Introduction

It has been recognized that the notion of triple [12] (or monad [8]) provides a unified simplicial method for defining homology and cohomology in categorical setting (Godement [15], Huber [19], Eilenberg-Moore [12], Dold-MacLane-Oberst [8], Beck [5]).

It has been shown that many known, classical or special, (co-)homology theories of groups, modules and algebras (Eilenberg-MacLane [10], Cartan-Eilenberg [6], Hochschild [17], Harrison [16], Shukla [27], etc.) are triple cohomologies (Barr-Beck [1], Barr [2], [3], Iwai [20]).

In the former announcement [26], we treated triple cohomologies viewing them as derived functors (in a functor category) in the sense of relative homological algebra [11], [25]. Since then such interpretations have appeared (Dold-MacLane-Oberst [8], Dubuc [9]). Therefore we will not discuss this subject here.

We will treat the calculation or interpretation of triple cohomology of an algebra with coefficients in a module [5]. The 0th
and 1st cohomology groups $H^0$ and $H^1$ (the dimension indices in triple cohomology being one less than usual) were discussed by J. M. Beck in his dissertation [5]. The purpose of the present paper is to interpret the second cohomology, $H^2(A,M)$, of an algebra $A$ with coefficients in an $A$-module $M$ as the set of equivalence classes after Yoneda [29] of two term extensions of $A$ by $M$ (see §3, Lichtenbaum-Schlessinger [23] or Gerstenhaber [13] for two term extensions).

Our interpretation appears to be more direct than those through classical obstruction theory for algebra extensions (MacLane [24], [10], Hochschild [18], Shukla [27], Barr [4]) and suggests a close relationship between $H^n$ and $n$ term extensions for $n \geq 2$ (see §4). In fact, such an interpretation of $H^n$ has been obtained by A. Iwai, one of the present authors, and is to appear in his subsequent paper [21].

In the sequel we choose and deal with a specific category, the category of Lie algebras over a commutative ground ring. The argument is functorial, at least in substance, so is applicable to other categories of algebraic systems with tripleable underlying object functors [5].

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§ 1. Preliminaries

Recall the notion of cotriple [12]. A cotriple $G=(G, \varepsilon, \delta)$ in a category $\mathcal{C}$ consists of a (covariant) functor $G: \mathcal{C} \to \mathcal{C}$ and natural transformations $\varepsilon: G \to I_\mathcal{C}$ ($I_\mathcal{C}$ denotes the identity functor) and $\delta: G \to G \circ G$ satisfying

$$\varepsilon G \circ \delta = G \varepsilon \circ \delta = 1_G$$

($1_G$ denotes the identity natural transformation),

$$G \delta \circ \delta = \delta G \circ \delta,$$

where $\varepsilon G$ means the natural transformation defined by

$$\varepsilon G(A) = \varepsilon(G(A)): G(A) \to G(A),$$

and similarly for $G \varepsilon$, $G \delta$, and $\delta G$. Triples are defined dually.

A cotriple comes usually from an adjoint pair of functors $(S, U)$,
Let $S$ be the left adjoint of $U$ (notation $S \leftarrow U$). This means that there exists a natural equivalence $\lambda$ between the Hom set functors:

$$\lambda(C, A) : \text{Hom}_C (S(C), A) \cong \text{Hom}_C (C, U(A))$$

for $C \in \text{ob} \mathcal{C}$, $A \in \text{ob} \mathcal{A}$. Define natural transformations $\varepsilon$ and $\eta$ by

$$\varepsilon(A) = \lambda^{-1}(1_{U(A)}): SU(A) \to A, \quad \text{for } A \in \text{ob} \mathcal{A},$$

$$\eta(C) = \lambda(1_{S(C)}): C \to US(C), \quad \text{for } C \in \text{ob} \mathcal{C},$$

with abbreviation $\lambda = \lambda(U(A), A)$ resp. $\lambda = \lambda(C, S(C))$. Then we have

$$\lambda(\rho) = U(\rho) \circ \eta(C), \quad \text{for } \rho \in \text{Hom}_C (S(C), A),$$

with $\lambda = \lambda(C, A)$ and

$$\varepsilon S \circ S \eta = 1_S : S \to SUS \to S,$$

$$U \varepsilon \circ \eta U = 1_U : U \to USU \to U.$$

It follows that the adjoint pair $(S, U)$ yields a cotriple $(SU, \varepsilon, S \eta U)$ in $\mathcal{A}$ (dually a triple $(US, \eta, U \varepsilon S)$ in $\mathcal{C}$).

Conversely, given a cotriple $(G, \varepsilon, \delta)$ in $\mathcal{A}$. It is known ([12], [22]) that there exist a category $\mathcal{C}$ and an adjoint pair of functors $(S, U)$, $U : \mathcal{A} \to \mathcal{C}$, $S : \mathcal{C} \to \mathcal{A}$ with $S \leftarrow U$, inducing the cotriple $(G, \varepsilon, \delta)$ as above.

From now on we assume that all categories considered are pointed (i.e. have zero objects) with kernels and all functors are also pointed (i.e. $T(0)=0$), unless otherwise stated.

Given an adjoint pair of functors $U : \mathcal{A} \to \mathcal{C}$ and $S : \mathcal{C} \to \mathcal{A}$ with $S \leftarrow U$ which induces a cotriple $G = (G, \varepsilon, \delta)$ in $\mathcal{A}$. There is naturally defined a simplicial object $G_n = \{G_n; n \geq 0\}$ in the category of endofunctors $\mathcal{A} \to \mathcal{A}$, with $G_n = G^{n+1} = G \circ \cdots \circ G((n+1)$-fold iterated composition of $G$) and

$$\varepsilon' = G^i \varepsilon G^{n-i} : G_n \to G_{n-1}, \quad 0 \leq i \leq n \quad \text{(face operator)},$$

$$\delta' = G^i \delta G^{n-i} : G_n \to G_{n+1}, \quad 0 \leq i \leq n \quad \text{(degeneracy operator)}.$$

Here we have the usual commutation rule:

$$\varepsilon^i \delta^j = & \begin{cases} 
\varepsilon^{j-1} \varepsilon^i & (i < j) \\
\text{identity} & (i = j \text{ or } j + 1) \\
\delta^j \varepsilon^{i-1} & (i > j + 1)
\end{cases}$$
\[ \varepsilon'^j = \varepsilon'^{-1} \varepsilon^j \quad (i < j) \]
\[ \delta'^j = \delta'^{-1} \delta^j \quad (i \leq j) \]

with abbreviations \( \varepsilon'^n = \varepsilon'_n \) and \( \delta'^n = \delta'_n \). For any object \( A \) in \( \mathcal{G} \) the corresponding simplicial object \( G_\#(A) = \{ G_n(A) ; n \geq 0 \} \) in \( \mathcal{G} \) is called the standard simplicial complex over \( A \) associated with the cotriple \( G \). And the augmented simplicial complex \( G^j_\#(A) = \{ G_n(A) ; n \geq -1 \} \) with natural augmentation \( \varepsilon(A) : G_0(A) \to G_\#(A) = A \), i.e. the sequence

\[
\begin{align*}
\varepsilon^n & \to G_n(A) \to G_{n-1}(A) \to \cdots \to G_0(A) \to A, \\
& \varepsilon^n
\end{align*}
\]

together with degeneracy operators, is called the standard simplicial resolution of \( A \).

The following lemma is a slight modification of Theorem in [20].

**Lemma 1.4.** Suppose that \( U(A) \) is an abelian group object in \( \mathcal{C} \) for every object \( A \) in \( \mathcal{G} \). Then the augmented chain complex \( UG^\#_\#(A) \) with differential \( d_n(A) = \sum_{i=0}^n (-1)^i U\varepsilon^i(A) \) \( (n > 0) \) and augmentation \( U\varepsilon(A) \) is acyclic. Precisely, there exists a contracting homotopy \( s_n(A) : UG_n(A) \to UG_{n+1}(A) \) \( (n \geq -1) \) such that \( U\varepsilon \circ s_{-1} = 1 \) and

\[ d_{n+1} s_n + s_{n-1} d_n = 1 \quad (n \geq 0) \].

**Proof.** Define morphisms \( t_n(A) : UG_m(A) \to UG_{m+1}(A) \) and \( u_n(A) : UG_n(A) \to UG_{n+1}(A) \), for \( 0 \leq n \leq m \), as follows:

\[
\begin{align*}
t_0 &= 1, \quad u_{-1} = 0, \quad u_0 = 0 \\
t_n &= (1 - U\delta^0 \varepsilon^0)(1 - U\delta^1 \varepsilon^0) \cdots (1 - U\delta^{n-1} \varepsilon^0) \quad (n \geq 1) \\
u_n &= t_0 \circ U\delta^0 - t_1 \circ U\delta^1 + \cdots + (-1)^{n-1} t_{n-1} \circ U\delta^{n-1} \quad (n \geq 1).
\end{align*}
\]

Then we have

\[
U\varepsilon^i \circ t_n = \begin{cases} t_{n-1} \circ d_n & (i = 0) \\ 0 & (0 < i \leq n) \end{cases}
\]
\[
1 - t_n = d_{n+1} u_n + u_{n-1} d_n \quad (n \geq 0).
\]

Now define morphisms \( \eta_n(A) = \eta UG_n(A) : UG_n(A) \to UG_{n+1}(A) \), which satisfy
Using these we get the required contracting homotopy:

\[ s_n = \eta_n t_n + u_n \quad (n \geq 0) \]
\[ s_{-1} = \eta_{-1} = \eta U. \]

Note that all these \( t_n, u_n, \eta_n \) and \( s_n \) are natural transformations of functors \( G \to C \).

For later use, we give the following variant of the above lemma which will be similarly proved.

**Lemma 1.9.** Suppose that there is given another category \( C' \) and a functor \( U' : C \to C' \) such that \( U'U(A) \) (but not \( U(A) \)) is an abelian group object in \( C' \) for every object \( A \) in \( \mathfrak{A} \). Then the augmented chain complex \( U'U \mathfrak{G} \mathfrak{A}(A) \) with differential \( d_n(A) = \sum_{i=0}^{n} (-1)^i U'U \mathfrak{G}^i(A) \) \((n > 0)\) and augmentation \( U'U \mathfrak{G}(A) \) is acyclic.

For the purpose of defining cotriple cohomology, we first fix an object \( A \) in \( \mathfrak{A} \) and consider the comma category \( (\mathfrak{G}, A) \) [5]. By definition an object in \( (\mathfrak{G}, A) \) is a pair \( (B, \gamma) \) of an object \( B \) in \( \mathfrak{G} \) and a morphism \( \gamma : B \to A \) in \( \mathfrak{G} \), and a morphism \( (B, \gamma) \to (B', \gamma') \) is such a commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\varphi} & B' \\
\downarrow{\gamma} & & \downarrow{\gamma'} \\
A
\end{array}
\]

An object \( (B, \gamma) \) in \( (\mathfrak{G}, A) \) will be often denoted by \( B \to A \) or simply by \( B \). Note that the comma category \( (\mathfrak{G}, A) \) is not pointed for \( A = 0 \), but has kernels and the terminal object \( A = (A, 1_A) \).

An adjoint pair \( C \to \mathfrak{G} \to C \) induces canonically an adjoint pair \( (C, U(A)) \to (\mathfrak{G}, A) \to (C, U(A)) \) which will be denoted by the same symbols \( S \) and \( U \) as before, and similarly for the induced cotriple \( \mathcal{G} = (G, \varepsilon, \delta), \mathcal{G} = SU, \delta = S\eta U \). Note that \( G^\mathbb{A}(A) \) is regarded canonically as an object in \( (\mathfrak{G}, A) \) with the unique morphism \( G^\mathbb{A}(A) \to A \) expressed by a composition of face operators.
Now denote by $\text{Ab}$ the category of abelian groups and let $T : (\mathfrak{g}, A) \to \text{Ab}$ be a contravariant functor. Then we have a cochain complex $TG_*(A) = \{TG_n(A) ; n \geq 0\}$ with differential $d^*(A) = \sum (-1)^i \times T\ell^i(A)$. Its derived groups $H^n(TG_*(A))$ are, by definition, the cotriple cohomology groups of $A$ with initial group $H^0(TG_*(A)) = T(A)$.

For example, take an abelian group object $Y \to A$ in $(\mathfrak{g}, A)$ and define a functor $T : (\mathfrak{g}, A) \to \text{Ab}$ by $T(X) = \text{Hom}_{\mathfrak{g}, A}(X, Y)$. Then we have the cohomology groups $H^n_\mathfrak{g}(A, Y) = H^n(TG_*(A))$ with $H^0_\mathfrak{g}(A, Y) = \text{Hom}_{\mathfrak{g}, A}(A, Y)$.

§ 2. Lie Algebras and Cotriple

Let $K$ be a commutative ring with unit which we fix as ground ring. By a $K$-Lie algebra we mean a $K$-module $\Gamma$ with a $K$-bilinear product $[x, y] \in \Gamma$ such that

$\begin{align*}
[x, x] &= 0, \\
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0
\end{align*}$

for $x, y, z \in \Gamma$. By a $\Gamma$-module for a $K$-Lie algebra $\Gamma$ we mean a $K$-module $M$ with left operation of $\Gamma$ on $M$ such that $[x, y]m = xym - yxm$ for $x, y \in \Gamma$, $m \in M$.

Let $\mathcal{L}$ be the category of all $K$-Lie algebras with obvious morphisms, and $\mathcal{M}$ be the category of all $\Gamma$-modules. Then every abelian group object in the comma category $(\mathcal{L}, \Gamma)$ is known to be of the form of the split extension $\Gamma \ast M$ of $\Gamma$ by a $\Gamma$-module $M$. This may be called also the idealization of a $\Gamma$-module $M$ and is defined as a direct product $\Gamma \times M$ of $K$-modules with bracket product

$\begin{align*}
[[x, m], (y, n)] &= ([x, y], xn - ym)
\end{align*}$

for $x, y \in \Gamma$ and $m, n \in M$. With the injection $m \to (0, m)$ and the projection $(x, m) \to x$, and regarding $M$ as an abelian Lie algebra, we have a split exact sequence

$0 \to M \to \Gamma \ast M \to \Gamma \to 0$

in $\mathcal{L}$ (also in $(\mathcal{L}, \Gamma)$).

Denote by $\text{Ab}(\mathcal{L}, \Gamma)$ the full subcategory of $(\mathcal{L}, \Gamma)$ formed of all abelian group objects. Then we have an equivalence of categories
where $\text{Ker}$ denotes the kernel functor and $\Theta(M) = \gamma^*M$ [5].

For a $\Gamma$-module $M$ and an object $(L, \gamma)$ in $(\mathcal{L}, \Gamma)$, a $K$-derivation (or simply derivation) $f : L \rightarrow M$ is defined as a $K$-linear map such that

$$\text{(2.2)} \quad f[x, y] = \gamma(x)f(y) - \gamma(y)f(x).$$

The set of all such $K$-derivations $f : L \rightarrow M$ forms an abelian group denoted by $\text{Der}_M(L, \gamma)$ (or simply by $\text{Der}_M(L)$), and it defines the derivation functor $\text{Der}_M : (\mathcal{L}, \Gamma) \rightarrow \text{Ab}$. As is well known, there is a canonical isomorphism

$$\text{(2.3)} \quad \text{Der}_M(L) \cong \text{Hom}_{(\mathcal{L}, \Gamma)}(L, \Theta(M)).$$

Now we consider the cotriple cohomology $\text{H}^n(\mathcal{G}, \Theta(M)) = H^n(\text{Der}_M G^*(\Gamma'))$ with respect to a cotriple $\mathcal{G}$ in $(\mathcal{L}, \Gamma)$ as defined at the end of §1. For brevity we occasionally denote $H^n(\Gamma', \Theta(M))$ by $\text{H}^n(\Gamma', M)$ (or $H^n(\Gamma, M)$) and call this the $n$-th cohomology group of $\Gamma$ with coefficients in $M$.

To calculate explicitly $H^n(\Gamma', M)$, we shall choose some typical cotriples $\mathcal{G}$ in $\mathcal{L}$. Take $\mathcal{L}$ for $\mathcal{G}$ in the preceding section. Let $\mathcal{C}$ be either one of the following pointed categories: 1) the category $\mathcal{K}\mathcal{P}$ of $K$-modules, 2) the category $\mathcal{S}$ of pointed sets (i.e. sets with base points and base points preserving maps) and 3) the category $\mathcal{S}^\times$ of pointed sets with multiplications. More explanation is needed for the last category. An object in $\mathcal{S}^\times$ is a pointed set $(X, x_0)$ with multiplication (may be non-associative) $X \times X \rightarrow X$ such that $x_0 \cdot x = x \cdot x_0 = x_0$ for any $x \in X$ and a morphism $f : (X, x_0) \rightarrow (Y, y_0)$ is a multiplication preserving set map.

According to each of the above cases 1), 2) and 3), $\mathcal{C}$ will be also denoted by $\mathcal{C}_i$ ($i = 1, 2, 3$). Let $U_i : \mathcal{L} \rightarrow \mathcal{C}_i$ be the underlying object functor. It is clear that the $U_i$ are faithful and known to be tripleable in the sense of Beck [5] for all the above cases of $\mathcal{C}$. The left adjoint $S_i : \mathcal{C}_i \rightarrow \mathcal{L}$ of $U_i$ will be given as follows in respective cases.

In case $\mathcal{C}_i = \mathcal{K}\mathcal{P}$, $S_i$ is given by the functor $L$ described in [6],
that is, for a $K$-module $M$, $S_1(M) = L(M)$ is the quotient $K$-Lie algebra of the free non-associative algebra $A(M) = M + M \otimes M + (M \otimes M) \otimes M + M \otimes (M \otimes M) + \cdots$ by the two-sided ideal generated by elements of the form $m \otimes m$ and $m_i \otimes (m_s \otimes m_t) + m_t \otimes (m_s \otimes m_i) + m_s \otimes (m_t \otimes m_i)$ for $m, m_i \in M$.

In case $C_2 = S$, $S_2$ is given by $S_2(X, x_0) = L(F(X, x_0))$ where $F(X, x_0) = K(X)/K(x_0)$ is the free $K$-module generated by the set $X$ with identification $x_0 = 0$.

In case $C_3 = C_x$, the functor $S_3$ is given as follows: Let $(X, x_0)$ be an object in $S^x$. Construct first the free $K$-module $F(X, x_0)$ as above and introduce in it a unique $K$-bilinear multiplication induced from the multiplication of $X$. Then the $K$-Lie algebra $S_3(X, x_0)$ is defined as the quotient of the non-associative $K$-algebra $F(X, x_0)$ by the two-sided ideal generated by elements of the form $x \cdot x$ and $x \cdot (y \cdot z) + y \cdot (x \cdot z) + z \cdot (x \cdot y)$.

The corresponding cotriples $G$ and the cohomologies $H^*_G(\Gamma, M)$ will be denoted by $G_i$ and $H^*_G(\Gamma, M)$ in respective cases $i = 1, 2, 3$.

We remark that $H^*_G(\Gamma, M)$ is the Hochschild cohomology [17] and $H^*_S(\Gamma, M)$ is related to the cohomologies of Dixmier [7] and Shukla [28]. It is known in [5] and [1] that

$$H^*_G(\Gamma, M) \cong \begin{cases} \text{Der}_M(\Gamma) & (n=0) \\ \text{Ex}^1_M(\Gamma, M) & (n=1) \end{cases}$$

for $i = 1, 2, 3$, where $\text{Ex}^1_M(\Gamma, M)$ denotes the set of all isomorphism classes of singular $U$-split extensions of $\Gamma$ by $M$ in $\mathcal{L}$. The bijective correspondence $H^*_G \cong \text{Ex}^1$ becomes an isomorphism of $K$-modules if we introduce a suitable Baer sum in $\text{Ex}^1_M(\Gamma, M)$.

To conclude this section, we are situated in the following commutative diagram

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {$U_1$};
\node (B) at (2,0) {$U''$};
\node (C) at (0,-2) {$U_3$};
\node (D) at (2,-2) {$U'$};
\node (E) at (0,-4) {$U_2 = U''U_1 = U'U_3$};
\node (F) at (2,-4) {$S_2 = S.F = S_3S'$};
\node (G) at (0,-1) {$S_1$};
\node (H) at (2,-1) {$F$};
\node (I) at (0,-3) {$S_x$};
\node (J) at (2,-3) {$S'$};

\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (D) -- (A);
\draw[->] (G) -- (H);
\draw[->] (H) -- (I);
\draw[->] (I) -- (J);
\draw[->] (J) -- (G);
\end{tikzpicture}
\end{center}
where $U'$ and $U''$ are the forgetful functors and $S'$ is the left adjoint of $U'$.

§ 3. Two Term Extensions and Main Theorem

Let $M$ be a $\Gamma$-module which may be regarded as an abelian $K$-Lie algebra. By a $U$-split exact sequence in $\mathcal{L}$ we mean a sequence in $\mathcal{L}$ of which transformation by $U$ is split exact (in $\mathcal{C}$).

Definition. By a two term extension of $\Gamma$ by $M$ with respect to the underlying object functor $U_i$ we mean a $U_i$-split exact sequence in $\mathcal{L}$ ($i=1,2,3$):

$$(e) : \quad 0 \to M \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} \Gamma \to 0$$

where $X_0$ operates on $X_1$ in the following way:

1. $\varphi_2(\varphi_0(x)m) = x \cdot \varphi_2(m)$,
2. $\varphi_1(x \cdot u) = [x, \varphi_1(u)]$,
3. $\varphi_1(u) \cdot v = [u, v]$ for $x \in X_0$, $m \in M$, $u, v \in X_1$. Moreover, in the case $i=3$ (i.e. $\mathcal{C}=S^\times$), we put an extra condition: there exists a set map $\beta : X_0 \to X_1$ such that

$$(3.4) \quad \varphi_2\beta = 1 - \sigma \varphi_0 \beta,$$

$\beta[x, y] = x \beta(y) - \sigma \varphi_0 y \cdot \beta(x)$ for $x, y \in X_0$,

where $\sigma : \Gamma \to X_0$ is a morphism in $S^\times$ satisfying $\varphi_0 \sigma = 1$ (the existence of such a map $\sigma$ is ensured by the $U$-splitness of the sequence $(e)$).

The totality of all such two term extensions of $\Gamma$ by $M$ with respect to $U_i$ form a category $\mathcal{E}_i$ of which a morphism $(e) \to (e')$ is given by a commutative diagram:

$$
\begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \varphi_0 \\
0 & \to & X_1 \\
\downarrow \varphi_1 & & \downarrow \varphi_0 \\
0 & \to & X_0 \\
\downarrow \varphi_2 & & \downarrow \varphi_0 \\
0 & \to & \Gamma \\
\end{array}
$$

$$(e') : \quad 0 \to M \xrightarrow{\varphi_2'} Y_1 \xrightarrow{\varphi_1'} Y_0 \xrightarrow{\varphi_0'} \Gamma \to 0$$

where $\varphi_0$ and $\varphi_1$ are morphisms in $\mathcal{L}$ compatible with the operations of $X_0$ and $Y_0$ on $X_1$ and $Y_1$ ($i=1,2,3$). To extensions $(e)$ and $(e')$
are called equivalent (notation \((e) \sim (e')\)) if they are connected by a sequence of morphisms of both directions: e.g.,

\[(e) = (e_0) \leftarrow (e_1) \rightarrow (e_2) \rightarrow \cdots \leftarrow (e_n) = (e').\]

The set of all equivalence classes of two term extensions of \(\Gamma\) by \(M\) with respect to the underlying object functor \(U_i\) is denoted by \(\text{Ex}_i^2(\Gamma, M)\) \((i = 1, 2, 3)\).

Now we can state our main theorem.

**Theorem 3.5.** *There is a bijective correspondence*

\[H_i^2(\Gamma, M) \approx \text{Ex}_i^2(\Gamma, M)\]

*for each* \(i = 1, 2, 3\).

The proof of this theorem will be given in §5.

We remark that, for an object \(L\) in \(\mathcal{L}\), the underlying \(K\)-module \(U_i(L)\) and the underlying set \(U_3(L)\) are clearly abelian group objects in \(\mathcal{C}K\) and \(\mathcal{S}\) respectively, but \(U_2(L)\) is not so in \(\mathcal{S}^\times\), which is the reason why we need the exceptional condition (3.4) in the definition of two term extensions with respect to \(U_2\).

§ 4. Standard Two Term Extensions

Take any one of the cotriples \(G_i\) in \((\mathcal{L}, \Gamma), i = 1, 2, 3\), as before. The standard simplicial complex \(G^*(\Gamma)\) over \(\Gamma\) induces the underlying chain complex of \(K\)-modules, denoted by the same notation \(G_i^*(\Gamma)\), with differential \(d_n = \sum_{j=0}^n (-1)^j e^j(\Gamma)\) \((n \geq 1)\) and the augmented complex \(G_i^*(\Gamma)\) is acyclic by Lemma 1.4 in cases \(\mathcal{C} = \mathcal{K}\) or \(\mathcal{S}\), by Lemma 1.9 in case \(\mathcal{C} = \mathcal{S}^\times\).

Let \(M\) be a \(\Gamma\)-module. \(G^*(\Gamma)\) operates naturally on \(M\) via the canonical morphism \(G^*(\Gamma) \rightarrow \Gamma\) (see §1), so that \(M\) is considered as a \(G^*(\Gamma)\)-module for every \(n \geq 0\).

A derivation 2-cocycle \(f \in \text{Der}_M(G^*(\Gamma))\) is a \(K\)-linear map \(f : G^*(\Gamma) \rightarrow M\) and \(f[x, y] = xf(y) - yf(x)\) for \(x, y \in G^*(\Gamma)\) and \(d^2 = fd_3 = 0 : G^*(\Gamma) \rightarrow M\). Two such cocycles \(f\) and \(f'\) are \(D\)-cohomologous (notation \(f \sim f'\)) if there exists a derivation \(\omega : G^*(\Gamma) \rightarrow M\) and \(\omega d_3 = f - f'\).

Given a derivation 2-cocycle \(f \in \text{Der}_M(G^*(\Gamma))\). We shall construct
a $K$–Lie algebra $E_f$ as follows. Put $\tilde{G}_1(\Gamma) = \text{Ker}(\varepsilon : G(\Gamma) \to G(\Gamma))$. Let $\tilde{G}_1(\Gamma) \times M$ be the direct product of $K$–Lie algebras ($M$ being an abelian Lie algebra). Let $I$ be the ideal of $\tilde{G}_1(\Gamma) \times M$ generated by elements of the form $(-t, d_2y, f(t, y))$ for $y \in G(\Gamma)$ (see (1.5)). Define $E_f$ to be the quotient $K$–Lie algebra $\tilde{G}_1(\Gamma) \times M / I$. Then $E_f$ has a set-presentation $N(\Gamma) \times M$, where $N(\Gamma) = \text{Ker}(\varepsilon : G(\Gamma) \to \Gamma)$. To see this, we define set maps

$$\pi : \tilde{G}_1(\Gamma) \times M \to N(\Gamma) \times M$$

and

$$\kappa : N(\Gamma) \times M \to \tilde{G}_1(\Gamma) \times M$$

by

$$\pi(x, m) = (d_1x, m + f(\bar{x}))$$

and

$$\kappa(n, m) = (\bar{n}, m)$$

respectively, where $\bar{x} = \gamma_n(x)$ for $x \in UG_n(\Gamma)$ (see (1.7)).

Direct calculations show that

$$\pi(I) = (0, 0)$$

$$\pi \circ \kappa = \text{identity}$$

and

$$\kappa \circ \pi \equiv \text{identity mod } I.$$

It follows that $\pi$ induces a canonical one to one correspondence $E_f \cong N(\Gamma) \times M$ as set.

Denote by $(n, m)$ an element of $E_f$ for $n \in N(\Gamma)$ and $m \in M$ in this presentation. Then the $K$–Lie algebra structure of $E_f$ is given explicitly by

(4.1) $(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2 + f(\bar{n_1}, \bar{n_2}))$,

(4.2) $k(n, m) = (kn, km + f(kn))$, for $k \in K$,

(4.3) $[(n_1, m_1), (n_2, m_2)] = ([n_1, n_2], f(\bar{n_1}, \bar{n_2}))$,

where we used the notation

(4.4) $x = \gamma_n(x) \in UG_{n+1}(\Gamma)$

for $x \in UG_n(\Gamma)$ (see §1, (1.7)).

**Remark.** If $\mathcal{C} = \kappa \mathcal{N}$, then $E_f = N(\Gamma) \oplus M$ (direct sum) as $K$–
module. If $C=S^x$, then (4.3) reduces to

$$[(n_1, m_1), (n_2, m_2)] = ([n_1, n_2], 0).$$

Define morphisms $i: M \to E_f$ and $\varphi: E_f \to G(\Gamma)$ by $i(m) = (0, m)$ and $\varphi(n, m) = n$ respectively. Then we have a $U$-split exact sequence in $\mathcal{L}$:

(4.5) $$(e_f): 0 \to M \overset{i}{\to} E_f \overset{\varphi}{\to} G(\Gamma) \to \Gamma \to 0$$

which will be called a \textit{standard two term extension} of $\Gamma$ by $M$ (with respect to the underlying object functor $U: \mathcal{L} \to \mathcal{C}$). By (4.3) $M$ is contained in the centre of $E_f$. $G(\Gamma)$ operates on $E_f$ by

(4.6) $$x(n, m) = ([x, n], x m + f([x, n])), $$

so that we have

(4.7) $$i(xm) = x i(m),$$

(4.8) $$\varphi(xu) = [x, \varphi(u)]$$

and

(4.9) $$\varphi(u)v = [u, v]$$

for $x \in G(\Gamma)$, $m \in M$ and $u, v \in E_f$.

Further we have the following commutative diagram of $K$-modules:

(4.10) $$\begin{array}{c}
G'(\Gamma) \xrightarrow{d_2} G'(\Gamma) \xrightarrow{d_2} G'(\Gamma) \xrightarrow{d_1} G(\Gamma) \xrightarrow{\varphi} \Gamma \xrightarrow{i} E_f \xrightarrow{\varphi} G(\Gamma) \xrightarrow{\varphi} \Gamma \xrightarrow{j} 0 \\
0 \xrightarrow{i} M \xrightarrow{e_f} 0
\end{array}$$

where $\alpha$ is a canonical $K$-linear map $G'(\Gamma) \to E_f$ given as follows. Using the canonical decomposition (see Lemmas 1.4, 1.9)

$$x = d_s x + s d_t x, \quad \text{for} \quad x \in G'(\Gamma),$$

we define $\alpha$ by

(4.11) $$\alpha(x) = (d_t x, f s x).$$

Then we can verify that $\alpha$ is $K$-linear.

\textbf{Proposition 4.12.} The $K$-linear map $\alpha$ satisfies
\[ \alpha[x, y] = \varepsilon x \cdot \alpha(y) - \varepsilon y \cdot \alpha(x) \quad \text{for} \quad x, y \in G^2(\Gamma). \]

**Proof.** The right-hand side will be computed by using (4.10), (4.13) as follows:

\[ \varepsilon x \alpha(y) - \varepsilon y \cdot \alpha(x) = (d, [x, y], m_1 + m_2) \]

where

\[ m_1 = \varepsilon x \cdot f s, y - \varepsilon y \cdot f s, x, \]

\[ m_2 = f([\varepsilon x, d, y] - [\varepsilon y, d, x] + [\varepsilon x, d, y] - [\varepsilon y, d, x]). \]

Since \( f \) is a derivation, we have

\[ m_1 = f(t, s, y) = f(1 - s, t, y), \]

where \( t = 1 - s \cdot \varepsilon^i : G^r(\Gamma) \to G^r(\Gamma) \) (Cf. (1.5)). Therefore \( m_1 + m_2 \) is written as \( f(\xi) \) for a certain element \( \xi \in G^r(\Gamma) \), and a direct calculation shows that \( d \xi = d \xi [x, y] \). Since \( f \) is a cocycle and \( G^2(\Gamma) \) is acyclic as above, we obtain

\[ m_1 + m_2 = f(\xi) = f[x, y]. \]

The proposition follows from (4.13) and (4.14).

We have thus assigned canonically the standard two term extension \( (e, f) \) to each derivation 2-cocycle \( f : G^2(\Gamma) \to M \).

We show that the extension \( (e, f) \) belongs to \( E_i \) (see §3). This is clear in case of \( C = K \mathbb{R} \) or \( S \) by (4.6~8). In case of \( C = S^3 \), we define a set map \( \beta : G(\Gamma) \to E \) by

\[ \beta(x) = \alpha \eta_0(x) = \alpha(x). \]

Then, we have, by (1.7)

\[ \alpha \eta \beta = \alpha \alpha \eta_0 = d \eta_0 = 1 - \eta \alpha, \]

and by Prop. 4.12

\[ \beta[x, y] = \alpha[x, y] = \alpha[x, y] = \varepsilon x \cdot \alpha(y) - \varepsilon y \cdot \alpha(x) = x \beta(y) - \eta \varepsilon y \cdot \beta(x), \]

so that the condition (3.4) is satisfied.

**Proposition 4.15.** If derivation 2-cocycles \( f \) and \( f' \) are \( D \)-cohomologous \( (f \sim f') \), then we have \( (e, f) \sim (e, f'). \)
Proof. Suppose that $f' = f + \omega d_2$ for a derivation $\omega : G^2(\Gamma) \to M$. Define $\psi : E_f \to E_{f'}$ by

$$\psi(n, m) = (n, m - \omega(n)) \in E_{f'}.$$  

Then $\psi$ is a (bijective) morphism (in $\mathcal{L}$) and commutes with the operations of $G(\Gamma)$ on $E_f$ and $E_{f'}$. We have thus a morphism $(e_f) \to (e_{f'})$ in the category $\mathcal{C}_i$ of two term extensions of $\Gamma$ by $M$ (see §3).

Remark. In a parallel way as above, we can define what to be called the standard $n$-term extension $(e_f)$ of $\Gamma$ by $M$ for derivation $n$-cocycle $f : G^n(\Gamma) \to M$ ($n \geq 2$) as follows.

Define first the Moore subcomplex $\tilde{G}_k(\Gamma)$ of the chain complex $G_k(\Gamma)$ of $K$-modules by

$$(4.16) \quad \tilde{G}_k(\Gamma) = G_k(\Gamma),$$

$$\tilde{G}_k(\Gamma) = \bigcap_{i=1}^k \ker(\delta^i : G^{k+i}(\Gamma) \to G^k(\Gamma)) \quad \text{for } k > 0.$$  

Then $\tilde{G}_k(\Gamma)$ is an ideal of $G^{k+1}(\Gamma)$ and we have a commutative diagram of $K$-modules:

$$\begin{array}{cccccc}
\cdots & \to & G^{n+1}(\Gamma) & \overset{d_n}{\to} & G^n(\Gamma) & \to & \cdots \\
\downarrow t_n & & \downarrow t_{n+1} & & \downarrow t_1 & & \downarrow t_0 \\
\cdots & \to & \tilde{G}_n(\Gamma) & \overset{\delta_0}{\to} & \tilde{G}_{n-1}(\Gamma) & \to & \cdots & \tilde{G}_1(\Gamma) & \overset{\delta_0}{\to} & \tilde{G}_0(\Gamma) & \overset{\delta_0}{\to} & \Gamma & \to 0 \\
\end{array}$$

$$(4.17)$$

where

$$(4.18) \quad t_n = (1 - \delta^0\epsilon^1)(1 - \delta^1\epsilon^2) \cdots (1 - \delta^{n-1}\epsilon^n), \quad n \geq 0$$

are retractions and define a chain equivalence $G_k(\Gamma) \simeq \tilde{G}_k(\Gamma)$.

Now let $\tilde{G}_{n-1}(\Gamma) \times M$ be the direct product of $K$-Lie algebras ($M$ being an abelian Lie algebra). Let $I$ be the ideal of $\tilde{G}_{n-1}(\Gamma) \times M$ generated by elements of the form $(-t_{n-1}d_n(y), f(t_ny))$ for $y \in G^{n+1}(\Gamma)$. Define $E_f = \tilde{G}_{n-1}(\Gamma) \times M/I$ to be the quotient $K$-Lie algebra. Then $E_f$ has a canonical set presentation $N_{n-2}(\Gamma) \times M$, where $N_{n-2}(\Gamma) = \ker(\delta^i : \tilde{G}_n(\Gamma) \to \tilde{G}_{n-1}(\Gamma))$ for $n \geq 3$ and $N_0(\Gamma) = N(\Gamma) = \ker(\delta : G(\Gamma) \to \Gamma)$ as before. Using this set presentation, we can give the $K$-Lie algebra structure of $E_f$ explicitly by the same form of formulas as in (4.1), (4.2) and (4.3) (in fact, we have only to replace $N(\Gamma)$ by $N_{n-2}(\Gamma)$).
Then we have a $U$-split exact sequence $(e_f)$ in $\mathcal{L}$:

$$0 \to M \overset{t}{\to} E_f \overset{\varphi}{\to} \bar{G}_{n-2}(\Gamma) \to \cdots \to \bar{G}_0(\Gamma) \to \Gamma \to 0$$

(4.19) which we call a standard $n$ term extension of $\Gamma$ by $M$ with respect to the underlying object functor $U$.

Similarly as in (4.10), we have the following commutative diagram

$$
\begin{array}{ccc}
G^n(\Gamma) & \to & G^{n+1}(\Gamma) \\
\downarrow f & & \downarrow \alpha \\
0 & \to & M \\
\end{array}
$$

(4.20) where $\alpha$ is the canonical $K$-map $G^n(\Gamma) \to E_f$ explicitly given by

$$
\alpha(x) = (t_{n-2}, d_{n-1}x, f s_{n-1}x).
$$

(4.21)

§ 5. Proof of Theorem 3.5

In the last section we have defined a map $\Phi : H^i_{\Gamma}(\Gamma, M) \to \text{Ex}^i(\Gamma, M)$ ($i = 1, 2, 3$) (see Proposition 4.15). We shall prove the following two propositions of which the first asserts the ontoness of $\Phi$ and the second one asserts $\Phi$ to be $1$–$1$.

**Proposition 5.1.** Given a two term extension $(e) \in \mathcal{E}_i$ ($i = 1, 2, 3$). Then there exists a derivation 2-cocycle $f : G^2(\Gamma) \to M$ such that $(e_f) \to (e)$ in $\mathcal{E}_i$.

**Proposition 5.2.** If $(e_f) \to (e) \to (e_f')$ in $\mathcal{E}_i$ ($i = 1, 2, 3$), then $f$ and $f'$ are $D$-cohomologous (i.e. $f$ and $f'$ determine the same cohomology class $\in H^2_{\Gamma}(\Gamma, M)$).

**Proof of Proposition 5.1.** Given a two term extension $(e) \in \mathcal{E}_i$:

$$(e) : 0 \to M \overset{\varphi_2}{\to} X_1 \overset{\varphi_1}{\to} X_0 \overset{\varphi_0}{\to} \Gamma \to 0$$

which is $U_i$–split exact sequence in $\mathcal{L}$ and $X_0$ operates on $X_1$ as in (3.1), (3.2) and (3.3) with additional condition (3.4) in case of $\mathcal{E}_i = S^\times$.

There exists a morphism $\sigma : U(\Gamma) \to U(X_0)$ in $\mathcal{C}$ with $\varphi_0 \sigma = 1$, so that $\sigma$ determines a unique morphism $\tau = \lambda^{-1}(\sigma) : G(\Gamma) \to X_0$ in $\mathcal{L}$ by
(1.1). Then we have

$$\varphi_0 \tau = \varepsilon.$$ 

Define the idealization $X_0 \ast X_1$ of $X_0$-module $X_i$. That is, $X_0 \ast X_1$ is the direct sum of $X_0$ and $X_1$ as $K$-module and the bracket product is given by

$$(5.3) \quad [(x, u), (y, v)] = ([x, y], [u, v] + x \cdot v - y \cdot u)$$

for $x, y \in X_0$, $u, v \in X_1$.

We have a morphism $\beta : U(X_0) \to U(X_i)$ in $\mathcal{C}$ such that

$$\varphi_1 \beta = 1 - \sigma \varphi_0$$

in case of $\mathcal{C} = \kappa \mathbb{N}$ or $S$. In case of $\mathcal{C} = S^\times$, we must take such a $\beta$ as in (3.4).

Now define a morphism $\rho' : UG(\Gamma) \to U(X_0 \ast X_1)$ in $\mathcal{C}$ by

$$(5.4) \quad \rho'(x) = (\tau \eta x, \beta \tau x) = (\sigma \varphi_0 \tau x, \beta \tau x) \quad \text{for} \ x \in G(\Gamma).$$

This determines a unique morphism $\rho : G^2(\Gamma) \to X_0 \ast X_1$ in $\mathcal{L}$.

**Lemma 5.5.** The morphism $\rho : G^2(\Gamma) \to X_0 \ast X_1$ is expressed by

$$\rho(x) = (\tau \varepsilon x, g(x)),$$

where $g : G^2(\Gamma) \to X_1$ is $K$-linear and satisfies:

$$\varphi_1 g = \tau d_1$$

and

$$g[x, y] = \tau \varepsilon \cdot g(y) - \tau \varepsilon \cdot g(x)$$

for $x, y \in G^2(\Gamma)$.

**Proof.** Define two morphisms $\theta^0, \theta^1 : X_0 \ast X_1 \to X_0$ in $\mathcal{L}$ by

$$\theta^0(x, u) = x + \varphi_1(u)$$

and

$$\theta^1(x, u) = x.$$

Then using (1.3), we have

$$\theta^0 \rho(x) = \theta^0 \rho'(x) = \tau \eta \varepsilon(x) + \varphi_1 \beta \tau x = \tau x = \tau \varepsilon(x)$$

and

$$\theta^1 \rho(x) = \theta^1 \rho'(x) = \tau \eta \varepsilon(x) = \tau \varepsilon(x).$$
Using again (1.3), we conclude that

$$\theta^p \rho = \tau \epsilon^p$$

and

$$\theta^q \rho = \tau \epsilon^q,$$

as morphisms $G^q(\Gamma) \to X_0$ in $\mathcal{L}$. The lemma follows from these properties of $\rho$.

Returning to the proof of Prop. 5.1, consider the following commutative diagram:

$$
\begin{array}{c}
G^3(\Gamma) \xrightarrow{d_2} G^2(\Gamma) \xrightarrow{d_1} G(\Gamma) \xrightarrow{\epsilon} \Gamma \to 0 \\
\uparrow f \hspace{1cm} \uparrow g \hspace{1cm} \uparrow \tau \\
0 \to M \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} \Gamma \to 0
\end{array}
$$

where $f = \varphi_2^{-1} g d_1$ is a derivation 2-cocycle by Lemma 5.5. We have now a morphism $(e_f) \to (e)$ in $\mathcal{L}$:

$$
\begin{array}{c}
(e_f) : 0 \to M \xrightarrow{\ell} E_f \xrightarrow{\varphi} G(\Gamma) \xrightarrow{\epsilon} \Gamma \to 0 \\
\uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \\
(e) : 0 \to M \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} \Gamma \to 0
\end{array}
$$

where $\phi : E_f \to X_1$ is a morphism in $\mathcal{L}$ defined by

$$\phi(n, m) = \varphi_2(m) + g(\bar{n})$$

q. e. d.

**Proof of Proposition 5.2.** Suppose that there is given a commutative diagram.

$$
\begin{array}{c}
(e_f) : 0 \to M \xrightarrow{\ell} E_{f'} \xrightarrow{\varphi} G(\Gamma) \xrightarrow{\epsilon} \Gamma \to 0 \\
\uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \\
(e) : 0 \to M \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} \Gamma \to 0 \\
\uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \\
(e_{f'}) : 0 \to M \xrightarrow{\ell'} E_{f'} \xrightarrow{\varphi'} G(\Gamma) \xrightarrow{\epsilon'} \Gamma \to 0
\end{array}
$$

Define a morphism $\rho' : U(\Gamma) \to U(X_0 \ast X_1)$ by

$$\rho'(x) = (\tau(x), \beta \tau'(x)).$$
Then \( \rho' \) determines a unique morphism \( \rho : G(\Gamma) \to X \ast X \) in \( \mathcal{L} \).

**Lemma 5.6.** The morphism \( \rho \) is expressible in the form

\[
\rho(x) = (\tau(x), \omega(x)),
\]

where \( \omega : G(\Gamma) \to X \) is \( K \)-linear and satisfies:

\[
\varphi_1 \omega = \tau' - \tau
\]

and

\[
\omega[x, y] = \tau'(x) \cdot \omega(y) - \tau(y) \cdot \omega(x), \quad \text{for } x, y \in G(\Gamma).
\]

The proof is similar as in that of Lemma 5.5 and hence omitted.

Now consider a \( K \)-linear map

\[
(5.7) \quad \omega_i = \phi' \alpha' - \phi \alpha - \omega d_i : G^i(\Gamma) \to X,
\]

where \( \alpha' : G^i(\Gamma) \to E_{f'} \) and \( \alpha : G^i(\Gamma) \to E_f \) are defined in (4.11). Then

\[
\varphi_1 \omega_i = \tau' \phi' \alpha' - \tau \phi \alpha - (\tau' - \tau) d_i = 0.
\]

Therefore \( \omega_i \) is regarded as a map \( G^i(\Gamma) \to M \). A straightforward calculation shows that

\[
\omega_i[x, y] = \tau' \epsilon x \cdot \omega_i(y) - \tau \epsilon' y \cdot \omega_i(x)
\]

\[
= \epsilon \epsilon x \cdot \omega_i(y) - \epsilon \epsilon' y \cdot \omega_i(x).
\]

This means that \( \omega_i : G^i(\Gamma) \to M \) is a derivation. And clearly we have

\[
\varphi_1 \omega_i d = \phi' \alpha'_ d - \phi \alpha d = \phi' f' - \phi f = \varphi_1 (f' - f),
\]

that is, \( \omega_i d = f' - f \). q. e. d.

**References**


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