CLASSIFICATION OF SMALL SPIN MODELS

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Abstract

Symmetric spin models were introduced by Jones to obtain invariants of links. In his paper he proposed to obtain the classification of models (X, w_+, w_-) with $|X|=4, 5, 6$ and 7 . In the present paper we complete this task by showing that the only spin models of these sizes are the Potts models, those coming from cyclic groups for $5 \leq n \leq 7$, and for $n=4$ some other models obtained by the product construction of de la Harpe. Furthermore, we also classify the non-symmetric Jones-type spin models introduced by Kawagoe, Munemasa and Watatani, for $n=4, 5$.

1. Introduction

Symmetric spin models were introduced by Jones [8] as a method to obtain link invariants. Later Jaeger [7] and de la Harpe [6] developed the connection between these models and association schemes. Recently several new constructions were given and a non-symmetric generalization was investigated; see Bannai and Bannai [1], Nomura [12] and Kawagoe, Munemasa and Watatani [10]. Even more recently, Bannai and Bannai have invented a far reaching generalization of the concept [2].

Definition 1.1 *Let a be a non-zero complex number, n be a positive integer, and D be one of its square roots. A (non-symmetric Jones-type) spin model of size n with loop variable D and modulus a is a triple (X, w_+, w_-) , where X is a finite set of size $n=D^2$ and w_+, w_- are complex-valued functions on $X \times X$ which satisfy the following properties for all α, β, γ in X :*

1. $w_+(\alpha, \alpha) = a, w_-(\alpha, \alpha) = a^{-1}$.

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2. $\sum_{x \in X} w_+(\alpha, x) = \sum_{x \in X} w_+(x, \alpha) = Da^{-1}$,
 $\sum_{x \in X} w_-(\alpha, x) = \sum_{x \in X} w_-(x, \alpha) = Da$.
3. $w_+(\alpha, \beta)w_-(\beta, \alpha) = 1$
4. $\sum_{x \in X} w_+(\alpha, x)w_-(x, \beta) = n\delta_{\alpha, \beta}$ (where δ is the Kronecker symbol),
5. $\sum_{x \in X} w_+(\alpha, x)w_+(x, \beta)w_-(x, \gamma) = Dw_+(\alpha, \beta)w_-(\beta, \gamma)w_-(\alpha, \gamma)$.

The spin model is called *symmetric* if $w_+(\alpha, \beta) = w_+(\beta, \alpha)$ and $w_-(\alpha, \beta) = w_-(\beta, \alpha)$ hold.

Remark The above equations are not independent (see [2, 8, 10]). We need the following proposition of [2].

Proposition 1.2 Under conditions (3) and (4) of Definition 1.1, (5) is equivalent to

$$\sum_{x \in X} w_+(\alpha, x)w_+(x, \beta)w_-(\gamma, x) = Dw_+(\alpha, \beta)w_-(\gamma, \alpha)w_-(\gamma, \beta).$$

Definition 1.1 can be interpreted using $n \times n$ matrices W_+ and W_- (see [7]). Let $W_{\pm} = (w_{\pm}(\alpha, \beta))_{\alpha \in X, \beta \in X}$ and let $\circ$ denote the Hadamard product of matrices (i.e., the entry-wise product of two matrices of the same size). Furthermore, let us define for $(\beta, \gamma) \in X \times X$ the column vector $\mathbf{Y}_{\beta\gamma}$ indexed by X as

$$\mathbf{Y}_{\beta\gamma}(x) = w_+(x, \beta)w_-(x, \gamma) \quad \forall x \in X.$$

Proposition 1.3 (X, w_+, w_-) is a spin model with loop variable D and modulus a if and only if the following properties hold:

1. $I \circ W_+ = aI$, $I \circ W_- = a^{-1}I$.
2. $W_+J = JW_+ = Da^{-1}J$, $W_-J = JW_- = DaJ$.
3. $W_+ \circ W_-^T = J$
4. $W_+W_- = nI$.
5. For every $(\beta, \gamma) \in X \times X$, $W_+\mathbf{Y}_{\beta\gamma} = Dw_-(\beta, \gamma)\mathbf{Y}_{\beta\gamma}$.

Here I denotes the identity matrix and J denotes the matrix whose entries are all 1's. The spin model is called *symmetric* if W_+ and W_- are symmetric. ■

We shall use a third interpretation, too. Namely, we can write

$$W_+ = \sum_{i=0}^t a_i A_i,$$

where a_0 is the modulus of the model, $A_0 = I$, $\sum_{i=0}^t A_i = J$, $a_i \neq a_j$ for $1 \leq i < j \leq t$ and the A_i 's are adjacency matrices of edge-disjoint simple digraphs (graphs) on the vertex set X . Thus, $A_i \circ A_j = \delta_{ij} A_i$. We denote the graph whose adjacency matrix is A_i by G_i . The above determined t is called the *degree* of the model (X, W_+, W_-) . There exist models of degree *one* for any size $n \geq 2$ as follows [6, 8].

Definition 1.4 (Potts model) Let n be an integer, $n \geq 2$, and let $b \in \mathbb{C}$ be such that.

$$(b^2 + b^{-2})^2 = n.$$

Then we obtain a spin model of modulus $a = -b^3$ and loop variable $D = -b^2 - b^{-2}$ by taking $X = \{1, 2, \dots, n\}$, $W_+ = -b^{-3} I + b(J - I)$, $W_- = -b^3 I + b^{-1}(J - I)$.

Let us mention that the case when $(X, A_i : 0 \leq i \leq t)$ is an association scheme is interesting for its own sake (see [5, 4] for definitions).

The following is a fundamental result concerning the classification problem.

Lemma 1.5 ([8], Proposition 2.16.) For each $z \in \mathbb{C}$ let k_z be the number of ordered pairs (α, β) for which $w_+(\alpha, \beta) = z$. Then k_z is a multiple of n . ■

As an immediate consequence of this lemma we obtain that the degree of a spin model on an n element set is at most $n-1$. However, we shall need a stronger result in order to reduce the number of cases to be checked. This is Theorem 2.1 of the next section.

It is known [1] that there exist spin models associated to the cyclic group $\mathbb{Z}/n\mathbb{Z}$ for any n . The aim of this paper is to prove the following theorems.

Theorem 1.6 Let (X, W_+, W_-) be a symmetric spin model of size n . Furthermore, let us assume that it is of degree $t > 1$, i.e.,

$$W_+ = a_0 I + a_1 A_1 + \dots + a_t A_t \quad \text{with } a_i \neq a_j \text{ for } 1 \leq i < j \leq t.$$

Then for $5 \leq n \leq 7$ $(X, \{A_i\}_{0 \leq i \leq t})$ is the symmetric association scheme coming from the cyclic group $\mathbb{Z}/n\mathbb{Z}$. In particular, we have $t = \lfloor \frac{n}{2} \rfloor$. If $n=4$ the model is either coming from the cyclic group or it is a product of two (Potts) models of size two.

Theorem 1.7 Let (X, W_+, W_-) be a non-symmetric Jones-type spin model of size n .

Then for $n=4, 5$ it can be written as

$$W_+ = a_0 I + a_1 A_1 + \dots, a_{n-1} A_{n-1},$$

where $(X, \{A_i\}_{0 \leq i \leq n-1})$ is the association scheme coming from the cyclic group $\mathbf{Z}/n\mathbf{Z}$. For $n=4$ all a_i 's are different for $i \neq 0$.

In Theorem 1.7 we do not state that for $n=5$ the a_i 's are all different for $i \neq 0$. All possible solutions will be given in a subsequent paper by Bannai, Bannai and Jaeger.

In Section 2 we formulate general results and in Section 3 we turn to the symmetric case, while Section 4 deals with non-symmetric models. We will use the three possible interpretations simultaneously, always switching to the one which is most convenient to formulate the statement in question.

We consider two spin models as equivalent if one can be obtained from the other by simultaneous permutation of rows and columns, i.e. conjugation of W_+ and W_- by a permutation matrix. In the graph representation this means simultaneous renumbering of the vertices of each graph. Furthermore, we allow renumbering of the G_i 's when we use the graph interpretation.

2 General results

In this section we prove the following strengthening of Lemma 1.5 (using essentially the same idea as in [8]).

Theorem 2.1 *Let (X, W_+, W_-) be a spin model of degree t . Then for $i=1, 2, \dots, t$ there exists a positive integer k_i such that each vertex in G_i has in-degree and out-degree k_i , or equivalently, $JA_i = A_i J = k_i J$.*

Proof of Theorem 2.1

Let $X = \{1, 2, \dots, n\}$ and let $\Delta_\gamma = \text{diag}(w_-(1, \gamma), w_-(2, \gamma), \dots, w_-(n, \gamma))$ for $\gamma \in X$. We claim that

$$W_+ \Delta_\gamma W_+ = D \Delta_\gamma W_+ \Delta_\gamma, \quad \forall \gamma \in X. \quad (1)$$

Indeed, the (α, β) -entry of the left hand side of (1) is

$$\sum_{x \in X} w_+(\alpha, x) [w_-(x, \gamma) w_+(x, \beta)] \quad (2)$$

and the (α, β) -entry of the right hand side of (1) is

$$Dw_-(\alpha, \gamma)w_+(\alpha, \beta)w_-(\beta, \gamma). \quad (3)$$

The equality of (3) and (2) is equivalent to (5) of Definition 1.1. Note that Δ_γ is invertible by (3) of Proposition 1.3. Now (1) is equivalent to

$$\Delta_\gamma^{-1} W_+ \Delta_\gamma W_+ W_- = DW_+ \Delta_\gamma W_- \quad (4)$$

i.e.,

$$n\Delta_\gamma^{-1} W_+ \Delta_\gamma = DW_+ \Delta_\gamma W_- = DnW_-^{-1} \Delta_\gamma W_-. \quad (5)$$

Hence, Δ_γ and $D^{-1}W_+$ are conjugate.

It follows that the spectrum of Δ_γ does not depend on the choice of $\gamma \in X$. Equivalently, the columns of W_- are permutations of each other. Using (3) of Proposition 1.3 we obtain the same for the rows of W_+ .

Using Proposition 1.2 we obtain in a similar way that the columns of W_+ are permutations of each other. Then simple counting shows that the in- and out-degrees of G_i must coincide, for each i . ■

The following theorem was independently proved by Jaeger [7] and de la Harpe [6], but the proof given here is simpler.

Theorem 2.2 *Let (X, W_+, W_-) be a symmetric spin model of degree 2. Then G_1 (and consequently G_2) is a strongly regular graph.*

Proof of Theorem 2.2

We have to establish the existence of the parameters k, λ and μ , such that G_1 is k -regular and the number of common neighbors of any pair of adjacent (resp.: non-adjacent) vertices is λ (resp.: μ).

The regularity follows from Theorem 2.1. Now we prove the existence of λ and μ . Let α_{ij} be the number of common a_1 entries of rows i and j . Furthermore, suppose that (i, j) and (r, s) are edges of the graph G_1 . We have to prove that $\alpha_{ij} = \alpha_{rs}$. Using that the (i, j) (resp. (r, s)) entry of $W_+ W_-$ is 0, we obtain

$$a_0 a_1^{-1} + a_0^{-1} a_1 + n - 2k + 2\alpha_{ij} + (k - 1 - \alpha_{ij}) (a_1 a_2^{-1} + a_1^{-1} a_2) = 0$$

and

$$a_0 a_1^{-1} + a_0^{-1} a_1 + n - 2k + 2\alpha_{rs} + (k - 1 - \alpha_{rs}) (a_1 a_2^{-1} + a_1^{-1} a_2) = 0.$$

Taking the difference of these two equations we obtain

$$(\alpha_{ij} - \alpha_{rs}) (2 - a_1 a_2^{-1} - a_1^{-1} a_2) = 0.$$

If $\alpha_{ij} \neq \alpha_{rs}$, then $a_1 a_2^{-1} + a_1^{-1} a_2 = 2$, which implies $a_1 = a_2$, a contradiction. The existence of μ can be proved in exactly the same way. ■

The next theorem is a natural extension of Theorem 2.2.

Theorem 2.3 *Let (X, W_+, W_-) be a symmetric spin model of degree 3. Then $\{A_0, A_1, A_2, A_3\}$ are the adjacency matrices of a symmetric 3-class association scheme*

Proof of Theorem 2.3

Note first that $W_+^T = W_+ = \sum_{i=0}^3 a_i A_i$ and (3) of Proposition 1.3 imply that the matrices A_i are symmetric and $W_- = \sum_{i=0}^3 a_i^{-1} A_i$. Let $\mathcal{M}$ be the algebra generated by $\{J, W_+\}$ with respect to the ordinary matrix product, and let $\mathcal{H}$ be the algebra generated by $\{I, W_-\}$ with respect to the Hadamard product, as introduced in [7]. Furthermore, let $\mathcal{A}$ be the algebra generated by $\{A_0, A_1, A_2, A_3\}$ with respect to the Hadamard product. Since $A_i \circ A_j = \delta_{ij} A_i$, $\{A_0, A_1, A_2, A_3\}$ is a basis of $\mathcal{A}$. It is clear that $\mathcal{H} \subseteq \mathcal{A}$. Now, I, J, W_+ and W_- are in $\mathcal{H}$ (see [7]). The matrix that expresses $\{I, J, W_+, W_-\}$ in the basis $\{A_0, A_1, A_2, A_3\}$ of $\mathcal{A}$ is

$$\begin{bmatrix} 1 & 1 & a_0 & a_0^{-1} \\ 0 & 1 & a_1 & a_1^{-1} \\ 0 & 1 & a_2 & a_2^{-1} \\ 0 & 1 & a_3 & a_3^{-1} \end{bmatrix}.$$

Its Vandermonde type determinant is non-zero, because the a_i 's are distinct for $i = 1, 2, 3$. Thus, $\{I, J, W_+, W_-\}$ is also a basis of $\mathcal{A}$, i.e. $\mathcal{H} = \mathcal{A}$. Consequently, $\mathcal{H}$ is of dimension 4. Using that $\mathcal{H} \cong \mathcal{M}$ (see [7]), we obtain that $\mathcal{M}$ is also of dimension four. However, $\{I, J, W_+, W_-\} \subseteq \mathcal{M}$ yields $\mathcal{M} = \mathcal{H}$. Now, applying Proposition 4 of [7], we obtain that $\mathcal{M}$ is the Bose-Mesner algebra of a self-dual association scheme. ■

3 Symmetric models

In this section we turn to the classification of small symmetric spin models. It is easy to see that for any n , the only spin model of size n and degree 1 is the Potts model of [8]. So we shall always assume that the degree of the model is at least 2.

Also it is clear that the degree of the model is at most $n-1$ for n even and $\frac{n-1}{2}$

for n odd (since then only even degrees of graphs G_i can occur). For the sake of completeness we begin with the case $n=4$.

3.1 $n=4$

Degree 2 By Theorem 2.1 G_1 and G_2 are 1 and 2-regular graphs respectively. It is obvious that the two-regular graph must be the four-cycle. This case is solved in [6, 7]: there are two one-parameter families of solutions given by $W_+ = a_0 I - a_0 A_1 \pm a_0^{-1} A_2$.

Degree 3 Now G_1 , G_2 and G_3 are all perfect matchings. We have

$$\begin{aligned} W_+ &= a_0 I + a_1 A_1 + a_2 A_2 + a_3 A_3, \\ W_- &= a_0^{-1} I + a_1^{-1} A_1 + a_2^{-1} A_2 + a_3^{-1} A_3, \\ A_i^2 &= I \text{ for } i=1, 2, 3, \\ A_i A_j &= A_k \text{ for } \{i, j, k\} = \{1, 2, 3\}. \end{aligned}$$

Writing $x = a_0/a_1$, $y = a_0/a_2$ and $z = a_0/a_3$ we obtain the following set of three equations from the condition $W_+ W_- = nI$:

$$\begin{aligned} x + 1/x + y/z + z/y &= 0 \\ y + 1/y + x/z + z/x &= 0 \\ z + 1/z + x/y + y/x &= 0 \end{aligned}$$

It is easy to check that the only solutions of this system with x, y, z distinct are such that one of the variables is equal to 1 and the other two are negatives of each other. Thus, we may assume that W_+ is

$$W_+ = \begin{bmatrix} a & a & b & -b \\ a & a & -b & b \\ b & -b & a & a \\ -b & b & a & a \end{bmatrix}.$$

Now (2) of Definition 1.1 gives $a^2 = \frac{D}{2} \in \{-1, 1\}$ and (5) of Definition 1.1 with $\alpha=1$, $\beta=2$, $\gamma=3$ gives $a^{-2} b^4 = \frac{D}{2}$, so that $b^4 = \left(\frac{D}{2}\right)^2 = 1$. All these cases are covered by the direct product construction of de la Harpe ([6], Section 5.3).

3.2 $n=5$

By Theorem 2.1 the model is either of degree 1 or degree 2. Thus, if (X, W_+, W_-) is not the Potts model, then G_1 and G_2 are both 5-cycles. This is the well-studied case of the pentagon [6, 7, 9].

3.3 $n=6$

In this case a model could be of degree 2, 3, 4 or 5.

3.3.1 Degree 2

By Theorem 2.2 the graph G_1 is strongly regular. Furthermore, we may assume that it is 1- or 2-regular, otherwise we may exchange G_1 and G_2 . However, in any case G_1 is disconnected and this contradicts condition (34) of [7].

3.3.2 Degree 3

Now we can apply Theorem 2.3 to obtain that $\{X, A_0, A_1, A_2, A_3\}$ is a symmetric three-class association scheme. It is known [11] that the only such association scheme is the scheme of the cyclic group $\mathbf{Z}/\mathbf{Z}_6$.

3.3.3 Degree 4

According to Theorem 2.1 there are two possible cases.

Case 1. G_1, G_2 and G_3 are matchings and G_4 is a 6-cycle. There is only one such decomposition, because the complement of a 6-cycle has a unique decomposition into perfect matchings. The corresponding W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_1 \\ a_1 & a_0 & a_1 & a_4 & a_2 & a_3 \\ a_2 & a_1 & a_0 & a_1 & a_3 & a_4 \\ a_3 & a_4 & a_1 & a_0 & a_1 & a_2 \\ a_4 & a_2 & a_3 & a_1 & a_0 & a_1 \\ a_1 & a_3 & a_4 & a_2 & a_1 & a_0 \end{bmatrix}.$$

Taking the (1, 5) and (5, 1) entries of $W_+ W_-$, we obtain the following two equations :

$$\begin{aligned} a_0 a_4^{-1} + a_1 a_2^{-1} + a_3 a_1^{-1} + a_2 a_3^{-1} + a_0^{-1} a_4 + 1 &= 0 \\ a_0^{-1} a_4 + a_1^{-1} a_2 + a_2^{-1} a_3 + a_3^{-1} a_1 + a_0 a_4^{-1} + 1 &= 0 \end{aligned}$$

Subtracting the second one from the first and multiplying by $a_1 a_2 a_3$ we obtain

$$a_1^2 a_3 - a_2^2 a_3 + a_2^2 a_1 - a_3^2 a_1 + a_3^2 a_1 - a_2^2 a_2 = 0$$

that is equivalent to

$$(a_2 - a_1)(a_3 - a_1)(a_3 - a_2) = 0.$$

However, this contradicts the assumption. We call the above type equation *the cyclic permutation equation*.

Case 2. G_1, G_2 and G_3 are matchings and G_4 is a union of two triangles. It is easy to see that if we assume that the two triangles are on vertices $\{1, 2, 3\}$ and $\{4, 5, 6\}$, respectively, then taking the $(1, 2)$ and $(2, 1)$ entries of $W_+ W_-$ we obtain the cyclic permutation equation, hence a contradiction.

3.3.4 Degree 5

Now we have a decomposition of K_6 into five matchings. There is only one way to decompose K_6 into five matchings up to permutation of the matchings, because two of these matchings form a 6-cycle and the complement of a 6-cycle has a unique decomposition into perfect matchings. The resulting matrix W_+ is as follows.

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_5 & a_4 & a_2 \\ a_1 & a_0 & a_2 & a_4 & a_3 & a_5 \\ a_3 & a_2 & a_0 & a_1 & a_5 & a_4 \\ a_5 & a_4 & a_1 & a_0 & a_2 & a_3 \\ a_4 & a_3 & a_5 & a_2 & a_0 & a_1 \\ a_2 & a_5 & a_4 & a_3 & a_1 & a_0 \end{bmatrix}$$

Taking the difference of the $(1, 2)$ entry of $W_+ W_-$ with the $(3, 4)$, $(4, 3)$, $(5, 6)$ and $(6, 5)$ entries, respectively, we obtain the following four equations.

$$\begin{aligned} a_3 a_2^{-1} + a_2 a_5^{-1} + a_5 a_4^{-1} - a_2 a_4^{-1} - a_3 a_5^{-1} - a_5 a_2^{-1} &= 0 \\ a_3 a_2^{-1} + a_4 a_3^{-1} + a_5 a_4^{-1} - a_3 a_4^{-1} - a_4 a_2^{-1} - a_5 a_3^{-1} &= 0 \\ a_3 a_2^{-1} + a_4 a_3^{-1} + a_2 a_5^{-1} - a_2 a_3^{-1} - a_4 a_2^{-1} - a_3 a_5^{-1} &= 0 \\ a_4 a_3^{-1} + a_2 a_5^{-1} + a_5 a_4^{-1} - a_2 a_4^{-1} - a_5 a_3^{-1} - a_4 a_5^{-1} &= 0 \end{aligned}$$

Reducing we obtain

$$\begin{aligned}
(a_5 - a_2)(a_3 a_4 - a_5 a_4 - a_2 a_4 + a_5 a_2) &= 0 \\
(a_3 - a_4)(a_5 a_2 + a_3 a_4 - a_3 a_2 - a_2 a_4) &= 0 \\
(a_2 - a_3)(a_3 a_2 + a_5 a_4 - a_5 a_2 - a_3 a_5) &= 0 \\
(a_4 - a_5)(a_3 a_2 + a_5 a_4 - a_3 a_4 - a_3 a_5) &= 0.
\end{aligned}$$

Using that a_i 's are different and subtracting the first and second (respectively : third and fourth) equations we get $a_2 a_3 = a_4 a_5$ (respectively : $a_3 a_4 = a_2 a_5$).

Hence $\frac{a_3}{a_5} = \frac{a_4}{a_2} = \frac{a_5}{a_3}$ and $a_3^2 = a_5^2$. This yields $a_3 = -a_5$ and $a_4 = -a_2$. Then the first equation gives $a_2 = 3a_3$ and the third gives $a_3 = -3a_2$, a contradiction. Thus, the case $n=6$ is finished.

3.4 $n=7$

Applying again Theorem 2.1 we obtain that the number of different offdiagonal entries of W_+ is at most three. If (X, W_+, W_-) is of degree two, then G_1 should be a strongly regular graph by Theorem 2.2. However, strongly regular graphs on 7 vertices do not exist. So, we may assume that the model is of degree three. Now G_1, G_2 and G_3 are all 2-regular graphs. Furthermore, by Theorem 2.3 the A_i 's are adjacency matrices of an association scheme. However, it is well known folklore that the only association scheme on 7 points with $k_1 = k_2 = k_3 = 2$ is the scheme of the 7-gon. To see this, observe that since each edge of G_i is in p_{ii}^i triangles of G_i , each G_i is a 7-cycle. Now $A_1^2 - 2I$ must be one of A_2, A_3 . Suppose without loss of generality that $A_1^2 = 2I + A_2$. Then $A_3 = J - I - A_1 - (A_1^2 - 2I)$ and the multiplication table of the A_i 's can easily be derived from the properties of A_1 . Thus the only case here is the cyclic group case. The proof of Theorem 1.6 is now complete.

4 Non-symmetric models

In this section G_i 's are oriented graphs. Furthermore, we always assume that the models are non-symmetric, i.e., there exists at least one G_i and an edge $(k, l) \in E(G_i)$ such that $(l, k) \notin E(G_i)$. By Theorem 2.1 in-degree = out-degree = k_i for every vertex in $G_i, i=1, 2, \dots$. The model is of degree at least 2 (otherwise it is symmetric and it is a Potts model).

4.1 $n=4$

4.1.1 Degree 2

We may assume that $k_1=1$ and $k_2=2$. Now G_1 either consists of two independent edges directed in both ways, or it is a directed four-cycle. In the first case we obtain a symmetric model. In the second case we may assume that the directed four cycle is (1234) , i.e., W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_2 \\ a_2 & a_0 & a_1 & a_2 \\ a_2 & a_2 & a_0 & a_1 \\ a_1 & a_2 & a_2 & a_0 \end{bmatrix}.$$

Taking the $(1, 2)$ and $(2, 1)$ entries of $W_+ W_-$ we obtain a cyclic permutation equation in variables a_0, a_1, a_2 , i.e., $(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0$. This implies that $a_0 = a_1$ or $a_0 = a_2$. In the first case we obtain

$$W_+ = \begin{bmatrix} a_0 & a_0 & a_2 & a_2 \\ a_2 & a_0 & a_0 & a_2 \\ a_2 & a_2 & a_0 & a_0 \\ a_0 & a_2 & a_2 & a_0 \end{bmatrix}.$$

Taking the $(1, 2)$ and $(1, 3)$ entries of $W_+ W_-$ we obtain

$$a_0 a_2^{-1} + a_0^{-1} a_2 + 2 = 0 = 2(a_0 a_2^{-1} + a_0^{-1} a_2),$$

a contradiction.

In the second case we have

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_0 & a_0 \\ a_0 & a_0 & a_1 & a_0 \\ a_0 & a_0 & a_0 & a_1 \\ a_1 & a_0 & a_0 & a_0 \end{bmatrix}.$$

Considering the $(1, 2)$ entry of $W_+ W_-$ we get $a_1 = -a_0$. Property (2) of Definition 1.1 yields $a_0^2 = \frac{D}{2}$ while Property (5) with $\alpha = \beta = 2$, $\gamma = 1$ yields

$a_0^2 = \frac{-D}{2}$. Hence this case does not occur.

4.1.2 Degree 3

G_1 , G_2 and G_3 all have both in-degree and out-degree 1. If the model is non-symmetric, we may assume that G_1 is a directed four-cycle. This implies that the other two graphs must be the reverse four-cycle and the graph consisting of the diagonals oriented in both ways. This is the general cyclic group case. Some solutions are given in [1].

4.2 $n=5$

4.2.1 Degree 2

Now we have two cases to be distinguished: $k_1=1$, $k_2=3$ and $k_1=k_2=2$.

$k_1=1$ In this case G_1 cannot be symmetric. There are two possibilities for G_1 . One is a union of a directed triangle and an edge oriented both ways, the other is the directed five-cycle. In the first case assuming that the directed triangle is (123) we have

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_2 & a_2 \\ a_2 & a_0 & a_1 & a_2 & a_2 \\ a_1 & a_2 & a_0 & a_2 & a_2 \\ a_2 & a_2 & a_2 & a_0 & a_1 \\ a_2 & a_2 & a_2 & a_1 & a_0 \end{bmatrix}.$$

From the (1, 2) and (2, 1) entries of $W_+ W_-$ we obtain the cyclic permutation equation $(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0$. This implies that either $a_0 = a_1$ or $a_0 = a_2$. In the first case, the (4, 5) entry of $W_+ W_-$ would be 5, a contradiction.

In the second case, (5) of Definition 1.1 gives $2a_0 + 2a_1 + a_0^2 a_1^{-1} = D a_0 a_1^{-2}$ when $\alpha = \beta = 2$, $\gamma = 1$ and gives $2a_0 + 2a_1 + a_0^2 a_1^{-1} = D a_1^{-1}$ when $\alpha = 2$, $\beta = 4$, $\gamma = 1$. Hence $a_1 = a_0 = a_2$, a contradiction.

If G_1 is a directed five-cycle, then the situation is very similar. The (1, 2) and (2, 1) entries of $W_+ W_-$ again yield a cyclic permutation equation, which shows that either $a_0 = a_1$ or $a_0 = a_2$. In the first case, the (1, 2) and (1, 3) entries of $W_+ W_-$ yield incompatible values for $a_0 a_2^{-1} + a_2 a_0^{-1}$.

In the second case, (5) of Definition 1.1 for $(\alpha, \beta, \gamma) = (2, 1, 1)$ and for $(\alpha, \beta, \gamma) = (2, 2, 1)$ imply $a_0 = a_1$. Hence, the case $k_1=1$, $k_2=3$ does not occur.

$k_1=2$ Because the model is non-symmetric, we may assume that $(1, 2) \in E(G_1)$ and $(2, 1) \in E(G_2)$. Now the first two rows of W_+ look like

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & x & y \\ a_2 & a_0 & a_1 & z & u \end{array}$$

after suitable rearrangement of the last three rows and columns, where the part $\begin{array}{cc} x & y \\ z & u \end{array}$ stands for either $\begin{array}{cc} a_1 & a_2 \\ a_2 & a_1 \end{array}$ or $\begin{array}{cc} a_1 & a_2 \\ a_1 & a_2 \end{array}$. In any case, from the (1, 2) and (2, 1) entries of $W_+ W_-$ we obtain again the cyclic permutation equation $(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0$, which implies that either $a_0 = a_1$ or $a_0 = a_2$. By symmetry reasons we may assume that $a_0 = a_2$. Now the product of row i of W_+ and column j of W_- for $i \neq j$ is either $2(a_0 a_1^{-1} + a_0^{-1} a_1) + 1$ or $a_0 a_1^{-1} + a_0^{-1} a_1 + 3$. However, both cannot occur at the same time. If all products are $2(a_0 a_1^{-1} + a_0^{-1} a_1) + 1$, no column of W_+ could contain two a_1 's and this is clearly impossible. If all products are $a_0 a_1^{-1} + a_0^{-1} a_1 + 3$, any two rows have an a_1 entry in the same position, and it is easy to see that this is also impossible. Hence there do not exist non-symmetric spin models of size 5 and degree 2.

4.2.2 Degree 3

We may assume that $k_1 = k_2 = 1$ and $k_3 = 2$. If G_1 is a union of a directed triangle and an edge directed in both ways, then we may assume that the triangle is (123). By symmetry, and the regularity of the G_i 's we may assume that the last two rows of W_+ are as follows:

$$\begin{array}{ccccc} a_3 & a_3 & a_2 & a_0 & a_1 \\ a_2 & a_3 & a_3 & a_1 & a_0 \end{array}$$

Now G_1 contains the triangle (123) and by the regularity of the G_i 's, W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_3 & a_2 \\ a_3 & a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_0 & a_3 & a_3 \\ a_3 & a_3 & a_2 & a_0 & a_1 \\ a_2 & a_3 & a_3 & a_1 & a_0 \end{bmatrix} \quad \text{or} \quad W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_3 & a_2 \\ a_1 & a_2 & a_0 & a_3 & a_3 \\ a_3 & a_3 & a_2 & a_0 & a_1 \\ a_2 & a_3 & a_3 & a_1 & a_0 \end{bmatrix}.$$

Taking the (3, 5) and (5, 3) entries of $W_+ W_-$ we obtain a cyclic permutation equation in variables a_1, a_2 and a_3 , a contradiction. Thus, we may assume that both G_1 and G_2 are directed five-cycles. For a digraph G let us denote by $-G$ the digraph obtained from G by reversing all edges. If $G_1 = -G_2$ or $G_1 \cap -G_2 =$

$\emptyset$, then we have instances of the cyclic group case. Thus, we may assume that G_1 is the five-cycle (12345) and that $(2, 1) \in G_3$ and $(3, 2) \in G_2$. Now we have to consider two cases.

Case 1. $(2, 5) \in G_2$. It is easy to see that this implies that G_2 is the cycle (13254). So W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_3 \\ a_3 & a_0 & a_1 & a_3 & a_2 \\ a_3 & a_2 & a_0 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_0 & a_1 \\ a_1 & a_3 & a_3 & a_2 & a_0 \end{bmatrix}.$$

Considering the entries (3, 5) and (5, 3) of the product $W_+ W_-$, we obtain the cyclic permutation equation in the variables a_1, a_2 , and a_3 , a contradiction.

Case 2. $(2, 4) \in G_2$. This gives $G_2 = (15324)$, and

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_3 & a_2 \\ a_3 & a_0 & a_1 & a_2 & a_3 \\ a_3 & a_2 & a_0 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_0 & a_1 \\ a_1 & a_3 & a_2 & a_3 & a_0 \end{bmatrix}.$$

In a similar way as in Case 1 we obtain a contradiction from the cyclic permutation equations given by the (1, 3) and (3, 1) entries of the product $W_+ W_-$.

4.2.3 Degree 4

First we must observe that if $G_i = -G_j$ for some $i \neq j$, then we have an instance of the cyclic group case. Next, if G_i is a union of a directed triangle and an edge directed in both ways, then assuming that the triangle is (123), we have

$$W_+ = \begin{bmatrix} a_0 & a_1 & * & * & * \\ * & a_0 & a_1 & * & * \\ a_1 & * & a_0 & * & * \\ a_{i_1} & a_{i_2} & a_{i_3} & a_0 & a_1 \\ a_{j_1} & a_{j_2} & a_{j_3} & a_1 & a_0 \end{bmatrix}.$$

Here (i_1, i_2, i_3) and (j_1, j_2, j_3) are permutations of $\{2, 3, 4\}$ so that $i_k \neq j_k$ for $k=1, 2, 3$.

2, 3. This immediately implies a cyclic permutation equation in the variables a_2 , a_3 and a_4 , a contradiction.

Thus, we may assume that all four G_i 's are five-cycles. If $G_1 \cap -G_2 = \emptyset$, it is easy to see that again we have an instance of the cyclic group case. So we assume that $G_1 = (12345)$, $(2, 1) \notin G_2$ and $(3, 2) \in G_2$. Similarly to the degree 3 case we obtain that either $G_2 = (13254)$ or $G_2 = (15324)$. However, it is not hard to verify that in neither of these cases we can decompose the remaining edges into two directed five-cycles. This completes the proof of Theorem 1.7. ■

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