Normal Positive Linear Mappings of Norm 1 from a von Neumann Algebra into Its Commutant and Its Application

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Abstract

Let $M$ and $N$ be von Neumann algebras such that $N \subseteq M'$. Let $Z = N \cap M$ and $\rho$ be any normal positive linear functional of $(M \cup N)''$. There exists a unique mapping $F_\rho^{\vee M}$ from $M$ into $N$ satisfying

$$(1/2)\rho(F_\rho^{\vee M}(Q_1)Q_2 + Q_2F_\rho^{\vee M}(Q_1)) = \rho(Q_1Q_2)$$

for all $Q_1 \in M$, $Q_2 \in N$ and $s(F_\rho^{\vee M}(Q_1)) \leq s^N(\rho)$, where $s$ denotes the support and $s^N$ denotes the support in $N$. The mapping $F_\rho^{\vee M}$ is $Z$-linear, positive and transposed-$\alpha$-positive, of norm 1 and continuous on the unit ball weakly and strongly.

As an application, a generalization of a clustering theorem for an asymptotically abelian case is given.

§ 1. Preliminaries

We consider two von Neumann algebras $M$ and $N$ such that $N \subseteq M'$ and a normal positive linear functional $\rho$ of $(M \cup N)''$. $H_\rho$, $\pi_\rho$, and $Q_\rho$ denote a Hilbert space, a representation of $(M \cup N)''$ and a cyclic vector canonically associated with $\rho$ through $\rho = \omega_\rho$ where $\omega_\rho$ denotes the expectation functional by the vector $Q$ (called a vector state if $\omega_\rho(1) = 1$).

$s(A)$ for an operator $A$ on a Hilbert space denotes the support of $A$, namely the smallest projection $E$ satisfying $EA = AE = A$. $s(A)$ is in the von Neumann algebra generated by $A$ and $A^*$ and hence the notation $s(A)$ is also used for an element of von Neumann algebra. $s^N(\rho)$ denotes

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the support of \( \rho \) relative to \( N \), namely the smallest projection \( E \) in \( N \) such that \( \rho(E) = \rho(1) \). \( s^N(\Omega) \) denotes \( s^N(\omega_\Omega) \).

Our tool is the following version of the Radon-Nikodym theorem by Sakai [6].

**Lemma 1.** Let \( \mu \) and \( \nu \) be normal positive linear functionals of a von Neumann algebra \( N \) such that \( \mu \geq \nu \). There exists a unique \( h_0 \in N \) satisfying

1. \( \nu(Q) = (1/2)\mu(h_0 Q + Qh_0) \), \( Q \in N \),
2. \( s(h_0) \leq s^N(\mu) \),
3. \( 0 \leq h_0 \leq 1 \).

**Proof.** The existence of \( h_0 \) satisfying (1) and (3) is in [6]. Since \( 0 \leq \nu(1-s^N(\mu)) \leq \mu(1-s^N(\mu)) = 0 \), we have \( s^N(\nu) \leq s^N(\mu) \). Setting \( Q = s^N(\mu)h_0(1-s^N(\mu)) \), we obtain from (1)

\[
0 = \nu(Q) = (1/2)\mu(Qh_0 + h_0 Q) = (1/2)\mu(QQ^*) .
\]

Since \( s^N(\mu)QQ^*s^N(\mu) = QQ^* \), we obtain \( QQ^* = 0 \), i.e. \( Q = Q^* = 0 \). Hence

\[
h_0 = h'_0 + h''_0
\]

where \( h'_0 = s^N(\mu)h_0 s^N(\mu) \) and \( h''_0 = (1-s^N(\mu))h_0(1-s^N(\mu)) \). Since

\[
\mu(h_0 Q + Qh_0) = \mu(h'_0 Q + Qh'_0)
\]

\( h'_0 \in N \) satisfies (1), (2) and (3).

The uniqueness holds in the following slightly more general form. Q.E.D.

**Lemma 2.** Let \( \mu \) and \( \nu \) be normal linear functionals of \( N \) and \( \mu \) be positive. An operator \( h_0 \in N \) satisfying (1) and (2) of Lemma 1 is unique, if it exists.

**Proof.** Suppose \( h_0 \) and \( h'_0 \) satisfy (1) and (2). Then \( h = h_0 - h'_0 \) satisfy \( \mu(hQ + Qh) = 0 \) for all \( Q \in N \). Substituting \( Q = h^* \), we have
\[ 0 \leq \mu(h^*h) \leq \mu(hh^* + h^*h) = 0 \]

and hence \( s^N(\mu)h^*hs^N(\mu) = 0 \). Since \( s(h) \leq s^N(\mu) \), we have \( h^*h = 0 \) and hence \( h_0 - h_0' = h = 0 \).

Q.E.D.

We use Lemma 1 in the following complex form.

**Lemma 3.** Let \( \mu \) and \( \nu \) be normal linear functionals of \( N \),

\[ \nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4), \]

\( \mu, \nu_1, \nu_2, \nu_3, \) and \( \nu_4 \) be positive and \( \nu_k \leq \lambda \mu, \ k = 1, 2, 3, 4, \ \lambda > 0 \). There exists a unique \( h_0 \in N \) satisfying the conditions (1) and (2) of Lemma 1.

**Proof.** Immediate from Lemmas 1 and 2. Q.E.D.

A linear mapping \( F \) from a von Neumann algebra \( M \) into \( N \) is called \( n \)-positive if the mapping \( F \otimes 1 \) from \( M \otimes \mathcal{B}(C^n) \) to \( N \otimes \mathcal{B}(C^n) \) is positive, where \( C^n \) is an \( n \)-dimensional Hilbert space, \( \mathcal{B}(C^n) \) is the set of all linear operators on \( C^n \) and \( (F \otimes 1)(Q \otimes Q') = F(Q) \otimes Q' \) for \( Q \in M, Q' \in \mathcal{B}(C^n) \). If \( F \) is \( n \)-positive for all positive integers \( n \), \( F \) is called completely positive.

\( F \) is called transposed-\( n \)-positive if \( F \otimes t \) from \( M \otimes \mathcal{B}(C^n) \) to \( N \otimes \mathcal{B}(C^n) \) is positive where \( t \) is any transposition of matrices relative to any fixed orthonormal basis. The positivity of \( F \otimes t \) does not depend on \( t \) because two transpositions \( t \) and \( t' \) relative to different orthonormal bases are always related by \( t'(Q) = ut(Q)u^* \) for some unitary \( u \in \mathcal{B}(C^n) \).

If \( F \) is \( n \)-positive or transposed-\( n \)-positive, then \( Q \geq 0 \) implies \( Q \otimes 1 \geq 0 \) and hence \( F(Q) \otimes 1 \geq 0 \) and hence \( F(Q) \geq 0 \). (More generally it is \( n' \)-positive or transposed-\( n' \)-positive for \( n' \leq n \).) Considering \( F((z+Q)^* (z+Q)) \geq 0 \) for \( z=1 \) and \( i \), we then have the selfadjointness \( F(Q)^* = F(Q^*) \).

**Lemma 4.** If a linear map \( F \) from \( M \) into \( N \) is 2-positive and satisfies \( F(1)F(Q) = F(Q), \ Q \in M \), then

\[ F(Q^*Q) \geq F(Q)^*F(Q), \quad Q \in M. \]
If a linear map $F$ from $M$ into $N$ is transposed-2-positive and satisfies $F(1)F(Q) = F(Q)$, $Q \in M$, then

\[(1.2) \quad F(Q^*Q) \geq F(Q)F(Q)^*, \quad Q \in M.\]

**Proof.** Consider

\[\hat{Q} = \begin{pmatrix} 1 & Q \\ Q^* & Q^*Q \end{pmatrix} \in M \otimes \mathcal{B}(C^2)\]

for $Q \in M$ relative to a fixed orthonormal basis $e_1$ and $e_2$ in $C^2$. Let $x_1$ and $x_2$ be vectors in defining Hilbert space of $M$ and $N$ and $x = x_1 \otimes e_1 + x_2 \otimes e_2$. Then

\[(x, \hat{Q}x) = \|x_1 + Qx_2\|^2 \geq 0\]

and hence $\hat{Q} \succeq 0$.

If $F$ is 2-positive then

\[0 \leq (x, (F \otimes 1)(\hat{Q})x) = (x_1, F(1)x_1) + 2\text{Re}(x_1, F(Q)x_2) + (x_2, F(Q^*Q)x_2)\]

where we have used $F(Q)^* = F(Q^*)$. Setting $x_1 = -F(Q)x_2$, we have

\[0 \leq (x_2, F(Q^*Q)x_2) - (x_2, F(Q)^*F(Q)x_2)\]

for any $x_2$ where we have used $F(1)F(Q) = F(Q)$. Hence we have (1.1).

If $F$ is transposed-2-positive, we have

\[0 \leq (x, (F \otimes t)(\hat{Q})x) = (x_1, F(1)x_1) + 2\text{Re}(x_1, F(Q)x_2) + (x_2, F(Q^*Q)x_2)\]

Hence, by setting $x_1 = -F(Q)^*x_2$, we obtain (1.2). Q.E.D.

For a cyclic and separating vector $Q$ for $M$, the polar decomposition

\[\hat{S} = J_\rho A_\rho^{1/2}\]

of the closure $\hat{S}$ of the operator $S$ defined on $M\Omega$ by

\[SQ\Omega = Q^*\Omega, \quad Q \in M\]

defines the modular operator $A_\rho$, which is a strictly positive selfadjoint
operator satisfying $A^* Q = Q$ and $J_\alpha A_\alpha = A_\alpha^* J_\alpha$, and the modular conjugation $J_\alpha$ which is an antiunitary involution satisfying $J_\alpha Q = Q$.

If $Q$ is not a cyclic and separating vector, we consider the restrictions of $M$ and $M'$ to $s^M(Q) s^M(Q) H$, and define $J_\alpha$ and $A_\alpha$ on $s^M(Q) s^M(Q) H$ as above and 0 on $(1 - s^M(Q) s^M(Q)) H$. The mapping

$$\tau_\alpha(t) Q = \frac{1}{2} i QA_\alpha^* s^M(Q) s^M(Q)$$

maps $M$ onto $s^M(Q) s^M(Q) M s^M(Q)$ and $M'$ onto $s^M(Q) s^M(Q) M' s^M(Q)$. It is an automorphism of $s^M(Q) s^M(Q) M s^M(Q)$ and $s^M(Q) s^M(Q) M' s^M(Q)$.

We denote

$$j_\alpha(Q) = J_\alpha Q J_\alpha.$$

It brings $M$ onto $s^M(Q) M' s^M(Q) s^M(Q)$ and $M'$ onto $s^M(Q) Ms^M(Q) s^M(Q)$.

For a normal positive linear functional $\rho$ on $M$, we denote $J_\rho$, $A_\rho$, $\tau_\rho(t)$, $j_\rho$ for $\pi_\rho(M)$ and $Q = \Omega_\rho$ by $J_\rho$, $A_\rho$, $\tau_\rho(t)$ and $j_\rho$. We sometimes denote the expectation functional of $B(H_\rho)$ by the vector $\Omega_\rho$ again by $\rho$.

We need the following.

**Lemma 5.** Let $\rho$ be a normal positive linear functional of $M$ and $Z_\rho$ be the set of $x \in M$ such that $\rho(x Q) = \rho(Q x)$ for all $Q \in M$. Then for every $z \in Z_\rho$, $[s^M(\rho), z] = 0$, $[A_\rho, \pi_\rho(z)] = 0$ and

$$\tau_\rho(t) \pi_\rho(z) = \pi_\rho(z s^M(\rho)).$$

If $z \in M \cap M'$, then

$$j_\rho(\pi_\rho(z)) = \pi_\rho(z^* s^M(\rho)).$$

**Proof.** Substituting $Q s^M(\rho)^\perp$ into $Q$ of $\rho(x Q) = \rho(Q x)$, we obtain $\rho(Q s^M(\rho)^\perp x) = 0$ where $s^M(\rho)^\perp = 1 - s^M(\rho)$. Hence $\pi_\rho(s^M(\rho)^\perp x) \Omega_\rho = 0$. Multiplying $\pi_\rho(M')$, we obtain $0 = \pi_\rho(s^M(\rho)^\perp x) s^* s^M(\rho)(\Omega_\rho) = \pi_\rho(s^M(\rho)^\perp x s^M(\rho))$. Substituting $s^M(\rho)^\perp Q$ into $Q$ of $\rho(x Q) = \rho(Q x)$, we also obtain $\pi_\rho(s^M(\rho) x s^M(\rho)^\perp) = \pi_\rho(s^M(\rho)^\perp x^* s^M(\rho)) = 0$. Hence $\pi_\rho([x, s^M(\rho)]) = 0$. Hence $s_\rho([x, s^M(\rho)]) = 0$ where $s_\rho$ is the central support of $\rho$. Since $[1 - s_\rho] s^M(\rho) = 0$, we have $[x, s^M(\rho)] = 0$. 


Since $\mathcal{Q}_\rho$ is cyclic for $R = \pi_\rho(M), s^R(\mathcal{Q}_\rho) = 1$. Since $\tau_\rho(t)\pi_\rho(z) = \tau_\rho(t)\pi_\rho(zs^M(\rho))$ and $j_\rho(\pi_\rho(z)) = j_\rho(\pi_\rho(z)s^M(\rho))$ by definitions of $\tau_\rho$ and $j_\rho$, it is enough to prove

$$\tau_\rho(t)\pi_\rho(z) = \pi_\rho(z)$$
$$j_\rho(\pi_\rho(z)) = \pi_\rho(z)^*$$

for $z \in Zs^M(\rho)$ on $\pi_\rho(s^M(\rho))H_\rho = H_\rho'$. Since $\mathcal{Q}_\rho$ is cyclic and separating for $R = \pi_\rho(s^M(\rho), Ms^M(\rho))$ on $H_\rho$, the first equation is known. [8] It implies $\mathcal{A}_\rho = \pi_\rho(z) = 0$. From $j_\rho(z)\mathcal{Q}_\rho = A_\rho^{\frac{1}{2}}z^*\mathcal{Q}_\rho = z^*\mathcal{Q}_\rho$ we have $j_\rho(z) = z^*$ for $\bar{z} = \pi_\rho(z), z \in M \cap M's^M(\rho)$. Q.E.D.

§ 2. Mapping $F^{NM}_\rho$ from a von Neumann Algebra $M$ into $M'$

**Theorem 1.** Let $M$ and $N$ be von Neumann algebras such that $N \subseteq M'$. Let $\rho$ be a normal positive linear functional of $(M \cup N)'$. There exists a unique mapping $F^{NM}_\rho$ from $M$ into $N$ satisfying

(2.1) $\rho(Q'Q') = \rho(F^{NM}_\rho(Q)Q' + Q'F^{NM}_\rho(Q))/2$

for all $Q \in M, Q' \in N$, and

(2.2) $s(F^{NM}_\rho(Q)) \leq s^N(\rho)$.

It has the following properties:

1. $F^{NM}_\rho$ is $(M \cap N)$-linear. $F^{NM}_\rho(Q)^* = F^{NM}_\rho(Q^*)$.

2. $F^{NM}_\rho(1) = s^N(\rho)$.

3. $F^{NM}_\rho$ is transposed-$n$-positive for all positive integers $n$. (In particular, $F^{NM}_\rho$ is positive and $F^{NM}_\rho(Q^*) = F^{NM}_\rho(Q^*)$.)

4. $\|F^{NM}_\rho\| = 1$ for $\rho \neq 0$.

5. $F^{NM}_\rho$ is $\sigma$-weakly continuous (i.e. normal). It is continuous on the unit ball relative to the strong topology on $M$ and * strong topology on $N$. 

(6) For any automorphism \( \tau \) of \((M \cup N)''\) satisfying \( \tau(M) = M \) and \( \tau(N) = N \),
\[
F_{\rho}^{NM}(\tau Q) = \tau F_{\rho}^{NM}(Q)
\]
where \( \tau^* \rho \) is defined by \((\tau^* \rho)(Q) = \rho(\tau Q)\). In particular, if \( u \in M \) is unitary,
\[
F_{\rho}^{NM}(u Q u^*) = F_{\rho u^* \rho}^{NM}(Q)
\]
and if \( v \in N \) is unitary,
\[
v F_{\rho}^{NM}(Q) v^* = F_{v \rho^* v}^{NM}(Q)
\]
where \((t_1 t_2)(Q) = \rho(t_2 Q t_1)\).

(7) For any \( A \in M \cap N \), \( A \geq 0 \),
\[
F_{A_{\rho}}^{NM}(Q) = F_{\rho}^{NM}(Q) s(A).
\]
(8) If \( \lim \|\rho_n - \rho\| = 0 \) and \( \lim s^N(\rho_n) = s^N(\rho) \), then
\[
\lim F_{\rho_n}^{NM}(Q) = F_{\rho}^{NM}(Q), \quad \lim F_{\rho_n}^{NM}(Q)^* = F_{\rho}^{NM}(Q)^*
\]
uniformly for a bounded set of \( Q \). (If \( s^N(\rho_n) \leq s^N(\rho) \), then \( \lim \|\rho_n - \rho\| = 0 \) implies \( \lim s^n(\rho_n) = s^N(\rho) \).)

Proof. Let \( Q \in M \) and \( Q' \in N \). Consider
\[
f_{Q}(Q') = \rho(Q Q').
\]
If \( Q \geq 0 \), then
\[
f_{Q}(Q') = \rho(Q^{1/2} Q' Q^{1/2})
\]
is normal positive linear functional on \( N \). If \( Q' \geq 0 \) in addition,
\[
f_{Q}(Q') = \rho(Q^{1/2} Q' Q^{1/2}) \leq \|Q\| \rho(Q').
\]
Hence \( f_{Q} \leq \|Q\| \rho \).
For general $Q$, we have

\[(2.8)\quad Q = Q_1 - Q_2 + i(Q_3 - Q_4)\]

where $Q_1$ and $Q_2$ are positive and negative parts of $(Q + Q^*)/2$, $Q_3$ and $Q_4$ are positive and negative parts of $(Q - Q^*)/(2i)$. Then

\[f_a = f_{a_1} - f_{a_2} + i(f_{a_3} - f_{a_4})\]

where $f_{a_k} \leq ||Q_k||\rho$.

By Lemma 3, there exists a unique $h_0 = F^{NM}_\rho(Q) \in N$ such that

\[(2.9)\quad f_0(Q') = \rho(F^{NM}_\rho(Q)Q' + Q'F^{NM}_\rho(Q))/2\]

for all $Q' \in N$ and

\[(2.10)\quad s(F^{NM}_\rho(Q)) \leq s^N(\rho).\]

This shows the existence and uniqueness of $F^{NM}_\rho$.

(1) Let $z_1, z_2 \in M \cap N$ and $Q_1, Q_2 \in M$. Note that $M \cap N$ is in the center of $(N \cup M)'$ by $N \subseteq M'$. We have, for $Q = z_1Q_1 + z_2Q_2$,

\[
\rho(F^{NM}_\rho(Q)Q' + Q'F^{NM}_\rho(Q))/2 = \rho(QQ')
\]

\[
= \rho(Q_1z_1Q') + \rho(Q_2z_2Q')
\]

\[
= \rho(F^{NM}_\rho(Q_1)z_1Q' + z_1Q'F^{NM}_\rho(Q_1))/2
\]

\[
+ \rho(F^{NM}_\rho(Q_2)z_2Q' + z_2Q'F^{NM}_\rho(Q_2))/2
\]

\[
= \rho(F'Q' + Q'F')/2
\]

where

\[F' = z_1F^{NM}_\rho(Q_1) + z_2F^{NM}_\rho(Q_2).\]

Since $s(F^{NM}_\rho(Q_k)) \leq s^N(\rho)$, $k = 1, 2$, we also have $s(F') \leq s^N(\rho)$. By the uniqueness, we have

\[F' = F^{NM}_\rho(z_1Q_1 + z_2Q_2).\]
From $\rho(Q^*Q) = \rho(Q(Q')^*)$ and the uniqueness, we obtain $F_{p}^{N,M}(Q)^* = F_{p}^{N,M}(Q^*)$.

(2) The substitution of $Q=1$ and $F_{p}^{N,M}(Q) = s^N(\rho)$ into (2.1) and (2.2) immediately proves this statement.

(3) If $Q \geq 0$, then $F_{p}^{N,M}(Q) \geq 0$ from Lemma 1. Hence $F_{p}^{N,M}$ is positive.

To prove transposed-$n$-positivity for $n > 1$, let $e_1, \ldots, e_n$ be an orthonormal basis of $C^*$, $\Omega = n^{-1/2} \sum_{k=1}^{n} e_k \otimes e_k \in C^n \otimes C^n$.

$J_\Omega$ be the modular conjugation for $\Omega$ ($J_\Omega \sum c_{ij} e_i \otimes e_j = \sum \overline{c}_{ij} e_j \otimes e_i$), and the transposition $t$ be chosen to be

$$tQ = J_\Omega Q^* J_\Omega.$$  \hfill (2.11)

which maps $Q \in \mathcal{A}(C^n) \otimes 1$ onto $1 \otimes \mathcal{A}(C^n)$. Consider (on $H \otimes (C^n \otimes C^n)$)

$$\bar{M} = M \otimes (\mathcal{A}(C^n) \otimes 1),$$

$$\bar{N} = N \otimes (1 \otimes \mathcal{A}(C^n)),$$

$$\bar{\rho} = \rho \otimes \omega_\rho.$$

Then $F_{p}^{N,M} \otimes t$ from $\bar{M}$ to $\bar{N}$ coincides with $F_{p}^{N,M}$ due to the following computation and hence is positive by our earlier result. Let $Q_1 \in M$, $Q'_1 \in N$, $Q_2 \in \mathcal{A}(C^n) \otimes 1$, $Q'_2 \in 1 \otimes \mathcal{A}(C^n)$. Then

$$\rho((Q_1 \otimes Q_2)(Q'_1 \otimes Q'_2)) = \rho(Q_1 Q'_1)(\Omega, Q_2 Q'_2 \Omega)$$

$$= \rho(F_{p}^{N,M}(Q_1)Q'_1)(Q'_2 \Omega, Q'_2 \Omega)/2$$

$$+ \rho(Q_1 F_{p}^{N,M}(Q_1))(\Omega, Q'_2 Q_2 \Omega)/2$$

$$= \rho(F_{p}^{N,M}(Q_1)Q'_1)(j_\rho(Q_2) \Omega, Q'_2 \Omega)/2$$

$$+ \rho(Q_1 F_{p}^{N,M}(Q_1))(\Omega, Q'_2 j_\rho(Q'_2) \Omega)/2$$

where we have used the fact that the modular operator for a faithful
trace vector $\mathcal{Q}$ is 1 and hence $J_2\mathcal{Q} = J_3\mathcal{Q} = J_4\mathcal{Q} = \mathcal{Q}$. Substituting the definition of \( \mathcal{Q} \), we have

$$\rho((Q_1 \otimes Q_2)Q') = \rho(F_{p}^{NM}(Q_1) \otimes Q')/2$$

$$+ \rho(Q'F_{p}^{NM}(Q_1) \otimes Q)/2$$

for $Q' = Q_1 \otimes Q_2$. Since such $Q'$ linearly span $N \otimes (1 \otimes A(C^*))$, the same equation holds for all $Q'$ in $N$. Since $s^{N}(\rho) = s^{A}(\rho) \otimes 1$ because $\mathcal{Q}$ is cyclic for $1 \otimes A(C^*)$, we have $s(F_{p}^{NM}(Q_1) \otimes Q_2) \leq s(F_{p}^{NM}(Q_1)) \otimes 1 \leq s^{A}(\rho)$. Hence

(2.12) \[ F_{p}^{NM}(Q_1 \otimes Q_2) = (F_{p}^{NM} \otimes t)(Q_1 \otimes Q_2). \]

(4) From Lemma 1 (3) and (2.7), we have

$$||F_{p}^{NM}(Q)|| \leq ||Q||$$

for $Q \geq 0$. Due to Lemma 4, we have

$$||F_{p}^{NM}(Q)||^2 = ||F_{p}^{NM}(Q)F_{p}^{NM}(Q)||^2 \leq ||F_{p}^{NM}(Q)\mathcal{Q}|| \leq ||\mathcal{Q}|| = ||Q||^2$$

for arbitrary $Q$. From (2), we obtain $||F_{p}^{NM}|| = 1$ if $\rho \neq 0$.

(5) Assume that a net $Q_{\alpha} \in M$ has a weak limit $Q$ and $||Q_{\alpha}|| \leq 1$. Then

(2.13) \[ \lim_{\alpha} \rho(F_{p}^{NM}(Q_{\alpha})Q' + Q'F_{p}^{NM}(Q_{\alpha})) = \rho(F_{p}^{NM}(Q)Q' + Q'F_{p}^{NM}(Q)). \]

Since $||F_{p}^{NM}(Q_{\alpha})|| \leq ||Q_{\alpha}|| \leq 1$, the set of accumulation points

(2.14) \[ \bigcap_{\beta} (\bigcup_{\alpha \geq \beta} F_{p}^{NM}(Q_{\alpha})) \text{ - (weak)} \]

is non-empty due to the weak compactness. Let $\tilde{Q}$ be in this set. Then from (2.13), we have

$$\rho(F_{p}^{NM}(Q)Q' + Q'F_{p}^{NM}(Q)) = \rho(\tilde{Q}Q' + Q'\tilde{Q}).$$

From the uniqueness in Lemma 2, we have
\[ \hat{Q} = F^N_{\rho}(Q) \]

and hence the set (2.14) consists of a single point \( F^N_{\rho}(Q) \). Thus

\[ \text{w-lim}_\alpha F^N_{\rho}(Q_\alpha) = F^N_{\rho}(\text{w-lim}_\alpha Q_\alpha). \]

The weak continuity on bounded sets implies the normality and the \( \sigma \)-weak continuity for a positive linear mapping.

Next, we assume that a net \( Q_\alpha \in M \) has a strong limit \( Q \) and \( \| Q_\alpha \| \leq 1 \). Then \( \| F^N_{\rho}(Q_\alpha - Q) \| \leq \| Q_\alpha - Q \| \leq 2 \). Hence

\[ \lim_\alpha \rho(F^N_{\rho}(Q_\alpha - Q))^*(Q_\alpha - Q) = 0. \]

By using (2.1) with \( Q = (Q_\alpha - Q)^* \), \( Q' = F^N_{\rho}(Q_\alpha - Q) \), we have

\[ 0 \leq \rho(F^N_{\rho}(Q_\alpha - Q))^*F^N_{\rho}(Q_\alpha - Q) + \rho(F^N_{\rho}(Q_\alpha - Q)F^N_{\rho}(Q_\alpha - Q))^* \]

\[ = 2\rho(F^N_{\rho}(Q_\alpha - Q))^*(Q_\alpha - Q) \rightarrow 0 \]

and hence

\[ \lim_\pi \rho(F^N_{\rho}(Q_\alpha - Q))^*Q_\rho = 0, \]

\[ \lim_\pi \rho(F^N_{\rho}(Q_\alpha - Q))^*Q_\rho = 0. \]

Multiplying \( \hat{Q} \in \pi_\rho(N') \), we have

\[ \lim_\pi \rho(F^N_{\rho}(Q_\alpha - Q))^*Q_\rho = 0, \]

\[ \lim_\pi \rho(F^N_{\rho}(Q_\alpha - Q))^*Q_\rho = 0. \]

for \( \Psi = \hat{Q}Q_\rho \). Since \( \| F^N_{\rho}(Q_\alpha - Q) \| \leq 2 \), the same hold on the closure of \( \pi_\rho(N')Q_\rho \), which is \( \pi_\rho(s^N(\rho))H_\rho \). Hence

\[ \lim_\pi \rho(F^N_{\rho}(Q_\alpha - Q)s^N(\rho)) = 0, \]

\[ \lim_\pi \rho(F^N_{\rho}(Q_\alpha - Q)^*s^N(\rho)) = 0. \]

Since \( \pi_\rho \) is faithful at least on \( s^N(\rho)Ns^N(\rho) \), \( \pi_\rho^{-1} \) is continuous on \( s^N(\rho)Ns^N(\rho) \) and

\[ 0 = \lim_\alpha s^N(\rho)F^N_{\rho}(Q_\alpha - Q)s^N(\rho) = \lim_\alpha (F^N_{\rho}(Q_\alpha) - F^N_{\rho}(Q_\alpha)), \]
\[ 0 = \lim_{a} s^N(\rho)F_{\rho}^{NM}(Q_a - Q)^*s^N(\rho) = \lim_{a} \{F_{\rho}^{NM}(Q_a)^* - F_{\rho}^{NM}(Q)^*\}, \]
due to (2.2) and (1).

(6) For \( Q \in M \) and \( Q' \in N \), we have

\[
\rho(\tau(Q)Q') = \rho(\{Q\tau^{-1}Q'\}) = \tau^*\rho(Q\tau^{-1}Q') = \tau^*\rho(F_{\tau\rho}^{NM}(Q)\tau^{-1}Q')/2 + \tau^*\rho(\{\tau^{-1}Q'\}F_{\tau\rho}^{NM}(Q))/2 = \rho(\{\tau F_{\tau\rho}^{NM}(Q)Q'\})/2 + \rho(Q'\tau F_{\tau\rho}^{NM}(Q))/2.
\]

We also have

\[
s(\tau F_{\tau\rho}^{NM}(Q)) = \tau \{s(F_{\tau\rho}^{NM}(Q))\} \leq \tau \{s^N(\tau^*\rho)\} = s^N(\rho).
\]

Hence (2.3) holds by the uniqueness.

(2.4) and (2.5) are special cases of (2.3) where \( \tau(A) = uAu^* \) and \( \tau(A) = vAv^* \) for \( A \in (N \cup M)'' \).

(7) Since \( N \subset M' \), \( M \cap N \) is in the center of \( (N \cup M)'' \). We have

\[
(A\rho)(QQ') = \rho(QQ'A) = \rho(F_{\rho}^{NM}(Q)Q'A)/2 + \rho(Q'AF_{\rho}^{NM}(Q))/2 = A\rho(s(A)F_{\rho}^{NM}(Q)Q' + Q's(A)F_{\rho}^{NM}(Q))/2.
\]

We also have

\[
s(A)F_{\rho}^{NM}(s) = s(A)s(F_{\rho}^{NM}(s)) \leq s(A)s^N(\rho) = s^N(A\rho).
\]

Hence, by uniqueness, we have

\[
F_{A\rho}^{NM}(Q) = F_{\rho}^{NM}(Q) s(A).
\]

(8) We have for \( \delta_n = F_{\rho}^{NM}(Q) - F_{\rho_n}^{NM}(Q) \) the following estimate

\[
|\rho(\delta_nQ' + Q'\delta_n)| \leq 2|\rho(QQ') - \rho_n(QQ')|.
\]
VON NEUMANN ALGEBRA 451

\[ + |\rho_n(F_{p_n}^{NM}(Q)Q' + Q'F_{p_n}^{NM}(Q)) - \rho(F_{p_n}^{NM}(Q)Q' + Q'F_{p_n}^{NM}(Q))| \]

\[ \leq 4||Q|| ||Q'|| ||\rho - \rho_n||. \]

Setting \( Q' = \delta_n^* \) and using \( ||\delta_n|| \leq 2||Q|| \), we have

\[ 0 \leq \rho(\delta_n\delta_n^*) \leq \rho(\delta_n\delta_n^* + \delta_n^*\delta_n) \leq 8||Q||^2 ||\rho - \rho_n|| \]

\[ 0 \leq \rho(\delta_n\delta_n^*) \leq 8||Q||^2 ||\rho - \rho_n||. \]

Hence we have

\[ \lim_{n \to \infty} \pi_{\rho_n}(\delta_n) = 0, \quad \lim_{n \to \infty} \pi_{\rho_n}(\delta_n^*) = 0, \]

for \( \mathcal{F} = \Omega_{\rho} \) and hence for \( \mathcal{F} = Q', \Omega_{\rho}, Q' \in \pi_{\rho}(N)' \). Since \( ||\pi_{\rho}(\delta_n)|| \leq 2||Q|| \) is uniformly bounded, the same holds for \( \mathcal{F} \in s_N(\Omega_{\rho})H_{\rho} \) and hence

\[ \lim_{n \to \infty} \pi_{\rho_n}(\delta_n s_N^N(\rho)) = 0, \quad \lim_{n \to \infty} \pi_{\rho_n}(\delta_n^* s_N^N(\rho)) = 0, \]

uniformly for a bounded set of \( Q \). Since \( \pi_{\rho}^{-1} \) is continuous on \( N s_N^N(\rho) \), where \( s_N^N(\rho) \) is the central support of \( s_N^N(\rho) \), we have

\[ \lim_{n \to \infty} \{ F_{p_n}^{NM}(Q) - F_{p_n}^{NM}(Q)s_N^N(\rho) \} = 0, \]

\[ \lim_{n \to \infty} \{ F_{p_n}^{NM}(Q)^* - F_{p_n}^{NM}(Q)^*s_N^N(\rho) \} = 0. \]

If \( \lim s_N^N(\rho_n) = s_N^N(\rho) \), then as \( ||F_{p_n}^{NM}(Q)|| \leq ||Q|| \) we have

\[ \lim_{n \to \infty} \{ F_{p_n}^{NM}(Q)s_N^N(\rho) - F_{p_n}^{NM}(Q) \} = \lim_{n \to \infty} F_{p_n}^{NM}(Q)(s_N^N(\rho) - s_N^N(\rho_n)) = 0 \]

and we obtain

\[ \lim_{n \to \infty} F_{p_n}^{NM}(Q) = \lim_{n \to \infty} F_{p_n}^{NM}(Q) \]

uniformly for a bounded set of \( Q \). Similar equation for adjoint also holds.

If \( s_N^N(\rho_n) \leq s_N^N(\rho) \), then

\[ |\rho(1 - s_N^N(\rho_n))| = |\rho(1 - s_N^N(\rho_n)) - \rho_n(1 - s_N^N(\rho_n))| \leq ||\rho - \rho_n|| \]
and hence
\[ \lim \pi_\rho(1 - s^N(\rho_n)) Q_\rho = 0. \]
As before, we have
\[ \lim (s^N(\rho) - s^N(\rho_n)) = 0. \quad \text{Q.E.D.} \]

The proof of (3) implies the following corollaries.

**Corollary 1.** If \( M' = N, \rho = \omega_\beta \) and \( \Omega \) is a faithful trace vector for \( M \) as well as for \( N \), then
\[
F^{NM}_\rho(Q) = J_\rho Q^* J_\beta.
\]

**Corollary 2.** Let \( M = M_1 \otimes M_2, N = N_1 \otimes N_2, \rho = \rho_1 \otimes \rho_2. \) If \( \rho_1 \) is a trace on \( N_1 \) or if \( \rho_2 \) is a trace on \( N_2 \), then
\[
F^{NM}(Q_1 \otimes Q_2) = F^{N_1}_{\rho_1}(Q_1) \otimes F^{N_2}_{\rho_2}(Q_2)
\]
for all \( Q_1 \in M_1, Q_2 \in M_2 \). (In particular, if either \( N_1 \) or \( N_2 \) is abelian then this holds for any normal states \( \rho_1 \) and \( \rho_2 \).)

**Remark 1.** \( F^{NM}_\rho(Q) = F^{NM'}_{\rho M}(Q) \) for \( Q \in M \), where \( \rho_M \) is the restriction of \( \rho \) (which is a functional on \( (N \cup N')'' \)) to \( (M \cup N)''. \) In this sense, the case \( M = N' \) is most canonical and we shall study it from different viewpoint in the next section.

**Remark 2.** In order to define \( F^{NM}_\rho(Q) \), \( \rho \) need not be normal on the whole \( (M \cup N)'', \) but it is sufficient that \( \rho \) is normal on \( N \). The uniqueness and existence together with properties (1), (2), (3), (4), (6), (7) and (8) hold for such non-normal \( \rho \). Note that \( f_Q \) defined by (2.6) is normal due to (2.7) if \( \rho \) is normal on \( N \).

**Remark 3.** Theorem 1 holds also for the case where \( N \) is a weakly closed * subalgebra of \( M' \) even if the unit in \( N \) is not the identity operator in \( M' \).
§3. Mapping $G^M_\rho$ from a von Neumann Algebra $M$ into Itself

**Theorem 2.** Let $\rho$ be a normal positive linear functional of $M$. There exists a unique mapping $G^M_\rho$ from $M$ into $s^M(\rho)M s^M(\rho)$ satisfying

$$ (3.1) \quad (\Omega_\rho, \pi_\rho(Q)A^{1/2}\pi_\rho(Q')\Omega_\rho) = \rho(G^M_\rho(Q)Q' + Q'G^M_\rho(Q))/2 $$

for all $Q, Q' \in M$.

It has the following properties:

1. $G^M_\rho$ is $Z_\rho$-linear, where $Z_\rho$ is the set of $x \in M$ such that $\rho(xQ) = \rho(Qx)$ for all $Q \in M$, and $M$ is considered as two-sided $Z_\rho$ module. In particular, $G^M_\rho$ is $Z$-linear for the center $Z = M \cap M'$.

2. $G^M_\rho(1) = s^M(\rho)$.

3. $G^M_\rho$ is completely positive. (In particular, it is positive and $G^M(Q)^* = G^M(Q^*)$.)

4. $\|G^M_\rho\| = 1$ for $\rho \neq 0$.

5. $G^M_\rho$ is $\sigma$-weakly continuous (i.e. normal). It is continuous on the unit ball relative to the strong topology for $Q$ and $*$ strong topology for $G^M_\rho(Q)$.

6. If $\tau$ is an automorphism of $M$ and $\tau^*\rho = \rho$, then

$$ G^M_\rho(\tau Q) = \tau G^M_\rho(Q). $$

7. If $z \in Z$, $z \geq 0$, then

$$ G^M_{z\rho}(Q) = G^M(Q) s^M(z). $$

8. The kernel of $G^M_\rho$ is

$$ s^M(\rho)M(1 - s^M(\rho)) + (1 - s^M(\rho))M, $$

which implies

$$ G^M_\rho(Q) = G^M_\rho(s^M(\rho)Qs^M(\rho)). $$
The image of $G^M_{\rho}$ is strongly dense in $s^M(\rho)Ms^M(\rho)$.

Proof. Let $R=\pi_\rho(M)$ and $Q, Q' \in R$. From the formula $j_\rho(Q)\Omega = A_{1/2}^{\rho}Q^*\Omega$ and (2.1), we have

\begin{equation}
(Q, QA_{1/2}^{\rho}Q'\Omega) = (Q, Qj_\rho(Q'^*)\Omega) + (Q, F_{\rho}^{\rho R}(Q)j_\rho(Q'^*)\Omega)/2 + (j_\rho(Q')\Omega, F_{\rho}^{\rho R}(Q')\Omega)/2
\end{equation}

where $\rho$ is also used for $\rho(Q)=(Q, Q\Omega), Q \in (R \cup R)''$, in writing $F_{\rho}^{\rho R}$. Since $(J_\rho x, y) = (J_\rho^2 J_\rho x, y) = (J_\rho x, J_\rho y) = (x, J_\rho y) = (x, J_\rho x)$ where $J_\rho^2 = s^R(\Omega) = (s^R(\Omega) = 1$ due to the cyclicity of $(\Omega)$), and since $J_\rho \Omega = \Omega$, we have

\begin{equation}
(Q, QA_{1/2}^{\rho}Q'\Omega) = (Q'^*\Omega, j_\rho(F_{\rho}^{\rho R}(Q'^*))\Omega) + (j_\rho(Q')\Omega, F_{\rho}^{\rho R}(Q')\Omega)/2.
\end{equation}

Since $s^R(j_\rho(F_{\rho}^{\rho R}(Q))) \subseteq s^R(\Omega) = \pi_\rho(s^M(\rho))$, there exists $G \in s^M(\rho)Ms^M(\rho)$ such that

\begin{equation}
\pi_\rho(G) = j_\rho(F_{\rho}^{\rho R}(Q'^*))
\end{equation}

From (3.2) and (3.3), $G^M_{\rho}(Q)=G$ satisfies (3.1) for all $Q' \in M$. Hence the existence is proved.

If $G^M_{\rho}(Q)=G$ and $G'$ both satisfy (3.1), then $G-G'$ also satisfies $\rho((G-G')Q' + Q'(G-G')) = 0$ for all $Q' \in M$. In particular, we have $\rho((G-G')^*(G-G')) = 0$ for $Q'=(G-G')^*$. Since $\rho$ is faithful on $s(\rho)Ms(\rho)$, we have $G-G'=0$ and hence the uniqueness.

(1) From (3.4) and Theorem 1 (1), $G^M_{\rho}$ is linear. If $z \in Z_\rho$, then $\tilde{z} = \pi_\rho(z)$ commutes with $A_{\rho}$ (Lemma 5) and we have

\begin{equation}
(Q, Q\tilde{z}A_{1/2}^{\rho}Q'\Omega) = (Q, QA_{1/2}^{\rho}\tilde{z}Q'\Omega) = \rho(G^M_{\rho}(Q)zQ' + zQ'G^M_{\rho}(Q))/2 = \rho(G^M_{\rho}(Q)zQ' + Q'G^M_{\rho}(Q)z)/2
\end{equation}
for $Q = \pi_p(Q)$ and $Q' = \pi_p(Q')$, $Q \in M$, $Q' \in M$. Since $z$ commutes with $s(\rho)$ by Lemma 5, $s(G^M_p(Q)z) \leq s(\rho)$ and hence

$$G^M_p(Qz) = G^M_p(Q)z.$$ 

Since $[j_\rho(\bar{z}), A] = j_\rho([\bar{z}, A^{-1}]) = 0$, we also have

$$(\mathcal{Q}_\rho, z\bar{Q}A^{1/2}\bar{Q}'\mathcal{Q}_\rho) = (z^*\mathcal{Q}_\rho, \bar{Q}A^{1/2}\bar{Q}'\mathcal{Q}_\rho)$$

$$= (j_\rho(z)\mathcal{Q}_\rho, \bar{Q}A^{1/2}\bar{Q}'\mathcal{Q}_\rho)$$

$$= (\mathcal{Q}_\rho, \bar{Q}A^{1/2}\bar{Q}'j_\rho(z^*)\mathcal{Q}_\rho)$$

$$= (\mathcal{Q}_\rho, \bar{Q}A^{1/2}\bar{Q}'z\mathcal{Q}_\rho)$$

$$= \rho(G^M_p(Q)Q'z + Q'zG^M_p(Q))/2$$

$$= \rho(zG^M_p(Q)Q' + Q'zG^M_p(Q))/2.$$ 

Hence we have

$$G^M_p(zQ) = zG^M_p(Q).$$ 

(2), (4) and (5) follow from the corresponding results in Theorem 1 and (3.4).

(3) Let $Q_{ij} \in R$ such that $\sum_{i,j} (x_i, Q_{ij}x_j) \geq 0$ for any $x_j \in H_\rho$ where the indices $i, j$ run from 1 to $n$. By Theorem 1 (3),

$$\sum (x_i, F^{R,R}_\rho(Q_{ij})x_j) \geq 0$$

for any vectors $x_j \in H_\rho$. Hence

$$\sum (x_i, \pi_\rho(G^M_p(Q_{ij})))x_i$$

$$= \sum (x_i, J_\rho^2J_\rho F^{R,R}_\rho(Q_{ij})x_i)$$

$$= \sum (J_\rho^2x_i, J_\rho F^{R,R}_\rho(Q_{ij})x_i)$$

$$= \sum (F^{R,R}_\rho(Q_{ij})x_i, J_\rho x_i)$$

$$= \sum (J_\rho x_i, F^{R,R}_\rho(Q_{ij})J_\rho x_i) \geq 0.$$
Since \( \pi_\rho \) is faithful on \( s^M(\rho)M s^M(\rho) \), this proves \( n \)-positivity of \( G_\rho^M \).

(6) If \( \tau^* \rho = \rho \), there exists a unitary operator \( U_\rho(\tau) \) on \( H_\rho \) such that

\[
U_\rho(\tau) \pi_\rho(Q) \Omega_\rho = \pi_\rho(\tau Q) \Omega_\rho.
\]

Applying \( S \), we have

\[
SU_\rho(\tau) \pi_\rho(Q) \Omega_\rho = \pi_\rho(\tau Q^*) \Omega_\rho
\]

\[= U_\rho(\tau) S \pi_\rho(Q) \Omega_\rho.\]

Hence \( U_\rho(\tau) \) also commutes with closure \( \hat{S} \) and hence with \( A_\rho \) and \( J_\rho \).

We also have \( \tau s^M(\rho) = s^M(\rho) \). From Theorem 1 (6), we now have, for \( \tau Q \equiv U_\rho(\tau)Q U_\rho(\tau)^* \),

\[
\pi_\rho(G_\rho^M(\tau Q)) = j_\rho(F^R_\rho(\tau Q^*))
\]

\[= j_\rho(\tau F^R_\rho(Q^*))
\]

\[= \tau j_\rho(G_\rho^M(Q))
\]

\[= \pi_\rho(\tau G_\rho^M(Q)).\]

Since \( s^M(\tau G_\rho^M(Q)) \leq s^M(\tau \rho) = s^M(\rho) \), we have (6).

(7) It follows from Theorem 1 (7) and \( j_\pi(\pi_\rho(s^M(z)^*)) = \pi_\rho(s^M(z)). \)

The latter equation is due to Lemma 5.

(8) From \( G_\rho^M(Q) = 0 \) and (3.1), we obtain

\[0 = (\Omega_\rho, \pi_\rho(Q)A_\rho^{1/2} \pi_\rho(Q') \Omega_\rho)
\]

\[= (\Omega_\rho, \pi_\rho(Q)j_\rho(\pi_\rho(Q')^*) \Omega_\rho)
\]

\[= (j_\rho(\pi_\rho(Q')) \Omega_\rho, \pi_\rho(Q) \Omega_\rho).
\]

Since \( j_\rho(\pi_\rho(M)) \Omega_\rho = \pi_\rho(M)^* \Omega_\rho \) span \( \pi_\rho(s^M(\rho))H_\rho \) \( (= s^R(\Omega_\rho)H_\rho \) \), we have

\[
\pi_\rho(s^M(\rho) Q) \Omega_\rho = \pi_\rho(s^M(\rho)) \pi_\rho(Q) \Omega_\rho = 0.
\]

By multiplying \( Q' \in \pi_\rho(M)' \), we obtain
\[ \pi_\rho(s^M(\rho)Qs^M(\rho))\mathcal{F} = 0 \]

for \( s^R(\mathcal{Q})\mathcal{F} = Q'\mathcal{Q} \) and hence for all \( \mathcal{F} \). Therefore

\[ \pi_\rho(s^M(\rho)Qs^M(\rho)) = 0 \]

and hence \( s^M(\rho)Qs^M(\rho) = 0 \). Thus \( Q \) must be in \( s^M(\rho)M(1-s^M(\rho)) + (1-s^M(\rho))M \). On the other hand, if \( Q \) is in this set, (3.1) vanishes and hence by the uniqueness of \( G^M_p(Q) \), we have \( G^M_p(Q) = G^M_p(0) = 0 \).

To prove that the image of \( G^M_p \) is strongly dense in \( s^M(\rho)Ms^M(\rho) \), it is enough to prove that the image of \( G^M_p \) is strongly dense in \( M \) for faithful \( \rho \) because \( \rho \) is faithful on \( s^M(\rho)Ms^M(\rho) \). Assume that \( \rho \) is faithful on \( M \).

Let \( \bar{Q} \in \pi_\rho(M) \) and

\[ \bar{Q}_\beta \equiv \int_0^1 \tau_\rho(t)\bar{Q} \exp(-t^2/\beta) dt/((\beta\pi)^{1/2}). \]

It satisfies \( ||\bar{Q}_\beta|| \leq ||Q|| \), \( \lim_{\beta \to 0} \bar{Q}_\beta = \bar{Q} \). Furthermore,

\[ \tau_\rho(t)\bar{Q}_\beta = \int_0^t \tau_\rho(s)\bar{Q} \exp(-(t-s)^2/\beta) ds/((\beta\pi)^{1/2}) \]

is analytic for all \( t \). Hence, for \( Q' \in \pi_\rho(M) \), we have

\[ (\mathcal{Q}_\rho, (\bar{Q}_\beta Q' + Q' \bar{Q}_\beta)\mathcal{Q}_\rho) = (\mathcal{Q}_\rho, (\bar{Q}_\beta + \tau_\rho(i\bar{Q}_\beta)) Q' \mathcal{Q}_\rho) \]

\[ = (\mathcal{Q}_\rho, \tau_\rho(-i/2)\bar{Q}_\beta + \tau_\rho(i/2)\bar{Q}_\beta) A^1_p Q' \mathcal{Q}_\rho \]

where the first equality is due to KMS condition. Hence we have for \( Q \in M, \bar{Q} = \pi_\rho(Q), Q_\beta = \pi_\rho(Q_\beta) \)

\[ G^M_p((\tau_\rho(-i/2)Q_\beta + \tau_\rho(i/2)Q_\beta)) = 2Q_\beta. \]

Thus the image of \( G^M_p \) is strongly dense in \( M \) for faithful \( \rho \).

Q.E.D.
§4. Projections of a von Neumann Algebra into Its Center

Theorem 3. Let $Z$ denote the center of $M$ and $N \subseteq Z$. Then $F_{\rho}^{NM}$ has the following properties besides the properties (1)–(8) of Theorem 1.

(9) $F_{\rho}^{NM}$ is a projection from $M$ onto $Ns^N(\rho)$.

(10) Define $\rho$ and $\rho'$ to be $N$-equivalent if $s^N(\rho) = s^N(\rho')$ and $\rho'$ is in the norm closure of the set of all $A_\rho$, $A \in N$, $A \geq 0$. It is an equivalence relation and $F_{\rho}^{NM} = F_{\rho'}^{NM}$ if and only if $\rho$ is $N$-equivalent to $\rho'$.

(11) Let $s^N(\Omega_\rho)$ be the projection on the closure of $\pi_\rho(N)\Omega_\rho$. The mapping from $Q \in Ns^N(\rho)$ to $s^N(\Omega_\rho)\pi_\rho(Q) \in s^N(\Omega_\rho)\pi_\rho(s^N(\rho)M)$ is bijective. Let the inverse mapping be $\alpha$. Then

\[ F_{\rho}^{NM}(Q) = \alpha s^N(\Omega_\rho)\pi_\rho(Q)s^N(\Omega_\rho). \]

(12) If $K \subseteq N$, then $F_{p}^{KN}F_{\rho}^{NM} = F_{p}^{KM}$.

Proof. (9) $F_{\rho}^{NM}(Q) = Qs^N(\rho) = Q$ for $Q \in Ns(\rho)$ due to Theorem 1 (1) and (2). Hence $F_{\rho}^{NM}$ is a projection onto $Ns^N(\rho)$.

(10) If $\rho$ is $N$-equivalent to $\rho'$, then $\rho'$ is a norm limit of $A_n\rho$, where we may restrict $s^N(\rho)A_n\rho = \rho_n$. Then by Theorem 1 (7) and (8), we have $F_{\rho}^{NM}(Q) = \lim F_{\rho_n}^{NM}(Q) = F_{\rho}^{NM}(Q)$.

Next assume that $F_{\rho}^{NM} = F_{\rho'}^{NM}$. From Theorem 1 (2), we have

\[ s^N(\rho) = s^N(\rho'). \]

By the Radon-Nikodym theorem, there exists a non-negative self-adjoint operator $A$ affiliated with $N$ such that $s(A) = s^N(\rho)$ and

\[ \rho(QA) = \rho'(Q), \quad Q \in N. \]

Let $E_\lambda^A$ be the spectral projection of $A$ and $A_n = AE_n^A \in N$, $\rho_n = A_n\rho$. Let $\overline{\rho} = A\rho = \lim A_n\rho$ which exists as a state of $M$, because $0 \leq \rho(A_nQ) - \rho(A_mQ) \leq ||Q||\rho(A_n - A_m) \rightarrow 0$ for $Q \geq 0$, $Q \in M$ and $n \geq m$. Then the restriction of $\rho$ to $N$ is the same as the restriction of $\rho'$ to $N$. By what we have already proved, $F_{\rho}^{NM} = F_{\rho'}^{NM} = F_{\rho'}^{NM}$. Hence we have
\[ \rho(QQ') = \rho'(QQ') \]
for all \( Q \in M \) and \( Q' \in N \). Setting \( Q' = 1 \), we have \( \rho = \rho' \) as a functional on \( M \). This shows that \( \rho \) is \( N \)-equivalent to \( \rho' \).

\( F_{\rho}^{NM} = F_{\rho'}^{NM} \) is certainly an equivalence relation for \( \rho \) and \( \rho' \).

(11) Since \( \rho \) is faithful on \( Ns^N(\rho) \), \( s^{N'}(\Omega_{\rho})\pi_{\rho}(Q) = 0 \) for \( Q \in Ns^N(\rho) \) implies \( ||s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)\Omega_{\rho}||^2 = \rho(Q^*Q) = 0 \) and hence \( Q = 0 \). Thus \( Q \rightarrow s^{N'}(\Omega_{\rho})\pi_{\rho}(Q) \) is bijective from \( Ns^N(\rho) \) to \( s^{N'}(\Omega_{\rho})\pi_{\rho}(Ns^N(\rho)) \).

We have, for \( Q \in M, Q' \in N, \)

\[ \rho(QQ') = (\Omega_{\rho}, \pi_{\rho}(Q)\pi_{\rho}(Q')\Omega_{\rho}) \]
\[ = (\Omega_{\rho}, s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)s^{N'}(\Omega_{\rho})\pi_{\rho}(Q')\Omega_{\rho}). \]

If we prove that

(4.2) \[ s^{N'}(\Omega_{\rho})\pi_{\rho}(M)s^{N'}(\Omega_{\rho}) = \pi_{\rho}(Ns^N(\rho))s^{N'}(\Omega_{\rho}), \]

then we have

\[ \rho(QQ') = \rho(\{\alpha s^{N'}(\Omega_{\rho})\pi_{\rho}(Q)s^{N'}(\Omega_{\rho})\})Q'. \]

Due to the commutativity of elements of \( N \), we have (4.1).

To prove (4.2), we note that \( \Omega_{\rho} \) is a cyclic vector for abelian \( \pi_{\rho}(N) \) on \( s^{N'}(\Omega_{\rho})H_{\rho} \) by definition and hence maximal abelian there. Furthermore, \( \pi_{\rho}(1-s^{N}(\rho))Q\Omega_{\rho} = 0 \) for \( Q \in \pi_{\rho}(N) \) by the commutativity and hence \( s^{N'}(\Omega_{\rho})\pi_{\rho}(s^{N}(\rho)) = s^{N'}(\Omega_{\rho}) \). Thus any \( Q \in \mathfrak{B}(s^{N'}(\Omega_{\rho})H_{\rho}) \) satisfying \([Q, Q_1] = 0\) for all \( Q_1 \in \pi_{\rho}(N) \) belongs to \( \pi_{\rho}(Ns^N(\rho))s^{N'}(\Omega_{\rho}) \).

Since \( s^{N'}(\Omega_{\rho}) \in \pi_{\rho}(N)' \) and \( N \) commutes with \( M, Q \in s^{N'}(\Omega_{\rho})\pi_{\rho}(M) \), \( s^{N'}(\Omega_{\rho}) \) commutes with any \( Q_1 \in \pi_{\rho}(N) \). Hence

\[ s^{N'}(\Omega_{\rho})\pi_{\rho}(M)s^{N'}(\Omega_{\rho}) \subseteq \pi_{\rho}(Ns^N(\rho))s^{N'}(\Omega_{\rho}). \]

Since \( M \supseteqNs^N(\rho) \), the equality holds.

(12) This is immediate from the defining equations (2.1) and (2.2) and the abelian property of \( N \).

Q.E.D.

**Corollary.** (10), (11) and (12) of Theorem 3 holds if \( N \subseteq M' \) and \( N \) is abelian.
Proof. Let \( R = (N \cup M)' \). Then \( N \) is in the center of \( R \). Furthermore

\[
F^N_M(Q) = F^N_R(Q), \quad Q \in M.
\]

Hence by applying Theorem 3 (10) and (11) to \( F^N_R \), we obtain (10) and (11) for \( F^N_M \). Note that \( F^N_R(Q) \) for \( Q \in M \) determines \( F^N_R \) due to the property (1) of Theorem 1.

Q.E.D.

Remark. If \( N \) is abelian, \( Q \in N \) can be identified with continuous function on its spectrum and any normal linear function on \( N \) with a Radon measure on its spectrum. Denoting the measure corresponding to the normal linear functional \( \rho(QQ') = f_Q(Q') \) for \( Q' \in N \) and \( Q \in M \) by \( \mu_Q \), \( F^N_M \) is given by the Radon-Nikodym derivative:

\[
F^N_M(Q) = d\mu_Q/d\mu_1
\]

where we define \( d\mu_Q/d\mu_1 = 0 \) outside the support of \( s^N(\rho) \).

\( F^N_M(Q) \) for an abelian \( N \) has been introduced through the equation (4.1) by D. Ruelle \[5\] in his theory of decomposition of state. If \( \mu_\rho \) denotes the measure on the spectrum \( \Xi_N \) of \( N \), corresponding to the restriction of \( \rho \) to \( N \), then

\[
\rho(Q) = \int_{\Xi_N} \xi(F^N_M(Q))d\mu_\rho(\xi)
\]

is his decomposition.

§ 5. Asymptotically Abelian System

A net \( Q_\alpha \) of elements of a von Neumann algebra \( M \) is called weakly central if there exists a weakly total selfadjoint subset \( M_0 \) of \( M \) such that

\[
[x, Q_\alpha] \to 0
\]

in the weak topology for every \( x \in M_0 \). If (5.1) holds with the strong limit, then \( Q_\alpha \) is called strongly central.
The following result is an extension of Proposition 4 of \([1]\) to non-factors.

**Theorem 4.** If \(Q_a\) is a uniformly bounded weakly central net in \(M\), then

\[
\text{w-lim}_a (Q_a - F_{\rho}^{Z,M}(Q_a))s^Z(\rho) = 0
\]

for any normal positive linear functional \(\rho\) on \(M\), where \(Z = M \cap M'\).

For any two normal positive linear functionals \(\rho\) and \(\rho'\),

\[
\text{w-lim}_a (F_{\rho}^{Z,M}(Q_a) - F_{\rho'}^{Z,M}(Q_a))(s^Z(\rho) \wedge s^Z(\rho')) = 0.
\]

In particular, if \(s^Z(\rho) = s^Z(\rho')\),

\[
\text{w-lim}_a (F_{\rho}^{Z,M}(Q_a) - F_{\rho'}^{Z,M}(Q_a)) = 0.
\]

When \(s^Z(\rho') \leq s^Z(\rho)\), let \(A^Z(\rho'/\rho)\) be the Radon-Nikodym derivative of \(\rho'\) by \(\rho\) relative to \(Z\), namely,

\[
A^Z(\rho'/\rho) = \int \lambda dE_\lambda, \quad E_\lambda \in Z,
\]

\[
s(A^Z(\rho'/\rho)) = s^Z(\rho'),
\]

\[
\rho'(z) = \rho(zA^Z(\rho'/\rho)), \quad z \in Z,
\]

where \(A^Z(\rho'/\rho)\) can be unbounded and \(\rho(zA^Z(\rho'/\rho)) = \int \lambda d(\rho(zE_\lambda))\).

If \(s^Z(\rho') \leq s^Z(\rho)\), then

\[
\lim_a \{\rho'(Q_a) - \rho(Q_a A(\rho'/\rho))\} = 0.
\]

In particular, if \(A(\rho'/\rho) = 1\) (i.e. if \(\rho | Z = \rho' | Z\)), then

\[
\lim_a \{\rho'(Q_a) - \rho(Q_a)\} = 0.
\]

**Proof.** Consider \(H_\rho, \pi_\rho, \Omega_\rho\) canonically associated with \(\rho \neq 0\). Let \(\overline{Q}_\rho = \pi_\rho(Q_a)\). Let \(R_0\) be the linear hull of \(\pi_\rho(M_0), R = \pi_\rho(M) = \pi_\rho(M_0)' = \overline{R_0}, Z = \pi_\rho(Z), s' = s^Z(\Omega_\rho)\).
Given $\varepsilon > 0$ and vectors $\Phi_j \in H, j = 1, \ldots, n, \Phi_j \neq 0$, there exist $Q''_k \in R_0$ and $Q'_l, \ldots, Q'_k \in R'$ such that

$$P_\varepsilon = \sum_{j=1}^{k} Q'_j Q'_j$$

satisfies

$$\|P_\varepsilon \Omega_\rho - \Omega_\rho\| = \|\{P_\varepsilon - s'\} \Omega_\rho\|$$

$$\leq \{\sup_{\alpha} \|Q_\alpha\|\}^{-1} \{\sup_{\alpha} \|\Phi_j\|\}^{-1} \varepsilon / 4,$$

$$\|\{P^*_\varepsilon - s'\} \Phi_j\| \leq \{\sup_{\alpha} \|Q_\alpha\|\}^{-1} \|\Omega_\rho\|^{-1} \varepsilon / 4,$$

because $s' \in R'$ and linear hull of $R_0^0 R_0'$ is * strongly dense in $Z'$.

For this set of operators, there exists $\alpha_\varepsilon$ such that for all $\alpha > \alpha_\varepsilon$,

$$|\langle \Phi_j, [Q_\alpha, P_\varepsilon] \Omega_\rho \rangle| < \varepsilon / 2,$$

due to the weakly central property.

Then for $\alpha > \alpha_\varepsilon$, we have

$$|\langle \Phi_j, [Q_\alpha, s'] \Omega_\rho \rangle|$$

$$\leq |\langle \Phi_j, [Q_\alpha, P_\varepsilon] \Omega_\rho \rangle| + \varepsilon / 2$$

$$< \varepsilon.$$

Hence

(5.7) \hspace{1cm} \text{w-lim}_{\alpha} [Q_\alpha, s'] \Omega_\rho = 0.

By Theorem 3 (11), we have

(5.8) \hspace{1cm} s' Q_\alpha s' = \pi_\rho (F_\rho^{Z_\rho}(Q_\alpha)) s'.

Since $s' \Omega_\rho = \Omega_\rho$, we obtain from (5.7) and (5.8)

$$\text{w-lim}_{\alpha} \pi_\rho (Q_\alpha - F_\rho^{Z_\rho}(Q_\alpha)) \Omega_\rho = 0.$$

Take any $Q'' \in R_0, Q' \in R_0'$. Since $\pi_\rho (F_\rho^{Z_\rho}(Q_\alpha)) \in Z$, it commutes with
$Q'Q'$. By weakly central property,

$$\text{w-lim}_a [\pi_{\rho}(Q_\alpha), Q'Q']_{\mathcal{F}} = 0.$$ 

Hence

$$(5.9) \quad \text{w-lim}_a \pi_{\rho}(Q_\alpha - F_{\rho}^{Z\mathcal{M}}(Q_\alpha))_{\mathcal{F}} = 0$$

for $\mathcal{F} = Q'Q'_{\mathcal{F}}$. Since $Q_\alpha$ is assumed to uniformly bounded and $\|F_{\rho}^{Z\mathcal{M}}\| = 1$, (5.9) holds for all $\mathcal{F}$ in the closure of $Z'_{\mathcal{Q}}$, which is $s^Z(\mathcal{F})_{\mathcal{F}} = \pi_\rho(s^Z(\rho))_{\mathcal{F}}$. Hence

$$\text{w-lim}_a \pi_{\rho}(\{Q_\alpha - F_{\rho}^{Z\mathcal{M}}(Q_\alpha)\} s^Z(\rho)) = 0.$$ 

Since $\pi_{\rho}$ is faithful on $s^Z(\rho)_{\mathcal{M}}$, we have (5.2).

From (5.2) for $\rho$ and $\rho'$, we have (5.3) and in the special case $s^Z(\rho') = s^Z(\rho)$, we obtain (5.4), where we use $F_{\rho}^{Z\mathcal{M}}(Q_\alpha) s^Z(\rho) = F_{\rho}^{Z\mathcal{M}}(Q_\alpha)$.

If $s^Z(\rho') \leq s^Z(\rho)$, we obtain from (5.2)

$$(5.10) \quad \lim_a \{\rho'(Q_\alpha) - \rho'(F_{\rho}^{Z\mathcal{M}}(Q_\alpha))\} = 0.$$ 

Using the definition of $A^Z(\rho'/\rho)$ and (2.1) with $Q' = 1$, we obtain

$$\rho'(F_{\rho}^{Z\mathcal{M}}(Q_\alpha)) = \int \lambda d\rho(F_{\rho}^{Z\mathcal{M}}(Q_\alpha) E_\lambda)$$

$$= \int \lambda d\rho(Q_\alpha E_\lambda) = \rho(Q_\alpha A^Z(\rho'/\rho)).$$

This proves (5.5). (5.6) then follows. Q.E.D.

If a subset $\mathcal{H}$ of a von Neumann algebra $\mathcal{M}$ and a net of * automorphisms $\tau_\alpha$ of $\mathcal{M}$ satisfy the property that $\tau_\alpha Q$ for every $Q \in \mathcal{H}$ is weakly (or strongly) central, then $\mathcal{H}$ is called weakly (or strongly) $\tau_\alpha$ central in $\mathcal{M}$.

**Corollary.** If $\mathcal{H}$ is weakly $\tau_\alpha$ central in $\mathcal{M}$ and $\rho$ is a $\tau_\alpha$ invariant normal positive linear functional on $\mathcal{M}$, then

$$(5.11) \quad \text{w-lim}_a (\tau_\alpha Q - \tau_\alpha F_{\rho}^{Z\mathcal{M}}(Q)) s^Z(\rho) = 0$$
for all $Q \in \mathfrak{A}$ where $Z$ is the center of $M$.

If $\rho'$ is another normal positive linear functional on $M$ and $s^Z(\rho') \subseteq s^Z(\rho)$, then

\[
\lim_{\alpha} \{ \rho'(\tau_\alpha Q) - \rho(Q \tau_\alpha^{-1} A^Z(\rho'/\rho)) \} = 0
\]

(5.12)

for all $Q \in \mathfrak{A}$ where $\tau_\alpha^{-1} A^Z(\rho'/\rho) = \int \lambda d(\tau_\alpha^{-1} E_\lambda)$. In particular if $\rho'(z) = \rho(z)$ for all $z \in Z$, then

\[
\lim_{\alpha} \rho'(\tau_\alpha Q) = \rho(Q), \quad Q \in \mathfrak{A}.
\]

(5.13)

**Proof.** Since $||\tau_\alpha Q|| = ||Q||$, $\tau_\alpha Q$ is uniformly bounded. By (5.2), (2.3) and $\tau_\alpha^* \rho = \rho$, we have (5.11). (5.12) follows from (5.5) and the invariance of $\rho$. (5.13) is a special case of (5.12) where $s^Z(\rho') = s^Z(\rho)$ and $A^Z(\rho'/\rho) = 1$. Q.E.D.

**Remark.** If $Q_\alpha$ is weakly central and uniformly bounded, then $w$-$\lim [x, Q_\alpha] = 0$ for all $x \in M$, because it holds for any $x$ in the linear hull $M_1$ of $M_0$, which, being a weakly dense linear subset, is * strongly dense in $M$, and hence for given $x \in M$, $\varepsilon > 0$, $\Psi_j$, $\Phi_j$, there exist $x' \in M_1$ and $\alpha_0$ such that $||Q_\alpha|| < \varepsilon$,

\[
||\Psi_j||L||x - x'||\Phi_j|| < \varepsilon/3, \quad ||(x - x')^*\Psi_j||L||\Phi_j|| < \varepsilon/3
\]

and

\[
|([\Psi_j, [x', Q_\alpha]]\Phi_j)| < \varepsilon/3 \quad \text{for } \alpha > \alpha_0
\]

which imply $|([\Psi_j, [x, Q_\alpha]]\Phi_j)| < \varepsilon, j = 1, \ldots, n$.

Hence, if $\mathfrak{A}$ is weakly $\tau_\alpha$ central, then the norm closure $\mathfrak{A}_1$ of the linear hull of $\mathfrak{A} \cup \mathfrak{A}^*$ is obviously weakly $\tau_\alpha$ central and (5.2)–(5.6) for $Q_\alpha = \tau_\alpha Q$ and (5.11)–(5.13) hold for any $Q \in \mathfrak{A}_1$.

(5.11)–(5.13) hold for $\sigma$-weak coiser of $\mathfrak{A}_1$ if $s^Z(\rho)$ and $s^Z(\rho') \subseteq s^Z(\rho)$ are replaced by $s^M(\rho)$ and $s^M(\rho') \subseteq s^M(\rho)$, because (5.11) implies

\[
w$-$\lim \pi_\rho(\tau_\alpha Q - \tau_\alpha F^M_\rho(Q)) \Psi = 0
\]
for $\mathcal{F} = \mathcal{Q}_\rho$ and $Q$ in the strong closure of the unit ball of $\mathcal{A}_1$ and hence for $\mathcal{F} \in \mathcal{K}\mathcal{Q}_\rho$ and $Q$ in the $\sigma$-weak closure of $\mathcal{A}_1$.

The next theorem has an application in [2].

**Description of situation.** A von Neumann algebra $M$, a net of $*$ automorphisms $\tau_\alpha$, a faithful normal positive linear functional $\rho \neq 0$ on $M$, invariant under all $\tau_\alpha$ and a $C^*$ subalgebra $\mathfrak{A}$ of $M$ are given. Let $U_\alpha$ be the unique unitary operator on $H_\rho$ satisfying $U_\alpha \pi_\rho(Q) \mathcal{Q}_\rho = \pi_\rho (\tau_\alpha Q) \mathcal{Q}_\rho$ for all $Q \in M$. Let $\tau_\alpha Q = U_\alpha Q U_\alpha^*$ for all $Q \in \mathfrak{B}(H_\rho)$. Let $J_\rho$ be the modular conjugation operator for the cyclic and separating $\mathcal{Q}_\rho$ relative to $\pi_\rho(M)$ and $j_\rho(Q) = J_\rho Q J_\rho, Q \in \mathfrak{B}(H_\rho)$. Let $\hat{\mathfrak{A}}$ be the $C^*$ algebra generated by

$$\pi_\rho(\mathfrak{A}) j_\rho \{ \pi_\rho(\mathfrak{A}) \}$$

and $\hat{\mathcal{R}} = (\pi_\rho(M) \cap \pi_\rho(M)')''$.

**Theorem 5.** Assume that $\mathfrak{A}$ is strongly $\tau_\alpha$ central in $M$. For any normal positive linear functional $\rho'$ on $\mathfrak{B}(H_\rho)$, all $Q \in \hat{\mathfrak{A}}$ satisfy

$$\lim_\alpha \{ \rho' (\tau_\alpha Q) - (\mathcal{Q}_\rho, Q \pi_\rho^{-1} A^{\pi}(\rho' / \rho) \mathcal{Q}_\rho) \} = 0$$

where $\rho = \omega_\alpha, \pi_\rho(M) \cap \pi_\rho(M)'$ which is the center of $\hat{\mathcal{R}}$ and $A^{\pi}(\rho' / \rho)$ is as in Theorem 4. In particular, if $\rho(z) = \rho'(\pi_\rho(z))$ for all $z$ in the center of $M$, then

$$\lim_\alpha \rho'(\tau_\alpha Q) = \rho(Q), \quad Q \in \hat{\mathfrak{A}}.$$

**Proof.** Let $S_\rho = J_\rho^\dagger A_\rho^{\dagger/2}$. We have

$$U_\alpha S_\rho Q \mathcal{Q}_\rho = U_\alpha Q^* \mathcal{Q}_\rho = (\tau_\alpha Q)^* \mathcal{Q}_\rho = S_\rho (\tau_\alpha Q) \mathcal{Q}_\rho = S_\rho U_\alpha Q \mathcal{Q}_\rho.$$

where $Q \in \pi_\rho(M)$, which implies $\tau_\alpha Q \in \pi_\rho(M)$. Thus $U_\alpha$ commutes with $S_\rho$ and hence with $A_\rho = S_\rho^* S_\rho$ and $J_\rho$.

Let $Q, Q' \in \mathfrak{A}, Q_0 \in M_0(Q), Q_0' \in M_0(Q')$ where $M_0(Q)$ is a selfadjoint
total subset of $M$ such that (5.1) is satisfied in the strong topology for all $x \in M_0(Q)$ and $Q_\alpha = \tau_\alpha(Q)$ and $M_0(Q')$ is the same for $Q'$.

(5.17) \[
[\pi_\rho(Q_0)j_\rho\{\pi_\rho(Q_0')\}, \tau_\alpha(\pi_\rho(Q)j_\rho\{\pi_\rho(Q')\})] \\
= \pi_\rho([Q_0, \tau_\alpha Q])j_\rho\{\pi_\rho(Q_0'\tau_\alpha Q')\} \\
+ \pi_\rho(\{\tau_\alpha Q\}Q_0)j_\rho\{\pi_\rho([Q_0', \tau_\alpha Q'])\}.
\]

Since both $[Q_0, \tau_\alpha Q]$ and $[Q_0', \tau_\alpha Q']$ tends to 0 strongly, and all operators are bounded uniformly in $\alpha$, (5.17) tends to 0 strongly.

Since $M_0(Q)$ and $M_0(Q')$ are selfadjoint and total,

$\hat{M}_0 = \pi_\rho(M_0(Q))j_\rho\{\pi_\rho(M_0(Q'))\}$

is also selfadjoint and total in $\hat{R}$. Hence (5.14) is strongly $\tau_\alpha$ central in $\hat{R}$.

By (5.12) and (5.13), we obtain (5.15) and (5.16) when $Q$ is in (5.14). Note that $\sigma^2(\rho) = 1$ because $\rho$ is assumed to be faithful. Note also that $Z = \pi_\rho(Z)$.

By Remark after Corollary to Theorem 4, $Q$ in (5.15) and (5.16) can be in the norm closure of the linear hull of (5.14), which is $\hat{A}$.

Q.E.D.

Remark. Let $\mathcal{A}_1$ be a $C^*$ algebra, $\rho$ be a state on $\mathcal{A}_1$ and $\tau_\alpha$ be a net of * automorphisms of $\mathcal{A}_1$. If

$\pi_\rho([Q_1, \tau_\alpha Q_2])$

tends to 0 weakly (or strongly) for all $Q_1, Q_2 \in \mathcal{A}_1$, then $\mathcal{A}_1$ is said to be weakly (or strongly) $\tau_\alpha$ asymptotically abelian. We can apply Theorem 4 to such a situation by taking $M_0 = \pi_\rho(\mathcal{A}_1)$, $M = \pi_\rho(\mathcal{A}_1)'$ and $Q_\alpha = \pi_\rho(\tau_\alpha Q)$ for $Q \in \mathcal{A}_1$. If $\rho$ is $\tau_\alpha$ invariant and $\mathcal{A}_1$ is strongly $\tau_\alpha$ asymptotically abelian, then we can apply Theorem 5.

Method of big translation in [3] can be formulated as follows. (See Theorem 6).
Lemma 6. Let $\rho_\alpha$ be a net of (not necessarily normal) positive linear functionals on $(M \cup N)'$ such that $\lim_\alpha \rho_\alpha = \rho$ (i.e. $\lim_\alpha \rho_\alpha(Q) = \rho(Q)$ for each $Q \in (M \cup N)'$). Assume that the restriction of $\rho_\alpha$ to $N$ is normal and independent of $\alpha$. Assume also that $N$ is abelian. Then

\[(5.18) \quad \text{w-lim}_\alpha F_{\rho_\alpha}^{NM}(Q) = F_{\rho}^{NM}(Q), \quad Q \in M.\]

(See Remark 2 of §2.)

Proof. Let the restriction of $\rho_\alpha$ to $N$ be denoted by $\sigma$ which is independent of $\alpha$ by assumption. Then we obtain, from $\lim_\alpha \rho_\alpha = \rho$,

$$\lim_\alpha \sigma(F_{\rho_\alpha}^{NM}(Q)Q_1Q_2) = \sigma(F_{\rho}^{NM}(Q)Q_1Q_2)$$

for all $Q_1 \in N$, $Q_2 \in N$. Setting $x_\alpha = (F_{\rho_\alpha}^{NM} - F_{\rho}^{NM})(Q)$, we have

$$\lim_\alpha (\mathcal{F}, \pi_\alpha(x_\alpha)\Theta) = 0$$

for $\mathcal{F} = \pi_\alpha(Q_1)\Omega_\sigma$ and $\Theta = \pi_\alpha(Q_2)\Omega_\sigma$. Since $\|x_\alpha\| \leq 2\|Q\|$, we have w-lim $\pi_\alpha(x_\alpha) = 0$. Since $s(x_\alpha) \leq s^N(\sigma)$, we obtain w-lim $x_\alpha = 0$.

Theorem 6. Let $\mathcal{A}$ be a weakly $\tau_\alpha$ central $C^*$ subalgebra of a von Neumann algebra $M$. Assume that the center $Z$ of $M$ is elementwise $\tau_\alpha$ invariant and has a faithful normal state $\rho$. Let $\overline{\mathcal{A}}$ be the $C^*$ algebra generated by $\mathcal{A}$ and $Z$. Then there exists a subnet $\tau_{\alpha(\beta)}$ such that

\[(5.19) \quad L(Q) = \text{w-lim}_\beta \tau_{\alpha(\beta)}Q\]

exists for all $Q \in \overline{\mathcal{A}}$, where $L$ is $Z$-linear, completely positive projection of norm 1 from $\overline{\mathcal{A}}$ onto $Z$ and $L(1) = 1$. If $Z$ is trivial, then $L(Q) = \omega(Q)1$ for a state $\omega$ on $\mathcal{A}$.

Proof. Let $\tilde{\rho}$ be any extension of $\rho$ to a state on $M$. By weak compactness, there exists a subnet $\alpha(\beta)$ such that

$$\lim \tau_{\alpha(\beta)}^* \beta = \rho_\infty$$

exists.
Since $Z$ is elementwise invariant under $\tau_a$, the restriction of $\tau^Z_a\rho$ to $Z$ is always $\rho$ and hence the restriction of $\rho_\infty$ to $Z$ is also $\rho$. Since $\rho$ is faithful on $Z$, $s^Z(\rho) = 1$. By (5.2), (2.3) and Lemma 6,

$$w\text{-}\lim_{\beta} \tau_a(\beta)Q = w\text{-}\lim_{\beta} F^Z_F(\tau_a(\beta)Q)$$

$$= w\text{-}\lim_{\beta} F^Z_M(\tau_a(\beta)Q)$$

$$= F^Z_M(Q).$$

Hence (5.19) holds with $L = F^Z_F$. The properties of $L$ follow from Theorem 1 applied for $F^Z_F$ (see Remark 2 of § 2) except possibly for the complete positivity.

Since $Z$ is abelian, $J_\rho z^*J_\rho = z, z \in \pi_\rho(Z)$ for a faithful state $\rho$. Hence $Q \rightarrow Q = J_\rho Q J_\rho$ is a transposition on $\mathcal{A}(H_\rho)$ leaving $Z$ invariant. Hence if $L$ is transposed-$n$-positive then

$$L \otimes 1 = (\pi^{-1}_\rho \otimes 1_n)(t \otimes t_n)(\pi_\rho \otimes 1_n)(L \otimes t_n)$$

is also positive and hence $F$ is $n$ positive. Here $1_n$ and $t_n$ denote the identity mapping and a transposition of $n \times n$ matrices. Q.E.D.

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**References**


