SOME VARIANTS OF A THEOREM OF MAHLER*

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§ 1. Introduction

Throughout this paper Z_p , Q_p , and C_p will respectively denote the ring of p -adic rational integers, the field of p -adic rational numbers and the completion of the algebraic closure of Q_p . Let J be the set N of natural numbers and 0 with the p -adic valuation $|\cdot|$, which is so normalized that $|p| = p^{-1}$, hence J is regarded as a dense subset of the p -adic integer ring Z_p . We consider any function $f(x)$ defined on J taking its values in C_p and Mahler's expansion

$$f^*(x) = \sum_{n=0}^{\infty} \binom{x}{n} \Delta^n f(0).$$

Here a linear difference operator Δ to f is, by definition, $\Delta f(x) = f(x+1) - f(x)$. Applying the n -th linear difference operator, we have

$$\Delta^n f(0) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} f(i).$$

Moreover, if $f(x)$ is a continuous function, then $\Delta^n f(0) \rightarrow 0$ (as $n \rightarrow \infty$) and $f^*(x) = f(x)$ ([3]). Recently several problems which include Mahler's expansion as a special or limiting case are treated by L. Van Hamme ([6]) in a simple and elegant way. It is known that a C_p -valued function on Z_p is S-Lipschitz (function) if and only if there exists a constant $k \geq 0$ such that $|\Delta^n f(0)| \leq kS([\log_p n])$ for all $n \geq 1$, where $S(\rho)$, defined on N , has the monotone nonincreasing property, namely we take any real valued function $S(\rho) \geq 0$ such that $S(\rho) \geq S(\rho+1)$ ([2]). The purpose of this paper is to prove a few results which include Mahler's expansion as a special case and another property of S-Lipschitz functions.

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§ 2. Theorems

In this section we prove a formula which is a generalization of Mahler's expansion. We can observe the followings. Let U, V be two indeterminates. The equality $(V(U-1) - U)^n = ((V-1)(U-1) - 1)^n$ yields

$$\sum_{k=0}^n \binom{n}{k} (-1)^k V^k (U-1)^k U^{n-k} = \sum_{k=0}^n \binom{n}{k} (-1)^k (V-1)^k (U-1)^k.$$

We consider a linear map

$$L_1 : C_p[U] \mapsto C_p : U^k \mapsto f(k),$$

which maps $(U-1)^k$ to $\Delta^k f(0)$.

Here, a short calculation gives us

$$\begin{aligned} \Delta^k f(n-k) &= \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} f(n-k+j) \\ &= \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} L_1(U^{n-k+j}) \\ &= L_1\left(\left\{\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} U^j\right\} U^{n-k}\right) = L_1(U^{n-k}(U-1)^k). \end{aligned}$$

Then define a linear map similar to L_1 by making use of V instead of U

$$L_2 : C_p[V] \mapsto C_p : V^k \mapsto g(k).$$

We have

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} (-1)^k g(k) \Delta^k f(n-k) &= \sum_{k=0}^n \binom{n}{k} (-1)^k L_2((V-1)^k) L_1((U-1)^k) \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k \Delta^k g(0) \Delta^k f(0). \end{aligned}$$

Consequently we have

$$\sum_{k=0}^n \binom{n}{k} (-1)^k g(k) \Delta^k f(n-k) = \sum_{k=0}^n \binom{n}{k} (-1)^k \Delta^k g(0) \Delta^k f(0).$$

Thus we obtain the following

Theorem 1. *If $f: Z_p \mapsto C_p$ is a continuous function and $g: J \mapsto C_p$ is a bounded sequence, then*

$$\sum_{n=0}^{\infty} \binom{x}{n} (-1)^n \Delta^n g(0) \Delta^n f(0) = \sum_{n=0}^{\infty} \binom{x}{n} (-1)^n g(n) \Delta^n f(x-n)$$

Proof. Since $\Delta^n f(x) \rightarrow 0$ (as $n \rightarrow \infty$) uniformly, the series on the left hand side and on the right hand side are uniformly convergent, which means that their sums are continuous functions of x , hence it is sufficient to prove the theorem for $x=m \in J$.

This is a generalization of Mahler's expansion.
Because for

$$g(n) = \begin{cases} 0 & n \geq 1 \\ 1 & n = 0, \end{cases}$$

we have $\Delta^n g(0) = (-1)^n$ for $n \geq 0$, the formula yields Mahler's expansion.

There is the third theorem of Van Hamme ([6]), which gives a Mahler expansion of a remainder term with convolution product.

Definition 1 ([4]). *The convolution of two sequences $f(n)$ and $g(n)$ is, by definition, the sequence given by*

$$f \otimes g(n) = \sum_{k=0}^n f(k) g(n-k).$$

Now we see by the above definition, for $x \in J$

$$\begin{aligned} \Delta(f \otimes g)(x) &= f \otimes g(x+1) - f \otimes g(x) \\ &= \sum_{i=0}^{x+1} f(i) g(x+1-i) - \sum_{i=0}^x f(i) g(x-i) \\ &= \sum_{i=0}^x f(i) g(x+1-i) - \sum_{i=0}^x f(i) g(x-i) + f(x+1) g(0) \\ &= \sum_{i=0}^x f(i) \Delta g(x-i) + f(x+1) g(0) \\ &= f \otimes \Delta g(x) + f(x+1) g(0). \end{aligned}$$

Here we notice

$$\begin{aligned} \Delta^2(f \otimes g)(x) &= \Delta(f \otimes g)(x+1) - \Delta(f \otimes g)(x) \\ &= \sum_{i=0}^{x+1} f(i) \Delta g(x+1-i) + f(x+2) g(0) - \sum_{i=0}^x f(i) \Delta g(x-i) \end{aligned}$$

$$\begin{aligned}
& -f(x+1)g(0) \\
& = \sum_{i=0}^x f(i) \{ \Delta g(x+1-i) - \Delta g(x-i) \} + f(x+1) \Delta g(0) \\
& \quad + \{ f(x+2) - f(x+1) \} g(0) \\
& = \sum_{i=0}^x f(i) \Delta^2 g(x-i) + f(x+1) \Delta g(0) + \Delta f(x+1) g(0) \\
& = \sum_{i=0}^x f(i) \Delta^2 g(x-i) + \sum_{j=0}^1 \Delta^j f(x+1) \Delta^{1-j} g(0) \\
& = f \otimes \Delta^2 g(x) + \sum_{j=0}^1 \Delta^j f(x+1) \Delta^{1-j} g(0).
\end{aligned}$$

Continuing this process, we have

$$\Delta^{n+1}(f \otimes g)(x) = f \otimes \Delta^{n+1} g(x) + \sum_{j=0}^n \Delta^j f(x+1) \Delta^{n-j} g(0).$$

Thus we obtain the following

Theorem 2. *Let $C(Z_p, C_p)$ be the space of all continuous functions on Z_p with values in C_p . For $f, g \in C(Z_p, C_p)$ and $n \in J$ we have*

$$\Delta^{n+1}(f \otimes g)(x) = (f \otimes \Delta^{n+1} g)(x) + \sum_{j=0}^n \Delta^j f(x+1) \Delta^{n-j} g(0).$$

Moreover,

$$\Delta^{n+1}((\Delta^m f) \otimes g)(x) = (\Delta^m f \otimes \Delta^{n+1} g)(x) + \sum_{j=0}^n \Delta^{j+m} f(x+1) \Delta^{n-j} g(0).$$

The above Theorem 2 becomes

$$\Delta^{n+1}\left(\binom{x}{n} \otimes f(x)\right) = \sum_{k=0}^n \binom{x+1}{k} \Delta^k f(0) + \left(\binom{x}{n} \otimes \Delta^{n+1} f(x)\right).$$

Remark. Set

$$\|f\| = \sup_{x \in Z_p} |f(x)|.$$

We observe that

$$\begin{aligned}
& \|f \otimes g\| \leq \|f\| \cdot \|g\| \\
& \|\Delta^{n+1}(f \otimes g)\| \leq \|f\| \cdot \|\Delta^{n+1} g\| + \max_{0 \leq j \leq n} \|\Delta^j f\| \cdot \|\Delta^{n-j} g\|,
\end{aligned}$$

$$\|\Delta^m f \otimes \Delta^{n+1} g\| \leq \|\Delta^m f\| \cdot \|\Delta^{n+1} g\|.$$

Lemma 1. *If the sequences $f(n)$ and $g(n)$ are the restrictions to J of two continuous functions from Z_p to C_p , then*

$$\Delta^n(f \otimes g)(0) \rightarrow 0 \text{ (as } n \rightarrow \infty \text{)}.$$

This is clear by Theorem 2.

We select a sequence of functions $\gamma_k(x) \in C_p[x]$ such that

$$\binom{x}{n} = \sum_{k=0}^n \alpha_k^{(n)} \gamma_k(x)$$

with $\alpha_k^{(n)} \in O_{C_p}$, the integer ring of C_p , and $\alpha_k^{(k)} \in O_{C_p}^\times$, the group of units in C_p .

Thus we obtain

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \binom{x}{n} \Delta^n f(0) \\ &= \sum_{n=0}^{\infty} \Delta^n f(0) \sum_{k=0}^n \alpha_k^{(n)} \gamma_k(x) \\ &= \sum_{k=0}^{\infty} \gamma_k(x) \sum_{n=k}^{\infty} \alpha_k^{(n)} \Delta^n f(0). \end{aligned}$$

Let β_k be an element of C_p such that

$$\beta_k = \sum_{n=k}^{\infty} \alpha_k^{(n)} \Delta^n f(0).$$

Then we have the following

Proposition 1. *For any $f \in C(Z_p, C_p)$ we have*

$$f(x) = \sum_{k=0}^{\infty} \beta_k \gamma_k(x).$$

In particular, for a fixed $y \in Z_p$ we take

$$\gamma_k(x) = \frac{x}{x+yk} \binom{x+yk}{k} \in C_p[x] \text{ and } \alpha_k^{(n)} = \binom{-yk}{n-k}.$$

Thus

$$\binom{x}{n} = \sum_{k=0}^n \alpha_k^{(n)} \gamma_k(x).$$

This identity follows from the more general identity

$$\binom{x+z+ny}{n} = \sum_{k=0}^n \gamma_k(x) \binom{z+(n-k)y}{n-k}$$

by taking $z+ny=0$. This identity is known as Riordan's formula, and can be proved easily in the cases $y=0$, $y=1$ ([6]). Consequently we have

$$\begin{aligned} \beta_k &= \sum_{n=k}^{\infty} \alpha_k^{(n)} \Delta^n f(0) = \sum_{n=k}^{\infty} \binom{-yk}{n-k} \Delta^n f(0) \\ &= \sum_{l=0}^{\infty} \binom{-yk}{l} \Delta^{l+k} f(0) = \Delta^k (1+\Delta)^{-y} f(0) \\ &= \Delta^k f(-yk). \end{aligned}$$

Therefore we obtain the following

Corollary 1 ([6]). *For $f \in (Z_p, C_p)$ we have*

$$f(x) = f(0) + \sum_{k=1}^{\infty} \frac{x}{x+yk} \binom{x+yk}{k} \Delta^k f(-yk).$$

For a fixed $y \in Z_p$ we take

$$\gamma_k(x) = \binom{x-y}{k}, \quad \alpha_k^{(n)} = \binom{y}{n-k}.$$

Thus we have

$$\begin{aligned} \beta_k &= \sum_{n=k}^{\infty} \binom{y}{n-k} \Delta^n f(0) = \sum_{l=0}^{\infty} \binom{y}{l} \Delta^{l+k} f(0). \\ &= \Delta^k (1+\Delta)^y f(0) = \Delta^k f(y). \end{aligned}$$

For $f \in C(Z_p, C_p)$, this gives the theorem of Mahler for the translation function $f_y(x) = f(x+y)$ with fixed element y

$$f_y(x) = \sum_{k=0}^{\infty} \Delta^k f_y(0) \binom{x}{k}.$$

Definition 2 ([2]). *We take any real valued function $S(\rho) \geq 0$ such that $S(\rho) \geq S(\rho+1)$ and $\|\Delta_{p^\rho} f(x)\| = \sup_{x \in J} |f(x+p^\rho) - f(x)| \leq S(\rho)$ for any $\rho \in \mathbb{N}$. We call such function $S(\rho)$ a p -difference bounded (function) for f . Let*

$S(\rho)$ be real valued non-negative monotone non-increasing function on N with $S(\rho) \leq pS(\rho+1)$, $S(\rho) \rightarrow 0$ (as $\rho \rightarrow \infty$). Then any difference bounded function is called an S -Lipschitz function.

Lemma 2 ([2]). *It is necessary and sufficient for a bounded function $f(x)$ on J to be S -Lipschitz that there exists a constant $C \geq 0$ such that $|\Delta^n f(0)| \leq CS([\log_p n])$ holds for all $n \geq 1$.*

In the above we gave the definition of a convolution $\otimes$ of two continuous (or bounded) functions. Then we know the convolution $f \otimes g \in C(Z_p, C_p)$ if $f, g \in C(Z_p, C_p)$. In the case where we consider any S -Lipschitz function instead of a continuous function we have the following variant.

Theorem 3. *If the sequence f and g are two S -Lipschitz functions from Z_p to C_p , then $f \otimes g$ is a S -Lipschitz function from Z_p to C_p .*

Proof. Setting $f_1(x) = f(x+1)$, by Theorem 2

$$\Delta^{n+1}(f \otimes g(x)) = f \otimes \Delta^{n+1}g(x) + \sum_{j=0}^n \Delta^j f_1(x) \Delta^{n-j}g(0).$$

Thus we obtain

$$\Delta^n(f \otimes g(x)) = f \otimes \Delta^n g(x) + \sum_{j=0}^{n-1} \Delta^j f_1(x) \Delta^{n-j-1}g(0).$$

If f is an S -Lipschitz function, then f is a continuous function. Hence $|\Delta^j f_1(x)| \leq M_2$ and $\|f\| \leq M_1$ for some constants M_1, M_2 . Consequently we have

$$\begin{aligned} |\Delta^n f \otimes g(0)| &\leq \|f\| \cdot |\Delta^n g(0)| + \max_{0 \leq j \leq n-1} |\Delta^j f_1(x)| \cdot |\Delta^{n-j-1}g(0)| \\ &\leq M_1 |\Delta^n g(0)| + \max_{0 \leq j \leq n-1} M_2 |\Delta^{n-j-1}g(0)| \end{aligned}$$

for some constants M_1, M_2 .

Let M_3 and M^* be constants such that $|\Delta^n g(0)| \leq M_3 S([\log_p n])$ and $M^* \geq (M_1 + pM_2)M_3$, then we have

$$\begin{aligned} |\Delta^n f \otimes g(0)| &\leq M_1 |\Delta^n g(0)| + \max_{0 \leq j \leq n-1} M_2 |\Delta^{n-j-1}g(0)| \\ &\leq M_1 M_3 S([\log_p n]) + \max_{0 \leq j \leq n-1} M_2 M_3 S([\log_p (n-j-1)]) \\ &\leq M_1 M_3 S([\log_p n]) + M_2 M_3 S([\log_p (n-1)]) \end{aligned}$$

$$\begin{aligned}
&\leq M_1 M_3 S([\log_p n]) + \left(\frac{M_2 M_3}{p^{n-1}} \right) (p^{n-1} S([\log_p (n-1)])) \\
&\leq M_1 M_3 S([\log_p n]) + \left(\frac{M_2 M_3}{p^{n-1}} \right) (p^n S([\log_p n])) \\
&\leq (M_1 + p M_2) M_3 S([\log_p n]) \\
&\leq M^* S([\log_p n]).
\end{aligned}$$

References

- [1] R. AHLWEDE and R. BOJANIC, *Approximation of continuous functions in p -adic analysis*, J. Approx. Theory, **15** (1975), 190–205.
- [2] K. IKEDA, T. KIM and K. SHIRATANI, *On p -adic bounded functions*, Mem. Fac. Sci. Kyushu univ., **46** (1992), 341–349.
- [3] K. MAHLER, *An interpolation series for continuous functions of a p -adic variable*, J. reine angew. Math., **199** (1958), 23–34.
- [4] K. SHIRATANI and S. YAMAMOTO, *On a p -adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu univ., **39** (1985), 113–125.
- [5] K. SHIRATANI and S. YOKOYAMA, *An application of p -adic convolutions*, Mem. Fac. Sci. Kyushu univ., **36** (1982), 73–83.
- [6] L. VAN HAMME, *Three generalization of Mahler's expansion for continuous functions on Z_p* , (p -adic Analysis) Lecture Notes in Mathematics., **1454** (1990), 356–361.

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