x, D) p(x, D)(x, \xi) - q(x, \xi)$, where $R = \text{ar}(A_{0}) A^{1/\kappa}$, $a \geq 1$, $q_{j}(x, \xi) = \sum_{|\alpha| = j} \omega^{\alpha} (A; x, \xi) e^{A(x, \xi)}$ and $r(A_{0})$ is the constant in Lemma 2.10. Lemma 2.10 implies that $q(x, \xi)$ and $r(x, \xi)$ satisfy (2.17) and (2.18), respectively. The symbol $r'_{\Lambda}(x, \xi) \equiv \sigma(r(x, D) R(e^{-A})(x, D))$ can be written as

\[r'_{\Lambda}(x, \xi) = Os - \int e^{-i\nu_{0} \gamma_{0} (x, \xi + \eta)} e^{-A(x, \xi + \eta + \nu)} d\nu d\eta\]

if $|\lambda_{0} + \lambda_1 + C_{0} + \delta| < a^{-1} \delta_{A_0} A^{-1/\kappa}$ and $|\lambda_{0} + C_{0}| < a^{-1} \delta_{A_0} A^{-1/\kappa}$, where $\delta_{A_0}$ is the constant in Lemma 2.10. Then we have
where $M = [n/2] + 1$. A simple calculation gives

$$|f_{a, b}(x, \xi, \eta)| \leq C_{A, A_0, \delta}(|\alpha|, C) (2^\alpha A)^{\beta^1} |\beta|^\xi |\gamma|^\tau |d|^\tau |$$

if $d > 0$, $|\lambda_0 + \lambda_1 + 2C_0| < a^{-1}\delta A_0 A^{-1/6}$, $c_{A_0} \leq a^{-1}\delta A_0 / 2$, $|\lambda_0 + \lambda_1 + C_0 + \delta| < a^{-1}\delta A_0 A^{-1/6}$ and $|\lambda_0 + C_0| < a^{-1}\delta A_0 A^{-1/6}$. This gives

$$|f_{a, b}(x, \xi, \eta)| \leq C_{A, A_0}(|\alpha|, C) (2^\alpha A)^{\beta^1} |\beta|^\xi \times \exp[(\delta - c_{A_0} A^{-1/6}) \langle \xi \rangle_h^{1/6} - |\eta|^{1/6}],$$

(2.25)

$$|r_{a, b}(x, \xi)| \leq C_{A, A_0}(|\alpha|, C) (2^\alpha A)^{\beta^1} |\beta|^\xi \times \exp[(\delta - c_{A_0} A^{-1/6}) \langle \xi \rangle_h^{1/6}],$$

if $|\delta| < a^{-1}\delta A_0 A^{-1/6}/2$, $C_0 < 2^{-4}a^{-1}\delta A_0 A^{-1/6}$ and $c_{A_0} \leq a^{-1}\delta A_0 / 2$. Put

$$p_A(x, \xi) = \sum_{j=0}^{2^\alpha A^1} (\xi) \sum_{|\alpha| = j} [q(x, \xi) (e^{-A(x, \xi)}_{|\alpha|}) (\alpha)],$$

$$r^*(x, \xi) = \sigma(q(x, D)^R (e^{-A}(x, D)) (x, \xi)) - p_A(x, \xi),$$

where $R = ar(A_0) A^{1/6}$, $a \geq a_0 (\geq 2)$, $a_0$ is a constant satisfying $a_0 r(A_0) \geq 2^{3/3}r_0 A_0 A^{-1/6}$ and $r_0$ is the constant in Proposition 2.8. Then it follows from Proposition 2.8 that

(2.26)

$$|\xi^{(a, b)}_{A, A_0}(x, \xi)| \leq C_{\rho, A, A_0}(|\alpha|, C) (2^\alpha A)^{\beta^1} |\beta|^\xi \times \exp[(\rho - a^{-1}\delta A_0 A^{-1/6}) \langle \xi \rangle_h^{1/6}]$$

if $\rho \geq \lambda_0 + \lambda_1 + 2C_0 + \delta$ and $|\lambda_0 + \lambda_1 + 2C_0 + \delta| < a^{-1}\delta A_0 A^{-1/6}$, where $\delta A_0 = 2^{-2}a_0 \delta A(A_0 A^{-1/6})$ is the constant in Proposition 2.8. In fact,

$$|q(x, \xi) (e^{-A(x, \xi)}_{|\alpha|})| \leq C_{\rho, A}(C, A_0 / A) (2^\alpha A)^{\beta^1}$$

$$\times (2^\alpha A)^{|a|} (8A_0)^{|\tau|} |\alpha|^! |\beta|^! |\gamma|^! |\langle \xi \rangle_h^{1/6} \exp[\rho \langle \xi \rangle_h^{1/6}]$$

if $\rho \geq \lambda_0 + \lambda_1 + 2C_0 + \delta$. (2.25), (2.26) and Propositions 2.3 and 2.12 imply that $r_A(x, \xi) = r_A(x, \xi) + r_A(x, \xi)$ satisfies (2.22)–(2.24) if $c_{A_0} \leq \min(2^{-4}a^{-1}\delta A_0, a^{-1}\delta A_0 / 6, 2^{-4}k^n - 1, 2^{-4}k^0)$, $C_0 < c_{A_0} A^{-1/6}, |\delta| < c_{A_0} A^{-1/6}$ and $a \geq a_0$. A simple calculation yields

$$|\sum_{a = 0}^{n} \mathbb{G}_{a, b} \cdot (x, \xi)| \leq C(C, A_0 / A) (8nA_0)^{j+k} A^{-1/6} A^{1/6} (|\alpha|+1|\beta|+j+k)! \langle \xi \rangle_h^{1/6} \times \exp[\delta \langle \xi \rangle_h^{1/6}] (\sum_{a=0}^{n} \mathbb{G}_{a} \langle \xi \rangle_h^{1/6} / l) (\sum_{a=0}^{n} \mathbb{G}_{a} \langle \xi \rangle_h^{1/6} / l),$$
where \( g_{a,b}(x, \xi) = (\alpha! \beta!)^{-1} \delta_{\{\alpha, \beta\}}(x, \xi) \omega^a(H; x, \xi) \omega_b(-H; x, \xi) \). Here we have used the inequalities that
\[
\sum_{n=|\alpha|+|\beta|, n_1+n_2=n} \mu_1^{n_1} \mu_2^{n_2} = |\alpha|! |\beta|! \nu_1^{n_1} \nu_2^{n_2} + j! |\alpha|! |\beta|! \nu_1^{n_1} \nu_2^{n_2} \nu_3^{n_3} \\
\times \left( |\alpha| + |\beta| + j + k \right) (|\alpha| + |\beta| + j + k)! \\
\leq 2^{j+k} |n|^{1-\varepsilon} j! \nu_1^{n_1} \nu_2^{n_2} \nu_3^{n_3} \left( \mu_1^{n_1} \mu_2^{n_2} \right)^{1-\varepsilon} \\
\times A_0 |n|^{1-\varepsilon} \leq 2^{j+k} A_0.
\]

Since \( \langle \xi \rangle^N \geq 2(N-1)R \) if \( j+k < N \), \( \varphi^a R(\xi) = 1 \) and \( \varphi^b R(\xi) = 1 \), we have
\[
|\sum_{|\alpha|+|\beta| \geq n} \left[ \varphi^a_{\{\alpha\}}(\xi) \varphi^b_{\{\beta\}}(\xi) g_{a,b}(x, \xi) \right]^{(a)} - g_{a,b}(x, \xi) |^{(b)} | \\
\leq C(C, A_0/A, N, C_0, R, A_0A) (2^a A)^{|a|+|\beta|} \langle \xi \rangle^N \left( |\alpha| + |\beta| \right)^{1-\varepsilon} \\
\times \exp \left[ \langle \xi \rangle^N \right].
\]

Moreover, we have
\[
|\sum_{|\alpha|+|\beta| \geq n} \left[ \varphi^a_{\{\alpha\}}(\xi) \varphi^b_{\{\beta\}}(\xi) g_{a,b}(x, \xi) \right]^{(a)} |^{(b)} | \\
\leq C' (C, A_0/A, N, R) (2^a A)^{|a|+|\beta|} \langle \xi \rangle^N \left( |\alpha| + |\beta| \right)^{1-\varepsilon} \\
\times \exp \left[ \langle \xi \rangle^N \right].
\]

if \( 2^{j+k} \varepsilon^2 n A_0 A R^{1-\varepsilon} \max(C_0, 1/R) \leq 1 \). Thus we obtain (2.21) if \( a \) is chosen large enough and if \( C_{\alpha_0} \) is chosen small enough.

Q. E. D.

**Lemma 2.14.** There are symbols \( q(x, \xi), q(x, \xi), r(x, \xi) \) and \( \bar{r}(x, \xi) \) such that
\[
(R (e^{-\alpha})(x, D) (e^{-\alpha})(x, D) = 1 + q(x, D) + r(x, D), \\
(e^{-\alpha})(x, D) R (e^{-\alpha})(x, D) = 1 + q(x, D) + \bar{r}(x, D),
\]
where \( \sigma(1)(x, \xi) = 1 \),
\[
(q^{(\alpha)}_{\{\beta\}}(x, \xi) | \leq C_{\alpha_0, \delta} (C_0) d^{\alpha \beta} \langle \xi \rangle^{|\alpha|+|\beta|} \langle \xi \rangle^N, \\
(r^{(\alpha)}_{\{\beta\}}(x, \xi) | \leq C_{\rho, \delta} (C_0) d^{\alpha \beta} \langle \xi \rangle^N \langle \xi \rangle^N \exp[-\rho \langle \xi \rangle^N].
\]

if \( d > 0 \) and \( \rho \in R \), and \( q(x, \xi) \) and \( r(x, \xi) \) satisfy the same estimates as (2.27) and (2.28), respectively. Moreover we have
(2.29) 
\[ (1 + q(x, D) + r(x, D)) R(e^{-A}) (x, D) \]
\[ = R(e^{-A}) (x, D) (1 + q(x, D) + r(x, D)). \]

**Remark.** With obvious notations, we have \( q(x, \xi) \sim \sum_{\alpha > 0} \phi_{\alpha} \xi (\xi - \Lambda; x, \xi) / \alpha ! \) and \( q(x, \xi) \sim \sum_{\alpha > 0} \phi_{\alpha} \xi (\xi - \Lambda; x, \xi) / \alpha ! \), and we can define \( q(x, \xi) \), \( q(x, \xi) \), \( r(x, \xi) \) and \( r(x, \xi) \) as analytic symbols (see [26]).

**Proof.** From Proposition 2.8 and Lemmas 2.4 and 2.9 the lemma easily follows. Q. E. D.

**Lemma 2.15.** Assume that a symbol \( p(x, \xi, y, \eta) \) satisfies
\[ |p_{(x, y)}^{(\xi, \eta)} (x, \xi, y, \eta)| \leq C_{\tau} A^{n\xi + |\beta| + |\delta|} |\alpha| |\beta| |\delta|, \]
\[ \times \langle \xi \rangle^{-1/\kappa} \exp[\delta_2 (\gamma) \langle \eta \rangle^{1/\kappa}] \text{ for } (x, \xi), (y, \eta) \in T^* R^*, \]
\[ |p_{(x, y)}^{(\xi, \eta)} (x, \xi + \eta, x + y, \eta)| \leq C_{\alpha, \gamma} (\beta^x A)^{|\beta| + |\delta|} |\beta| |\delta|^{1/\kappa} \]
\[ \times \exp[-a (\xi)^{1/\kappa}] \text{ if } |\eta| \leq c_1 \langle \xi \rangle \text{ and } |y| \leq c_2, \]
where \( a \in R, c_1 \) and \( c_2 \) are positive constants. Then there are \( d_0 > 0 \) and \( d_1 > 0 \) such that \( p(x, D_x, y, D_y) \) maps continuously \( \mathcal{S}_{\varepsilon, \eta} \) to \( \mathcal{S}_{\varepsilon, \eta + \rho} \) for \( |\varepsilon + \rho| < \kappa (nA)^{-1/\kappa} / 2 \) and \( H_{\varepsilon, \eta} \) to \( H_{\varepsilon, \eta + \rho} \) for \( |\varepsilon + \rho| < \varepsilon_0 A^{-1/\kappa} / 2 \) if \( |\delta| < d_0 \) \( A^{-1/\kappa} \) and \( \rho = \min (a, d_1 A^{-1/\kappa} - \delta_1 - \delta_2) \), where \( \varepsilon_0 \) is the constant in Proposition 2.12.

**Proof.** By the same argument as in the proof of Lemma 2.5, we have
\[ p(x, D_x, y, D_y) u(x) = q(x, D) u(x) \quad \text{for } u \in \mathcal{D}^{(x)} \]
if \( \delta_1 < \kappa (nA)^{-1/\kappa} \), where
\[ q(x, \xi) = \text{Os} - \int e^{-i\gamma x} p(x, \xi + \eta, x + y, \xi) dy d\eta. \]
We may assume that \( 0 < c_1 < 1 \). Choose \( \chi (\xi) \in \mathcal{D}^{(x)} (1 < \kappa < \kappa) \) so that \( \chi (\xi) = 1 \) for \( |\xi| \leq 1/2 \) and \( \chi (\xi) = 0 \) for \( |\xi| \geq 1 \). Put
\[ q_1 (x, \xi) = \int e^{-i\gamma x} p(x, \xi + \eta, x + y, \xi) \chi (\eta / (c_1 \langle \xi \rangle)) \chi (y / c_2) dy d\eta, \]
\[ q_2 (x, \xi) = \text{Os} - \int e^{-i\gamma x} p(x, \xi + \eta, x + y, \xi) \chi (\eta / (c_1 \langle \xi \rangle)) \]
\[ \times (1 - \chi (y / c_2)) dy d\eta, \]
\[ q_3 (x, \xi) = q(x, \xi) - q_1 (x, \xi) - q_2 (x, \xi). \]
Applying the same arguments as in the proofs of Lemmas 2.6 and 2.7, we have
where $M = [n/2] + 1$. Similarly, we have
\[ |q_{\beta}^{(g)}(x, \xi)| \leq C_{\alpha, \rho}(2^e A)^{|\beta|} |\xi|^{\ell} \exp[-a(\xi)^{1/\kappa}], \]
\[ |q_{\beta}^{(g)}(x, \xi)| \leq \inf_{\gamma = 0.1, 2, \ldots} \left| \left( \int e^{-ixf} |y|^{1 - 2N} y^{2M} \right. \right| \times (y \cdot D_\gamma)^N \partial^{N}_{\xi} D^{2e}_{\gamma} \right| \rho(x, \xi + y, \xi)(\eta/(c_1 \langle \xi \rangle)) \times (1 - \chi(y/c_2)) \right) dy \leq C_{\alpha, \rho}(2^e A)^{|\beta|} |\xi|^{\ell} \exp[\rho(\xi)^{1/\kappa}]
\] if $\rho > \delta_1 + \delta_2 + c_1^2 |\delta_1| - \kappa(1 - c_1)^{1/\kappa}c_2^2 e (2nA)^{-1/\kappa}$.

where $\phi(\xi)$ is the characteristic function of $\{\xi \in R; |\xi| \geq 1/2\}$. Therefore, taking $d_0 = \min (2^{-1/\kappa} \kappa (1 - c_1)/c_2, (n c_1)^{1/\kappa})$ and $d_1 = \min (2^{-1/\kappa} \kappa (1 - c_1)/c_2, n^{1/\kappa})$, the lemma follows from Propositions 2.3 and 2.12. Q. E. D.

**Corollary 1.** Let $1 < \kappa_1 \leq \kappa$, and assume that $p(x, \xi)$ satisfies (A-1). Then we have
\[ \mathrm{WF}_*(p(x, D) u) \subset \mathrm{WF}_*(u) \quad \text{for } u \in \mathcal{D}^{*1}, \]
where $* = (\kappa)$ if $*1 = (\kappa_1)$ and $* = [\kappa]$ if $*1 = [\kappa_1]$. Q. E. D.
Corollary 2. Let $\mathcal{C}$ be a conic subset of $T^*\mathbb{R}^n \setminus 0$, and let $\chi(x, \xi) \in \mathcal{E}^{*1}$ be a positively homogeneous function of degree 0 for $|\xi| \geq 1$ such that $\chi(x, \xi) = 1$ near $\mathcal{C} \cap \{||\xi|| \geq 1\}$ and $\{x \in \mathbb{R}^n; (x, \xi) \in \text{supp} \chi \text{ for some } \xi \in \mathbb{R}^n\}$ is compact. Then, $WF(u) \cap \mathcal{C} = \emptyset$ if $\chi(x, D)u \in \mathcal{E}^*$ and $u \in \mathcal{D}^{*1'}$.

Proof. Let $\chi_1(x) \in \mathcal{D}^{*1}$ and $\phi_1(\xi) \in \mathcal{E}^{*1}$ be functions such that $\chi_1(x) \phi_1(\xi) \subseteq \chi(x, \xi), \text{ i.e., } \chi_1(x) \phi_1(\xi)$ is positively homogeneous of degree 0 for $|\xi| \geq M$ and supp $\chi_1(x) \phi_1(\xi) \cap \{||\xi|| = M\} \subseteq \{(x, \xi) \in T^*\mathbb{R}^n; \chi(x, \xi) = 1\}$ for a sufficiently large $M$. Since $\phi_1(D) \chi_1(x) u = \phi_1(D) \chi_1(x) (\chi(x, D) - 1) u$ and $\phi_1(D) \chi_1(x) (\chi(x, D) - 1) u \in \mathcal{E}^{*1}$, we have $\phi_1(D) \chi_1(x) u \in \mathcal{E}^*$ if $\chi u \in \mathcal{E}^*$. This proves the lemma. Q. E. D.

Corollary 3. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be conic subsets of $T^*\mathbb{R}^n \setminus 0$ such that $\mathcal{C}_1 \subset \mathcal{C}_2$, i.e., $\mathcal{C}_1 \cap \{||\xi|| = 1\} \subset \mathcal{C}_2 \cap \{||\xi|| = 1\}$, and let $\chi(x, \xi) \in \mathcal{E}^{(e)}$ be a function such that $\chi(x, \xi)$ is positively homogeneous of degree 0 in $\xi$ for $|\xi| \geq 1$, $\chi(x, \xi) = 1$ near $\mathcal{C}_1 \cap \{||\xi|| \geq 1\}$ and supp $\chi(x, \xi) \cap \{||\xi|| \geq 1\} \subseteq \mathcal{C}_2$. Assume that symbols $p(x, \xi)$ and $q(x, \xi)$ satisfy

$$ |p(\xi)(x, \xi)| \leq C A^{||\alpha||} |\xi|^{||\beta||} \exp[\delta <\xi>^{1/\alpha}], $$
$$ |q(\xi)(x, \xi)| \leq C' A^{||\alpha||} |\xi|^{||\beta||} \exp[\delta <\xi>^{1/\alpha}], $$

and supp $q(x, \xi) \cap \{||\xi|| \geq 1\} \cap \mathcal{C}_2 \cap \mathcal{C}_1 = \emptyset$. Then there is $\xi_1 > 0$ such that $q(x, D) [p(x, D), \chi(x, D)] f \in L^2_{\xi_1}$ for $f \in L^2_{\xi_1}$ if $|\xi| \leq \xi_1 \equiv \xi_1 A^{-1/\alpha}$ and $|\delta| + |\delta'| \leq \xi_1$, where $[A, B] = AB - BA$.

Proof. We can write

$$ q(x, D)[p, \chi] = q(1 - \chi) p \chi - q \chi p (1 - \chi). $$

Let $\chi_j(x, \xi) \in \mathcal{E}^{(e)}(T^*\mathbb{R}^n)$ ($j = 1, 2$) satisfy $\chi_j(x, \xi) \subseteq \chi(x, \xi) \subseteq \chi_0(x, \xi), \chi_1(x, \xi) = 1$ near $\mathcal{C}_1 \cap \{||\xi|| \geq 1\}$ and supp $\chi_0(x, \xi) \cap \{||\xi|| \geq 1\} \subseteq \mathcal{C}_2$, and put $q_1(x, \xi) = q(x, \xi), \chi_1(x, \xi), p_1(x, \xi) = p(x, \xi) \chi_1(x, \xi), q_2(x, \xi) = q(x, \xi) - q_1(x, \xi)$ and $p_2(x, \xi) = p(x, \xi) - p_1(x, \xi)$. Then it follows from Lemma 2.15 that $q_1(x, D) (1 - \chi(x, D)) f, q_2(x, D) \chi(x, D) f, q_2(x, D) p_1(x, D) f$ and $p_2(x, D) \chi(x, D) f$ belong to $L^2_{\xi_1}$ if $f \in L^2_{\xi_1}$, $|\xi| \leq \xi_1 \equiv \xi_1 A^{-1/\alpha}$ and $|\delta| + |\delta'| \leq \xi_1$, where $\xi_1 > 0$. This proves that $q(1 - \chi) p \chi f \in L^2_{\xi_1}$ if $f \in L^2_{\xi_1}$, $|\xi| \leq \xi_1 \equiv \xi_1 A^{-1/\alpha}$ and $|\delta| + |\delta'| \leq \xi_1$, where $\xi_1 > 0$. We
can also apply the same argument to \( q \chi p(1 - \chi) \) and prove the assertion.

Q. E. D.

To end this section we have to remark that calculus of pseudo-differential operators in the space of real analytic functions and Gevrey classes has been studied by many authors (see [1], [4], [8], [22-26]).

§ 3. Hypoellipticity

To prove Theorem 1.4 we shall prepare several lemmas on construction of parametrices in this section. As a consequence of the lemmas, we shall prove that operators satisfying the so-called \((H)\)-condition are hypoelliptic in some Gevrey classes, which was essentially proved by Taniguchi [25].

Let \( 0 < \delta \leq 1 - \frac{1}{\kappa}, \frac{1}{3} \leq \rho < 1, h \geq 1 \) and \( m, m' \in \mathbb{R} \). We say that a symbol \( p(x, \xi) \) satisfies the condition \((H; C, A, d_0, d_1, B, N_0)\), where \( C, A, d_0, d_1, B \geq 0 \) and \( N_0 \) is a non-negative integer, if

\[
|p^{(a)}(x, \xi)| \leq CA^{\alpha + \beta}(|\alpha| + |\beta|)!\xi^{\kappa - |\alpha|} \quad \text{for any } \alpha \text{ and } \beta,
\]

\[
|p(x, \xi)| \geq d_0 \xi^{\kappa'},
\]

\[
|p^{(a)}(x, \xi)/p(x, \xi)| \leq d_1 B^{\alpha + \beta} \xi^{\delta + |\alpha|} \quad \text{for } |\alpha|, |\beta| \leq N_0.
\]

**Lemma 3.1.** Assume that \( p(x, \xi) \) satisfies the condition \((H; C, A, d_0, d_1, B, N_0)\) and that \( A(x, \xi) \) satisfies (2.13). Then there are positive constants \( \varepsilon_{A_1}, A_2, A_3, B, C, \) satisfying (C) and \( C_0 \) and symbols \( p^a_A(x, \xi) \) and \( r^a_A(x, \xi) \) for \( 0 \leq a \leq a_A A^{-1/\kappa} \) such that

\[
(p^{(a)}(x, D)p(x, D) r^{(\alpha)}(x, D)) \xi = p^a_A(x, D)^r + r^a_A(x, D),
\]

\[
|r^{(a)}(x, \xi)| \leq CA \xi^{a_A}(|\alpha|, C) (2^{a_A})^{b/|\beta|} \xi^{|\beta|} \xi^{b/\kappa} \exp[-3C_0 \xi^b],
\]

\[
r^a_A(x, D) \text{ maps continuously } L^2_{2\kappa} \text{ to } L^2_{2\kappa} \text{ if } |\xi| \leq \varepsilon_0 \equiv \varepsilon_0 A^{-1/\kappa}, \text{ and } p^a_A(x, \xi) \text{ satisfies the condition } \((H; C_A, A_0, c_A, A_0, 2^{\varepsilon} A, d_0/2, C(d), B, N_0 - r)\) \text{ if } h \geq a_A (a_1, 1/d_0, d_1, B, N_0) \text{ and if } a c_A A_0 B \leq c_A \text{ when } \delta = 1 - 1/\kappa \text{ or } \rho = 1/\kappa, \text{ where }
\]

\[
r = [(m - m') + 1)/(1 - 1/\kappa)].
\]

Here \( a_A \) is a constant depending on \( A_0 \) and \( C_0 \) and \( A_0 \) and \( h, A_0, A_0, \ldots \) is a constant depending on \( A, A_0, C_0, \ldots \).

**Remark.** When \( \delta = 0 \) or \( \rho = 1 \), we can also obtain similar results,
which is not necessary in this paper.

**Proof.** Applying Proposition 2.13 with \( A(x, \xi) \) replaced by \( aA(x, \xi) \), we obtain \( \hat{p}_{a}^{\alpha}(x, \xi) \) and \( r_{a}^{2}(x, \xi) \). It is obvious that \( r_{a}^{2}(x, \xi) \) has the properties in the lemma if \( \xi_{a} \) and \( a_{a} \) are chosen suitably. \( \hat{p}_{a}^{\alpha}(x, \xi) \) can be written as

\[
\hat{p}_{a}^{\alpha}(x, \xi) = \sum_{|\alpha|+|\beta| \leq r} (\alpha!\beta!)^{-1} \{p_{(\beta)}(x, \xi) \omega^{\alpha}(aA; x, \xi) \times r_{a}^{\alpha}(aA; x, \xi)\}^{(\alpha)} + \hat{p}_{a}^{\alpha}(x, \xi),
\]

\[
|\hat{p}_{a}^{\alpha}(x, \xi)| \leq C_{A, a_{0}}(C) (2\pi A)^{|\alpha|+|\beta|} (|\alpha| + |\beta|)! e^{-\langle \xi \rangle_{h}^{-|\alpha|-1-\gamma-\gamma-|\alpha|}}.
\]

It is easy to see that

\[
\left| \left\{ \omega^{\alpha}(aA; x, \xi) - (a\mathcal{A}_{\alpha}(x, \xi))^{(\alpha)} \right\}^{(\alpha)} \right| 
\leq C_{(C_{0}, A_{0}, a_{0}, \alpha, \beta, \alpha)} \langle \xi \rangle_{h}^{-1-\gamma-1-\gamma-|\alpha|},
\]

\[
\left| \left\{ \omega^{\alpha}(aA; x, \xi) - (a\mathcal{A}_{\alpha}(x, \xi))^{(\alpha)} \right\}^{(\alpha)} \right| 
\leq C_{(C_{0}, A_{0}, a_{0}, \alpha, \beta, \alpha)} \langle \xi \rangle_{h}^{-1-\gamma-1-\gamma-|\alpha|}.
\]

Therefore, we have

\[
(3.1) \quad |\hat{p}_{a}^{\alpha}(x, \xi)/p(x, \xi) - p^{(\alpha)}(x, \xi)/p(x, \xi)| 
\leq \left\{ \sum_{\alpha} \sum_{|\alpha|+|\beta| = d_{1}} B^{1+|\alpha|+|\beta|} (aC_{0}A_{0})^{\rho} \times \langle \xi \rangle_{h}^{\delta+1-\gamma-1-\gamma-|\alpha|} + \langle \xi \rangle_{h}^{-1-\gamma-|\alpha|} \times \langle \xi \rangle_{h}^{\delta+1-\gamma-1-\gamma-|\alpha|} \right\} \text{ if } |\alpha|, |\beta| \leq N_{0} - r.
\]

This shows that there are positive constants \( h_{p, a}(a, 1/d_{0}, d_{1}, B, N_{0}) \) and \( c_{d_{1}} \) such that \( |\hat{p}_{a}^{\alpha}(x, \xi)| \geq |p(x, \xi)|/2 \) if \( h \geq h_{p, a}(a, 1/d_{0}, d_{1}, B, N_{0}) \) and if \( aC_{0}A_{0}B \leq c_{d_{1}} \) when \( \delta = 1-1/\kappa \) or \( \rho = 1/\kappa \). (3.1) also gives

\[
|\hat{p}_{a}^{\alpha}(x, \xi)/p(x, \xi) - p^{(\alpha)}(x, \xi)/p(x, \xi)| \leq 2 (1 + d_{1}) B^{1+1/\gamma} \langle \xi \rangle_{h}^{\delta-1-\gamma-\gamma-|\alpha|} \text{ if } h \geq h_{p, a}(a, 1/d_{0}, d_{1}, B, N_{0}) \) \text{ and } |\alpha|, |\beta| \leq N_{0} - r, \text{ and if } aC_{0}A_{0}B \leq c_{d_{1}} \text{ when } \delta = 1-1/\kappa \text{ or } \rho = 1/\kappa, \text{ modifying } h_{p, a}(a, 1/d_{0}, d_{1}, B, N_{0}). \text{ This proves the lemma.}
\]

Q. E. D.

**Lemma 3.2.** Assume that \( p(x, \xi) \) satisfies the condition \( (H; C, A, d_{0}, d_{1}, B, N_{0}) \). Then there is \( C(d_{1}, N_{0}) > 0 \) such that

\[
|1/p(x, \xi)|^{(\alpha)} \leq C(d_{1}, N_{0}) B^{1+1/\gamma} \langle \xi \rangle_{h}^{\delta-1-\gamma-\gamma-|\alpha|} / |p(x, \xi)| \text{ for } |\alpha|, |\beta| \leq N_{0}.
\]
Proof. It is sufficient to show that

\[
| (1/p(x, \xi))^{(\alpha)} | \leq C(d_0, |\alpha| + |\beta|) B^{[\alpha] + [\beta]}
\times \langle \xi \rangle_{h^{[\beta]} - \rho [\alpha]}^{[\beta] - |\beta[\alpha] |} |p(x, \xi)|.
\]

Using the identity \( \{p(x, \xi) \partial (1/p(x, \xi))^{(\alpha)} = - \{\partial p(x, \xi) / p(x, \xi)\}^{(\alpha)} \), (3.2) can be proved by induction on \(|\alpha| + |\beta|\). Q. E. D.

Lemma 3.3. Assume that \( p(x, \xi) \) satisfies the condition \( (H; C, A, d_0, d_1, B, N_0) \) and that \( 0 \leq \rho \). Put \( q(x, \xi) = \sigma (p(x, D) \cdot (1/p) (x, D)) (x, \xi) - 1 \) and \( q(x, \xi) = \sigma((1/p) (x, D) \cdot p(x, D)) (x, \xi) - 1 \). Then

\[
|q^{(\alpha)}_{(*)}(x, \xi)| \leq \left\{ C(d_1, N_0, B^2 h^{-\rho}) B^2 h^\rho \right\} C(C, 1/d_0, d_1, B, N_0) \langle \xi \rangle_{h^{[\beta] - \rho [\alpha]}}^{[\beta] - |\beta[\alpha]|}
\]

for \(|\alpha|, |\beta| \leq N_0 \) and \( N_0, N_0 - r - 2L(N_0) - 1, \) where \( r = \left[ (m - m' + n + 1) / (1 - \delta) \right] \) and \( L(N_0) = \left[ (\rho N_0 + \delta (r + 1) + |m| - m' + n + 1) / (2 - 2\delta) \right] + 1, \) and \( q(x, \xi) \) also satisfies the estimates (3.3) for \(|\alpha| \leq N_0 - 2[n/2] - r - 3 \) and \(|\beta| \leq N_0 - r, \) where \( r = \left[ (m - m' + 1) / \rho \right]. \)

Proof. Let \( \phi \in \mathcal{D} (r^*) \) \((1 < r^* < r)\) be a function such that \( \phi(\xi) = 1 \) for \(|\xi| \leq 1/4 \) and \( \phi(\xi) = 0 \) for \(|\xi| \geq 1/2 \), and write

\[
q(x, \xi) = q_1(x, \xi) + \int_0^1 (1 - \theta)^r \{q_2(x, \xi, \theta) + q_3(x, \xi, \theta)\} d\theta,
\]

where \( q_1(x, \xi) = \sum_{1 \leq |\alpha| \leq r} q^{(\alpha)}(x, \xi) (1/p(x, \xi)) (x, \xi, \theta) / \alpha! \) and \( q_2(x, \xi, \theta) = \sum_{|\alpha| = r + 1} q^{(\alpha)}(x, \xi, \theta) = 0 \) for \(|\alpha| + |\beta| \leq N_0 - r, \)

\[
|q^{(\alpha)}_{(*)}(x, \xi)| \leq \left\{ C(d_1, N_0, B^2 h^{-\rho}) B^2 h^\rho \right\} C(C, 1/d_0, d_1, B, N_0) \langle \xi \rangle_{h^{[\beta] - \rho [\alpha]}}^{[\beta] - |\beta[\alpha]|}
\]

for \(|\alpha| \leq N_0 \) and \(|\beta| \leq N_0 - r - 1, \)

A simple calculation yields

\[
|q^{(\alpha)}_{(*)}(x, \xi, \theta)| \leq \int |\langle \xi \rangle^{-2L} \sum_{|\gamma| = r + 1} \sum_{a} \alpha^a q^{(\alpha)}(x, \xi, \theta) (1 - \phi(\gamma/\langle \xi \rangle_{h^{[\beta] - \rho [\alpha]}})) | | \langle \xi \rangle^2 D^{(\alpha)}_x \langle \xi \rangle^2 D^{(\beta)}_x \langle \xi \rangle^{-2M} \langle y \rangle^{-2M} / \langle x + \theta y, \xi \rangle | \eta(d\eta)
\]

for \(|\alpha|, |\beta| \leq N_0 \) and \( N_0, N_0 - r - 2L - 1, \) where \( L = L(N_0) \) and \( M = \left[ n/2 \right] + 1. \) (3.4)–(3.6) prove (3.3). Write
where \( q_1(x, \xi) = \sum \frac{1}{i!} \gamma_i x^i \) and \( q_2(x, \xi) = \sum \frac{1}{i!} \gamma_i x^i \).

Then, it is obvious that \( q_1(x, \xi) \) satisfies the estimates (3.4) for \( |\alpha|, |\beta| \leq N_0 - \tau \). Moreover, we have

\[
|q_2^{(\alpha)}(x, \xi)| \leq \sum_{|\gamma| = 1} |\gamma|!^{-1} \left( \frac{1}{|\gamma|!} \right)^2 |D_x^{\alpha} D_\xi^{\beta} D_\eta^{2M} D_\gamma^{2N} \right) \left( \frac{1}{|\gamma|!} \right)^2 |D_x^{\alpha} D_\xi^{\beta} D_\eta^{2M} D_\gamma^{2N} \right) |d\gamma d\eta|
\]

for \( |\alpha| \leq N_0 - 2M - \tau - 1 \) and \( |\beta| \leq N_0 \), where \( L = \left[ \left( (\delta + \rho) N_0 + (1 - \rho) \right) \right] + 1 \), which proves the lemma.

Q.E.D.

**Proposition 3.4.** Let \( N_0 \) be a sufficiently large positive integer, and assume that \( p(x, \xi) \) satisfies the condition \( (H; C, A, d_0, d_1, B, N_0) \) and that \( \delta \leq \rho \). Then there are positive constants \( \delta_0, h_p(1/d_0, d_1, B, C) \) and \( B(d_1) \) and an operator \( Q \) such that \( Q \) maps continuously \( L^{2}_{\delta, \rho} \) to \( H^{2}_{\delta, \rho} \) and satisfies \( Q p(x, D) = I \) on \( H^{2}_{\delta, \rho} \) and \( p(x, D) Q = I \) on \( H^{2}_{\delta, \rho} \) if \( \rho = d_1 \) and \( \rho \leq d_1 \).

Here \( I \) denotes the identity operator, and \( h_p(\cdots) \) is a constant depending on \( A, C, \cdots \). Let \( C_1 \) and \( C_2 \) be conic sets in \( T^* R^d \) such that the distance between \( C_1 \cap \{ |\xi| = 1 \} \) and \( C_2 \cap \{ |\xi| = 1 \} \) is not less than \( d > 0 \). If \( \chi_j \in \mathcal{C}^{(\alpha)}(T^* R^d) \), supp \( \chi_j(x, \xi) \subseteq C_j \), and \( |\chi_j^{(\alpha)}(x, \xi)| \leq C_d \delta^{2M+|\alpha|} \) for any \( d > 0 \), then there are positive constants \( \delta_0, h_p(1/d_0, d_1, B) \), \( \delta_{0, \delta_0, \rho} \) and \( B'(d_1) \) such that \( \chi_j(x, D) Q \chi_j(x, D) \) maps continuously \( L^{2}_{\delta, \rho} \) to \( L^{2}_{\delta, \rho} \) if \( \rho \leq \delta \) and \( h \geq h_p(1/d_0, d_1, B) \), and if \( \rho \leq \delta \) when \( \rho = d \).

**Proof.** From Lemma 3.1 with \( aA(x, \xi) \) replaced by \( \delta^{2M} \) it follows that there are \( \delta \geq 0 \) and symbols \( p_\epsilon(x, \xi) \) and \( r_\epsilon(x, \xi) \) for \( |\epsilon| \leq \epsilon_0 \) such that

\[
\exp[\epsilon(D)^{2M}] p_\epsilon(x, D) \exp[-\epsilon(D)^{2M}] = p_\epsilon(x, D) + r_\epsilon(x, D),
\]

\[
|r_\epsilon^{(\alpha)}(x, \xi)| \leq C_d \delta^{2M+|\alpha|} \exp[-\epsilon(D)]^{2M},
\]

if \( |\epsilon| \leq \epsilon_0 \), and \( p_\epsilon(x, \xi) \) satisfies the condition \( (H; C_d, 2^M A, d_0, d_1, B, C(d_1), B, N_0 - \tau) \), if \( \epsilon \) and \( h \) satisfy the following conditions:

\[
|\epsilon| \leq \epsilon_0 \text{ and } h \geq h_p(1/d_0, d_1, B), \text{ and } |\epsilon| \leq \epsilon_0 \text{ when } \epsilon = 1
\]

\(-1/\kappa \) or \( \rho = 1/\kappa \).
where 

\[ r = \left[ (m - m' + 1)/(1 - 1/\varepsilon) \right] \] and 

\[ C_A (|\alpha|, |\beta|, C), C_A (C), C (d_i), \]

\[ h_\varepsilon (1/d_\varepsilon, d_i, B) \]

and \( \varepsilon_d \) are positive constants. We set 

\[ g_\varepsilon (x, \xi) = \sigma ((p_\varepsilon (x, D) + r_\varepsilon (x, D)) (1/p_\varepsilon) (x, D)) (x, \xi) = 1. \]

Applying Lemmas 3.2 and 3.3, we can see that \( g_\varepsilon (x, \xi) \) satisfies the same estimates as (3.3) if \( C (C, A, 1/d_\varepsilon, d_i, B, N_0) \) is replaced by \( C_\varepsilon (1/d_\varepsilon, d_i, B, N_0) \) and \( C_\varepsilon \) and \( h \) satisfy (3.7) and if \( |\alpha|, |\beta| \leq \hat{N}_{m-m'} \)

and \( N_0 \geq \hat{N}_{m-m'} + r + l_1 + l_2, \) where \( l_1 = \left[ (m - m' + n + 1)/(1 - \delta) \right], \)

\[ l_2 = 2\left( (a \hat{N}_{m-m'} + \delta (l_1 + 1) + |m| - m' + n + 1)/(2 - 2\delta) \right) \]

and \( \hat{N}_{m-m'} \) is the constant in Lemma 2.11. In fact, \( \sigma (r_\varepsilon (x, D) (1/p_\varepsilon) (x, D)) (x, \xi) = O \)

\[ \int \int e^{-ir_\varepsilon (x, \xi + \eta)} p_\varepsilon (x + \eta, \xi)^{-1} dy d\eta \]

can be estimated similarly.

Therefore, it follows from Lemma 2.11 that there is the inverse 

\( (1 + q_\varepsilon (x, D))^{-1} \)

of \( (1 + q_\varepsilon (x, D)) \) such that \( (1 + q_\varepsilon)^{-1} (1 + q_\varepsilon) = (1 + q_\varepsilon)^{-1} = I \) on \( H_0^{m-m'} \) if \( \varepsilon \) and \( h \) satisfy (3.7), and if \( B \leq B (d_i) \) when \( \rho = \delta, \) where \( h_\varepsilon (1/d_\varepsilon, d_i, B) \)

and \( h (d_i) \) are suitable positive constants. Put 

\[ Q_\varepsilon = \exp \left[ -\varepsilon (D)^{1/\varepsilon} \right] (1/p_\varepsilon) (x, D) (1 + q_\varepsilon (x, D))^{-1} \exp \left[ \varepsilon (D)^{1/\varepsilon} \right] \]

Then \( Q_\varepsilon \) maps continuously \( H_0^{m-m'} \) to \( H_0^{m-m'} \) and satisfies 

\( p (x, D) Q_\varepsilon = I \) on \( H_0^{m-m'} \) if \( |\varepsilon| \leq \varepsilon_0. \) Here we have assumed that \( N_0 \geq \hat{N}_{m-m'} + r, \)

and applied Lemma 2.11 to \( (1 + q_\varepsilon (x, D)) \) \( (x, \xi). \) Put \( q_\varepsilon (x, \xi) = \sigma ((1/p_\varepsilon) (x, D) (p_\varepsilon (x, D) + r_\varepsilon (x, D))) (x, \xi)^{-1}. \) Similarly \( (1 + q_\varepsilon (x, D)) \)

has the inverse \( (1 + q_\varepsilon (x, D))^{-1} \) on \( H_0^{m-m'} \) if \( \varepsilon \) and \( h \) satisfy (3.7), and if \( B \leq B (d_i) \) when \( \rho = \delta, \) modifying the constants. If we set 

\( \tilde{Q}_\varepsilon = \exp \left[ -\varepsilon (D)^{1/\varepsilon} \right] (1 + q_\varepsilon (x, D))^{-1} (1/p_\varepsilon) (x, D) \exp \left[ \varepsilon (D)^{1/\varepsilon} \right], \)

then \( \tilde{Q}_\varepsilon \) maps continuously \( L_2^{m-m'} \) to \( H_0^{m-m'} \) and satisfies 

\( \tilde{Q}_\varepsilon p (x, D) = I \) on \( H_0^{m-m'} \) if \( |\varepsilon| \leq \varepsilon_0. \) Here we have assumed that \( N_0 \geq \hat{N}_{m-m'} + r + l_3+2[n(2)]+3, \)

where \( l_3 = \left[ (m - m' + 1)/\rho \right]. \) It is easy to see that \( Q_\varepsilon = Q_\varepsilon \) on \( H_0^{m-m'} \) if \( \varepsilon \) and \( \tilde{Q}_\varepsilon = Q_\varepsilon \) on \( H_0^{m-m'} \), which proves the first part of the proposition.

Choose a symbol \( A (x, \xi) \) satisfying

\[ |A (\xi A (x, \xi) | \leq C_\varepsilon A (|a| + |\beta|) |\xi|^{1/\varepsilon - 1/|a|}, \]

\[ \inf_{L>0} \sup_{|\xi| > L} |A (x, \xi) | |\xi|^{-1/\varepsilon} < 2, \]

\[ \inf_{L>0} \sup_{|x|, \xi = \varepsilon' ||L| \geq L} A (x, \xi) |\xi|^{-1/\varepsilon} < 1, \]

\[ \sup_{L>0} \inf_{|x|, \xi = \varepsilon' ||L| \geq L} A (x, \xi) |\xi|^{-1/\varepsilon} > 1. \]

For example, let \( \varphi (x, \xi) \) be a function in \( C^1 (T^* R^N) \) such that the first order derivatives of \( \varphi \) are bounded, and \( |\varphi (x, \xi) | < 5/3, \)

\( \varphi (x, \xi) = -4/3 \) in \( C_1 \cap \{ |\xi| \geq 1/2 \} \) and \( \varphi (x, \xi) = 4/3 \) in \( C_2 \cap \{ |\xi| \geq 1/2 \}. \) Put
\[ \varphi_0(x, \xi) = E_j \varphi(x, \xi) \text{ and } \Lambda(x, \xi) = \varphi_0(x, \xi/\langle \xi \rangle_0)\langle \xi \rangle_0^{1/k}, \]

where \( E_j(x, \xi) = j^*(4\pi)^{-n/2} \exp[-j (|x|^2 + |\xi|^2)/4]. \) If \( j \) is sufficiently large, then \( \Lambda(x, \xi) \) satisfies (3.8)-(3.11). Let symbols \( \varphi_0(x, \xi) \) and \( \varphi_0(x, \xi) \) be as defined in Lemma 3.1 for \( a \leq a \leq A^{-1/k} \), where \( \Phi \) is a positive constant. By Lemma 3.1 \( \varphi_0(x, \xi) \) maps continuously \( L^2_{x, \xi} \) to \( L^2_{x, \xi} \), for \( |\xi| \leq e_1 \equiv \xi_0 A^{-1/k} \), where \( \xi_0 \) is a positive constant. Applying the same argument as in the first part of the proof to \( \varphi_0(x, \xi) \) instead of \( \varphi_0(x, \xi) \), we can show that there is an operator \( Q_0 \) which maps continuously \( L^2_{x, \xi} \) to \( H^m_{x, \xi} \), and satisfies \( Q_0 \varphi_0 = I \) on \( H^m_{x, \xi} \) if \( |\xi| < e_0 / 4 \) and \( h \geq h_p, d \). By Lemma 2.10, Proposition 2.12 and (3.10), we have \( \varphi_0(\xi, \xi) \) maps continuously \( L^2_{x, \xi} \) to \( H^m_{x, \xi} \). Similarly, Propositions 2.8 and 2.12 and (3.11) imply that \( \varphi_0(\xi, \xi) \) maps continuously \( H^m_{x, \xi} \) to \( H^m_{x, \xi} \). Thus we have

\[ \chi \varphi \chi, \chi_0 \varphi \chi_0 \varphi^a(x, \xi) Q_0, \varphi \varphi^a - u \]

which completes the proof. Q.E.D.

Let \( (x^0, \xi^0) \in T^* \mathcal{R} \setminus 0 \) and \( |\xi^0| = 1 \), and let \( \mathcal{C} \) be a convex conic neighborhood of \( (x^0, \xi^0) \). Choose a neighborhood \( U \) of \( x^0 \) and a conic neighborhood \( \Gamma \) of \( \xi^0 \) so that \( U \times \Gamma \subset \mathcal{C} \). Moreover, let \( \mathcal{C}_1 \) be a conic neighborhood of \( (x^0, \xi^0) \) such that \( \mathcal{C}_1 \subset U \times \Gamma \). Choose \( \phi_0(x) \) for a fixed \( \epsilon' < \epsilon \) so that \( \phi_2(\xi) \) is positively homogeneous of degree 0 for \( |\xi| \geq 1 \), \( 0 \leq \phi_0(x) \phi_2(\xi) \leq 1 \), supp \( \phi_0(x) \phi_2(\xi) \) \( \phi_2(\xi) \cap \{|\xi| = 1\} \subset U \times \Gamma \) and \( \phi_0(x) \phi_2(\xi) = 1 \) on \( \mathcal{C}_1 \cap \{|\xi| \geq 1\} \). Let
\(\sigma(\xi) \in \mathcal{C}^{(x)}\) be a function such that \(\sigma(\xi) = 0\) for \(|\xi| \leq 1\) and \(\sigma(\xi) = 1\) for \(|\xi| \geq 2\), and write \(\sigma_h(\xi) = \sigma(\xi/h)\) for \(h \geq 1\). We set
\[
X(x) = (1 - \phi_1(x)) x^2 + \phi_1(x) x,
\]
\[
E_h(\xi) = [h(1 - \sigma_h(\xi)) + \sigma_h(\xi)(1 - \phi_2(\xi))] |\xi| x^2 + \sigma_h(\xi) \phi_2(\xi) \xi,
\]
\[
\tilde{f}_h(x, \xi) = p(X(x), E_h(\xi)).
\]
Then it is obvious that \(\tilde{f}_h(x, \xi) = p(x, \xi)\) if \((x, \xi) \in \mathcal{C}^1\) and \(|\xi| \geq 2h\).

**Lemma 3.5.** Assume that \(h \geq 1\) and that
\[
|\lambda x^2 + \xi| \geq (\lambda + |\xi|)/2\quad \text{for } \lambda > 0\text{ and } (x, \xi) \in \mathcal{C}^1.
\]
Then we have \((X(x), E_h(\xi)) \in \mathcal{C}^1\), \(|E_h(\xi)| \geq h/2\) and
\[
(2|\xi|)^{-1} < \xi > h \leq |E_h(\xi)| \leq (2|\xi|)^{-1} < \xi > h.
\]

**Lemma 3.6.** Assume that (3.12) is satisfied. If a symbol \(p(x, \xi)\) satisfies
\[
|p_{(\beta)}^{(a)}(x, \xi)| \leq C A_{a + |\beta|} (|\alpha| + |\beta|)!|\xi|^{m-|\alpha|} \quad \text{for } (x, \xi) \in \mathcal{C}^1
\]
with \(|\xi| \geq h_0\) (\(\geq 1\)),
then
\[
|\tilde{f}_h^{(a)}(x, \xi)| \leq C(C, 1/A_1) A_{a + |\beta|} (|\alpha| + |\beta|)!|\xi|^{m-|\alpha|} \quad \text{for } h \geq 2h_0\text{ and } (x, \xi) \in T^* R^1, \text{ where } A_1 \equiv C_{a, \phi_1, \phi_2} A \text{ and } C_{a, \phi_1, \phi_2} > 0.
\]

**Proof.** It is sufficient to verify that
\[
|\partial_x D_x p_{(\beta)}^{(a)}(X(x), E_h(\xi))| \leq (2|\xi|)^{m+1} C A_{a + |\beta|} (2|\xi|)^{\rho_1 + |\alpha|}
\]
\[
\times (\rho_1 + |\beta| + |\gamma| + |\nu|)!|\xi|^{m-|\alpha|} \sum_{k=0}^{a + |\beta|} b^k/k!^{r-1}
\]
for \(h \geq 2h_0\), where \(b \equiv b(1/A_1)\). (3.13) can be proved by induction on \(|\alpha| + |\beta|\) if \(A_1 \geq C_{a, \phi_1, \phi_2} A\). Q. E. D.

**Lemma 3.7.** Assume that (3.12) is satisfied. If a symbol \(p(x, \xi)\) satisfies
\[
|p_{(\beta)}^{(a)}(x, \xi)| \geq d_0|\xi|^{m},
\]
\[
|p_{(\beta)}^{(a)}(x, \xi)/p(x, \xi)| \leq d_1 B^{|\alpha| + |\beta|} |\xi|^{\rho_1 + |\alpha|}
\]
for \((x, \xi) \in \mathcal{C}^1\) with \(|\xi| \geq h_0\) (\(\geq 1\)) and \(|\alpha|, |\beta| \leq N_0\), then
\[
|\tilde{f}_h^{(a)}(x, \xi)| \geq (2|\xi|)^{m-1} d_0|\xi|^{m'}
\]
for \((x, \xi) \in T^*\mathbb{R}^n\) and \(h \geq 2h_0\),

\[
|\hat{p}(\xi)(x, \xi) / \hat{p}(x, \xi)| \leq d_1 |a| + |\beta| \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma}
\]
for \((x, \xi) \in T^*\mathbb{R}^n\), \(|\alpha| \leq N_0\) and \(h \geq h(1/B_1, h_0, N_0)\),

where \(B_1 = C_{\alpha, \phi, \phi_2} B\), \(C_{\alpha, \phi, \phi_2} > 0\) and \(h(1/B_1, h_0, N_0) > 0\).

**Proof.** (3.16) is obvious. We can prove by induction on \(|\alpha| + |\beta|\) that

\[
|\partial^\alpha \partial^\beta \hat{p}(\xi)(X(x), \mathcal{E}_A(\xi)) / \hat{p}(x, \xi)|
\]
\[
\leq d_2 |a| + |\beta| (2\sqrt{B}) |\gamma| + |\nu| \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma} \langle \xi \rangle^{\gamma}
\]
if \(h \geq h(1/B_1, h_0, N_0)\), \(|\alpha| + |\gamma| \leq N_0\) and \(|\beta| + |\nu| \leq N_0\), using \(\rho < 1\) and \(\delta > 0\). This proves (3.17). Q.E.D.

**Proposition 3.8.** Let \((x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0\), and let \(\mathcal{C}\) be a conic neighborhood of \((x^0, \xi^0)\). Assume that \(0 < \delta \leq 1 - 1/\kappa\), \(1/\kappa \leq \rho < 1\) and \(\delta < \rho\), and that \(p(x, \xi) \in \mathcal{E}_A\) satisfies (3.14) and (3.15) for \((x, \xi) \in \mathcal{C}\) with \(|\xi| \geq h_0(\geq 1)\) and \(|\alpha|, |\beta| \leq N_0\), and \(p(x, D)\) is properly supported, where \(N_0\) is a sufficiently large positive integer, \(m, m', m'' \in \mathbb{R}\) and \(h_0, d_0, d_1\) and \(B\) are positive constants. Moreover assume that \(\delta < 1 - 1/\kappa\) and \(\rho > 1/\kappa\) if \(* = (\kappa)\). Then there is an operator \(Q\), which maps continuously \(\mathcal{D}^*\) to \(\mathcal{D}^*\), such that

\[
(x^0, \xi^0) \in WF_*(pQf - f) \cup WF_*(Qpf - f) \quad \text{for } f \in \mathcal{D}^*,
\]

\[
(x^0, \xi^0) \in WF_*(Qf) \quad \text{if } (x^0, \xi^0) \in WF_*(f) \quad \text{and } f \in \mathcal{D}^*.
\]

**Remark.** (i) Taniguchi [25] essentially proved the proposition by his method of multi-products of pseudo-differential operators, and constructed \(Q\) as a pseudo-differential operator. (ii) The proposition implies that \(p(x, D)\) has a microlocal parametrix at \((x^0, \xi^0)\) modulo \(\mathcal{E}^*\) and, therefore, \(p(x, D)\) is hypoelliptic at \((x^0, \xi^0)\) (in \(\mathcal{C}\)) with respect to \(\mathcal{E}^*\). (iii) When \(\rho = 1\) or \(\delta = 0\), the proposition is valid, modifying \(\rho\) and \(\delta\).

**Proof.** We may assume that for any \(A > 0\) there is \(C \equiv C_A > 0\) (resp. there are \(A > 0\) and \(C > 0\)) such that

\[
|p^{(\alpha)}(x, \xi)| \leq CA^{\alpha|a| + |\beta|} |(\alpha| + |\beta|)|^{\kappa} \langle \xi \rangle^{\kappa - |a|} \quad \text{for } (x, \xi) \in T^*\mathbb{R}^n
\]
if \(* = (\kappa)\) (resp. if \(* = \{\kappa\}\)), and that \(|\xi^0| = 1\) and (3.12) is satisfied.
By Lemmas 3.5–3.7, Proposition 3.4 can be applicable and there is an operator $\tilde{Q}_h$, which maps continuously $L^2_{x, \varepsilon}$ to $H^m_{x, \varepsilon}$ and $H^{m-\nu}_{x, \varepsilon}$ to $H^{m-\nu}_{x, \varepsilon}$ and satisfies $\tilde{Q}_h \tilde{p}_h = I$ on $H^m_{x, \varepsilon}$ and $\tilde{p}_h \tilde{Q}_h = I$ on $H^{m-\nu}_{x, \varepsilon}$ if $\varepsilon$ and $h$ satisfy the following conditions;

(3.18) $|\varepsilon| < \varepsilon_1 \equiv \varepsilon_{U \times \Gamma, \xi} A^{-1/\kappa}$ and $h \geq h_{B, U \times \Gamma, \xi} (1/d_0, d_1, h_0)$, and

$|\varepsilon| B \leq \varepsilon_{d_1, U \times \Gamma, \xi}$ when $\delta = 1 - 1/\kappa$ or $\rho = 1/\kappa$.

Fix $h > 0$ so that $h$ satisfies (3.18). If $* = (\kappa)$, then $A$ can tend to zero. So, for any $\varepsilon \in \mathbb{R}$ we can define

$$\tilde{Q} f = \tilde{Q}_h f - \tilde{Q}_h (\tilde{p}_h - \tilde{p}_k) \tilde{Q}_h f$$

for $f \in H^{m-\nu}_{x, \varepsilon}$, where $h' (\geq h)$ is sufficiently large according to $|\varepsilon|$, when $* = (\kappa)$. If $f = \phi$, then $\tilde{Q} f$ can tend to zero. So, for any $\varepsilon \in \mathbb{R}$ we can define

$$\tilde{Q} f = \tilde{Q} \phi - \tilde{Q}_h (\tilde{p}_h - \tilde{p}_k) \tilde{Q} \phi$$

for $f = \phi$. This implies that $\tilde{Q}$ does not depend on $h'$. When $* = [\varepsilon]$, we define

$$\tilde{Q} = \tilde{Q}_h$$

Then we have

(3.19) $\tilde{Q} f = \tilde{Q}_h f - \tilde{Q}_h (\tilde{p}_h - \tilde{p}_k) \tilde{Q}_h f$ for $f \in H^{m-\nu}_{x, \varepsilon}$

if $f \in H^{m-\nu}_{x, \varepsilon}$, $0 < \varepsilon < \varepsilon_1$ and $|\varepsilon| < \varepsilon_1$, and if $h' (\geq h)$ is sufficiently large according to $\varepsilon_1$ (or $A^{-1}$) when $* = (\kappa)$, and if $\varepsilon$ and $h' (\geq h)$ satisfy (3.18) when $* = [\varepsilon]$, modifying $\varepsilon_{U \times \Gamma, \xi}$. Let $\varphi (x) \in D^\infty$ and $\chi (x, \xi) \in \mathcal{E}^{(\gamma)} (T^* \mathbb{R}^n)$ be functions such that $\varphi (x) = 1$ in a neighborhood of $U$, $\chi (x, \xi)$ is positively homogeneous of degree 0 in $\xi$ for $|\xi| \geq 1$, $0 \leq \chi (x, \xi) \leq 1$, $\supp \chi (x, \xi) \cap \{ |\xi| = 1 \} \subset C_1$ and $\chi (x, \xi) = 1$ if $(x, \xi)$ belongs to a conic neighborhood of $(x^0, \xi^0)$ and $|\xi| \geq 1$. We write $\chi \in C^0_1$ for $(x^0, \xi^0)$ if $\chi (x, \xi)$ has the above properties. Let $\chi_1 (x, \xi) \in \mathcal{E}^{(\gamma)} (T^* \mathbb{R}^n)$ satisfy $\chi_1 \subset \mathcal{X}$ for $(x^0, \xi^0)$, i.e., $\chi_1 \subset \{ (x, \lambda \xi) ; \chi (x, \xi) = 1, |\xi| \geq 1 \}$ and $\lambda > 0$ for $(x^0, \xi^0)$. We set $Q f = \tilde{Q} (x, D) \varphi (x) f$. Then it follows from (3.19) that $Q$ maps continuously $D^* \subset D^* \subset D^* \subset D^* \subset D^*$ to $H^m_{x, \varepsilon}$ if $* = (\kappa)$, $|\varepsilon| < \varepsilon_1$ and $h'$ is sufficiently large. We have also

$$\chi \varphi p Q f - \chi \varphi f = \chi \varphi (\chi \varphi - 1) f + \chi \varphi (p - \tilde{p}_h) Q f \in \mathcal{E}^*$$

for $f \in D^*$. Taking $h'$ sufficiently large according to $\chi_1$ and $\chi$, it follows from Proposition 3.4 (pseudo-locality of $\tilde{Q}_h$) and (3.19) that

$$\chi_1 Q_p f - \chi_1 f = \chi_1 (\tilde{Q}_h (\chi \varphi p - \tilde{p}_h) f + \chi_1 (\tilde{Q} - \tilde{Q}_h) (\chi \varphi p - \tilde{p}_h) f \in L^2_{x, \varepsilon_1}$$
if \( f \in L^2_{x,\varepsilon} \) and \( |\varepsilon| < \varepsilon_1 \), and if \( |\varepsilon| \) is sufficiently small according to \( B, d, \chi_1 \) and \( \chi \) when \( \delta = 1 - 1/\kappa \) or \( \rho = 1/\kappa \), modifying \( \varepsilon_{0 \times \varepsilon_1} \). This implies that \( \chi_Q \eta f - \chi, f \in \mathcal{E}^\star \) for \( f \in \mathcal{D}^\star \), since \( \delta < 1 - 1/\kappa \) and \( \rho > 1/\kappa \) when \( *= (\kappa) \). Assume that \( f \in \mathcal{D}^\star \) and \( (x^0, \xi^0) \in WF_\star (f) \). We may assume that \( f \in \mathcal{E}^\star \). Then there is \( \chi_2 (x, \xi) \in \mathcal{E}^{(\kappa)} (T^*R^\nu) \) such that \( \chi_2 \subset \mathcal{C}_1 \) for \( (x^0, \xi^0) \) and \( \chi_2 (x, D) f \in \mathcal{E}^\star \). In fact, by definition there is \( \chi (x, \xi) \in \mathcal{E}^{(\kappa)} (T^*R^\nu) \) such that \( \chi \subset \mathcal{C}_1 \) for \( (x^0, \xi^0) \) and \( \mathcal{R}_2 (x, D) f \in \mathcal{E}^\star \). If \( \chi_2 \subset \mathcal{C}_2 \) for \( (x^0, \xi^0) \), Lemma 2.15 implies that \( \chi_2 (x, D) \mathcal{R}_2 (x, D) f \in \mathcal{E}^\star \). Let \( \chi_3 (x, \xi) \in \mathcal{E}^{(\kappa)} (T^*R^\nu) \) satisfy \( \chi_3 \subset \mathcal{C}_2 \) for \( (x^0, \xi^0) \). Taking \( h' \) sufficiently large according to \( \chi_2 \) and \( \chi_3 \), Lemma 2.15 and Proposition 3.4 give
\[
\chi_3 (x, D) \mathcal{R}_3 (x, D) \varphi (x, D) (1 - \chi_2 (x, D)) f \in L^2_{x, \varepsilon_1 (h')},
\]
where \( \varepsilon_1 (h') > 0 \) and \( \varepsilon_1 (h') \to \infty \) as \( h' \to \infty \) when \( *= (\kappa) \). In fact, we can write
\[
\chi \varphi (1 - \chi_2) f = b_1 \varphi (1 - \chi_2) f + b_2 \varphi (1 - \chi_2) f,
\]
b_1 (x, \xi) = \chi (x, \xi) (1 - \chi_4 (x, \xi)), \quad b_2 (x, \xi) = \chi (x, \xi) \chi_4 (x, \xi),
\]
where \( \chi_4 (x, \xi) \in \mathcal{E}^{(\kappa)} (T^*R^\nu) \) satisfies \( \chi_4 \subset \mathcal{C}_2 \) for \( (x^0, \xi^0) \). On the other hand, we have \( \chi_3 (x, D) \mathcal{R}_3 (x, D) f \in \mathcal{E}^\star \). Therefore, by (3.19) we have \( \chi_3 (x, D) \mathcal{R}_3 (x, D) f \in \mathcal{E}^\star \). Corollary 2 of Lemma 2.15 implies that \( (x^0, \xi^0) \in WF_\star (\mathcal{Q} f) \).

\[Q.E.D.\]

\section{The Microlocal Cauchy Problem}

Modifying \( \rho (x, D) \) and using pseudo-differential operators of infinite order, we shall reduce the problem in the Gevrey classes to the problem in the Sobolev spaces and construct the inverses of the reduced operators in this section. Then we can construct microlocal parametrices of the microlocal Cauchy problem in Gevrey classes and prove microlocal well-posedness (see Theorem 4.11 below). Theorem 1.4 easily follows from Theorem 4.11 (see \S 5). In this section we assume that \( \rho (x, \xi) \) satisfies the conditions (A-1) and (A-2) with \( \kappa \) replaced by \( \kappa \) (>1). Let \( \zeta^0 = (x^0, \xi^0) \in T^*R^\nu \setminus 0 \), \( |\xi^0| = 1 \) and \( \mathcal{Y} \in T^*R^\nu \), and assume that \( \rho (x, \xi) \) is microhyperbolic with respect to \( \mathcal{Y} \) at \( \zeta^0 \).

\textbf{Lemma 4.1.} ([30], [33]). Let \( M \subset \Gamma (\rho m^0, \mathcal{Y}) \). Then there is a
neighborhood $\mathcal{U}$ of $z^0$ in $T^*\mathbb{R}^n \setminus 0$ such that $p_m$ is microhyperbolic with respect to $\mathfrak{b} \in M$ at $z \in \mathcal{U}$, and $M \subset \Gamma(p_{m^2}, \mathfrak{b})$ for $z \in \mathcal{U}$.

Define for $v \in T_{v}(T^*\mathbb{R}^n)$

$$p_m(x, \xi; v, t) = \sum_{j=0}^{l} (-it)^j p_m(x, \xi) / j!,$$

where $l = \mu(z^0)$ and $v$ is regarded as a vector field. By definition there are a neighborhood $\mathcal{U}$ of $z^0$ in $T^*\mathbb{R}^n \setminus 0$ and positive constants $c$ and $t_0$ such that

$$|p_m(x, \xi; \theta, t)| \geq ct^l \text{ for } (x, \xi) \in \mathcal{U} \text{ and } 0 \leq t \leq t_0. \quad (4.1)$$

**Lemma 4.2.** Let $M$ be a compact subset of $\Gamma(p_{m^2}, \mathfrak{b})$. Modifying $\mathcal{U}$, $c$ and $t_0$, we have

$$|p_m(x, \xi; v, t)| \geq ct^l \quad (4.2)$$

$$|\partial_{\xi}^\alpha p_m^{(v)}(x, \xi; v, t) / p_m(x, \xi; v, t)| \leq C(\alpha, \beta) t^{-|\alpha|-|\beta| - j} \quad (4.3)$$

if $(x, \xi) \in \mathcal{U}$, $v \in M$ and $0 < t \leq t_0$.

**Remark.** Without applying the Malgrange preparation theorem, we can also prove the lemma if only $p_m(x, \xi) \in C^{1+\delta}(T^*\mathbb{R}^n)$ and $0 < \delta < 1$.

**Proof.** Let $\chi(x, \xi) \in C_{0}^{\infty}(T^*\mathbb{R}^n)$ satisfy $\text{supp} \chi \subset \{|x| + |\xi| \leq 2h\}$ and $\chi(x, \xi) = 1$ for $|x| + |\xi| \leq h$, where $h > 0$, and define an almost analytic extension of $p_m(x, \xi)$ by

$$p_m(x + iy, \xi + i\eta) = \sum_{\alpha, \beta} (\alpha! \beta!)^{-1} (-i\eta)^{\alpha} (-y)^{\beta} p_m^{(\alpha)}(x, \xi) \times \chi(b_{|\alpha| + |\beta|, y, b_{|\alpha| + |\beta|} \eta}) \text{ for } (x, \xi) \in \mathcal{U} \text{ and } (y, \eta) \in \mathbb{R} \times \mathbb{R},$$

where $b_0 = 1$ and $\{b_j\} \subset \mathbb{R}$ is a rapidly increasing sequence. Then we have

$$|\partial_{\xi}^\alpha \partial_{\eta}^\beta D_x^y \{p_m((x, \xi) - itv) - p_m(x, \xi; v, t)\}| \leq c_0 |t|^{l + 1 - j} \quad (4.4)$$

if $t \in \mathbb{R}$, $|t| \leq t_0$, $(x, \xi) \in \mathcal{U}$, $j \leq l$ and $|\alpha| + |\beta| \leq l$. From Lemma 2.6 in [33] it follows that

$$|p_m((x, \xi) - itv)| \geq c_1 t^l \text{ for } (x, \xi) \in \mathcal{U}, v \in M \text{ and } 0 \leq t \leq t_0,$$

modifying $\mathcal{U}$ and $t_0$ if necessary, which proves (4.2). Applying the Malgrange preparation theorem, there are a neighborhood $\mathcal{U}'$ of $z^0$, $\delta > 0$, $e(z, v, t) \in C^{\infty}(\mathcal{U}' \times M \times [-\delta, \delta])$ and $a_{j}(z, v) \in C^{\infty}(\mathcal{U}' \times M) (1 \leq j \leq l)$ such that $a_{j}(z^0, v) = 0$ for $v \in M$ and $1 \leq j \leq l$, $p_m(z + tv) = e(z, v, t)$.
×g(z, v, t) and e(z, v, t) ≠ 0 for (z, v, t) ∈ \bar{U}_1 × M × [−Δ, Δ], and g(z, v, t) ≡ t^i + a_1(z, v) t^{i-1} + \cdots + a_i(z, v) ≠ 0 for (z, v, t) ∈ \bar{U}_1 × M × C and \text{Im} t < 0. In fact, by Mather's proof of the Malgrange preparation theorem we can obtain the above assertion without dividing M (see [30]). Applying Theorem 2 in [32] to g(z, v, t), we have

\[ |\partial g^{(a)}(x, \xi; v, t)/g(x, \xi; v, t)| ≤ C'(\alpha, \beta) |\text{Im} t|^{-|\alpha|−|\beta|−1} \]

if (x, \xi) ∈ \mathcal{U}_1, v ∈ M, t ∈ C and −1 ≤ \text{Im} t < 0. On the other hand, (4.4) yields

\[ |\partial g^{(a)}(x, \xi; v, t)| ≤ C_0 |t|^{1+1−j} \]

for t ∈ R, |t| ≤ t_0, (x, \xi) ∈ \mathcal{U}_1, j ≤ l and |\alpha| + |\beta| ≤ l, modifying \mathcal{U} if necessary, where e(x, \xi, v, t + ir) is an almost analytic extension of e(x, \xi, v, t) in t. This proves (4.3) for j ≤ l and |\alpha| + |\beta| ≤ l. It is obvious that (4.3) is valid for j + |\alpha| + |\beta| ≥ l. Q. E. D.

Corollary. Let M be a compact subset of \Gamma(p_{m0}, g). Then there are a conic neighborhood \mathcal{C} of z^0 and positive constants c and t_0 such that

\[ |p_m(x, \xi; v(\xi), t |\xi|)| ≥ c t^i |\xi|^n, \]

\[ |p^{(a)}_{m}(x, \xi)/p_m(x, \xi; v(\xi), t |\xi|)| ≤ C(\alpha, \beta) t^{-|\alpha|−|\beta|} |\xi|^{-|\alpha|} \]

if (x, \xi) ∈ \mathcal{C}, v ∈ M and 0 < t ≤ t_0, where v(\xi) = (v_x/|\xi|, v_t) for v = (v_x, v_t) ∈ R^n × R^n.

Proof. It is easy to see that

\[ p^{(a)}_{m0}(x, \xi/|\xi|) = \sum_{j=0}^{l} (-t)^j \partial p^{(a)}_{m}(x, \xi/|\xi|; v, t)/j!. \]

On the other hand,

\[ p_m(x, \xi/|\xi|; v, t) = p_m(x, \xi; v(\xi), t |\xi|)|\xi|^{-n}. \]

Therefore, we have

\[ |p^{(a)}_{m}(x, \xi)/p_m(x, \xi; v(\xi), t |\xi|)| ≤ |\xi|^{-|\alpha|} \sum_{j=0}^{l} t^j |\partial p^{(a)}_{m}(x, \xi/|\xi|; v, t)|/j! \]

\[ ≤ C'(\alpha, \beta) t^{-|\alpha|−|\beta|} |\xi|^{-|\alpha|} \]

if (x, \xi/|\xi|) ∈ \mathcal{C}, v ∈ M and 0 < t ≤ t_0. This completes the proof. Q. E. D.

Now assume that 1 < \kappa ≤ \kappa_0 ≡ \min \{2, \mu(z^0)/(\mu(z^0) − 1)\} if * = (\kappa) and 1 < \kappa ≤ \kappa_0 if * = \{\kappa\}. Let \varphi(x, \xi) ∈ C^1(\Gamma^* R^n \backslash 0) be a real-valued
positively homogeneous function of degree 0 in $\xi$ such that $\varphi(\xi) = 0$
and $-H_\varphi(\xi) \equiv -\sum_{i=1}^n \{(\partial \varphi / \partial \xi_i)(\xi) (\partial / \partial x_i) - (\partial \varphi / \partial x_i)(\xi) (\partial / \partial \xi_i)\} \in \Gamma(p_{m^0}, \theta)$. Choose a compact subset $M$ of $\Gamma(p_{m^0}, \theta)$ so that $\theta \in M$
and $-H_\varphi(\xi) \in M$. Then there is a neighborhood $\mathcal{U}$ of $\xi$ such that
$M \subset \Gamma(p_{m^0}, \theta)$ for $\xi \in \mathcal{U}$. For given $f \in \mathcal{D}^*$ with $WF_+(f) \cap \{\varphi(x, \xi) < 0\} \cap \mathcal{U} = \emptyset$, we shall consider the microlocal Cauchy problem at $\xi$

$$(MCP) \quad \{p(x, D)u = f, \quad WF_+(u) \cap \{\varphi(x, \xi) < 0\} \cap \mathcal{U} = \emptyset,$$
where $u \in \mathcal{D}^*$.

**Lemma 4.3.** Let $M_1$ be a compact convex subset of $\Gamma(p_{m^0}, \theta)$ such
that $M \subset M_1 \subset \Gamma(p_{m^0}, \theta)$ for $\xi \in \mathcal{U}$. Then there are symbols $A_h(x, \xi)$
($h \geq 1$), a convex conic neighborhood $\mathcal{C}$ of $\xi$, and positive constants $\varepsilon$, $C_0$,
$A_0$, $\alpha_1$, and $\alpha_2$ such that

$$|A_h(x, \xi)| \leq \varepsilon_h^{\frac{\alpha}{\alpha_1}} (x, \xi) \in T^* \mathbb{R}^n,$$

$$|A_h(\xi, \xi)| \leq C_0 A_0^{\alpha + |\beta|} (x, \xi) |\xi - \xi^0|^{\alpha - |\beta|},$$

$$|A_h(\xi, \xi)| \leq C_0 A_0^{\alpha + |\beta|} (x, \xi) |\xi - \xi^0|^{\alpha - |\beta|},$$

$$|A_h(x, \xi)| \leq \varepsilon_h^{\frac{\alpha}{\alpha_1}} (x, \xi) \in \mathcal{C} \text{ and } |\xi| \geq h,$$

$$|A_h(x, \xi)| \leq \varepsilon_h^{\frac{\alpha}{\alpha_1}} (x, \xi) \in \mathcal{C} \text{ and } |\xi| \geq h,$$

$$\langle \xi \rangle^{\alpha - |\beta|} \sum_{i=1}^n \{(\partial \varphi / \partial x_i)(x, \xi) (\partial / \partial x_i)$$
$$- (\partial \varphi / \partial x_i)(x, \xi) (\partial / \partial \xi_i)\} + \sum_{i=1}^n \{y_i (\partial / \partial x_i) + \eta_i (\partial / \partial \xi_i)\} \in M_1$$

for $(x, \xi) \in \mathcal{C}$, $|\xi| \geq h$ and $|y|^2 + |\eta|^2 \leq \varepsilon^2$.

**Proof.** Put

$$\varphi_1(x, \xi) = (x - x_0) \cdot \mathbb{P}_x \varphi(x_0) + \xi \cdot \mathbb{P}_\xi \varphi(x_0)$$
$$+ B_\varphi(|x - x_0|^2 + |\xi - \xi_0|^2),$$

$$\varphi_2(x, \xi) = \varphi_1(x, \xi) (1 + \varphi_1(x, \xi) \xi^0)^{-1/2},$$

and choose $B_\varphi \in \mathbb{R}$ so that $\varphi(x, \xi) \leq \varphi_1(x, \xi) - 2(|x - x_0|^2 + |\xi - \xi_0|^2)$
for $|x - x_0|^2 + |\xi - \xi_0|^2 \leq 1$ and $|\xi| = 1$. If we set $A_h(x, \xi) = -\varphi_2(x, \xi / \langle \xi \rangle^\theta) \langle \xi \rangle^\theta$ and $\theta (0 < \theta \leq 1)$ is chosen appropriately, we can show
that $A_h(x, \xi)$ satisfies (4.5)-(4.9). It is obvious that $A_h(x, \xi)$ satisfies (4.5). Noting that $\varphi_1(x, \eta)$ is a polynomial of $(x, \eta)$, there
are $L > 0$ and $(1 \gg \varepsilon) > 0$ such that $\text{Re } \varphi_1(x, \eta)^2 \geq 0$ if $(x, \eta) \in C^* \times$
$C^\alpha. \quad |\text{Re } x - x_0| \geq L, \quad |\text{Im } x| \leq \varepsilon |\text{Re } x - x_0| \quad \text{and} \quad |\eta| \leq 3$. Then Cauchy's estimates yield

$$|\varphi_{(\alpha)}^{(\beta)}(x, \eta) | \leq C_\alpha A_\beta^{1+|\beta|}(|\alpha| + |\beta|)!$$

for $(x, \eta) \in T^*\mathbb{R}^n$ with $1/2 \leq |\eta| \leq 2$. From Lemma 2.2 (4.6) easily follows, where $A_0$ depends on $\theta$. (4.7) is trivial. It is easy to see that

$$\varphi_2(x, \xi/\langle \xi \rangle_\theta) - \varphi(x, \xi) \geq \varphi_1(x, \xi/|\xi|) - \varphi_1(x, \xi/|\xi|) - \varphi_2(x, \xi/\langle \xi \rangle_\theta) - \varphi_2(x, \xi/\langle \xi \rangle_\theta),$$

$$|\varphi_1(x, \xi/|\xi|) - \varphi_1(x, \xi/\langle \xi \rangle_\theta) | \leq h^2 |\varphi(\xi_\theta) | \langle \xi \rangle_\theta^{-1} |\xi|^{-1/2}$$

for $\xi \neq 0$, and that there are $\delta_\alpha > 0$ and $\theta_\alpha > 0$ such that $\theta_\alpha \leq 1$ and

$$|\varphi_1(x, \xi/\langle \xi \rangle_\theta) - \varphi_2(x, \xi/\langle \xi \rangle_\theta) | \leq |\varphi_1(x, \xi/\langle \xi \rangle_\theta) |^3/2$$

$$\leq |x - x_0|^2 + |\xi/|\xi| - \xi_0|^2 + c_\alpha h^2 |\xi|^{-2}$$

if $0 < \theta \leq \theta_\alpha, \quad |x - x_0|^2 + |\xi/|\xi| - \xi_0|^2 \leq \delta_\alpha$ and $|\xi| \geq h$, where $c_\alpha > 0$. Here we have used the inequality that $\varphi_1(x, \xi/\langle \xi \rangle_\theta) - \varphi_1(x, \xi/\langle \xi \rangle_\theta) | \leq \theta h^2 |\xi|^{-1/2}$. This proves (4.8), taking $\theta \leq \theta_\alpha$. A simple calculation gives

$$\varphi_1^{(\alpha)}(x, \xi) = \varphi_1^{(\alpha)}(x, \xi) (1 + \varphi_1(x, \xi/\xi)^2)^{-3/2} \quad \text{for } |\alpha| + |\beta| = 1,$$

$$\varphi_1^{(\alpha)}(x, \xi) = \varphi_1^{(\alpha)}(\xi_\theta) + 2B_\xi (|\beta|/|x - x_0|) / i + |\alpha| (\xi - \xi_0)^n$$

for $|\alpha| + |\beta| = 1,$

$$A_h^{(e_\xi)}(x, \xi) = -\sum_{k=1}^{1} \varphi_2^{(e_\xi)}(x, \xi/\langle \xi \rangle_\theta) \langle \delta_\alpha \xi/\langle \xi \rangle_\theta \rangle \langle \xi_\theta \rangle_\xi \langle \xi \rangle_\xi^{-1} - \xi_\xi \langle \xi \rangle_\xi^{-1} \langle \xi \rangle_\xi^{-1}$$

$$\varphi_2(x, \xi/\langle \xi \rangle_\theta) \xi_\xi \langle \xi \rangle_\xi^{-2} / \kappa,$$

$$A_h^{(e_\xi)}(x, \xi) = -\varphi_2^{(e_\xi)}(x, \xi/\langle \xi \rangle_\theta) \xi_\xi \langle \xi \rangle_\xi^{-1}$$

where $e_\xi = (0, \ldots, 1, \ldots, 0) \in (\mathcal{N} \cup \{0\})^n$. Moreover, with $C_\alpha > 0$, we have

$$|\varphi_1(x, \xi/\langle \xi \rangle_\theta) | \leq C_\alpha (|x - x_0|^2 + |\xi/\langle \xi \rangle_\theta - \xi_0|^2)^{1/2}$$

$$|\sum_{k=1}^{1} \varphi_2^{(e_\xi)}(x, \xi/\langle \xi \rangle_\theta) \xi_\xi \langle \xi \rangle_\xi^{-1} | \leq (|\varphi(\xi_\theta) | + 2|B_\xi|)$$

$$\times \langle \xi/\langle \xi \rangle_\theta - \xi_0 \rangle,$$

since $\xi_\theta \cdot \varphi(\xi_\theta) = 0$. Therefore, we have

$$|\langle \xi \rangle_\xi^{-1/2} \xi | \langle \xi \rangle_\xi^{1/2} (x, \xi) + \varphi^{(e_\xi)}(\xi_\theta) | + |\langle \xi \rangle_\theta^{-1/2} A_h^{(e_\xi)}(x, \xi) + \varphi^{(e_\xi)}(\xi_\theta) |$$

$$\leq C_\alpha \{(|x - x_0|^2 + |\xi/|\xi| - \xi_0|^2)^{1/2} + \theta h^2 |\xi|^{-2}\},$$
if \( \theta \leq 1/2 \), \(|x - x^0|^2 + |\xi|/|\xi| - \xi^0|^2 \leq 1/2 \) and \(|\xi| \geq h\). Taking \( \psi \) and \( \theta \) sufficiently small, \( A_h(x, \xi) \) satisfies (4.9).

**Q.E.D.**

**Lemma 4.4.** Modifying a conic neighborhood \( \mathcal{C} \) of \( \xi^0 \), there are symbols \( W_h(x, \xi) (h \geq 1) \) and positive constants \( C'_0 \) and \( A'_0 \) such that

\[
W_h(x, \xi) \leq 2<\xi>^{1\alpha},
\]

\[
|W_{k\beta}^{(\alpha)}(x, \xi)| \leq C'_0 A_0^{1|\alpha| + |\beta|} (|\alpha| + |\beta|) |<\xi>^{1\alpha - |\alpha|},
\]

\[
<\xi>^{2/\alpha} \{ |\xi|^2 |\partial_x W_h(x, \xi)|^2 + |\partial_\xi W_h(x, \xi)|^2 \} \leq \varepsilon^2
\]

for \((x, \xi) \in \mathcal{C}\) with \(|\xi| \geq h\),

\[
\varepsilon<\xi>^{1\alpha}/2 + |A_h(x, \xi)| < W_h(x, \xi) + c_1 h^{1\alpha}
\]

for \(|\xi| \geq h\), where \( \varepsilon_1 (<2) \) and \( c_1 \) are the constants in Lemma 4.3.

**Proof.** There is a conic neighborhood \( \mathcal{C} \) of \( \xi^0 \) such that \( \mathcal{C} \subset \mathcal{C}' \) and

\[
A_h(x, \xi) \leq \varepsilon_2<\xi>^{1\alpha}/\theta + c_1 h^{1\alpha}
\]

for \((x, \xi) \in \mathcal{C}'\) with \(|\xi| \geq h\), modifying \( \mathcal{C} \). Choose \( w(x, \xi) \in C^3(T^*\mathbb{R}^*) \) so that the first and second order derivatives of \( w \) are bounded, and \( 3\varepsilon_2/4 \leq w(x, \xi) \leq 2 \), \( w(x, \xi) = 3\varepsilon_2/4 \) for \((x, \xi) \in \mathcal{C}'\) with \(|\xi| \geq 1/2 \) and \( w(x, \xi) = 2 \) for \((x, \xi) \in \mathcal{C}'\) with \(|\xi| \geq 1/2 \). Put \( w_j(x, \xi) = E_j w(x, \xi) \) and \( W_h(x, \xi) = w_j(x, \xi/<\xi>_\alpha) \times <\xi>^{1\alpha} \), where \( E_j(x, \xi) = j^*(4\pi)^{-1} \exp[-j(|x|^2 + |\xi|^2)/4] \). If \( j \) is sufficiently large, then \( W_h(x, \xi) \) satisfies the conditions in the lemma.

**Q.E.D.**

We may assume that for any \( A > 0 \) there is \( C \equiv C_A > 0 \) (resp. there are \( A > 0 \) and \( C > 0 \)) such that

\[
|\rho^0_{(\alpha)}(x, \xi)| \leq CA^{1|\alpha| + |\beta|} (|\alpha| + |\beta|) |<\xi>^{1\alpha - |\alpha|}
\]

if \(* = (\alpha)\) (resp. if \(* = [\alpha]\)). From (4.1) it follows that

\[
(4.11) \quad \text{Re}\{\left(\frac{\varepsilon_0}{|c_0|}\right) p_m(x, \xi ; \theta, t)\} \geq c_0 |t|^{1/2}
\]

for \((x, \xi) \in \mathcal{C}'\) and \( t_0/3 \leq t \leq t_0 \), where \( c_0 = p_m(x^0, \xi^0; \theta, t) \times t^{-1} \), modifying \( \mathcal{C} \) if necessary. Let \( \mathcal{C} \) and \( \mathcal{C}_j (j = 1, 2) \) be conic neighborhoods of \( \xi^0 \) such that \( \mathcal{C} \cap \{|\xi| = 1\} \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}' \) and Lemmas 4.3 and 4.4 hold for \( \mathcal{C} \). Let \( \chi_j(x, \xi) \in \mathcal{E}(\mathbb{R}^*) (T^*\mathbb{R}^*)^0 (j = 1, 2) \) be positively homogeneous functions of degree 0 such that \( 0 \leq \chi_j(x, \xi) \)
\[ \chi_i(x, \xi) = 1 \text{ on } \mathcal{C}_i, \supp \chi_i \subseteq \mathcal{C}_2 \text{ and } \supp \chi_2 \subseteq \mathcal{C}, \text{ and let } \sigma(\xi) \in \mathcal{E}^{(c)} \text{ be a function such that } 0 \leq \sigma(\xi) \leq 1, \sigma(\xi) = 1 \text{ for } |\xi| \geq 1 \text{ and } \sigma(\xi) = 0 \text{ for } |\xi| \leq 1/2. \text{ We set for } h \geq 1
\]
\[ \tilde{p}_{m, h}(x, \xi) = \sigma_h(\xi) \chi_2(x, \xi) p_m(x, \xi) \theta(\xi), \quad t_0 \{(1 - \sigma_{2h}(\xi)) h \\
+ (1 - \chi_1(x, \xi)) |\xi|/3 + c_0(t_0/3)^i |\xi|^{m-1} \sigma_h(\xi) (1 - \chi_2(x, \xi))/2 \\
+ c_0(t_0/3)^i h_m (1 - \sigma_h(\xi)), \}
\]
\[ \tilde{p}_h(x, \xi) = \tilde{p}_{m, h}(x, \xi) + \sigma_h(\xi) (p(x, \xi) - p_m(x, \xi)), \]
where \( \sigma_h(\xi) = \sigma(\xi/h), \theta = (\theta_1, \theta_2) \) and \( \theta(\xi) = (\theta_1/|\xi|, \theta_2) \). Note that \( \tilde{p}_{m, h}(x, \xi) = p_m(x, \xi) \) and \( \tilde{p}_h(x, \xi) = p(x, \xi) \) for \( (x, \xi) \in \mathcal{C}_1 \) with \( |\xi| \geq 2h \), and that \( \tilde{p}_h(x, \xi) = \tilde{p}_{h'}(x, \xi) \) if \( h' \geq h \) and \( |\xi| \geq 2h' \).

**Lemma 4.5.** There are positive constants \( C' \equiv C'_{x_1, x_2, r, A}(C) \) and \( d_0 \) and \( h_0 \geq 1 \) such that

\[
\begin{align*}
&|\tilde{p}_h(\xi, \xi)| \leq C' (2^h A)^{|\alpha| + |\beta|} (|\alpha| + |\beta|)! |\xi|^{n-|\alpha|}, \\
&|\tilde{p}_h(\xi, \xi)| \geq d_0 |\xi|^{n_0} \quad \text{if } (x, \xi) \in \mathcal{C}_2 \text{ or } |\xi| \leq h \text{ and if } h \geq h_0.
\end{align*}
\]

**Proof.** Since \( \langle \xi \rangle \leq \langle \xi \rangle \leq \sqrt{5} \langle \xi \rangle \) for \( h \geq 1 \) and \( |\xi| \geq h/2 \), (4.12) easily follows. If \( |\xi| \leq h/2 \), then \( \tilde{p}_{m, h}(x, \xi) = c_0(t_0/3)^i h_m. \) If \( h/2 \leq |\xi| \leq h \), then \( t_0/3 \leq t_0 \{h/|\xi| + (1 - \chi_1(x, \xi)) \}/3 \leq t_0. \) Therefore, (4.11) gives

\[ \Re \{ \langle \xi \rangle |c_0| \tilde{p}_{m, h}(x, \xi) \} \geq |c_0| \{(t_0/3)^i \{(h + (1 - \chi_1(x, \xi)) |\xi|) \\
\times |\xi|^{m-1} \sigma_h(\xi) \chi_2(x, \xi) \}/2 + |\xi|^{m-1} \sigma_h(\xi) (1 - \chi_2(x, \xi))/2 \\
+ h_m (1 - \sigma_h(\xi)) \} \geq 5^{m-1/2} |c_0| (t_0/3)^i \langle \xi \rangle^{n_0}/2 \text{ if } h \leq |\xi| \leq h, \]
where \( m_+ = \max(m, 0) \). Since \( t_0/3 \leq t_0 \{(1 - \sigma_{2h}(\xi)) h/|\xi| + 1 \}/3 \leq t_0 \) for \( |\xi| \geq h \), it follows from (4.11) that

\[ \Re \{ \langle \xi \rangle |c_0| \tilde{p}_{m, h}(x, \xi) \} \geq |c_0| (t_0/3)^i \\\n\times \{(1 - \sigma_{2h}(\xi)) (h + |\xi|)^i |\xi|^{m-1} \chi_2(x, \xi) + |\xi|^{m-1} (1 - \chi_2(x, \xi)) \}/2 \\
\geq 2^{1-m-1/2} |c_0| (t_0/3)^i \langle \xi \rangle^{n_0} \text{ if } |\xi| \geq h \text{ and } (x, \xi) \in \mathcal{C}_2. \]

So there is \( d_0 > 0 \) such that

\[ |\tilde{p}_{m, h}(x, \xi)| \geq 2d_0 \langle \xi \rangle^{n_0} \quad \text{if } (x, \xi) \in \mathcal{C}_2 \text{ or } |\xi| \leq h. \]

This proves (4.13). Q.E.D.

Applying Proposition 2.13, we can write
\[
\exp[aM_h^k] (x, D) \tilde{p}_h (x, D) e^{\exp \left[ -aM_h^k \right]} (x, D) = \tilde{p}_h^{a,b} (x, D) + r_h^{a,b} (x, D)
\]
if \(0 \leq a \leq d_0 A^{-1/\kappa}\) and \(-1 \leq b \leq 1\), where \(M_h (x, \xi) = \Lambda_h (x, \xi) + bW_h (x, \xi), d_0 > 0, \xi > 0\) and
\[
|\tilde{p}_h^{a,b} (x, \xi)| \leq C_A (C) (2^{2+2k} A) |\alpha| + |\beta| \leq \epsilon
\]
\[
\times \langle \xi \rangle^{\gamma \left( 1-1/k \right) (1+1)-|\alpha|},
\]
\[
\tilde{p}_h^{a,b} (x, \xi) = \tilde{p}_h^{a,b} (x, \xi) - \sum_{|\alpha| + |\beta| \leq 1} ( \alpha ! \beta !)^{-1}
\]
\[
\times \{ \tilde{p}_h^{a,b} (x, \xi) \omega^a (aM_h^k; x, \xi) \omega_a (-aM_h^k; x, \xi) \} ^{(a)},
\]
\[
|r_h^{a,b} (x, \xi)| \leq C_A (|\alpha|, C) (2^{2+4k} A) |\beta| \leq \epsilon ! \exp \left[ -\epsilon A^{-1/\kappa} \langle \xi \rangle^{1/\kappa} \right].
\]

**Lemma 4.6.** For a fixed \(N_0\) there are positive constants \(a_\rho, h_\rho (a), c, c_1\) such that \(\tilde{p}_h^{a,b} (x, \xi)\) satisfies the condition \((H; C_A (C), c_A, c_1)\) with \(m' = m - (1-1/k)l, \delta = 1-1/\kappa\) and \(\rho = 1/\kappa\) defined in §3 if \(0 < a \leq d_0 A^{-1/\kappa}\), \(-1 \leq b \leq 1\) and \(h \geq h_\rho (a)\), and if \(a \geq a_\rho\) when \(\kappa = \kappa_0\). Here \(a_\rho, h_\rho (a)\) and \(c_1\) do not depend on the choice of \(A\) when \(*1 = \kappa_0\).

**Proof.** When \((x, \xi) \in \mathcal{C}_2\) or \(|\xi| \leq h\), by Lemma 4.5 we can apply Lemma 3.1 with \(\delta = (1-1/k)/2\) and \(\rho = \delta + 1/\kappa\). So it is sufficient to estimate \(\tilde{p}_h^{a,b} (x, \xi)\) for \((x, \xi) \in \mathcal{C}_2\) with \(|\xi| \geq h\). Let \((x, \xi) \in \mathcal{C}_2\) and \(|\xi| \geq h\). Then we have
\[
\tilde{p}_h^{a,b} (x, \xi) = p_m (x, \xi) + s_h^{a,b} (x, \xi) + s_h^{a,b} (x, \xi),
\]
\[
\tilde{p}_m (x, \xi) = p_m (x, \xi + \theta (\xi), d_1 (x, \xi)),
\]
where
\[
p_m^{a,b} (x, \xi) = \sum_{|\alpha| + |\beta| \leq 1} (\alpha ! \beta !)^{-1} \tilde{p}_h^{a,b} (x, \xi)
\]
\[
\times (iaF_x A_h^a (x, \xi)) \omega^a (aM_h^k; x, \xi) \omega_a (-aM_h^k; x, \xi) \},
\]
\[
s_h^{a,b} (x, \xi) = \sum_{|\alpha| + |\beta| \leq 1} (\alpha ! \beta !)^{-1}
\]
\[
\times \{ \tilde{p}_h^{a,b} (x, \xi) \omega^a (aM_h^k; x, \xi) \omega_a (-aM_h^k; x, \xi) \} ^{(a)}
\]
\[
- \tilde{p}_m^{a,b} (x, \xi),
\]
\[
s_h^{a,b} (x, \xi) = \tilde{p}_h^{a,b} (x, \xi) + \sum_{|\alpha| + |\beta| \leq 1} (\alpha ! \beta !)^{-1} \{ (p_h^{a,b} (x, \xi)
\]
\[
- \tilde{p}_m^{a,b} (x, \xi) \} \omega^a (aM_h^k; x, \xi) \omega_a (-aM_h^k; x, \xi) \} ^{(a)},
\]
\[
d_1 (x, \xi) = t_0 \left( (1-\sigma_2h(\xi) - (1-\chi_2 (x, \xi)) |\xi| \right) / 3.
\]
Put
\[
\tilde{p}_m^{a,b} (x, \xi) = p_m (x - X (x, \xi), \xi - \xi (x, \xi), u_\epsilon (x, \xi), v_\epsilon (x, \xi), d (x, \xi)),
\]
\[
X (x, \xi) = a \{ A_h^a (x, \xi), d_1 (x, \xi) / |\xi| \} \theta_x,
\]
\[ \mathcal{E}(x, \xi) = a \{ A_h(x, \xi), \delta_k(x, \xi) \} \theta \],
\[ v_x(x, \xi) = d(x, \xi)^{-1} \{ d_t(x, \xi) \theta \} \]
\[ v_{\xi l}(x, \xi) = d(x, \xi)^{-1} \{ d_t(x, \xi) \theta \} - aF_{\xi} A_h(x, \xi) \],
\[ d(x, \xi) = d_t(x, \xi) + a\langle \xi \rangle \hat{v}, \]

where \( \{ f, g \} = \sum_{s=1}^{n} \{(\partial f / \partial \xi_s)(\partial g / \partial x_s) - (\partial f / \partial x_s)(\partial g / \partial \xi_s)\} \). Note that there is \( h(a) > 0 \) such that \( (x - X(x, \xi), \xi - \mathcal{E}(x, \xi)) \in \mathcal{W} \) if \( h \geq h(a) \) and \(-1 \leq b \leq 1\), \( (x, \xi) \in \mathcal{W}_2 \) and \( |\xi| \geq h \). From Lemmas 4.3 and 4.4 it follows that \( (|\xi| v_x(x, \xi), v_{\xi l}(x, \xi)) \in M_1 \). Therefore, by Corollary of Lemma 4.2 we have

\[ \begin{align*}
|\vec{p}_{\alpha,b}(x, \xi)| & \geq c d(x, \xi)^{l} |\xi|^{m-l}, \\
|\vec{p}_{\mu,(\beta)}(x, \xi)| & \leq C d(x, \xi)^{l} |\alpha|^{-|\beta|} |\xi|^{|\beta|} \\
& \leq C |\alpha|^{-|\beta|} |\xi|^{|\beta|} if h \geq h_p(a) and |\alpha| + |\beta| \leq l,
\end{align*} \]

if \( h \geq h_p(a) \), modifying \( h_p(a) \). A simple calculation gives

\[ \begin{align*}
|\vec{p}_{\alpha,b}(x, \xi)| & \leq C |\alpha|^{-|\beta|} |\xi|^{|\beta|} if h \geq h_p(a) and |\alpha| + |\beta| \leq l,
\end{align*} \]

where \( h_p(a) > 0 \). It is easy to see that

\[ \begin{align*}
|\vec{p}_{\mu,(\beta)}(x, \xi)| & \leq C |\alpha|^{-|\beta|} |\xi|^{|\beta|} if h \geq h_p(a) and |\alpha| + |\beta| \leq l,
\end{align*} \]

where \( N = |\alpha| + |\beta| \leq l \). Then we have

\[ \begin{align*}
|\Sigma_1|/|\vec{p}_{\mu,(\beta)}(x, \xi)| & \leq C_{\rho}(|\alpha|^{-|\beta|} |\xi|^{|\beta|} if h \geq h_p(a) and |\alpha| + |\beta| \leq l,
\end{align*} \]

where \( |\alpha| + |\beta| \leq l \). Then we have

\[ \begin{align*}
|\Sigma_1|/|\vec{p}_{\mu,(\beta)}(x, \xi)| & \leq C_{\rho}(|\alpha|^{-|\beta|} |\xi|^{|\beta|} if h \geq h_p(a) and |\alpha| + |\beta| \leq l,
\end{align*} \]
modifying $h_p(a)$. Therefore, we have
\[ p_{n,h}(x, \xi) = \sum_{l} a_x^{l+1} a_x^{l+1} \beta_l \leq (\alpha \beta_l)^{-1} \] 
\[ \times p_{m,h}(x, \xi) \left( -iX(x, \xi) \right)^{a} \left( -iX(x, \xi) \right)^{a} \] 
\[ \times (d(x, \xi) v_{l}(x, \xi)) \] 
\[ \times \frac{1}{\beta} \sum_{l} \tilde{p}_{n,h}(x, \xi) + \sum_{l} + \sum_{l}, \] 
where \[ |\sum_{l} \tilde{p}_{n,h}(x, \xi) | \leq a C_{p}^{*} \langle \xi \rangle^{\kappa} \] 
if $h \geq h_p(a)$ and
\[ |\sum_{l} \tilde{p}_{n,h}(x, \xi) | \leq a C_{p}^{*} \langle \xi \rangle^{\kappa} \] 
if $h \geq h_p(a)$. The same argument as in (4.15) yields
\[ |s_{n,h}^{\alpha}(x, \xi) / \tilde{p}_{n,h}(x, \xi) | \leq C_{p} a^{-1} |\beta| |\xi|^{1/\kappa - |a|/\kappa} \] 
if $h \geq h_p(a)$ and $|\alpha| + |\beta| \leq l$. This implies that
\[ |s_{n,h}^{\alpha}(x, \xi) / \tilde{p}_{n,h}(x, \xi) | \leq 1/6 \] 
if $h \geq h_p(a)$, since
\[ |\omega^{\alpha}(-a M_{h}^{\alpha}; x, \xi) \rangle \langle \xi | \leq C(\alpha, \bar{\alpha}, \bar{\beta}, a) \] 
\[ \times \langle \xi \rangle^{1/\kappa - |a|/\kappa}, \] 
\[ |\omega^{\beta}(-a M_{h}^{\beta}; x, \xi) \rangle \langle \xi | \leq C(\beta, \bar{\alpha}, \bar{\beta}, a) \] 
\[ \times \langle \xi \rangle^{1/\kappa - |\beta|/\kappa}, \] 
\[ |\omega^{\beta}(a M_{h}^{\beta}; x, \xi) \rangle \omega^{\alpha}(-a M_{h}^{\alpha}; x, \xi) \rangle \langle \xi | \leq C(\alpha, \beta, \bar{\alpha}, \bar{\beta}, a) \] 
\[ \times \langle \xi \rangle^{1/\kappa - (1/\kappa) - |a|/\kappa}, \] 
\[ |\omega^{\alpha}(a M_{h}^{\alpha}; x, \xi) \rangle \langle \xi | \leq C(\alpha, \beta, \bar{\alpha}, \bar{\beta}, a) \] 
\[ \times \langle \xi \rangle^{1/\kappa - (1/\kappa) - |a|/\kappa}. \] 
It is obvious that
\[ |s_{n,h}^{\alpha}(x, \xi) / \tilde{p}_{n,h}(x, \xi) | \leq 1/6 \] 
if $h \geq h_p(a)$ and $\kappa < \kappa_{0}$. When $\kappa = \kappa_{0}$, we have also
\[ |s_{n,h}^{\alpha}(x, \xi) / \tilde{p}_{n,h}(x, \xi) | \leq 1/6 \] 
if $h \geq h_p(a)$ and $a \geq a_{p}$,
where \( a_p > 0 \). Thus we have
\[
|\tilde{P}_h^{a,b}(x, \xi)| \geq |\tilde{P}_{m,h}^{a,b}(x, \xi)| / 2 \geq c'a'\langle \xi \rangle_h^{m-(1-1/\kappa)}
\]
if \( h \geq h_p(a) \), and if \( a \geq a_p \) when \( \kappa = \kappa_0 \), where \( c' > 0 \). Similarly, we have
\[
|\tilde{P}_h^{a,b}(x, \xi)/\tilde{P}_{m,h}^{a,b}(x, \xi)|
\leq C_h(\alpha, \beta) a^{-|\alpha|-|\beta|} \langle \xi \rangle_h^{(1-1/\kappa)|\beta|+|\alpha|/\kappa}
\]
if \( h \geq h_p(a) \), and if \( a \geq a_p \) when \( \kappa = \kappa_0 \). Here we have modified \( h_p(a) \), if necessary. This proves the lemma. Q. E. D.

From Proposition 3.4 (or its proof) and Lemma 4.6 it follows that \( \tilde{P}_h^{a,b}(x, D) + r_h^{a,b}(x, D) \) has the inverse \( \tilde{Q}_h^{a,b} \), i.e., \( \tilde{Q}_h^{a,b} \) maps continuously \( L^2 \) to \( H^m(\mathbb{R}^n) \) and \( H^{(1-1/\kappa)} \) to \( H^m \), and satisfies
\[
\tilde{Q}_h^{a,b}(\tilde{P}_h^{a,b}(x, D) + r_h^{a,b}(x, D)) = I \quad \text{on} \quad H^m \quad \text{and} \quad (\tilde{P}_h^{a,b}(x, D) + r_h^{a,b}(x, D)) \tilde{Q}_h^{a,b}
= I \quad \text{on} \quad H^{(1-1/\kappa)} \quad \text{if} \quad a, b \quad \text{and} \quad h \quad \text{satisfy the following conditions;}
\]
(4.16) \( 0 < a \leq a_0 A^{-1/\kappa}, -1 \leq b \leq 1 \quad \text{and} \quad h \geq h_{p, a}, \quad \text{and} \quad a \geq a'_p \quad \text{when} \quad \kappa = \kappa_0, \quad \text{where} \quad h_{p, a} \quad \text{and} \quad a'_p \quad \text{are positive constants}. \quad \text{By Lemma 2.14 and}\)
Proposition 3.4, for any \( \varepsilon > 0 \) there is \( h_\varepsilon(\varepsilon) > 0 \) such that \( 1 + q_h^{a,b}(x, D) \equiv \exp [-a_h^{a,b}(x, D) \exp [a_h^{a,b}(x, D)] \) has the inverse \( (1 + q_h^{a,b}(x, D))^{-1} \) which maps continuously \( L^2_\varepsilon \) to \( L^2_\varepsilon \) if \( |\varepsilon'| \leq \varepsilon, h \geq h_\varepsilon(\varepsilon), a \geq 0 \quad \text{and} \quad -1 \leq b \leq 1 \). Let us introduce the following spaces.

**Definition 4.7.** Let \( \Lambda(x, \xi) \) satisfy (2.13). For \( s \in \mathbb{R} \) we define \( H^s_\Lambda = \{ f \in H^1 : (e^a(x, D) f \in H^s \} \), and write \( L^2_\Lambda = H^0_\Lambda, H^s_\Lambda = H^s_\Lambda, b \) and \( L^2_\Lambda = H^0_\Lambda, b \).

**Lemma 4.8.** (i) \( f \in H^s_\Lambda \) if and only if \( R(e^{-a}) (x, D) f \in H^s_\Lambda \). (ii) If \( f \in H^s_\Lambda \) and \( k(x, \xi) \) satisfies
\[
|k^{(a)}(x, \xi)| \leq C_\Lambda A^{|a|+|\beta|} (|\alpha| + |\beta|) \langle \xi \rangle_h^{m-|a|},
\]
then \( k(x, D) f \in H^s_\Lambda \) for \( |a| \leq c_\Lambda A_\Lambda^{-1/\kappa} \), where \( c_\Lambda > 0 \).

**Proof.** By Lemma 2.14 we can write
\[
R(e^{-a})(x, D) (e^a(x, D) f = 1 + q_h^a(x, D) + r_h^a(x, D),
\]
\[
(e^a(x, D) R(e^{-a})(x, D) f = 1 + q_h^a(x, D) + r_h^a(x, D),
\]
where \( h' \geq 1 \) and
\[
|q_h^{(a)}(x, \xi)| \leq C_\Lambda A_\Lambda d^{|a|+|\beta|} (|\alpha| + |\beta|) \langle \xi \rangle_h^{m-1-|a|} \quad \text{for any} \quad d > 0,
\]
Lemma 4.9. Let $\Lambda(x, \xi)$ and $\Lambda'(x, \xi)$ satisfy (2.13), where $h \geq 1$, and assume that a symbol $q(x, \xi)$ satisfies

$$|q^{(a)}_{\beta}(x, \xi)| \leq C_d d^{a|+\beta|} |\alpha|!^{f_{\beta}}!^f(\xi)^{-|\alpha|}$$

for any $d > 0$. If $\inf_{L > 0} \sup_{(x, \xi) \in \text{supp } q(x, \xi)} (\Lambda(x, \xi) - \Lambda'(x, \xi)) < \xi^{-1/k} < \epsilon$, then $q(x, D) (e^{Ax}) (x, D) R (e^{-Ax}) (x, D)f$ and $(e^{Ax}) (x, D) R (e^{-Ax}) (x, D)f$ belong to $L^2_{k, a, -\epsilon}$ for $f \in L^2_{k, a}$.

Proof. By Proposition 2.8 we can write $q(x, D) (e^{Ax}) (x, D) q(x, D) + f(x, D)$, where $q(x, \xi) = \sum_{n=0}^{\infty} \sum_{|a| = n} d^{a|+\beta|} |\alpha|!^{f_{\beta}}!^f(\xi)^{-|\alpha|} \times (e^{Ax}) (x, D)f$ and $f(x, D) : L^2_{k, a} \rightarrow L^2_{k, a'}$ continuously for any $a, a' \in \mathbb{R}$. It follows from Corollary of Lemma 2.9 that

$$|q^{(a)}_{\beta}(x, \xi)| \leq C_{d, a, \beta} d^{a|+\beta|} |\alpha|!^{f_{\beta}}!^f(\xi)^{-|\alpha|}$$

and that $\text{supp } q(x, \xi) \subset \text{supp } q(x, \xi)$. Since $\Lambda(x, \xi)$ and $\Lambda'(x, \xi)$ satisfy (2.13), there are $c_1 > 0$ and $c_2 > 0$ such that $\inf_{L > 0} \sup \{(\Lambda(x, \xi + \gamma) - \Lambda'(x, \xi + \gamma)) \langle \xi \rangle^{-1/k}; (x, \xi + \gamma) \in \text{supp } q, |\xi| \geq L, |\gamma| \leq c_1 \langle \xi \rangle$ and $|\gamma| \leq c_2 \langle \xi \rangle \langle \xi \rangle^{-1/k} \leq \epsilon$. Applying Lemma 2.15 to $q(x, \xi) R (e^{-Ax}) (x, D)$, we have $q(x, D) R (e^{-Ax}) (x, D)f \in L^2_{k, a, -\epsilon}$ for $f \in L^2_{k, a}$, which proves that $q(x, D)$
\[ x(e^A)(x, D) R(e^{-A}) (x, D) f \in L^2_{\alpha, -\varepsilon} \text{ for } f \in L^2_{\alpha, \varepsilon}. \] Similarly, we can prove \( (e^A)(x, D) q(x, D) R(e^{-A}) (x, D) f \in L^2_{\alpha, -\varepsilon} \text{ for } f \in L^2_{\alpha, \varepsilon}. \) Q. E. D

Let us construct parametrices of (MCP). Define \( Q^{a,b}_h \) by
\[
Q^{a,b}_h f = R \exp[-aA_h^a] (x, D) Q^{a,b}_h \exp[aA_h^a] (x, D) f.
\]
Then \( Q^{a,b}_h \) maps continuously \( L^2_{\alpha, b} \) to \( H^-_{\alpha, b} \) and \( H^{(1-1/\alpha)} \) to \( H^0_{\alpha, b} \) and satisfies \( Q^{a,b}_h \bar{p}_h(x, D) = I \) on \( H^-_{\alpha, b} \) and \( \bar{p}_h(x, D) Q^{a,b}_h = I \) on \( H^{(1-1/\alpha)}_{\alpha, b} \) if \( a, b, h \) satisfy (4.16). In fact, by Lemma 2.14 and Proposition 3.4 we may assume that \( (1 + q^{a,b}_h(x, D))^{-1} \) which maps \( H_{\alpha, e} \) to \( H_{\alpha, e} \) if \( |s| \leq |m| + (1 - 1/\alpha)I, \) \( -1 \leq b \leq 1 \) and \( h \geq h_\alpha. \) We may also assume that \( (1 + q^{a,b}_h(x, D))^{-1} \) maps continuously \( L^2_{\alpha, e} \) to \( L^2_{\alpha, e} \) if \( |s| \leq 4a, \) \( -1 \leq b \leq 1 \) and \( h \geq h_\alpha. \) From Corollary of Lemma 2.9 and Proposition 2.12 it follows that \( R \exp[-aA_h^a] (x, D) g \in L^2_{\alpha, -\varepsilon} \) for \( g \in H^\alpha \) where \( s \in R, \) \( a > 0, \) \( -1 \leq b \leq 1 \) and \( h \geq h_\alpha. \) So, by (2.29) we have
\[
(4.17) \quad f = R \exp[-aA_h^a] (x, D) (1 + q^{a,b}_h(x, D))^{-1} \exp[aA_h^a] (x, D) f
\]
if \( f \in H^-_{\alpha, b}, \) \( s \in R, \) \( a \geq 0, \) \( -1 \leq b \leq 1 \) and \( h \geq h_\alpha. \) This implies that
\[
Q^{a,b}_h \bar{p}_h(x, D) f = R \exp[-aA_h^a] Q^{a,b}_h \exp[aA_h^a] \bar{p}_h R \exp[-aA_h^a] g = R \exp[-aA_h^a] (1 + q^{a,b}_h)^{-1} \exp[aA_h^a] f = f,
\]
\[
g = (1 + q^{a,b}_h)^{-1} \exp[aA_h^a] f \in H^0,
\]
if \( f \in H^-_{\alpha, b} \) and \( a, b, h \) satisfy (4.16). Similarly, we can prove \( \bar{p}_h(x, D) Q^{a,b}_h f = f \) if \( f \in H^{(1-1/\alpha)}_{\alpha, b} \) and \( a, b, h \) satisfy (4.16). From Lemma 4.9 and (4.17) it follows that \( H^-_{\alpha, b} \subset H^0_{\alpha, -\varepsilon} \) for \( s \in R, \) \( 0 \leq a \leq a', \) \( -1 \leq b \leq 1 \) and \( h \geq h_\alpha. \) Therefore, we have \( Q^{a,b}_h f = Q^{a,b}_h f \) if \( f \in H^{(1-1/\alpha)}_{\alpha, b} \) and \( a, b, \) and \( h \) satisfy (4.16). Fix \( h \) sufficiently large and define \( Q_h \) by
\[
Q_h f = Q^{a,b}_h f - Q^{a,b}_h (\bar{p}_h(x, D) - p_h(x, D)) Q^{a,b}_h f
\]
for \( f \in H^{(1-1/\alpha)}_{\alpha, b} \), where \( a' \) and \( h' \) satisfy (4.16). Then, by the same arguments as in the proof of Proposition 3.8 \( Q_h \) does not depend on the choice of \( a, a', \) and \( h' \), and satisfies \( \bar{p}_h(x, D) Q_h f = Q_h \bar{p}_h(x, D) f = f \) if \( f \in L^2_{\alpha, -1} \) and \( a' \) satisfies (4.16). Moreover we have
\[
Q_h f - Q^{a,b}_h f \in H^-_{\alpha, -1} \\ \text{if } f \in H^{(1-1/\alpha)}_{\alpha, b} \text{ and } a' \text{ and } h' \text{ satisfy (4.16).}
\]
Here we have modified \( d_0 \) if necessary. So \( Q_h \) maps continuously \( L^2_{\alpha, b} \) to \( H^{(1-1/\alpha)}_{\alpha, b} \) if \( a' \) and
Let \( \chi(x, \xi) \in \mathcal{E}^{(a)}(T^*\mathbb{R}^n) \) satisfy \( \chi \subseteq \mathcal{E}_1 \) for \( x \). Here we have used the notation in the proof of Proposition 3.8. Choose \( \phi(x) \in \mathcal{D}^{(a)} \) so that \( \phi(x) = 1 \) in a neighborhood of \( \{ x \in \mathbb{R}^n : (x, \xi) \in \mathcal{E}_1 \text{ for some } \xi \in \mathbb{R}^n \} \), and define \( Q \) by \( Q \psi = Q_\lambda \chi(x, D) \phi(x) \psi \). Let \( \chi_0(x, \xi) \in \mathcal{E}^{(a)}(T^*\mathbb{R}^n) \) satisfy \( \chi_0 \subseteq \chi \) for \( x \). Then we have

\[
\chi_0(x, D) p(x, D) Q \psi - \chi_0(x, D) f = \chi_0(\chi \phi - 1) f + \chi_0(\phi - \phi_0) Q_\lambda f \in L^2 \text{ if } f \in H^{(0, -1/4)}_n \text{ and } a' \text{ satisfies (4.16), where } \epsilon_0 = \epsilon_0' A^{-1/8} \text{ and } \epsilon_0'' > 0.
\]

Define \( A'_h(x, \xi) \), replacing \( \phi_0(x, \xi) \) by \( \phi_0(x, \xi) - 2 (|x - x_0|^2 + |\xi - \xi_0|^2)/3 \) in the construction of \( A_h(x, \xi) \) in the proof of Lemma 4.3. Then we may assume that \( A'_h(x, \xi) \) satisfies

\[
A'_h(x, \xi) \leq A'_0(x, \xi) - (|x - x_0|^2 + |\xi - \xi_0|^2)/2 + C_1(h)
\]

for \( (x, \xi) \in \mathcal{E}_1 \) and \( |\xi| \geq 1 \). Let us prove that \( Q \) is a left microlocal parametrix of \( p \) on \( H^{(0, -1/4)}_n(A'_h + bW_h) \). Let \( f \in H^{(0, -1/4)}_n(A'_h + bW_h) \), and let \( \chi(x, \xi) \in \mathcal{E}^{(a)}(T^*\mathbb{R}^n) \) be a positively homogeneous function of degree 0 in \( \xi \) for \( |\xi| \geq 1 \) such that \( \leq \chi(x, \xi) \leq 1 \), supp \( \chi \cap \text{supp } (1 - \chi) \cap \{ |\xi| = 1 \} \subseteq \{ (x, \xi) \in T^*\mathbb{R}^n : \chi(x, \xi) = 1 \} \) and supp \( \chi(x, \xi) \) is included in a small conic neighborhood of \( \text{supp } \chi \cap \text{supp } (1 - \chi) \cap \{ |\xi| = 1 \} \). Assume that \( a \) and \( h \) satisfy (4.16). Then it follows from Proposition 2.8, Corollary 3 of Lemma 2.15 and its proof that

\[
(1 - \chi(x, D)) \exp[a A'_h(x, D)] (x, D) [p, \chi \phi] f \in L^2
\]

if \( b, b' \in [-1, 1] \), modifying \( a_0 \). On the other hand, Lemma 4.9 and (4.17) yield

\[
\chi(x, D) \exp[a A'_h(x, D)] (x, D) [p, \chi \phi] f = \chi \exp[a A'_h(x, D)] \exp[-a (A'_h + bW_h)] [p, \chi \phi] f \in L^2
\]

if \( b' - b < 2 b_0 \equiv \inf_{(x, \xi) \in \text{supp } \chi} (|x - x_0 + \xi| + |\xi - \xi_0|^2)/4 \), where \( 1 + q_{x, b}^a \chi(x, D) = \exp[a (A'_h + bW_h)] (x, D) \exp[-a (A'_h + bW_h)] (x, D) \). We may assume that \( b_0 \leq 2 \). So we have \([p, \chi \phi] f \in L^2_{A'_h + bW_h} \) if \(-1 \leq b \leq 1 - b_0 \). This gives \( Q_\lambda [p, \chi \phi] f \in H^{(0, -1/4)}_{n, b_0/2} \) if \(|b| \leq b_0/2 \). Modifying \( \chi_0 \), we can assume that

\[
\inf_{L_{>0}} \sup_{(x, \xi) \in \text{supp } \chi} |x| \leq L \| A_h(x, \xi) \| \|\xi\|^{-1/8} \leq \epsilon_0 b_0/8.
\]

Then, by Lemma 4.9 and (4.17) we have
This implies that
\[ \chi(x, D)Q[p, x] f - \chi(x, D) f = -\chi(x, D) \varphi f + \chi(x, \varphi - 1) f \]
\[ + \chi(x, \varphi - 1) \varphi f \in L^2_{c, \varepsilon, b} \]
if \( |b| \leq b_0/2 \), where \( b_0 > 0 \). Now assume that \( f \in \mathcal{D}' \) and \( WF_{\varphi}(f) \cap \mathcal{C} \subseteq \{ \varphi(x, \xi) \geq 0 \} \). For a fixed \( b \) with \(-b_0/2 \leq b < 0\) there is \( a_f > 0 \) such that
\[ (4.18) \]
\[ \chi(x, D) \varphi f \in H_{-b}^{(1-\varepsilon)/2} \cap H_{\varepsilon}^{m}(A_{b}^{+} + b_{w}) \]
if \( a \geq a_f \) when \( *= (\kappa) \) and if \( 0 < a < a_f \) when \( *= [\kappa] \). In fact, let \( \chi_{2}(x, \xi) \in \mathcal{S}(\mathbb{R}^{n}) \) be a positively homogeneous function of degree 0 in \( \xi \) for \( |\xi| \geq 1 \) such that \( 0 \leq \chi_{2}(x, \xi) \leq 1 \), \( \chi_{2}(x, \xi) = 1 \) if \( |\xi| \geq 1 \) and \( \lim \sup_{|\xi| \to \infty} (A_{\delta}(x, \lambda \xi) + b W_{\delta}(x, \lambda \xi))/4 (\lambda |\xi|)^{-1/\kappa} \leq 0 \), and \( \chi_{2}(x, \xi) = 0 \) if \( |\xi| \geq 1 \) and \( \lim \inf_{|\xi| \to \infty} (A_{\delta}(x, \lambda \xi) + b W_{\delta}(x, \lambda \xi))/2 (\lambda |\xi|)^{-1/\kappa} \geq 0 \). Since
\[ \chi(x, D) \varphi f \in H_{-b}^{(1-\varepsilon)/2} \cap H_{\varepsilon}^{m}(A_{b}^{+} + b_{w}) \]
for \( (x, \xi) \in \text{supp} \chi_{2} \cap \mathcal{C} \) with \( |\xi| \geq 1 \), applying Lemma 4.9 and the Paley–Wiener theorem for \( \mathcal{S}(\mathbb{R}^{n}) \) (see, e.g., [18]), there is \( a_f > 0 \) such that \( \chi_{2}(x, \xi) \in \mathcal{S}(\mathbb{R}^{n}) \) is a positively homogeneous function of degree 0 in \( \xi \) for \( |\xi| \geq 1 \) such that \( 0 \leq \chi_{2}(x, \xi) \leq 1 \), \( \chi_{2}(x, \xi) = 1 \) if \( |\xi| \geq 1 \) and \( \lim \sup_{|\xi| \to \infty} (A_{\delta}(x, \lambda \xi) + b W_{\delta}(x, \lambda \xi))/4 (\lambda |\xi|)^{-1/\kappa} \leq 0 \), and \( \chi_{2}(x, \xi) = 0 \) if \( |\xi| \geq 1 \) and \( \lim \inf_{|\xi| \to \infty} (A_{\delta}(x, \lambda \xi) + b W_{\delta}(x, \lambda \xi))/2 (\lambda |\xi|)^{-1/\kappa} \geq 0 \). Since
\[ \chi(x, D) \varphi f \in H_{-b}^{(1-\varepsilon)/2} \cap H_{\varepsilon}^{m}(A_{b}^{+} + b_{w}) \]
and if \( a_f > 0 \) when \( *= (\kappa) \) and if \( 0 < a < a_f \) when \( *= [\kappa] \). On the other hand, we have \( (1 - \chi_{2}(x, D)) \chi(x, D) \varphi f \in \mathcal{S}(\mathbb{R}^{n}) \), since \( \{ \chi_{2}(x, \xi) \neq 1 \text{ and } |\xi| = 1 \} \cap \mathcal{C} \subseteq \{ \varphi(x, \xi) < 0 \} \). This proves \( (4.18) \), taking \( a_f > 0 \) small enough when \( *= (\kappa) \). Noting that \( A \) can tend to zero when \( *= (\kappa) \) and, therefore, \( a \) can tend to \( \infty \), we have \( \chi_{2}(x, \xi) \varphi f - \chi_{2} f \in \mathcal{S}(\mathbb{R}^{n}) \) and \( \chi_{2}(x, \xi) \varphi f - \chi_{2} f \in \mathcal{S}(\mathbb{R}^{n}) \). In particular, if \( f \in \mathcal{D}' \) and \( WF_{\varphi}(f) \cap \mathcal{C} = \emptyset \), then there is \( a_f > 0 \) such that \( \chi(x, D) \varphi f \in L^{2}_{c, \varepsilon} \) for any \( a \geq 0 \) when \( *= (\kappa) \) and for \( 0 < a < a_f \) when \( *= [\kappa] \). This implies that \( \mathcal{S}(\mathbb{R}^{n}) \) if \( f \in \mathcal{D}' \) and \( WF_{\varphi}(f) \cap \mathcal{C} \subseteq \{ \varphi(z) \geq 0 \} \). Therefore, we have just proved the following microlocal version of Holmgren’s uniqueness theorem, which is necessary to prove that there is a conic neighborhood \( \mathcal{C}_{3} \) of \( z_{0} \) such that \( WF_{\varphi}(Q_{f}) \cap \mathcal{C}_{3} \subseteq \{ \varphi(z) \geq 0 \} \) for \( f \in \mathcal{D}' \) with \( WF_{\varphi}(f) \subseteq \{ \varphi(z) \geq 0 \} \).

**Proposition 4.10.** Assume that \( p(x, \xi) \) satisfies the condition \((A-1)\) and \((A-2)\) with \( \kappa \) replaced by \( \kappa \) (\( \kappa \)). Let \( z_{0} \in T^{*}R^{n} \setminus 0 \), and assume that \( \varphi(z) \in C^{2}(T^{*}R^{n} \setminus 0) \) is real-valued positively homogeneous of degree 0
in \( \xi \) and \( \varphi(z) = 0 \) and that \( p_m(z) \) is microhyperbolic with respect to \( -H_p(z^0) \) at \( z^0 \). Then \( z^0 \in WF_*(u) \) if \( u \in D^* \), \( z^0 \in WF_*(pu) \) and \( WF_*(u) \cap \mathcal{C} \cap \{ \varphi(z) < 0 \} = \emptyset \) for a conic neighborhood \( \mathcal{C} \) of \( z^0 \). Here \( \ast \) denotes \( (\kappa) \) or \( \{ \kappa \} \).

Let \( f \in D^* \) satisfy \( WF_*(f) \cap \mathcal{C} \subset \{ \varphi(z) \geq 0 \} \). Then \( Qf \in H_{s, \kappa}^{-1/2} \) for a fixed \( \kappa^* \) with \( -b/2 \leq \kappa^* < 0 \), and \( a \geq a_f \) when \( \ast = (\kappa) \) and \( 0 < a \leq a_f \) when \( \ast = \{ \kappa \} \), if \( a \) satisfies (4.16). Therefore, we have \( WF_*(Qf) \cap \{ \varphi_2(x, \xi) < 3b \} = \emptyset \), since \( \inf_{\xi > 0} \{ A_\kappa(x, \xi) |\xi|^{-1/\kappa} : \varphi_2(x, \xi) < 3b \text{ and } |\xi| \geq L \} \geq -b \), where \( \varphi_2(x, \xi) = \varphi_2(x, \xi/|\xi|) \) and \( \varphi_2(z) \) is defined by (4.10). We may assume that \( -H_\kappa(z) \) and \( -H_{\kappa_1}(z) \) belong to \( \Gamma(p_{ms}, \theta) \) for \( z \in \mathcal{C}_4 \equiv \{(x, \xi); |x-x^0|^2 + |\xi|/|\xi|-\xi^0|^2 \leq r_0^2 \} \) and \( WF_*(PQf - f) \cap \mathcal{C}_3 = \emptyset \), where \( r_0 > 0 \). Let \( \zeta(x, \xi) \in C^2(T^*\mathbb{R}^n \setminus 0) \) be a real-valued positively homogeneous function in \( \xi \) such that \( 0 \leq \zeta(z, \xi) \leq 1 \), and \( \zeta(z, \xi) = 1 \) if \( |x-x^0|^2 + |\xi|/|\xi|-\xi^0|^2 \leq r_1^2 \) and \( \zeta(z, \xi) = 0 \) if \( |x-x^0|^2 + |\xi|/|\xi|-\xi^0|^2 \geq 4r_1^2 \), where \( 0 < r_1 \leq (r_0/2) \). Then we may assume that \( |H_\kappa(x, \xi)| = O(r_1^{-1}) \) for \( |\xi| \leq 1 \). Since \( |\varphi(x, \xi) - \varphi_2(x, \xi)| = O(|x-x^0|^2 + |\xi|/|\xi|-\xi^0|^2) \), we have \( -H_{\kappa_1}(z) \in \Gamma(p_{ms}, \theta) \) for \( z \in \mathcal{C}_4 \) and \( \theta \in [0, 1] \), if \( b \) and \( r_1 \) are sufficiently small, where \( \varphi_0(z) = \theta \{ \zeta(z) \varphi(z) + (1 - \zeta(z)) (\varphi_2(z) - 3b) \} + (1 - \theta) (\varphi_2(z) - 3b) \). Now assume that there is \( \theta \in [0, 1] \) such that \( WF_*(Qf) \cap \{ z \in \mathcal{C}_4; \varphi_0(z) = 0 \} \neq \emptyset \). We set \( \theta_0 = \inf \{ \theta \in [0, 1]; WF_*(Qf) \cap \{ z \in \mathcal{C}_4; \varphi_0(z) = 0 \} \neq \emptyset \} \). Then, \( WF_*(PQf - f) \cap \{ z \in \mathcal{C}_4; \varphi_0(z) = 0 \} = \emptyset \) and \( WF_*(Qf) \cap \{ z \in \mathcal{C}_4; \varphi_0(z) = 0 \} = \emptyset \), which contradicts the definition of \( \theta_0 \). This proves that \( WF_*(Qf) \cap \mathcal{C}_3 \subset \{ \varphi(z) \geq 0 \} \), where \( \mathcal{C}_3 = \{(x, \xi); |x-x^0|^2 + |\xi|/|\xi|-\xi^0|^2 \leq r_1^2 \} \). Thus we have the following

**Theorem 4.11.** Let the hypothesis of Proposition 4.10 be satisfied, and let \( \mathcal{C} \) be a conic neighborhood of \( z^0 \). Then there is a continuous operator \( Q : D^* \to D^* \) and a conic neighborhood \( \mathcal{C}_1 \) of \( z^0 \) such that

\[
\begin{align*}
z^0 & \in WF_*(pQf - f) \cup WF_*(Qp f - f), \\
WF_*(Qf) \cap \mathcal{C}_1 & \subset \{ \varphi(z) \geq 0 \},
\end{align*}
\]

if \( f \in D^* \) and \( WP_*(f) \cap \mathcal{C} \subset \{ \varphi(z) \geq 0 \} \). Moreover, \( z^0 \in WF_*(Qf) \) if \( z^0 \in WF_*(f) \).
Remark. (i) The theorem implies that (MGP) is microlocally well-posed in $\mathcal{D}^*$ at $z^0$ modulo $\mathcal{E}^*$. (ii) By Theorems 1.5 and 4.11, (MCP) can be solved globally modulo $\mathcal{E}^*$ under reasonable assumptions.

§ 5. Proof of Theorem 1.4 and Some Remarks

Let us begin with some remarks on existence of time functions.

Proposition 5.1. Let $z^0 = (x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0$ and $\mathcal{G} \in T_\rho(T^*\mathbb{R}^n)$, and assume that $p_m(x, \xi)$ is microhyperbolic with respect to $\mathcal{G}$ at $z^0$. Then the following conditions are equivalent: (i) There are a conic neighborhood $\mathcal{C}$ of $z^0$ and a time function for $p_m$ in $\mathcal{C}$. (ii) There is $\mathcal{G} \in T_\rho(T^*\mathbb{R}^n)$ such that $p_m$ is microhyperbolic with respect to $\mathcal{G}$ at $z^0$ and $\sigma(r_0, \mathcal{G}) = 0$, where $r_0 = \sum_{j=1}^n \xi_j^0 (\partial / \partial \xi_j)$. (iii) There is $\mathcal{G} \in T_\rho(T^*\mathbb{R}^n)$ such that $p_m$ is microhyperbolic with respect to $\mathcal{G}$ at $z^0$ and $\pm r_0 \in \Gamma(p_m\mathcal{G}, \mathcal{G})$.

Proof. Let $t(x, \xi)$ be a time function for $p_m$ in $\mathcal{C}$. Then $p_m$ is microhyperbolic with respect to $-H_t(z^0)$ at $z^0$ and $\sum_{j=1}^n (\partial / \partial \xi_j) (x, \xi) \xi_j = 0$, i.e., $\sigma(r_0, -H_t(z^0)) = 0$. This proves that the condition (i) implies the condition (ii). It is obvious that the condition (ii) implies the condition (iii). Assume that the condition (iii) holds. Then there are $\mathcal{G} \in \Gamma(p_m\mathcal{G}, \mathcal{G}) (j = 1, 2)$ such that $(-1)^j \sigma(\mathcal{G}, r^0) > 0$. Therefore, there is $\mathcal{G} \in \Gamma(p_m\mathcal{G}, \mathcal{G})$ such that $\sigma(\mathcal{G}, r^0) = 0$. Then $t(x, \xi) \equiv \sigma(\mathcal{G}, (x-x^0, |\xi^0|/|\xi| - \xi^0))$ is a time function for $p_m$ in a conic neighborhood of $z^0$.

Q. E. D.

We assume that the hypotheses of Theorem 1.4 be satisfied. We shall prove Theorem 1.4 by the same arguments as in [30], using Proposition 4.10 (and Theorem 4.11). If $p_m(z^0) \neq 0$, Proposition 3.8 implies that $z^0 \in WF_*(u)$ when $u \in \mathcal{D}^{* \dagger}$ and $z^0 \in WF_*(pu)$. So, in Theorem 1.4 $p_m(z)$ must vanish at $z^0$. If $\Gamma(p_m\mathcal{G}, \mathcal{G}(z^0))$ contains $r_0 = \sum_{j=1}^n \xi_j^0 (\partial / \partial \xi_j)$ or $-r_0$, then Theorem 1.4 is trivial.

Proposition 5.2. Let $z^0 = (x^0, \xi^0) \in \Omega$ and $|\xi^0| = 1$, and let $M$ be a compact subset of $\Gamma(p_m\mathcal{G}, \mathcal{G}(z^0))$. Assume that $p_m(z^0) = 0$ and $\pm r_0 \in$
\[ \Gamma(p_m, \mathcal{H}(\mathcal{V}^0)) \] and that \( \mathcal{H} \in \Gamma(p_m, \mathcal{H}(\mathcal{V}^0)) \) and \( \sigma(\mathcal{H}, r_0) = 0 \). Then there is \( t_0 > 0 \) such that

\[ WF_*(u) \cap \{(x, \xi) \in \mathcal{V}^0 - M^s ; \sigma((x-x_0, \xi/|\xi| - \xi^0), \mathcal{V}^0) = t \} \neq \emptyset \]

for \( 0 \leq t \leq t_0 \).

if \( u \in D^* \) and \( \mathcal{V}^0 \in WF_*(u) \setminus WF_*(pu) \).

**Proof.** We may assume that \( u \in D^* \) and that \( \mathcal{H} \in \mathcal{M} \), i.e., \( \sigma(\mathcal{H}, \delta z) > 0 \) for \( \delta z \in M^s \setminus \{0\} \). Let \( M_1 \) be a compact subset of \( \Gamma(p_m, \mathcal{H}(\mathcal{V}^0)) \) such that \( M \subseteq M_1 \). Then there are a neighborhood \( U \) of \( \mathcal{V}^0 \) and \( t_0 > 0 \) such that \( WF_*(pu) \cap U = \emptyset \). Let \( p_m \) is microhyperbolic with respect to \( U_0 \) for \( z \in U_0 \), \( t(x, \xi) \) is a time function for \( p_m \) in a conic neighborhood of \( U_0 \) and \( \{z \in \mathcal{V}^0 - M^s ; -t_0 \leq t(x, \xi) \leq 0 \} \in U_0 \), where \( t(x, \xi) = \sigma(\mathcal{H}, (x-x_0, \xi/|\xi| - \xi^0)) \). Now assume that \( WF_*(u) \cap \{z \in \mathcal{V}^0 - M^s ; t(x, \xi) = -t_1 \} = \emptyset \) for some \( t_1 \) with \( 0 < t_1 \leq t_0 \).

We can assume without loss of generality that \( \xi^0 = (0, \ldots, 0, 1) \). We denote by \( S^*R^n(\sim R^* \times S^{n-1}) \) the cosphere bundle over \( R^* \) and we use inhomogeneous local coordinates \((x, q)\). Let \( \tau : T^*R^n \setminus 0 \rightarrow S^*R^n \) be the canonical map defined as \((x, \xi) \mapsto (x, -\xi_1/\xi_n, \ldots, -\xi_{n-1}/\xi_n)\) for \((x, \xi) \in T^*R^n \setminus 0\) with \( \xi_n \neq 0 \). The map \( \tau \) induces a map \( d\tau_z : T_z(T^*R^n \setminus 0) \ni (\delta x, \delta \xi) \mapsto (\delta x, \delta q) \in T_{\tau(z)}(S^*R^n) \), where \( \delta q_j = -\xi_n^{-1}(\delta \xi_j + q_j \delta \xi_n) \) (\( 1 \leq j \leq n-1 \)), \( z = (x, \xi) \) and \( q_j = -\xi_j/\xi_n \) (\( 1 \leq j \leq n-1 \)). Since \( \pm r_0 \in M^s \) and \( M^s \) is a closed proper convex cone, modifying \( U \) if necessary, there is a closed convex cone \( K \) with its vertex at the origin in \( R^{n+1} \) (\( \sim T_{\tau(z)}(S^*R^n) \)) such that \( d\tau_z(M^s) \ni K \supset d\tau_z(M^s) \) for \( z \in U_0 \), \( \tau(z^0 - M^s) \cap \tau(U_0) \supset \tau(z^0 - K) \cap \tau(U_0) \).

Then there are \( \epsilon > 0 \) and \( \hat{z} \in \hat{K} \) such that \( \tau(WF_*(u)) \ni K' \cap \{z : -t_1 + \epsilon \geq t(z) \geq -t_1 \} = \emptyset \), where \( K' = \tau(z^0) + \hat{z} - K \). Let \( \phi(x, \xi) \in D^*(\tau(U_0)) \) be a positively homogeneous functions of degree 0 in \( \xi \) for \( |\xi| \geq 1/2 \) such that \( \phi(x, \xi) = 1 \) if \((x, \xi) \in \mathcal{U}, |\xi| \geq 1/2 \) and \( t(x, \xi) \geq -t_1 + \epsilon \), and \( \phi(x, \xi) = 0 \) if \((x, \xi) \in \mathcal{U}, |\xi| \geq 1/2 \) and \( t(x, \xi) \leq -t_1 \), where \( \mathcal{U} \) is a conic neighborhood of \( U \). We set \( v = \phi(x, D)u \) and \( g = \psi(x, D)v \), then \( [\psi, \phi]u \). Then,
We may assume that the boundary \( \partial (K' \cap T) \) of \( K' \cap T \) in \( S^* R^* \cap T \) is smooth, where \( T = \{ \tau (z) \mid t(z) = -t_1 \} \). Let \( S \) be a \( C^2 \) hypersurface in \( S^* R^* \) such that \( S \cap T = \partial (K' \cap T) \) and one of the normals \( (\delta x, \delta q) \) at each point on \( S \cap \tau (\mathbb{U}) \) belongs to \( K^* \), where \( K^* = \{ (\delta x, \delta q) \mid \delta x \cdot \delta \bar{x} + \delta q \cdot \delta \bar{q} \geq 0 \) for any \( (\delta x, \delta q) \in K \). The family of hypersurfaces \( S \) with the above properties sweeps out the region \( K^0 \cap \{ \tau (z) : t(z) \geq -t_1 \} \) (see [12]). Assume that \( \varphi \in C^2 (T^* R^* \setminus 0) \) is real-valued, positively homogeneous of degree 0 in \( \xi \), \( \tau^{-1} (S) \cap \mathbb{U} = \{ z \in \mathbb{U} ; \varphi (z) = 0 \} \), \( d_x (H_\varphi (z)) \in K^* \) on \( \tau^{-1} (S) \cap \mathbb{U} \) and \( WF^* (v) \cap \{ z \in \mathbb{U} ; \varphi (z) < 0 \} = \emptyset \). We need the following

**Lemma 5.3.** (Lemma 3.1 in [30]). For \( z^1 = (x^1, \xi^1) \in \tau^{-1} (S) \cap \mathbb{U} \), we have \(-H_\varphi (z^1) \in \Gamma (p_{m1}, \mathcal{O})\).

From Lemma 5.3 and Theorem 4.11 it follows that \( z^1 \in WF^* (Q g - v) \) for \( z^1 \in \tau^{-1} (S \cap K^0) \), where \( Q \) is as defined in Theorem 4.11, replacing \( z^0 \) and \( \kappa \) by \( z^1 \) and \( \kappa_1 \), respectively. From the proof of Theorem 4.11 it follows that \( Q : \mathbb{D}^* \rightarrow \mathbb{D}^* \) and \( Q \) satisfies the assertions in Theorem 4.11 with \( z^0 \) replaced by \( z^1 \). Therefore, we have \( z^1 \in WF^* (Q g) \) and \( WF^* (v) \cap \tau^{-1} (S \cap K^0) = \emptyset \). The method of sweeping out in [12] shows that \( z^0 \in WF^* (v) \). This proves Proposition 5.2.

**Q. E. D.**

From the same arguments as in the proof of Theorem 3.3 in [31], it follows that for every \( z^0 \in \Omega \) there are neighborhood \( \mathbb{U} (z^0) \) \((\subset \Omega)\) of \( z^0 \) and \( t(z^0) > 0 \) such that for any \( z^1 \in \mathbb{U} (z^0) \) there is a Lipschitz continuous function \( z (t) \) defined on \((-t(z^0), 0)\) with values in \( \Omega \) satisfying \( z (t) \in WF^* (u) \) for \( t \in (-t(z^0), 0) \), \( (d/dt) z (t) \in \Gamma (p_{m1}, \mathcal{O}) \) for \( a. e. \) \( t \in (-t(z^0), 0) \) and \( z (0) = z^1 \) if \( u \in \mathbb{D}^* \), \( z^1 \in WF^* (u) \) and \( WF^* (p u) \cap \Omega = \emptyset \). Therefore, by the same arguments as in the proof of extension theorem in theory of ordinary differential equations, we can prove Theorem 1.4.
References


