The Structure of Group $C^*$-algebras of the Generalized Dixmier Groups

By

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Abstract

In this paper we first analyze the algebraic structure of group $C^*$-algebras of the generalized Dixmier groups, and next consider that of group $C^*$-algebras of some Lie semi-direct products with multi-diagonal or diagonal actions. As an application, we estimate the stable rank and the connected stable rank of these $C^*$-algebras in terms of groups. Also, we show that some of these group $C^*$-algebras have no nontrivial projections.

§1. Introduction

Group $C^*$-algebras provide many important examples in some topics of the theory of $C^*$-algebras such as their representation theory, K-theory, extension theory, etc. (cf. [1], [2], [22]). The (algebraic) structure of group $C^*$-algebras in this paper means their composition series with well understood subquotients. The structure of group $C^*$-algebras for some connected Lie groups was examined by some mathematicians (cf. [5], [14], [18], [21] and [23]). In particular, the author [18] analyzed the structure of group $C^*$-algebras of the Lie semi-direct products $\mathbb{C}^n \rtimes_{\alpha} \mathbb{R}$ (we often omit the action’s symbol $\alpha$). However, the structure of group $C^*$-algebras for general Lie groups is still mysterious. On the other hand, the stable rank theory of $C^*$-algebras was initiated by M. A. Rieffel [12], who raised an interesting problem of determining the stable rank of group $C^*$-algebras of Lie groups in terms of groups. See [15], [18], [19] and [20] for some partial answers of this problem.

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This paper is organized as follows. First of all, we consider the structure of group $C^*$-algebras of the generalized Dixmier groups. For analysis of their subquotients we use a Green’s result [5, Corollary 15], a corollary of Green’s imprimitivity theorem [6, Corollary 2.10], a Dixmier-Douady’s result (cf. [4, Chapter 10]), and some techniques of Connes’ foliation $C^*$-algebras ([2], [9]). These known results are used frequently in this paper. As a corollary, we estimate the stable rank and the connected stable rank of these group $C^*$-algebras. Moreover, it is shown that these group $C^*$-algebras have no nontrivial projections. We next investigate the case of Lie semi-direct products of $C^n$ by connected Lie groups with multi-diagonal actions. Finally, we analyze the case of Lie semi-direct products of the product groups $\mathbb{R}^n \times C^n$ by connected Lie groups with diagonal actions.

Notation. Let $G$ be a Lie group, $C^*(G)$ its (full) group $C^*$-algebra (cf. [4, Part II]), and $\hat{G}_1$ the space of all 1-dimensional representations of $G$. Denote by $\mathfrak{A} \rtimes_\alpha G$ the $C^*$-crossed product of a $C^*$-algebra $\mathfrak{A}$ by $G$ with an action $\alpha$ (we often omit the symbol $\alpha$), (cf. [1]). Denote by $C_0(X)$ the $C^*$-algebra of all continuous complex-valued functions on a locally compact $T_2$-space $X$ vanishing at infinity. Set $C_0(X) = C(X)$ when $X$ is compact. We say that an action of $G$ on $X$ is wandering if any compact set of $X$ is wandering under the action [5]. Let $\mathbb{K} = \mathbb{K}(H)$ be the $C^*$-algebra of all compact operators on a separable Hilbert space $H$.

Denote by $\text{sr}(\mathfrak{A})$, $\text{csr}(\mathfrak{A})$ the stable rank and the connected stable rank of a $C^*$-algebra $\mathfrak{A}$ respectively [12]. $\lor$, $\land$ respectively mean the maximum and the minimum. Set $\dim_{\mathbb{C}}(X) = \lceil \dim(X)/2 \rceil + 1$ where $\dim X$ is the covering dimension of a space $X$ and $\lfloor x \rfloor$ means the greatest integer with $\lfloor x \rfloor \leq x$. Let $\mathbb{R}^+$ be the space of all nonzero positive real numbers, and $\mathbb{T}^k$ the $k$-torus group (or space).

Basic formulas of stable ranks.

(F1): For an exact sequence $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{J} \rightarrow 0$ of $C^*$-algebras,

\[
\text{sr}(\mathfrak{J}) \lor \text{sr}(\mathfrak{A}/\mathfrak{J}) \leq \text{sr}(\mathfrak{A}) \leq \text{sr}(\mathfrak{J}) \lor \text{sr}(\mathfrak{A}/\mathfrak{J}) \lor \text{csr}(\mathfrak{A}/\mathfrak{J}), \quad \text{csr}(\mathfrak{A}) \leq \text{csr}(\mathfrak{J}) \lor \text{csr}(\mathfrak{A}/\mathfrak{J}).
\]

(F2): For the $C^*$-tensor product $\mathfrak{A} \otimes \mathbb{K}$ for a $C^*$-algebra $\mathfrak{A}$,

\[
\text{sr}(\mathfrak{A} \otimes \mathbb{K}) = 2 \land \text{sr}(\mathfrak{A}), \quad \text{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2 \land \text{csr}(\mathfrak{A}).
\]

(F3): $\text{sr}(C_0(X)) = \dim_{\mathbb{C}} X^+$, where $X^+$ means the one-point compactification of a locally compact $T_2$-space $X$, and
\[ csr(C_0(\mathbb{R})) = 2, \quad csr(C_0(\mathbb{R}^2)) = 1, \quad \text{and} \quad csr(C_0(\mathbb{R}^n)) = [(n + 1)/2] + 1, \quad n \geq 3. \]

See [10], [12] and [15] for (F1), (F2) and (F3).

\section{Group $C^*$-algebras of the Generalized Dixmier Groups}

First of all, we review the structure of the generalized Heisenberg groups. Let $H_{2n+1}$ be the real $(2n + 1)$-dimensional generalized Heisenberg group of all the matrices:

\[ g = (c, b, a) = \begin{pmatrix} 1 & a & c \\ 0^t_n & I_n & b' \\ 0 & 0 & 1 \end{pmatrix} \]

with $c \in \mathbb{R}$, $b = (b_1, \ldots, b_n)$, $a = (a_1, \ldots, a_n)$, $0_n = (0, \ldots, 0) \in \mathbb{R}^n$, where $I_n$ means the $n \times n$ identity matrix and $0_n^t, b'$ respectively mean the transposes of $0_n, b$. The group $H_{2n+1}$ is a simply connected nilpotent Lie group isomorphic to the semi-direct product $\mathbb{R}^{n+1} \rtimes_{\alpha} \mathbb{R}^n$ with the action $\alpha$ defined by $\alpha_c(b) = (c + \sum_{i=1}^n a_ib_i, b)$. It is obtained by definition of crossed products and the Fourier transform that

\[ C^*(H_{2n+1}) \cong C^*(\mathbb{R}^{n+1}) \rtimes_{\alpha} \mathbb{R}^n \cong C_0(\mathbb{R}^{n+1}) \rtimes_{\bar{\alpha}} \mathbb{R}^n \]

where $\bar{\alpha}(l, m) = (l, (m_i + a_il))$ for $l \in \mathbb{R}$, $m = (m_i) \in \mathbb{R}^n$. Since $\{0\} \times \mathbb{R}^n$ is fixed under $\bar{\alpha}$ and closed in $\mathbb{R}^{n+1}$, the following exact sequence is obtained:

\[ 0 \to C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \times \mathbb{R}^n \to C_0(\mathbb{R}^{n+1}) \times \mathbb{R}^n \to C_0(\mathbb{R}^{2n}) \to 0. \]

Moreover, $\bar{\alpha}$ on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ is free and wandering. Green’s result [5] implies that

\[ C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \times \mathbb{R}^n \cong C_0(((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n)/\mathbb{R}^n) \otimes \mathbb{K}(L^2(\mathbb{R}^n)) \cong C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \]

where the orbit space $((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n)/\mathbb{R}^n$ is homeomorphic to $\mathbb{R} \setminus \{0\}$.

We now give the following definition:

**Definition.** Denote by $D_{6n+1}$ the real $(6n + 1)$-dimensional generalized Dixmier group defined by the semi-direct product $\mathbb{C}^{2n} \rtimes_\beta H_{2n+1}$ with the action $\beta$ as follows:

\[ \beta_g(z, z') = ((e^{ia_1}z_1, (e^{ib_1}z_{n+1})), \quad z = (z_i)_{i=1}^n, z' = (z_{n+i})_{i=1}^n \in \mathbb{C}^n, g \in H_{2n+1}, \]

\[ \beta_g = \begin{pmatrix} e^{ia_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & e^{ia_n} & 0 \end{pmatrix} \otimes \begin{pmatrix} e^{ib_1} & 0 \\ \vdots & \ddots \\ 0 & \cdots & e^{ib_n} \end{pmatrix} \in GL_{2n}(\mathbb{C}). \]
The group $D_{6n+1}$ is a simply connected solvable Lie group of non type I. When $n = 1$, $D_7$ is said to be the Dixmier group [3]. It is obtained by the Fourier transform that

$$C^*(D_{6n+1}) \cong C^*(C^{2n}) \rtimes_{\hat{\beta}} H_{2n+1} \cong C_0(C^{2n}) \rtimes_{\hat{\beta}} H_{2n+1},$$

where $\hat{\beta}(w, w') = ((e^{-i\varphi}w_1), (e^{-i\varphi}w_{n+1}))$ for $w = (w_1), w' = (w_{n+1}) \in C^n$. Since the origin $0_{2n} \in C^{2n}$ is fixed under $\hat{\beta}$ and closed in $C^{2n}$, we have that

$$0 \to C_0(C^{2n} \setminus \{0_{2n}\}) \rtimes H_{2n+1} \to C_0(C^{2n}) \rtimes H_{2n+1} \to C^*(H_{2n+1}) \to 0.

Moreover, since the subspace $C \setminus \{0\}$ in each direct factor of $C^{2n}$ is $\hat{\beta}$-invariant and closed in $C^{2n} \setminus \{0_{2n}\}$, it is obtained that

$$0 \to C_0(X_1) \rtimes H_{2n+1} \to C_0(C^{2n} \setminus \{0_{2n}\}) \rtimes H_{2n+1} \to \bigoplus_{n+1} C_0(C \setminus \{0\}) \rtimes H_{2n+1} \to 0

where $X_1$ means the complement of the disjoint union $\sqcup_{2n} C \setminus \{0\}$ of all $C \setminus \{0\}$ in $C^{2n} \setminus \{0_{2n}\}$. Since the direct products of either $C \setminus \{0\}$ or $\{0\}$ in direct factors of $C^{2n}$, homeomorphic to $(C \setminus \{0\})^k$ for $2 \leq k \leq 2n-1$ are invariant under $\hat{\beta}$, the following exact sequences ($2 \leq k \leq 2n - 1$) are obtained inductively:

$$0 \to C_0(X_k) \rtimes H_{2n+1} \to C_0(X_{k-1}) \rtimes H_{2n+1}

\to \bigoplus_{1 \leq i_1 < \cdots < i_k \leq 2n} C_0(C \setminus \{0\})^k \rtimes H_{2n+1} \to 0

with $X_{k-1} = \sqcup_{2n} (C \setminus \{0\})^k$ and $X_{2n-1} = (C \setminus \{0\})^{2n}$, where $\bigoplus_{1 \leq i_1 < \cdots < i_k \leq 2n}$ means the combination $\binom{2n}{k}$-direct sum. Since $\hat{\beta}$ on $(C \setminus \{0\})^k$ is the multi-rotation, $C_0((C \setminus \{0\})^k) \rtimes H_{2n+1}$ is isomorphic to $C_0(\mathbb{R}^k_+ \otimes C(\mathbb{T}^k)) \rtimes H_{2n+1}$. Moreover, the action $\beta$ on $\mathbb{T}^k$ is transitive. Thus Green’s result [6] implies that

$$C(\mathbb{T}^k) \rtimes H_{2n+1} \cong C(H_{2n+1}/(H_{2n+1})) \rtimes H_{2n+1} \cong C^*((H_{2n+1})_{1_k}) \otimes \mathbb{K}(L^2(\mathbb{T}^k))$$

where $(H_{2n+1})_{1_k}$ is the stabilizer of $1_k \in \mathbb{T}^k$.

Summing up the above argument, the following theorem is obtained:

**Theorem 2.1.** The $C^*$-algebra $C^*(D_{6n+1})$ has a finite composition series $(\mathcal{J}_j)_{j=1}^{2n+1}$ with each subquotient $\mathcal{J}_{2n+1-k}/\mathcal{J}_{2n-k}$ isomorphic to $C^*(H_{2n+1})$ for $k = 0$, and

$$\bigoplus_{1 \leq i_1 < \cdots < i_k \leq 2n} C_0(\mathbb{R}^k_+ \otimes C^*((H_{2n+1})_{1_k}) \otimes \mathbb{K}(L^2(\mathbb{T}^k))) \quad \text{for } 1 \leq k \leq 2n.$$

We next analyze the structure of group $C^*$-algebras of the stabilizers $(H_{2n+1})_{1_k}$ in the following. Note that $D_{6n+1} = (C^n \times C^n) \rtimes_{\hat{\beta}} H_{2n+1}$. 
Case 1. First suppose that $1_k$ is contained in $\mathbb{C}^n \times \{0_n\}$. Then we may have that $(H_{2n+1})_{1_n}$ is isomorphic to $\mathbb{R}^{n+1} \rtimes_{\alpha} (\mathbb{Z}^k \times \mathbb{R}^{n-k})$. It is obtained by the Fourier transform that

$$C^*((H_{2n+1})_{1_n}) \cong C^*(\mathbb{R}^{n+1}) \rtimes_{\alpha} (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \cong C_0(\mathbb{R}^{n+1}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^k \times \mathbb{R}^{n-k}),$$

where $\hat{\alpha}(l, m) = (l, (m_i + a_i l)_{i=1}^k)$ for $l \in \mathbb{R}, m = (m_i) \in \mathbb{R}^n, a \in \mathbb{Z}^k \times \mathbb{R}^{n-k}$. Since $\{0\} \times \mathbb{R}^n$ is fixed under $\hat{\alpha}$ and closed in $\mathbb{R}^{n+1}$, the following exact sequence is obtained:

$$0 \rightarrow C_0(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n \rtimes (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \rightarrow$$

$$C_0(\mathbb{R}^{n+1}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \rightarrow C_0(\mathbb{T}^k \times \mathbb{R}^{2n-k}) \rightarrow 0.$$

Moreover, the action of $\mathbb{Z}^k \times \mathbb{R}^{n-k}$ on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ is free and wandering, so that Green’s result [6] implies that

$$C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) \rtimes (\mathbb{Z}^k \times \mathbb{R}^{n-k})$$

$$\cong C_0(\mathbb{R} \setminus \{0\} \times \mathbb{R}^n) \rtimes (\mathbb{Z}^k \times \mathbb{R}^{n-k}) / (\mathbb{Z}^k \times \mathbb{R}^{n-k}) \otimes \mathbb{K}(L^2(\mathbb{Z}^k \times \mathbb{R}^{n-k})).$$

Furthermore, since the orbit of the point $(l, m) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ is parameterized with the point $(l, (m_i \text{ mod } l)_{i=1}^k)$, the orbit space $((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n) / (\mathbb{Z}^k \times \mathbb{R}^{n-k})$ has the fiber structure whose base space is $\mathbb{R} \setminus \{0\}$ and fibers are $\mathbb{T}^k$. This orbit space splits into the product space $(\mathbb{R} \setminus \{0\}) \times \mathbb{T}^k$ since any orbit in $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ has the same type.

Case 2. Next suppose that $1_k$ is contained in $\{0_n\} \times \mathbb{C}^n$. Then the stabilizer $(H_{2n+1})_{1_n}$ is isomorphic to $(\mathbb{R} \times (\mathbb{Z}^k \times \mathbb{R}^{n-k})) \rtimes_{\hat{\alpha}} \mathbb{R}^n$. By the Fourier transform,

$$C^*((H_{2n+1})_{1_n}) \cong C^*(\mathbb{R} \times (\mathbb{Z}^k \times \mathbb{R}^{n-k})) \rtimes_{\hat{\alpha}} \mathbb{R}^n \cong C_0(\mathbb{R} \times (\mathbb{T}^k \times \mathbb{R}^{n-k})) \rtimes_{\hat{\alpha}} \mathbb{R}^n,$$

where $\hat{\alpha}(l, m) = (l, (e^{im_i + a_i l})_{i=1}^k)_{i=1}^n, (m_i)_{i=k+1}^n) \in \mathbb{T}^k \times \mathbb{R}^{n-k}, l \in \mathbb{R}, a \in \mathbb{R}^n$. Since $\{0\} \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ is fixed under $\hat{\alpha}$ and closed in $\mathbb{R} \times \mathbb{T}^k \times \mathbb{R}^{n-k}$, the following exact sequence is obtained:

$$0 \rightarrow C_0(\mathbb{R} \setminus \{0\} \times \mathbb{T}^k \times \mathbb{R}^{n-k}) \times \mathbb{R}^n \rightarrow$$

$$C_0(\mathbb{R} \times \mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes_{\hat{\alpha}} \mathbb{R}^n \rightarrow C_0(\mathbb{T}^k \times \mathbb{R}^{2n-k}) \rightarrow 0.$$

Moreover, the above ideal is decomposed into $\oplus 2 C_0(\mathbb{R}_+ \times \mathbb{T}^k \times \mathbb{R}^{n-k}) \times \mathbb{R}^n$ since two connected components of $(\mathbb{R} \setminus \{0\}) \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ are $\hat{\alpha}$-invariant, and each direct factor is assumed to be the $C^*$-algebra of continuous fields over $\mathbb{R}_+$ with the fibers $C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \times \mathbb{R}^n$, and denoted by $C_0(\mathbb{R}_+, \cup_{\mathbb{R}_+} C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \rtimes_{\hat{\alpha}} \mathbb{R}^n$.
The action $\hat{\alpha}$ on $\mathbb{T}^k \times \mathbb{R}^{n-k}$ is transitive. Thus, it is obtained by Green's result [4] that

$$C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \times_{\hat{\alpha}} \mathbb{R}^n \cong C_0(\mathbb{R}^n/(\mathbb{R}^n_{(l,m)})) \times \mathbb{R}^n$$

$$\cong C^*((\mathbb{R}^n_{(l,m)}) \otimes \mathbb{K}(L^2(\mathbb{T}^k \times \mathbb{R}^{n-k}))) \cong C(\mathbb{T}^k) \otimes \mathbb{K}$$

where $(\mathbb{R}^n_{(l,m)})$ is the stabilizer of $(l,m)$, isomorphic to $\mathbb{Z}^k$. Since the cohomology group $H^3(\mathbb{R}, \mathbb{Z})$ vanishes, it is obtained by [4] that $C_0(\mathbb{R}+, \cup_{\mathbb{R}^+} C_0(\mathbb{T}^k \times \mathbb{R}^{n-k}) \times \mathbb{R}^n) \cong C_0(\mathbb{R} \times \mathbb{T}^k) \otimes \mathbb{K}$.

**Case 3.** We consider the other cases such that $1_k$ is not contained in $\mathbb{C}^n \times \{0_n\}$ and $(0_n) \times \mathbb{C}^n$. We may assume that $(H_{2n+1})_{1_k} \cong (\mathbb{R} \times \mathbb{Z}^{k_1} \times \mathbb{R}^{n-k_1}) \times (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$ for $k = k_1 + k_2$, where $1 \leq k_1 = k_2 \leq n$, or $1 \leq k_1 < k_2 \leq n$, or $n \geq k_1 > k_2 \geq 1$. In each case, it is obtained by the Fourier transform that

$$C^*((H_{2n+1})_{1_k}) \cong C^*(\mathbb{R} \times \mathbb{Z}^{k_1} \times \mathbb{R}^{n-k_1}) \times_{\hat{\alpha}} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$$

$$\cong C_0(\mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \times_{\hat{\alpha}} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$$

where the action $\hat{\alpha}$ is defined by $\hat{\alpha}_a(\ell, m) = (l, (e^{i(m_j+a_jl)})_{j=1}^{k_1})$ for $(\ell, m) = (l, (e^{im_j})_{j=1}^{k_1}, (m_j)_{j=k_1+1}^{n-k_1}) \in \mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}, a \in \mathbb{Z}^{k_1} \times \mathbb{R}^{n-k_2}$. Since $(\mathbb{R} \setminus \{0\}) \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_2}$ is fixed under $\hat{\alpha}$ and closed in $\mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$, it follows that

$$0 \to C_0((\mathbb{R} \setminus \{0\}) \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \times (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$$

$$\to C_0(\mathbb{R} \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \times (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2}) \to C_0(\mathbb{T}^k \times \mathbb{R}^{2n-k}) \to 0.$$

The above ideal is decomposed into $\bigoplus C_0(\mathbb{R}+, \cup_{\mathbb{R}^+} C_0(\mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \times (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$ since two connected components of $(\mathbb{R} \setminus \{0\}) \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$ are $\hat{\alpha}$-invariant. Then each direct factor of the above decomposition is regarded as the $C^*$-algebra of continuous fields over $\mathbb{R}^+$ with the fibers $C_0(\mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \times_{\theta} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})$, and denoted by

$$C_0(\mathbb{R}+, \cup_{\theta \in \mathbb{R}^+} C_0(\mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}) \times_{\theta} (\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2})),$$

where the action $\theta$ corresponds to the restriction of $\hat{\alpha}$ to $(\theta) \times \mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$. Since each direct factor of $\mathbb{Z}^{k_2} \times \mathbb{R}^{n-k_2}$ acts on one of direct factors of $\mathbb{T}^{k_1} \times \mathbb{R}^{n-k_1}$ componentwise, each fiber is isomorphic to one of the following tensor products:

$$\begin{align*}
(\otimes^{k_1} C(\mathbb{T}) \times_{\theta} \mathbb{Z}) & \otimes (\otimes^{n-k_1} C_0(\mathbb{R}) \times \mathbb{R}) & k_1 = k_2 \\
(\otimes^{k_1} C(\mathbb{T}) \times_{\theta} \mathbb{Z}) & \otimes (\otimes^{k_2-k_1} C_0(\mathbb{R}) \times \mathbb{Z}) \otimes (\otimes^{n-k_2} C_0(\mathbb{R}) \times \mathbb{R}) & k_1 < k_2 \\
(\otimes^{k_2} C(\mathbb{T}) \times_{\theta} \mathbb{Z}) & \otimes (\otimes^{k_1-k_2} C(\mathbb{T}) \times \mathbb{R}) \otimes (\otimes^{n-k_2} C_0(\mathbb{R}) \times \mathbb{R}) & k_1 > k_2,
\end{align*}$$
which is also proved by considering correspondence between generators of each fiber and those of tensor products. The above tensor factors have the following isomorphisms:

\[ C_0(\mathbb{R}) \times \mathbb{R} \cong \mathbb{K}, \quad C_0(\mathbb{R}) \times \mathbb{Z} \cong C(T) \otimes \mathbb{K}, \quad C(T) \times \mathbb{R} \cong C(T) \otimes \mathbb{K} \]

since each action is the shift, and \( C(T) \rtimes_\theta \mathbb{Z} \cong \mathfrak{A}_\theta \) is the irrational or rational rotation algebra. Thus, each fiber is isomorphic to one of the following:

\[
\begin{align*}
\otimes^n \mathfrak{A}_\theta & \text{ for } k = 2n, \text{ and } (\otimes^k \mathfrak{A}_\theta) \otimes \mathbb{K} \text{ for } k = 2k_1 < 2n - 2, \\
(\otimes^{k_1} \mathfrak{A}_\theta) \otimes C(T^{k_2-k_1}) \otimes \mathbb{K} & \text{ for } k_1 < k_2, \\
(\otimes^{k_2} \mathfrak{A}_\theta) \otimes C(T^{k_1-k_2}) \otimes \mathbb{K} & \text{ for } k_1 > k_2.
\end{align*}
\]

Summing up the above argument, the following theorem is deduced:

**Theorem 2.2.** The group \( C^*\)-algebras \( C^*(H_{2n+1})_{1+k} \) of the stabilizers \((H_{2n+1})_{1+k}\) have the following decompositions:

\[ 0 \to \mathfrak{L}_k \to C^*((H_{2n+1})_{1+k}) \to C(T^k) \otimes C_0(\mathbb{R}^{2n-k}) \to 0 \]

for \( 0 \leq k \leq 2n \) and \( \mathfrak{L}_k \) is isomorphic to

\[
\begin{align*}
C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} & \text{ for } k = 0, \text{ and } C_0((\mathbb{R} \setminus \{0\}) \times T) \otimes \mathbb{K} \text{ for } k = 1, \\
C_0((\mathbb{R} \setminus \{0\}) \times T^k) \otimes \mathbb{K} & \text{ or} \\
C_0(\mathbb{R} \setminus \{0\}, \cup_{\theta \in \mathbb{R} \setminus \{0\}} ((\otimes^n \mathfrak{A}_\theta) \otimes C(T^{2n}) \otimes \mathbb{K})) & \text{ for } 2 \leq k \leq n, \\
C_0(\mathbb{R} \setminus \{0\}, \cup_{\theta \in \mathbb{R} \setminus \{0\}} ((\otimes^n \mathfrak{A}_\theta) \otimes C(T^{2n}) \otimes \mathbb{K})) & \text{ for } n + 1 \leq k \leq 2n - 1, \\
C_0(\mathbb{R} \setminus \{0\}, \cup_{\theta \in \mathbb{R} \setminus \{0\}} \otimes^n \mathfrak{A}_\theta) & \text{ for } k = 2n
\end{align*}
\]

with \( s_1 \geq 1, s_2 \geq 0, 2s_1 + s_2 = k \).

**Remark.** Let \( \Gamma \) be the discrete central subgroup of both \( H_{2n+1} \) and \( D_{6n+1} \) defined by

\[
\Gamma = \left\{ \begin{pmatrix} 1 & 0 & 2\pi k \\ 0 & I_n & 0 \\ n & 0_n & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}.
\]

Then \( D_{6n+1}/\Gamma \cong \mathbb{C}^{2n} \rtimes (H_{2n+1}/\Gamma) \). If \( H_{2n+1} \) is replaced by \( H_{2n+1}/T \) in the above theorem, then \( C^*(H_{2n+1}/T)_{1+k} \cong (\otimes_{\mathbb{Z} \setminus \{0\}} C(T^k) \otimes \mathbb{K}) \oplus C_0(T^k \times \mathbb{R}^{2n-k}) \) for \( 0 \leq k \leq 2n \). It follows that \( C^*(D_{6n+1}/\Gamma) \) is of type I while \( C^*(D_{6n+1}) \) is of non type I (cf. [3]).
Taking a refinement of the composition series of Theorems 2.1 and 2.2, we obtain

**Theorem 2.3.** There exists a finite composition series \( \{ R_j \}_{j=1}^K \) of \( C^*(D_{6n+1}) \) with its subquotients \( R_j/R_{j-1} \) given by \( C_0(\mathbb{R}^{2n}) \) for \( j = K \), and

\[
\begin{cases}
C_0(\mathbb{R}) \otimes \mathbb{K}, & \text{or } C_0(\mathbb{T}_k \times \mathbb{R}^{2n}) \otimes \mathbb{K}, \text{ or } C_0(\mathbb{T}_k \times \mathbb{R}^{k+1}) \otimes \mathbb{K}, \text{ or } \\
C_0(\mathbb{R}^1) \otimes \mathbb{K} \otimes C_0(\mathbb{R} \setminus \{ 0 \}), & \text{or } \\
\cup_{\theta \in \mathbb{R} \setminus \{ 0 \}} ((\otimes^s \mathbb{A}_\theta) \otimes C(\mathbb{T}^{s^2}) \otimes \mathbb{K})
\end{cases}
\]

for \( 1 \leq j \leq K - 1 \) with \( 1 \leq k \leq 2n, s_1 \geq 1, s_2 \geq 0, 2s_1 + s_2 = k \).

**Remark.** The \( C^* \)-tensor product \( (\otimes^s \mathbb{A}_\theta) \otimes C(\mathbb{T}^{s^2}) \) is isomorphic to the crossed product \( C(\mathbb{T}^{s^1} \times \mathbb{T}^{s^2}) \times \mathbb{Z}^{s^1} \) which is a special case of noncommutative tori. We see that \( C^*(D_{6n+1}) \) has \( \mathbb{K} \) and \( \mathbb{K} \otimes (\otimes^s \mathbb{A}_\theta) \) for \( \theta \) irrational as simple subquotients (cf. [6], [11]).

Applying (F1), (F2), (F3) to the composition series of Theorem 2.3, it follows that

**Corollary 2.4.** For the group \( C^* \)-algebra \( C^*(D_{6n+1}) \), it holds that

\[ \text{sr}(C^*(D_{6n+1})) = n + 1 = \dim_{\mathbb{C}}(D_{6n+1})_{1}^{\wedge}, \quad \text{and} \quad 2 \leq \text{csr}(C^*(D_{6n+1})) \leq n + 1. \]

**Proof.** Note that Theorem 2.3 implies that the space \( (D_{6n+1})_{1}^{\wedge} \) of all 1-dimensional representations of \( D_{6n+1} \) is homeomorphic to \( \mathbb{R}^{2n} \). By Theorem 2.3 and (F2), it is obtained that \( \text{sr}(R_j/R_{j-1}) \leq 2 \) and \( \text{csr}(R_j/R_{j-1}) \leq 2 \) for \( 1 \leq j \leq K - 1 \). Inductively applying (F1) to the composition series of Theorem 2.3, \( \text{sr}(R_j) \leq 2 \) and \( \text{csr}(R_j) \leq 2 \) for \( 1 \leq j \leq K - 1 \). Therefore, it is obtained by (F1) and (F3) that

\[ \text{sr}(C_0(\mathbb{R}^{2n})) = n + 1 \leq \text{sr}(C^*(D_{6n+1})) \leq 2 \vee \text{sr}(C_0(\mathbb{R}^{2n})) \]
\[ \vee \text{csr}(C_0(\mathbb{R}^{2n})) = n + 1, \]
\[ \text{csr}(C^*(D_{6n+1})) \leq 2 \vee \text{csr}(C_0(\mathbb{R}^{2n})) = n + 1. \]

On the other hand, note that \( D_{6n+1} \) is isomorphic to \( ([\mathbb{R}^{4n}] \times \mathbb{R}^{n+1}) \times \mathbb{R}^n \), where \( (z, z', g) \mapsto ((z, z'), (c, b), a) \). Thus, \( C^*(D_{6n+1}) \cong (C_0([\mathbb{R}^{4n}] \times \mathbb{R}^{n+1}) \times \mathbb{R}^n) \). By using Connes’ Thom isomorphism for K-groups of \( C^* \)-algebras (cf. [2], [22]), \( K_1(C^*(D_{6n+1})) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \). By Hassan’s result [7], \( \text{csr}(C^*(D_{6n+1})) \geq 2 \). \( \square \)

**Remark.** For \( D_{6n+1}/\Gamma \) with \( \Gamma \) in the remark of Theorem 2.2, it is obtained that

\[ \text{sr}(C^*(D_{6n+1}/\Gamma)) = n + 1 = \dim_{\mathbb{C}}(D_{6n+1}/\Gamma)_{1}^{\wedge}, \quad \text{csr}(C^*(D_{6n+1}/\Gamma)) \leq n + 1. \]
It follows from the composition series of Theorem 2.3 that

**Corollary 2.5.** The group $C^*\text{algebra } C^*(D_{6n+1})$ has no nontrivial projections.

**Proof.** Notice that if a nontrivial projection exists in a $C^*$-algebra, its image in any quotient is a nontrivial projection or zero. On the other hand, each subquotient of $C^*(D_{6n+1})$ has no nontrivial projections since each subquotient $K_j/\mathfrak{K}_{j-1}$ $(1 \leq j \leq K)$ has a commutative $C^*$-algebra on a noncompact connected space as a tensor factor.

**Remark.** There exist nontrivial projections in $K$ and $K \otimes (\otimes^s \mathfrak{M}_\theta)$ for $\theta$ irrational.

§3. The Lie Semi-direct Products of $\mathbb{C}^n$ by Connected Lie Groups with Multi-diagonal Actions

Let $G$ be a connected Lie group defined by the semi-direct product $\mathbb{C}^n \rtimes \alpha N$ with $N$ a connected Lie group. The action $\alpha$ is also a Lie group homomorphism from $N$ to $GL_n(\mathbb{C})$. Denote by $\mathfrak{d}$ the differential of $\alpha$ from the Lie algebra $\mathfrak{N}$ of $N$ to the Lie algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$. Moreover, suppose that the action $\alpha$ is induced from the following commutative diagram:

$$
\begin{array}{ccc}
N & \longrightarrow & N/\mathfrak{N} \\
exp & & \exp \\
\mathfrak{N} & \longrightarrow & \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}] \\
\exp & & \exp \\
& & GL_n(\mathbb{C}) \\
& & M_n(\mathbb{C})
\end{array}
$$

where $\exp$ means the exponential map, and $[N, N]$, $[\mathfrak{N}, \mathfrak{N}]$ mean the commutators of $N$ and $\mathfrak{N}$ respectively. Then $N/\mathfrak{N}$ isomorphic to $\mathbb{R}^{l-m} \times \mathbb{T}^m$ and $\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ isomorphic to $\mathbb{R}^l$ for some $l \geq 0$ and $0 \leq m \leq l$. First suppose that $\alpha$ is a complex 1-dimensional, multi-diagonal action of the form:

$$
\alpha_t = \begin{pmatrix}
\ldots & 0 \\
\ldots & \lambda_n t_n
\end{pmatrix} \in GL_n(\mathbb{C})
$$

with $t = ((t_1)_i^{n-m}, (e^{it_j})_{j=n-m+1}^n) \in \mathbb{R}^{n-m} \times \mathbb{T}^m$, $\lambda_i \in \mathbb{C}$ $(1 \leq i \leq n-m)$, $\lambda_j \in i\mathbb{R}$ $(n-m+1 \leq j \leq n)$. We may assume that $\lambda_k = 0$ for $1 \leq k \leq n_0$, $\lambda_k \not\in i\mathbb{R}$ for $n_0 + 1 \leq k \leq n_1$, and $\lambda_k \in i(\mathbb{R} \setminus \{0\})$ for $n_1 + 1 \leq k \leq n$ with $n_0, n_1 \geq 0$. Note that if the action $\alpha$ of $N$ is diagonal on $\mathbb{C}^n$, it reduces to that of $N/\mathfrak{N}$ automatically.

Under the above situation, the following theorem is obtained:
Theorem 3.1. Let $G$ be a Lie semi-direct product of $\mathbb{C}^n$ by a connected Lie group $N$ with a complex 1-dimensional, multi-diagonal action. Then $C^*(G)$ has a finite composition series $\{J_i\}_{i=1}^{n+1}$ with $J_{n-n_0-k+1}/J_{n-n_0-k}$ isomorphic to $C_0(\mathbb{C}^{n_0}) \otimes C^*(N)$ for $k = 0$, and

$$
\oplus_{n_0+1 \leq i_1 < \cdots < i_k \leq n} \begin{cases} C_0(\mathbb{C}^{n_0} \times \mathbb{T}^k) \otimes \mathbb{K} & \text{for } 1 \leq k \leq n-n_0 \\ C_0(\mathbb{C}^{n_0} \times \mathbb{T}^{n_1} \times \mathbb{R}^{k_2}) \otimes C^*(N_{l_k}) \otimes \mathbb{K} & \end{cases}
$$

with $0 \leq n_0 \leq n$ and $k_2 \geq 1$, $k = k_1 + k_2$, where $\mathbb{C}^{n_0}$ is the fixed point subspace of $\mathbb{C}^n$ under the action of $N$, and the first alternative corresponds to that the action of $N$ on invariant subspaces $(\mathbb{C} \setminus \{0\})^k$ of $\mathbb{C}^n$ is free and wandering.

Proof. Since the action of $N$ on $\mathbb{C}^{n_0}$ is trivial, it follows that

$$C^*(G) \cong C^*(\mathbb{C}^n) \rtimes_\alpha N \cong C_0(\mathbb{C}^n) \rtimes_\alpha N \cong C_0(\mathbb{C}^{n-n_0}) \otimes (C_0(\mathbb{C}^{n-n_0}) \rtimes N)$$

where $\alpha_\alpha(z_i) = (e^{\lambda_{i+1}}z_i)$ for $(z_i) \in \mathbb{C}^n$. By the same argument before Theorem 2.1, we obtain a finite composition series $\{J_i\}_{i=1}^{n-n_0+1}$ of $C_0(\mathbb{C}^{n-n_0}) \rtimes N$ with subquotients $J_{n-n_0-k+1}/J_{n-n_0-k}$ isomorphic to $C^*(N)$ for $k = 0$, and

$$\oplus_{n_0+1 \leq i_1 < \cdots < i_k \leq n} C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_\alpha N \text{ for } 1 \leq k \leq n-n_0.$$ 

For $0 \leq k_1 \leq n_1$ and $0 \leq k_2 = l_1 + l_2 \leq n - n_0 - n_1$ with $k = k_1 + k_2$, the action of $N$ on each direct factor of $(\mathbb{C} \setminus \{0\})^{k_1}$ is free and wandering, and that on $(\mathbb{C} \setminus \{0\})^{k_2}$ is the multi-rotation by $\mathbb{R}^{l_2}$, and that on $(\mathbb{C} \setminus \{0\})^{l_1}$ is the multi-rotation by $\mathbb{T}^{l_1}$. If $k_2 = 0$, it is obtained by Green’s result [5] that

$$C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_\alpha N \cong C_0((\mathbb{C} \setminus \{0\})^k/N) \otimes \mathbb{K} \cong C(\mathbb{T}^k) \otimes \mathbb{K},$$

where the orbit space $(\mathbb{C} \setminus \{0\})^k/N$ is homeomorphic to $\mathbb{T}^k$. Next suppose $k_2 \geq 1$. Note that for the restriction of the action of $N$ to $(\mathbb{C} \setminus \{0\})^{k_1}$, the crossed product of $C_0((\mathbb{C} \setminus \{0\})^{k_1})$ by $N$ has the same structure for whether $\lambda_{ij} \not\in i\mathbb{R}$ ($1 \leq j \leq k_1$) are real or not. Thus we may assume that all $\lambda_{ij}$ ($1 \leq j \leq k_1$) are real. Then the action of $N$ on the circle direction of each direct factor of $(\mathbb{C} \setminus \{0\})^{k_1}$ is trivial, and that on the radius direction of each direct factor of $(\mathbb{C} \setminus \{0\})^{k_2}$ is also trivial. Hence it follows that

$$C_0((\mathbb{C} \setminus \{0\})^k) \rtimes_\alpha N \cong C_0(\mathbb{T}^{k_1} \times \mathbb{R}^{k_2}) \otimes (C_0(\mathbb{R}^{k_1} \rtimes \mathbb{T}^{k_2}) \rtimes N).$$

Since the action of $N$ on $\mathbb{R}^{k_1} \rtimes \mathbb{T}^{k_2}$ is transitive, it follows by Green’s theorem [4] that

$$C_0(\mathbb{R}^{k_1} \times \mathbb{T}^{k_2}) \rtimes N \cong C_0(N_{l_k}) \rtimes N \cong C^*(N_{l_k}) \otimes \mathbb{K}$$

where $N_{l_k}$ is the stabilizer of $1_k \in \mathbb{R}^{k_1} \times \mathbb{T}^{k_2}$. □
Remark. If \( N = [N, N] \) in the above setting, then \( C^*(G) \cong C_0(\mathbb{C}^n) \otimes C^*(N) \). Even if \( N \) is nilpotent, the structure of \( C^*(N_{1_k}) \) is still mysterious.

In the above setting, if \( G \) is a Lie semi-direct product of \( \mathbb{R}^n \) by a connected Lie group \( N \) with a real 1-dimensional, multi-diagonal action \( (\lambda_i \in \mathbb{R}) \), it is obtained that

**Theorem 3.2.** If \( G \) is a Lie semi-direct product of \( \mathbb{R}^n \) by a connected Lie group \( N \) with a real 1-dimensional, multi-diagonal action \( (\lambda_i \in \mathbb{R}) \), then \( C^*(G) \) has a finite composition series \( \{C_j\}_{j=1}^{n} \) with \( C_{n-n_0-k+1}/C_{n-n_0-k} \) isomorphic to \( C_0(\mathbb{R}^{n_0}) \otimes C^*(N) \) for \( k = 0 \), and

\[
\oplus_{n_0+1 \leq i_1, \ldots, i_k \leq n} C_0(\mathbb{R}^{n_0}) \otimes (\oplus^k \mathbb{K}) \quad \text{for} \quad 1 \leq k \leq n - n_0.
\]

**Proof.** Since the action of \( \mathbb{R} \) on \( \mathbb{R} \) is trivial or the translation, and that of \( T \) on \( \mathbb{R} \) is trivial, the action of \( N \) on each direct factor of \( (\mathbb{R} \setminus \{0\})^k \) is free and wandering. Thus Green’s result [5] implies that \( C_0((\mathbb{R} \setminus \{0\})^k) \rtimes N \cong C_0((\mathbb{R} \setminus \{0\})^k/N) \otimes \mathbb{K} \cong \oplus^k \mathbb{K} \).

As a special case of Theorem 3.1, let \( N = H_{2n+1} \) and \( G = \mathbb{C}^{2n} \rtimes \beta H_{2n+1} \).

We assume that the action \( \beta \) on \( \mathbb{C}^{2n} \) is the diagonal sum:

\[
\beta_g = \begin{pmatrix}
    e^{\lambda_1 b_1} & 0 &  &  \\
    & \ddots &  &  \\
    &  & e^{\lambda_n b_n} &  \\
    0 &  & 0 & e^{\mu_1 a_1} \\
    & \ddots &  &  \\
    &  & 0 & e^{\mu_n a_n}
\end{pmatrix} \in GL_{2n}(\mathbb{C})
\]

with \( g = (c, b, a) \in H_{2n+1}, \lambda_i, \mu_i \in \mathbb{C} \ (1 \leq i \leq n) \). Then it follows that

**Proposition 3.3.** If the action of \( H_{2n+1} \) on \( \mathbb{C}^{2n} \) is given as above, then group \( C^* \)-algebras of the stabilizers \( (H_{2n+1})_{1_k} \) \((0 \leq k \leq 2n)\) are isomorphic to the \( C^* \)-algebras of continuous fields over \( \mathbb{R} \) with the following fibers:

\[
\begin{align*}
    C_0(\mathbb{T}^{p+2q} \times \mathbb{R}^{2n-k}) \quad \theta = 0 \quad & \text{for} \ 0 \leq p + 2q \leq k, \theta \in \mathbb{R} \\
    C(\mathbb{T}^p) \otimes (\oplus^q \mathbb{A}_g) \otimes \mathbb{K} \quad \theta \neq 0 \quad & \text{for} \ \theta \in \mathbb{R}.
\end{align*}
\]

**Proof.** For \( 1_k \in (\mathbb{R}^{m_1} \times \mathbb{T}^{m_2}) \times (\mathbb{R}^{l_1} \times \mathbb{T}^{l_2}) \subset \mathbb{C}^{n} \times \mathbb{C}^{n} \) and \( k = m_1 + m_2 + l_1 + l_2 \), and \( m_i, l_i \geq 0 \ (i = 1, 2) \), we may assume that

\[
(H_{2n+1})_{1_k} \cong (\mathbb{R} \times \{0_{m_1}\}) \times \mathbb{Z}^{m_2} \times \mathbb{R}^{n-m_1-m_2} \rtimes \alpha (\{0_{l_1}\} \times \mathbb{Z}^{l_2} \times \mathbb{R}^{n-l_1-l_2}).
\]

By the Fourier transform,

\[
C^*((H_{2n+1})_{1_k}) \cong C_0(\mathbb{R} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n-m_1-m_2}) \rtimes \alpha(\mathbb{Z}^{l_2} \times \mathbb{R}^{n-l_1-l_2}).
\]
Since the subspace \( \{0\} \times T^{m_2} \times \mathbb{R}^{n-m_1-m_2} \) is fixed under \( \hat{a} \), it is obtained that

\[
0 \rightarrow C_0((\mathbb{R} \setminus \{0\}) \times T^{m_2} \times \mathbb{R}^{n-m_1-m_2}) \times_{\hat{a}} (\mathbb{Z}^2 \times \mathbb{R}^{n-l_1-l_2}) \\
\rightarrow C_0(\mathbb{R} \times T^{m_2} \times \mathbb{R}^{n-m_1-m_2}) \times_{\hat{a}} (\mathbb{Z}^2 \times \mathbb{R}^{n-l_1-l_2}) \\
\rightarrow C_0(T^{m_2+l_2} \times \mathbb{R}^{2n-k}) \rightarrow 0.
\]

By the similar reasons as before Theorem 2.2, the ideal in the above exact sequence is isomorphic to the 2-direct sum \( \oplus^2 C_0(\mathbb{R}_+ \cup \theta \in \mathbb{R}^+) (C(T^2) \otimes (\otimes^q \mathbb{A}_\theta) \otimes \mathbb{K}) \), where \( q \geq 0 \) is the cardinal number of the intersection \( \{m_1 + 1, \ldots, m_1 + m_2\} \cap \{l_1 + 1, \ldots, l_1 + l_2\} \), and \( p = m_2 + l_2 - 2q \geq 0 \).

Combining Proposition 3.3 with Theorem 3.1, it is obtained that

**Theorem 3.4.** Let \( G \) be a Lie semi-direct product of \( \mathbb{C}^{2n} \) by \( H_{2n+1} \) with a complex 1-dimensional, multi-diagonal action. Then \( C^*(G) \) has a finite composition series \( \{\mathcal{J}_j\}_{j=1}^K \) with \( \mathcal{J}_j/\mathcal{J}_{j-1} \) isomorphic to \( C_0(\mathbb{C}^{n_0} \times \mathbb{R}^{2n}) \) for \( j = K \), and \( C_0(\mathbb{C}^{n_0} \times (\mathbb{R} \setminus \{0\})) \otimes \mathbb{K} \) for \( j = K-1 \), and

\[
\begin{cases}
C_0(\mathbb{C}^{n_0} \times \mathbb{T}^k) \otimes \mathbb{K}, & \text{or} \\
C_0(\mathbb{C}^{n_0} \times \mathbb{T}^{k_1+m_2+l_2} \times \mathbb{R}^{2n-k_1}) \otimes \mathbb{K}, & \text{or} \\
C_0(\mathbb{C}^{n_0} \times \mathbb{T}^{k_1} \times \mathbb{R}^{k_2}) \otimes \mathbb{K} \otimes C_0(\mathbb{R}_+ \cup \theta \in \mathbb{R}^+) (C(T^2) \otimes (\otimes^q \mathbb{A}_\theta) \otimes \mathbb{K}) & \text{for } 1 \leq k \leq n - n_0 \text{ with } 0 \leq n_0 \leq n \text{ and } k_2 \geq 1, \ k = k_1 + k_2.
\end{cases}
\]

**Remark.** In the above statement, \( \otimes^q \mathbb{A}_\theta \) is regarded as a noncommutative torus of the form \( C(T^2) \rtimes_{\Theta} \mathbb{Z}^q \), where \( \Theta \) is the multi-rotation by the same angle \( \theta \) (cf. [13]).

As a corollary, it follows from the same argument of Corollary 2.4 that

**Corollary 3.5.** Under the same situation with Theorem 3.4, it is obtained that

\[
\text{sr}(C^*(G)) = n_0 + n + 1 = \dim_{\mathbb{C}} \hat{G}_1, \quad \text{and} \quad \text{csr}(C^*(G)) \leq n_0 + n + 1.
\]

To compute the stable rank and the connected stable rank of \( C^*(G) \) in Theorem 3.1, we need to compute the stable ranks of \( C^*(N) \). Fortunately, if \( N \) is a simply connected, nilpotent Lie group, then \( \text{sr}(C^*(N)) = \dim_{\mathbb{C}} \hat{N}_1 \) ([19]). Furthermore, this formula is extended to the connected case ([20]). On the other hand, it is obtained that
Proposition 3.6. Let \( \mathfrak{A} \) be a \( C^* \)-algebra of continuous fields over a locally compact \( T_2 \)-space \( X \) with fibers \( \mathbb{K}(H_x) \) on separable Hilbert spaces \( H_x \), \( x \in X \). Then \( \mathfrak{A} \) has continuous trace, and it is stable, i.e. \( \mathfrak{A} \cong \mathfrak{A} \otimes \mathbb{K} \).

Remark. By local triviality of continuous fields [4, Theorem 10.8.8], \( \mathfrak{A} \) in the statement is assumed to be an inductive limit of \( C_0(X_k) \otimes \mathbb{K} \) with \( \{X_k\}_{k=1}^{\infty} \) open subspaces of \( X \). This implies that \( \mathfrak{A} \) satisfies Fell’s condition ([4, Definition 10.5.7]). If necessary, by using Hjelmborg and Rørdam’s result [8, Corollary 4.1], the latter claim is obtained.

Combining [16, Theorem 3] with the above proposition and (F1), it follows that

Proposition 3.7. If \( N \) is a connected nilpotent Lie group, then

\[
\text{csr}(C^*(N)) \leq 2 \lor \text{csr}(C_0(\hat{N}_1)) = ([\dim \hat{N}_1 + 1]/2) + 1.
\]

Proof. If \( N \) is simply connected, we use the structure of \( C^*(N) \) in [19], Proposition 3.6 and (F1). Also, the inequality in the statement is valid in the connected case because if \( N \) is connected, then \( C^*(N) \) is regarded as a quotient of \( C^*(\hat{N}) \) of the universal covering group \( \hat{N} \) of \( N \), so that the structure of \( C^*(N) \) is inherited from that of \( C^*(\hat{N}) \).

Applying the above estimates and (F1-F3) to Theorem 3.1, it is obtained that

Corollary 3.8. In Theorem 3.1, if \( N \) is nilpotent, then

\[
\begin{align*}
\text{sr}(C^*(G)) &= \dim_{\mathbb{C}} \hat{G}_1 & &\text{if } \dim \hat{G}_1 \text{ is even,} \\
\dim_{\mathbb{C}} \hat{G}_1 &\leq \text{sr}(C^*(G)) \leq \dim_{\mathbb{C}} \hat{G}_1 + 1 & &\text{if } \dim \hat{G}_1 \text{ is odd, and} \\
\text{csr}(C^*(G)) &\leq 2 \lor \text{csr}(C_0(\hat{G}_1)) = ([\dim \hat{G}_1 + 1]/2) + 1.
\end{align*}
\]

Proof. In Theorem 3.1, notice that \( C_0(\mathbb{C}^{n_0}) \otimes C^*(N) \cong C^*(\mathbb{C}^{n_0} \times N) \), and \( \mathbb{C}^{n_0} \times N \) is a connected, nilpotent Lie group. By Theorem 3.1, \( \hat{G}_1 \) is homeomorphic to the product space \( \mathbb{C}^{n_0} \times \hat{N}_1 \). Thus it follows from the rank estimates given above that

\[
\begin{align*}
\text{sr}(C_0(\mathbb{C}^{n_0}) \otimes C^*(N)) &= \dim_{\mathbb{C}} (\mathbb{C}^{n_0} \times \hat{N}_1) = \dim_{\mathbb{C}} \hat{G}_1, \\
\text{csr}(C_0(\mathbb{C}^{n_0}) \otimes C^*(N)) &\leq 2 \lor \text{csr}(C_0(\mathbb{C}^{n_0} \times \hat{N}_1)) = ([\dim \hat{G}_1 + 1]/2) + 1.
\end{align*}
\]

On the other hand, it is obtained that
Corollary 3.9. In Theorem 3.1, if $N$ is a Lie semi-direct product $\mathbb{R}^m \rtimes \mathbb{R}$, then the same conclusion as Corollary 3.8 is obtained.

Proof. If $N = \mathbb{R}^m \rtimes \mathbb{R}$, the rank estimates of Corollary 3.8 hold for $C^*(N)$ ([18]).

Remark. As an example, let $M_5$ the Mautner group defined by the Lie semi-direct product $\mathbb{C}^2 \ltimes \mathbb{R}$ with $\gamma_t(z_1, z_2) = (e^{it}z_1, e^{it}z_2)$ for $t \in \mathbb{R}$, $z_1, z_2 \in \mathbb{C}$, and an irrational number $\pi$. Then $M_5/[M_5, M_5] \cong \mathbb{R}$. Define $\gamma$ by the Lie semi-direct product of $\mathbb{C}$ by $M_5$. If the action of $M_5$ on $\mathbb{C}$ is nontrivial, then $\text{sr}(\mathbb{C}^*(\gamma)) = 2$ and $\text{csr}(\mathbb{C}^*(\gamma)) = 2$ (cf. [18]). If the action is trivial, these stable ranks of $\mathbb{C}^*(\gamma)$ are 2 or 3.

The complex multi-dimensional case.

Next suppose that $G = \mathbb{C}^s \ltimes \alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with $\alpha$ a complex multi-dimensional, multi-diagonal action on a direct sum $\mathbb{C}^s = \bigoplus_{i=1}^n \mathbb{C}^s_i$, that is,

$$\alpha_t = (\oplus_{i=1}^{n-m} \alpha_i(t)) \oplus (\oplus_{j=n-m+1}^n \alpha_j(e^{it})) = \begin{pmatrix} \alpha_1(t_1) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \alpha_n(e^{it_n}) \end{pmatrix} \in GL_n(\mathbb{C})$$

with $t = ((t_1)_{i=1}^{n-m}, (e^{it})_{j=n-m+1}^n) \in \mathbb{R}^{n-m} \times \mathbb{T}^m$, where $\alpha_i (1 \leq i \leq n-m)$ and $\alpha_i (n-m+1 \leq i \leq n)$ are Lie actions of $\mathbb{R}$, $\mathbb{T}$ on $\mathbb{C}^s_i$ respectively. Then $G$ is isomorphic to the direct product $(\Pi_{i=1}^{n-m} (\mathbb{C}^s_i \ltimes \alpha_i, \mathbb{R})) \times (\Pi_{j=n-m+1}^n (\mathbb{C}^s_j \ltimes \alpha_j, \mathbb{T}))$. Then

$$\mathbb{C}^s(\gamma) \cong (\bigotimes_{i=1}^{n-m} \mathbb{C}^s(\mathbb{C}^s_i \ltimes \alpha_i, \mathbb{R})) \otimes (\bigotimes_{j=n-m+1}^n \mathbb{C}^s(\mathbb{C}^s_j \ltimes \alpha_j, \mathbb{T})).$$

By [18], the structure of $\mathbb{C}^s(\mathbb{C}^s_i \ltimes \alpha_i, \mathbb{R})$ is obtained from extensions by $\{R_{i,j}/ R_{i,j-1}\}_{j=1}^{K_i}$ isomorphic to $C_0(\mathbb{R}^{2u_i+1})$ for $j = K_i$, and

$$C_0(\mathbb{R}^{2u_i+v_i} \times \mathbb{T}^{w_i}) \otimes \mathbb{K}, \text{ or } C_0(\mathbb{R}^{2u_i+v_i}) \otimes \mathbb{K} \otimes \mathbb{A}_{o_{i,j}}$$

for $1 \leq j \leq K_i - 1$ with $u_i, v_i, w_i, o_{i,j} \geq 0$.

Thus we now consider the structure of $\mathbb{C}^s(\mathbb{C}^s_j \ltimes \alpha_j, \mathbb{T})$. Then it is obtained that

Proposition 3.10. Let $G = \mathbb{C}^n \ltimes \alpha \mathbb{T}$. Then there exists a finite composition series $\{I_j\}_{j=1}^{n-n_0+1}$ of $\mathbb{C}^s(\gamma)$ with $I_{n-n_0+1-k} \cong I_{n-n_0-k}$ isomorphic to $C_0(\mathbb{C}^{n_0} \times \mathbb{T})$ for $k = n - n_0 + 1$, and

$$\oplus_{1 \leq i_1 < \cdots < i_k \leq n-n_0} (C_0(\mathbb{C}^{n_0} \times \mathbb{R}_+^k \times \mathbb{T}^{k-1}) \otimes \mathbb{K}) \text{ for } 1 \leq k \leq n-n_0,$$

where $\mathbb{C}^{n_0}$ is the fixed point subspace under the action of $\mathbb{T}$. 

built up by a finite number of extensions by subquotients. Otherwise, we have the contradiction against compactness of orbits under $\alpha$. Then we may have the diagonal sum $\alpha(e^t) = \sum_{k=1}^{n_0} e^{i\theta_k t}$ with $\theta_k = 0$ for $1 \leq k \leq n_0$ with some $0 \leq n_0 \leq n$ and $\theta_k \in \mathbb{R} \setminus \{0\}$ for $n_0 + 1 \leq k \leq n$, where $\theta_k (n_0 + 1 \leq k \leq n)$ are linearly dependent over $\mathbb{Q}$. Then $C^*(\mathbb{C}^n \rtimes_\alpha \mathbb{T})$ is isomorphic to $C_0(\mathbb{C}^n) \otimes C_0(\mathbb{C}^{n-n_0} \rtimes \mathbb{T})$. Moreover, by the same way as Theorem 3.1, the tensor product on the right side has a finite composition series $\mathcal{J}_k^{n-n_0+1}$ such that

$$\mathcal{J}_{n-n_0+1-k}/\mathcal{J}_{n-n_0-k} \cong \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n-n_0} (C_0(\mathbb{C}^{n} \setminus \{0\})^k \times \mathbb{T}).$$

Each direct factor $C_0(\mathbb{C}^{n} \setminus \{0\})^k \times \mathbb{T}$ splits into $C_0(\mathbb{R}^k) \otimes (C(\mathbb{T}^k) \rtimes \mathbb{T})$. Since $\mathbb{T}^k$ is homeomorphic to $\mathbb{T}^{k-1} \times \mathbb{T}$ and an orbit of $\mathbb{T}$ is compatible with the action of $\mathbb{T}$, it follows that $C(\mathbb{T}^k) \rtimes \mathbb{T} \cong C(\mathbb{T}^{k-1}) \otimes (C(\mathbb{T}) \rtimes \mathbb{T}) \cong C(\mathbb{T}^{k-1}) \otimes \mathbb{K}$. \hfill \Box

**Remark.** The structure of group $C^*$-algebras of Lie semi-direct products $\mathbb{R}^n \rtimes_\alpha \mathbb{T}$ is obtained similarly by taking quotients of group $C^*$-algebras of $\mathbb{C}^n \rtimes_\beta \mathbb{T}$ with $\beta = \alpha + ia$.

The following theorem is obtained from the above argument:

**Theorem 3.11.** Let $G = \mathbb{C}^n \rtimes_\alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with $\alpha$ a complex multi-dimensional, multi-diagonal action. Then there exists a finite composition series $\{\mathcal{J}_j\}_{j=1}^K$ of $C^*(G)$ with $\mathcal{J}_j/\mathcal{J}_{j-1}$ isomorphic to $C_0(\mathbb{R}^{2u+n-m} \times \mathbb{Z}^m) = C_0(\tilde{G}_1)$ for $j = K$, and

$$\begin{align*}
C_0(\mathbb{R}^{2u_j+v_j} \times \mathbb{T}^{w_j}) \otimes \mathbb{K} \quad &\text{or} \\
C_0(\mathbb{R}^{2u_j+v_j} \times \mathbb{T}^{w_j}) \otimes \mathbb{K} \otimes (\bigotimes_{i=1}^{k_j} \mathfrak{A}_{\theta_i}) \quad &\text{or} \\
C_0(\mathbb{R}^{2u_j+v_j} \times \mathbb{T}^{w_j}) \otimes \mathbb{K} \otimes (\bigotimes_{i=1}^{k_j} \mathfrak{A}_{\theta_i}) \quad &\text{for } 1 \leq j \leq K - 1
\end{align*}$$

with $u, u_j, v_j, w_j \geq 0, 1 \leq k_j \leq n-1$, and $\mathfrak{A}_{\theta_i} \cong C(\mathbb{T}^{l_i}) \times \mathbb{Z}$ a noncommutative torus.

**Proof.** Note that $C^*(G)$ splits into the tensor product of $C_0(\mathbb{C}^n) \times \mathbb{R}$ ($1 \leq i \leq n-m$) and $C_0(\mathbb{C}^n) \rtimes \mathbb{T}$ ($n-m+1 \leq i \leq n$). Each tensor factor is built up by a finite number of extensions by subquotients $\{\mathcal{R}_{i,j}/\mathcal{R}_{i,j-1}\}_{j=1}^{K_i}$ given above. Then $C^*(G)$ is built up by a finite number of extensions by subquotients $\bigotimes_{i=1}^{n} (\mathcal{R}_{i,j}/\mathcal{R}_{i,j-1})$. \hfill \Box

**Remark.** This theorem is a generalization for the case $n = 1, m = 0$ obtained in [18]. If $\mathbb{C}^*$ is replaced by $\mathbb{R}^n$, the structure of $C^*(G)$ of $G =$
If the action of $N$ is nontrivial. Then by the same argument as in the proof of Corollary 3.8, the proof is complete.

**Remark.** The same result as above can be deduced in the case of Lie semi-direct products of $\mathbb{R}^n$ by connected nilpotent Lie groups or $\mathbb{R}^m \times \mathbb{R}$ (cf. Remark of Theorem 3.11).

**§4. The Lie Semi-direct Products of $\mathbb{R}^n \times \mathbb{C}^o$ by Connected Lie Groups with Diagonal Actions**

Let $G = (\mathbb{R}^n \times \mathbb{C}^o) \times_{\alpha} (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with a diagonal action $\alpha$. We may assume that $\alpha_g$ for $g = ((g_i)_{i=1}^{n-m}, (z^m_j)_{j=m+1}^{n}) \in \mathbb{R}^{n-m} \times \mathbb{T}^m$ is defined by the diagonal sum:

$$
\begin{pmatrix}
    e^{i(\sum_{j=1}^{n-m} g_{1j})} & 0 & \cdots & 0 \\
    0 & e^{i(\sum_{j=1}^{n-m} w_{1j} g_{1j})} & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & e^{i(\sum_{j=1}^{n-m} w_{uj} g_{uj})}
\end{pmatrix}
$$

with $g_{kj} \in \{g_i\}_{i=1}^{n-m}$ for $0 \leq j \leq p_k \leq n - m$ ($1 \leq k \leq u$), and $w_{kj} \in \mathbb{C}$, $g_{ki} \in \{g_i\}_{i=1}^{n}$ for $0 \leq j \leq q_k \leq n$ ($1 \leq k \leq v$). If $g_{kij} \in \{g_i\}_{i=n-m+1}^{n}$, then $w_{kij} = i$. Thus, we may assume that the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on each direct factor is nontrivial. Then
Theorem 4.1. Let $G$ be a Lie semi-direct product $(\mathbb{R}^n \times \mathbb{C}^u) \rtimes_\alpha (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ with a diagonal action $\alpha$. Then $C^*(G)$ has a finite composition series

$$\{J_j\}_{j=1}^K$$

such that

$$J_j/J_{j-1} \cong \begin{cases} C_0(\mathbb{R}^{u+n-m} \times \mathbb{C}^v \times \mathbb{Z}^m) & \text{for } j = K, \\ C_0(\mathbb{R}^{p_j} \times \mathbb{T}^0 \times \mathbb{Z}^0 \times \Omega_j) \otimes \mathbb{K}, & \text{or} \\ C_0(\mathbb{R}^{p_j} \times \mathbb{T}^0 \times \mathbb{Z}^0) \otimes \mathbb{A}_\alpha_j \otimes \mathbb{K} & \text{for } 1 \leq j \leq K-1 \end{cases}$$

with $p_j, q_j, r_j \geq 0$, where the fixed point subspace under $\hat{\alpha}$ is homeomorphic to $\mathbb{R}^u \times \mathbb{C}^v$, each $\Omega_j$ is an orbit subspace on whose preimage $\hat{\alpha}$ is wandering, and $\mathbb{A}_\alpha_j$ is a higher dimensional noncommutative torus.

Proof. By the similar argument as before Theorem 2.1, we obtain a finite composition series $\{J_j\}_{j=1}^{u+v+u+1}$ of $C^*(G)$ with subquotients $J_j/J_{j-1}$ isomorphic to

$$\oplus_{1 \leq k_1 < \cdots < k_u \leq u} \oplus_{1 \leq l_1 < \cdots < l_m \leq m} (C_0((\mathbb{R} \setminus \{0\})^{u_j} \times (\mathbb{C} \setminus \{0\})^{v_j} \times (\mathbb{R}^{n-m} \times \mathbb{T}^m))$$

with $u_j, v_j \geq 0$. From the analysis of actions of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on $\mathbb{C} \setminus \{0\}$ in the previous section, each direct factor is isomorphic to the direct sum of tensor products

$$\oplus 2^{\nu_j} (C_0(\mathbb{T}^{v_j1} \times \mathbb{R}^{+}) \otimes (C_0(\mathbb{R}^{u_j+v_j1} \times \mathbb{T}^{v_j2} \times (\mathbb{C} \setminus \{0\})^{v_j-v_j1-v_j2}) \times (\mathbb{R}^{n-m} \times \mathbb{T}^m))$$

with $0 \leq v_j1 + v_j2 \leq v_j$, where $\mathbb{R}^{n-m} \times \mathbb{T}^m$ acts on direct factors of $\mathbb{R}^{u_j+v_j1}$ by translation, on those of $\mathbb{T}^{v_j2}$ by rotation and on those of $(\mathbb{C} \setminus \{0\})^{v_j-v_j1-v_j2}$ transitively. Put $X_j = \mathbb{R}^{u_j+v_j1} \times \mathbb{T}^{v_j2} \times (\mathbb{C} \setminus \{0\})^{v_j-v_j1-v_j2}$. Note that if a direct factor of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ acts on $X_j$ trivially, $C_0(X_j) \times (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ has the tensor factor $C_0(\mathbb{R})$ or $C_0(\mathbb{Z})$. Thus we assume that each direct factor of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ acts on $X_j$ nontrivially.

Suppose that the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$ on $X_j$ is wandering. We can analyze the orbit space $\Omega_j = X_j/(\mathbb{R}^{n-m} \times \mathbb{T}^m)$ under the action of $\mathbb{R}^{n-m} \times \mathbb{T}^m$, and every orbit in this subspace has the same type. Thus $X_j$ is homeomorphic to the product space of $\Omega_j$ and an orbit. Thus, Green’s result [6] implies that $C_0(\Omega_j) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m)$ is isomorphic to

$$C_0(\Omega_j) \otimes C^*((\mathbb{R}^{n-m} \times \mathbb{T}^m)/(\mathbb{R}^{n-m} \times \mathbb{T}^m)_{1_{u_j+v_j}}) \rtimes (\mathbb{R}^{n-m} \times \mathbb{T}^m) \cong C_0(\Omega_j) \otimes C^*((\mathbb{R}^{n-m} \times \mathbb{T}^m)_{1_{u_j+v_j}}) \otimes \mathbb{K}$$

where $(\mathbb{R}^{n-m} \times \mathbb{T}^m)_{1_{u_j+v_j}}$ is the stabilizer of $1_{u_j+v_j} \in (\mathbb{R} \setminus \{0\})^{u_j} \times (\mathbb{C} \setminus \{0\})^{v_j}$, and it is isomorphic to a product group of either $\mathbb{R}$, $\mathbb{T}$ or $\mathbb{Z}$. 

Next suppose that the action of $\mathbb{R}^{n-m} \times T^m$ on $X_j$ is not wandering. Then $X_j = T^{v_2}$. If the action is 1-dimensionally multi-diagonal, then

$$C(X_{j}) \rtimes (\mathbb{R}^{n-m} \times T^m) = C(T^v) \rtimes (\mathbb{R}^{n-m} \times T^m) \cong (\otimes^{n-m} C(T) \rtimes \mathbb{R}) \otimes (\otimes^m C(T) \rtimes T) \cong C(T^{n-m}) \otimes K.$$ 

If the action is multi-dimensionally multi-diagonal, then

$$C(X_{j}) \rtimes (\mathbb{R}^{n-m} \times T^m) = C(\Pi_{k=1}^{n} T^{l_k}) \rtimes (\mathbb{R}^{n-m} \times T^m) \cong (\otimes^{n-m} C(T) \rtimes \mathbb{R}) \otimes (\otimes_{k=m+1}^{n} C(T) \rtimes T)$$

with $\sum_{k=1}^{n} l_k = v_2$. Moreover, each direct factor $C(T^{l_k}) \rtimes \mathbb{R}$ is assumed to be a foliation $C^*$-algebra. Thus $C(T^{l_k}) \rtimes \mathbb{R} \cong (C(T^{l_k-1}) \rtimes \mathbb{R}) \rtimes \mathbb{K}$, where $C(T^{l_k-1}) \rtimes \mathbb{R}$ is a special case of higher dimensional noncommutative tori, say $\mathfrak{A}_\Theta$ (cf. [2], [18]). For other direct factors, it is obtained that $C(T^{l_k}) \rtimes T \cong C(T^{l_k-1}) \rtimes (C(T) \rtimes T) \cong C(T^{l_k-1}) \rtimes \mathbb{K}$ since the action of $T$ on $T^{l_k}$ is periodic. More generally, since dimension of stabilizers of points of $X_j$ is fixed, $C(X_{j}) \rtimes (\mathbb{R}^{n-m} \times T^m)$ is also assumed to be a foliation $C^*$-algebra. If the action of $\mathbb{R}^{n-m} \times T^m$ on $X_j$ is transitive, we obtain the same conclusion as the case of wandering actions. The other cases can be treated the similar way as the case of multi-dimensionally multi-diagonal actions. In fact, since the action on each direct factor of $X_j$ is explicitly given, we can find an invariant torus $T^{v_j}$ transversal to every orbits under $\mathbb{R}^{n-m} \times T^m$ such that $C(X_{j}) \rtimes (\mathbb{R}^{n-m} \times T^m) = C(T^{v_j}) \rtimes (\mathbb{R}^{n-m} \times T^m)$ is isomorphic to $C(T^{v_j}) \rtimes (\mathbb{R}^{n-m} \times T^m) \rtimes (\mathbb{R}^{n-m} \times T^m)$ for some $n_1, n_2, n_3 \geq 0$, where $C(T^{v_j}) \rtimes \mathbb{Z}^{n_3}$ is a special case of $\mathfrak{A}_\Theta$.

Remark. The proof of this theorem suggests that each $\Omega_j$ is also homeomorphic to a product space $T^{k_j} \times \mathbb{R}^{s_j} \times \mathbb{Z}^{t_j}$ for some $k_j, s_j, t_j \geq 0$.

Similarly, it is obtained that

**Theorem 4.2.** Let $G$ be a Lie semi-direct product of $\mathbb{R}^n \times \mathbb{C}^v$ by a connected Lie group $N$ with a diagonal action. Then there exists a finite composition series $\{J_j\}_{j=1}^K$ of $C^*(G)$ such that

$$J_j/J_{j-1} \cong \begin{cases} C_0(\mathbb{R}^{q_j} \times \mathbb{C}^{r_j}) \otimes C^*(N) & \text{for } j = K, \\ C_0(\mathbb{R}^{p_j} \times T^{s_j} \times \mathbb{Z}^{r_j} \times \Omega_j) \otimes C^*(N_{q_j}) \otimes \mathbb{K} & \text{or} \\ C_0(\mathbb{R}^{p_j} \times T^{r_j} \times \mathbb{Z}^{s_j}) \otimes C^*_r(W_j) \otimes \mathbb{K} & \text{for } 1 \leq j \leq K - 1 \end{cases}$$

with $p_j, q_j, r_j \geq 0$, where $\mathbb{R}^{q_j} \times \mathbb{C}^{r_j}$ is the fixed point subspace under the dual action of $N$, each $\Omega_j$ is an orbit subspace on whose preimage the dual action of
$N$ is wandering, and $N_{z_j}$ means the stabilizer of a point $z_j$ of an $N$-invariant subspace of $\mathbb{R}^n \times \mathbb{C}^v$, and $C^*_r(W_j)$ means the reduced $C^*$-algebra of a reduced groupoid $W_j$ associated with orbits on an $N$-invariant torus.

Proof. Note that a diagonal action of $N$ is reduced to that of $N/[N,N]$. Thus we can use the setting of Theorem 4.1. It suffices to consider the crossed product $C_0(X_j) \rtimes N$. If $X_j$ is the fixed point subspace under the action of $N$, it is homeomorphic to $\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}$. Then $C_0(X_j) \rtimes N \cong C_0(\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}) \otimes C^*(N)$. If the action of $N$ on $X_j$ is wandering,

$$C_0(X_j) \rtimes N \cong C_0(X_j/N) \otimes (C^*(N/N_{z_j}) \rtimes N) \cong C_0(\Omega_j) \otimes C^*(N_{z_j}) \otimes K$$

where $N_{z_j}$ is the stabilizer of $z_j \in X_j$ and $\Omega_j = X_j/N$. If the action of $N$ on $X_j$ is not wandering, it follows from some techniques of foliation $C^*$-algebras that

$$C_0(X_j) \rtimes N \cong C^*_r(X_j \rtimes N) \cong C^*_r(W_j) \otimes K$$

where $C^*_r(X_j \rtimes N)$ means the reduced (foliation) $C^*$-algebra of the groupoid $X_j \rtimes N$ arising from the action of $N$ of $X_j$, and $C^*_r(W_j)$ means the reduced $C^*$-algebra of the reduced groupoid $W_j$ of $X_j \rtimes N$ (cf. [9]).

Remark. If $N = [N,N]$ in the above setting, then $C^*(G) \cong C_0(\mathbb{R}^n \times \mathbb{C}^v) \otimes C^*(N)$.

Corollary 4.3. In Theorem 4.2, if $N$ is a connected, nilpotent Lie group, or a Lie semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$, then the same rank estimates as in Corollary 3.8 hold.

Example 4.4. Let $G$ be a Lie semi-direct product of $\mathbb{R}^n \times \mathbb{C}^v$ by the Mautner group $M_5$ with a diagonal action. Note that $M_5/[M_5, M_5] \cong \mathbb{R}$. Then $C^*(G)$ has a finite composition series $\{\mathcal{J}_j\}_{j=1}^K$ with each subquotient $\mathcal{J}_j/\mathcal{J}_{j-1}$ isomorphic to

$$\begin{cases} C_0(\mathbb{R}^{u_0} \times \mathbb{C}^{v_0}) \otimes C^*(M_5) & \text{for } j = K, \\ C_0(\mathbb{R}^{p_1+2u_j} \times \mathbb{T}^{q_j}) \otimes C^*(\mathbb{C}^2) \otimes K & \text{or} \\ C_0(\mathbb{R}^{p_1+2u_j}) \otimes (C^*(\mathbb{T}^{q_j}) \rtimes M_5) & \text{for } 1 \leq j \leq K-1 \end{cases}$$

where the second, third cases respectively correspond to that the action of $M_5/[M_5, M_5]$ is free, the multi-rotation on an invariant subspace of $\mathbb{R}^n \times \mathbb{C}^v$. Moreover,

$$C^*(\mathbb{T}^{q_j}) \rtimes M_5 \cong \begin{cases} C^*(\mathbb{T}^{q_j}) \rtimes (\mathbb{C}^2 \rtimes \mathbb{R}) \cong C_0(\mathbb{T}^{q_j} \times \mathbb{C}^2) \rtimes \mathbb{R}, \\ C^*_r(\mathbb{T}^{q_j} \times M_5) \cong C^*_r(\mathbb{T}^{q_j} \times \mathbb{C}^2 \rtimes \mathbb{Z}) \otimes K \end{cases}$$
with \( C^*(\mathbb{T}^q \times C^2 \times \mathbb{Z}) \cong C_0(\mathbb{T}^q \times C^2) \times \mathbb{Z} \), where the lower isomorphism is obtained by some techniques of foliation \( C^* \)-algebras (cf. [9]). In the upper case, since \( \mathbb{T}^q \times \{0\} \) is invariant under the action of \( \mathbb{R} \), it is obtained that

\[
0 \to C_0(\mathbb{T}^q \times (C^2 \setminus \{0\})) \rtimes \mathbb{R} \to C_0(\mathbb{T}^q \times C^2) \rtimes \mathbb{R} \to C(\mathbb{T}^q) \times \mathbb{R} \to 0
\]

with the quotient isomorphic to \( (C(\mathbb{T}^q)^{-1} \times \mathbb{Z}) \otimes \mathbb{K} \), and

\[
0 \to C_0(\mathbb{T}^q \times (C \setminus \{0\})) \rtimes \mathbb{R} \to C_0(\mathbb{T}^q \times (C^2 \setminus \{0\})) \rtimes \mathbb{R}
\]

\[
\to \oplus^2 C_0(\mathbb{T}^q \times (C \setminus \{0\})) \rtimes \mathbb{R} \to 0
\]

where two direct factors of the quotient, and the ideal are respectively isomorphic to

\[
\left\{ C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}^q)^{+1}) \rtimes \mathbb{R}) \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}^q) \times \mathbb{Z}) \otimes \mathbb{K}
\right. \\
\left. C_0(\mathbb{R}_+^2) \otimes (C(\mathbb{T}^q)^{+2}) \rtimes \mathbb{R}) \cong C_0(\mathbb{R}_+^2) \otimes (C(\mathbb{T}^q+1) \times \mathbb{Z}) \otimes \mathbb{K}
\right.
\]

(cf. [18]). On the other hand, the structure of \( C^*(M_5) \) is given by [18]. Moreover, \( C^*(G) \) has no nontrivial projections.

Acknowledgement

The author would like to thank the referee for some suggestions for revision.

References