ON EXPLICIT FORMULAS OF $p$-ADIC $q$-L-FUNCTIONS

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§ 1. Introduction

Throughout this paper $Q$, $C$, $Q_p$ and $C_p$ will respectively denote the field of rational numbers, the complex number field, the field of $p$-adic rational numbers and the completion of the algebraic closure of $Q_p$.

Let $v_p$ be the normalized exponential valuation of $C_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in C$, or $p$-adic number $q \in C_p$. If $q \in C$, one normally assumes $|q| < 1$. If $q \in C_p$, one normally assumes $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Carlitz's $q$-Bernoulli numbers $\beta_k = \beta_k(q)$ can be determined inductively by

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta^k = 1 \quad \text{if} \quad k = 1$$

$$= 0 \quad \text{if} \quad k > 1,$$

with the usual convention of replacing $\beta^t$ by $\beta_t$.

The $q$-Bernoulli polynomials are defined by

$$\beta_k(x : q) = (q^x \beta + [x])^k,$$

where $[x] = [x : q] = \frac{1 - q^x}{1 - q}$. As $q \to 1$, we have the usual Bernoulli numbers and Bernoulli polynomials. In [8], Koblitz constructed a $q$-analogue of the $p$-adic $L$-function $L_{p,k}(s, \chi)$ which interpolates the $q$-Bernoulli numbers. In Section 2, we present $q$-Euler numbers occurring in the coefficients of some Stirling type series for $p$-adic analytic functions. In Section 3, we treat

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generalized Kummer congruences for $q$-Bernoulli numbers. In Section 4, we study the value of $L_{p,q}(s, \chi)$ at $s=1$.

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§2. $q$-analogue of $p$-adic log gamma functions

In the complex case, Carlitz defined $q$-Euler numbers and polynomials as

$$H_0(u : q) = 1, \ (qH+1)^k - uH_k(u : q) = 0 \text{ for } k \geq 1,$$

where $u$ is a complex number with $|u| > 1$; and for $k \geq 0$

$$H_k(u, x : q) = (q^x H + \lfloor x \rfloor)^k,$$

with replacing $H^i$ by $H_i(u, x : q)$.

For a Dirichlet character $\chi$ with the conductor $f$, we know the generalized $q$-Euler numbers as follows [1] for $k \geq 0$

$$H_{k,\chi}(u : q) = [f]^k \sum_{a=1}^{f} u^{-a} \chi(a) H_k\left(u^{\frac{a}{f}} ; q^f\right),$$

and $qL$-function is defined by

$$L_q(s, \chi) = \frac{2-s}{s-1} (q-1) \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^{s}},$$

for $s \in \mathbb{C}$ [9].

In this paper, we assume that $q \in \mathbb{C}_p$ with $1-q < p^{-\frac{1}{p-1}}$. Let $u \in \mathbb{C}_p$ such that $|1-u' |_p \geq 1$ for $f$. For $z \in \mathbb{Z}_p$, the $p$-adic integer ring, we denote $[z] = [z : q] = \frac{1-q^2}{1-q}$.

Let $a \in \mathbb{Z}$ with $0 \leq a \leq fp^n - 1$, $n \geq 0$. Then $q$-Euler measure is defined by

$$E_{u,q}(a + fp^n Z_p) = \frac{u^{fp^n-a}}{1-u^{fp^n}} [f^{fp^n}]^k H_k\left(u^{fp^n} ; \frac{a}{fp^n} ; q^{fp^n}\right).$$

Note that if $E_u = E_{u,q}$ then

$$E_u(a + fp^n Z_p) = \frac{u^{fp^n-a}}{1-u^{fp^n}},$$
where \( E_u \) is Koblitz's measure [7], [9], [12].

For this measure, we define

\[
G_{p,q,u}(x) = \int_{\mathbb{Z}_p} \{(x+[z])\log(x+[z])-(x+[z])\} dE_u(z).
\]

Then we easily see that \( G_{p,q,u}(x) \) is locally analytic on \( \mathbb{C} \setminus \mathbb{Z}_p \).

We know in [9]

\[
\int_{\mathbb{Z}_p} [z]^k dE_u(z) = \frac{u}{1-u} H_k(u : q),
\]

and

\[
(x+[z])\log(x+[z])-(x+[z])
= (x+[z])\left\{\log\left(1+\frac{[z]}{x}\right)+\log x\right\}-(x+[z])
= [z]+x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{[z]^{n+1}}{x^{n+1}} + (x+[z])\log x-(x+[z]).
\]

Thus we can find the formula of \( p \)-adic Stirling type series for \( q \)-Euler numbers.

\[
G_{p,q,u}(x) = \frac{u}{1-u} H_1(u : q) + x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{H_{n+1}(u : q)}{x^{n+1}} \frac{u}{1-u}
+ \left( \frac{u}{1-u} x + \frac{u}{1-u} H_1(u : q) \right) \log x - \left( \frac{ux}{1-u} + \frac{u}{1-u} H_1(u : q) \right)
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{H_{n+1}(u : q)}{x^{n}} \frac{u}{1-u} + \left( \frac{ux}{1-u} + \frac{uH_1(u : q)}{1-u} \right) \log x - \frac{ux}{1-u}.
\]

Therefore we obtain the following

**Proposition 1.** For \( x \in \mathbb{C} \) with \( |x|_p > 1 \), we have

\[
G_{p,q,u}(x) = \frac{u}{1-u} \left\{ (x+H_1(u : q))\log x - x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{H_{n+1}(u : q)}{x^{n}} \right\}.
\]

Remark ([8], [4]). This formula resembles the formula of \( G_{p,q}(x) \), that is,

\[
G_{p,q}(x) = (x+\beta_1)\log x - x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \beta_{n+1} x^{-n}.
\]

Let \( d \) be a fixed integer and let \( p \) be a fixed prime number. We set
\[ X = \lim_{N \to \infty} (Z/dp^N Z), \]
\[ X^* = \bigcup_{0 < a < dp} a + dpZ, \]
\[ a + dp^N Z = \{ x \in X \mid x \equiv a \mod dp^N \}, \]

where \( a \in Z \) lies in \( 0 \leq a < dp^N \).

For any positive integer \( N \)

\[ \mu_q(a + dp^N Z) = \frac{q^a}{[dp^N]} = \frac{q^a}{[dp^N : q]} \]

is known as a distribution on \( X \).

This distribution yields an integral for each non-negative integer \( m \) [4]

\[ \int_{Z_p} [a]^m d\mu_q(a) = \lim_{N \to \infty} \sum_{a=0}^{p^N-1} [a]^m \frac{q^a}{[p^N]}. \]

**Proposition 2.** For \( f \in UD(Z_p, C_p) \), let

\[ I_q(f([x])) = \int_{Z_p} f([x]) d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f([x]) - \frac{q^a}{[p^N]}. \]

Then we have

\[ I_q(f([x])) = I_q(f([x])) + f^*(0) + (q-1)f(0), \]

where \( f_1(x) = qf(x+1) \).

**Proof.** By the definition of \( I_q \) we see

\[ I_q(f([x])) = \lim_{\rho \to \infty} \frac{1}{[p^\rho]} \sum_{x=0}^{p^\rho-1} f([x+1])q^{x+1} \]

\[ = \lim_{\rho \to \infty} \frac{1}{[p^\rho]} \sum_{x=1}^{p^\rho} f([x])q^x \]

\[ = \lim_{\rho \to \infty} \frac{1}{[p^\rho]} \sum_{x=0}^{p^\rho-1} f([x])q^x + \lim_{\rho \to \infty} \frac{1}{[p^\rho]}(p^\rho f([p^\rho]) - f(0)) \]

\[ = \lim_{\rho \to \infty} \frac{1}{[p^\rho]} \sum_{x=0}^{p^\rho-1} f([x])q^x + \lim_{\rho \to \infty} \frac{1}{[p^\rho]}([p^\rho](q-1)f([p^\rho])) \]

\[ + f([p^\rho]) - f(0)) \]
\[ = I_q(f) + f'(0) + (q-1)f(0). \]

It is known in [4] that Carlitz's q-Bernoulli numbers can be represented by
\[ I_q([z]^m) = \int_{\mathbb{Z}_q} [z]^m d\mu_q(z) = \beta_m(q). \]

The Carlitz's q-Bernoulli polynomials are defined by
\[ \beta_m(x : q) = \int_{\mathbb{Z}_q} [x+t]^m d\mu_q(t). \]

Thus these can be written as
\[ \beta_n(x : q) = (q^x \beta(q) + [x])^n \] [4].

We recall the following distribution on \( X \) [8].
For \( k \geq 1 \), \( \mu_k = \mu_k : q \) is defined by
\[ \mu_k(a + dp^N Z_q) = [dp^N : q]^{k-1} q^a \beta_k\left(\frac{a}{dp^N : q^{dp^N}}\right). \]

Then \( \mu_k \) extends to a \( Q_p(q) \)-valued distribution on the compact open set \( U \subset X \).

Let \( \chi \) be a primitive Dirichlet character with conductor \( d \in \mathbb{Z} \). Then we also defined generalized Carlitz's q-Bernoulli number \( \beta_{m,k}(q) \) as follows [4]; for \( k \geq 0 \),
\[ \beta_{k,x}(q) = \int_X [a]^k \chi(a) d\mu_q(a). \]

From the definition
\[ \beta_{k,x}(q) = [d]^{k-1} \sum_{a=0}^{d-1} \chi(a) q^a \beta_k\left(\frac{a}{d} : q^d\right). \]

Indeed we have
\[ \int_X \chi(t \cdot t) \cdot d\mu_q(t) = \lim_{\rho \to \infty} \frac{1}{[d(p^\rho) \cdot q^d]} \sum_{n=0}^{p^{\rho-1} - 1} q^n \chi(n) [n]^k \]
\[ = \lim_{\rho \to \infty} \frac{1}{[d]} \frac{1}{[p^\rho : q^d]} \sum_{i=0}^{p^{\rho-1} - 1} \sum_{n=0}^{i+dN} q^{i+dn} [i+dn]^k \chi(i+dn) \]
\[ = \lim_{\rho \to \infty} \frac{1}{[d]} \frac{1}{[p^\rho : q^d]} \sum_{i=0}^{p^{\rho-1} - 1} q^n \chi(i) \sum_{n=0}^{p^{\rho-1} - 1} q^{dn} \left(\left[\frac{i+n}{d} : q^d\right][d]\right)^k \]
\[ = \frac{1}{[d]} \sum_{i=0}^{p^{\rho-1} - 1} q^n [d]^k \chi(i) \int_{\mathbb{Z}_q} \left[\frac{i+n}{d} + t : q^d\right]^k d\mu_q^*(t) \]
\[ = [d]^{k-1} \sum_{i=0}^{d-1} q^i \chi(i) \beta_k \left( \frac{i}{d} ; q \right). \]

The locally constant function \( \chi \) on \( X \) can be integrated against the distribution \( \mu_k \) for \( k \geq 1 \), that is,
\[ \int_X \chi(x) \, d\mu_k(x) = \beta_{k,x}. \]

Therefore we obtain the following

**Theorem 1.** For \( k \geq 0 \), we have
\[ d\mu_k(x) = [x]^k \, d\mu_q(x). \]

For \( \alpha \in X^* \), \( \alpha \neq 1 \), \( k \geq 1 \), \( \mu_{k,\alpha} \) is defined by
\[ \mu_{k,\alpha}(U) = \mu_{k,\alpha_q}(U) = \mu_{k,q}(U) - \alpha^{-1}[\alpha^{-1} ; q]^{k-1} \mu_{k,q}^\alpha(\alpha U), \]
where \( U \subseteq X \) is compact open. Thus we see
\[ \lim_{q \to 1} \mu_{k,\alpha_q} = \mu_{\text{Mazur},k,\alpha}. \]

where \( \mu_{\text{Mazur},k,\alpha} \) means Mazur's measure [8].

For \( x \in X \), we denote
\[ [[x]] = [[x : q]] = (x+1)q^2 - q. \]

It is known that for \( k \geq 1 \)
\[ \mu_{k,\alpha_q}(a + dp^N Z_p) \equiv [[k : q^a]] [[a : q]^{k-1} \mu_{\text{Mazur},1,\alpha}(a + dp^N Z_p) \mod p^{N-c}], \]
where the constant \( c \) depends on \( k \) but not on \( N \) or \( q \); \( d\mu_k(x) = ((k+1)q^{2x} - q^2)\, [x]^{k-1} \, d\mu_{\text{Mazur},1,\alpha}(x). \]

Now we define the operator \( \chi^x = \chi^{x:k,q} \) on \( f(q) \) by
\[ \chi^x f(q) = [y]^{k-1} \chi(y) f(q^y), \]
\[ \chi^x \chi^y = \chi^{x + y, k} \chi^{x:k,q}. \]

Thus \( \chi^x \chi^y = \chi^{xy}. \)

Due to Koblitz [8] we find, for fixed \( \alpha \in X^* \) and \( \alpha \neq 1 \)
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$$(1 - \chi^q) \left( 1 - \frac{1}{\alpha^q} \right) \beta_{t, x}(q) = \int_{x^*} \chi(x) d\mu_{t, x}(x).$$

§ 3. Congruences

We define $\langle x \rangle = \langle x : y \rangle = [x : q] / \omega(x)$ where $\omega(x)$ is the Teichmüller character.

Let $p$ be a prime number and let $\chi$ be a Dirichlet character with values contained in algebraic closure of $Q_p$. We set $p^* = p$ for $p > 2$, and $p^* = 4$ for $p = 2$, and denote by $\hat{f} = [f, p^*]$ the least common multiple of the conductor $f$ of $\chi$ and $p^*$.

Let $\beta_{n, x}(q)$ denote the $n$-th generalized Carlitz's $q$-Bernoulli number belonging to the character $\chi$.

Then we have a $q$-analogue of Witt's formula in the $p$-adic cyclotomic field $Q_p(\chi)$ as follows [4]:

$$\beta_{m, x}(q) = \lim_{n \to \infty} \frac{1}{[fp^n]} \sum_{x=1}^{fp^n} \chi(x) [x]^{m} q^x$$

holds for $n \geq 0$.

Herein as usual we set $\chi(x) = 0$ if $x$ is not prime to the conductor $f$. In this section we shall give a few simple formulas of the Kummer congruences for the generalized Carlitz's $q$-Bernoulli numbers. From the above formula of $\beta_{n, x}(q)$ we have

$$\beta_{n, x}(q) = \lim_{p \to \infty} \frac{1}{[fp^n]} \sum_{1 \leq x \leq fp^n} \chi(x) [x]^{n} q^x$$

$$+ \lim_{p \to \infty} \frac{1}{[fp^{p-1}:q^p]} \frac{1}{[p]} \sum_{x=1}^{fp^{p-1}} \chi(p) \chi(y) [p]^{n} [y : q^n] q^{px},$$

where $*$ means to take sum over the rational integers prime to $p$ in the given range.

Hence we have

$$\beta_{n, x}(q) = \lim_{n \to \infty} \frac{1}{[fp^n]} \sum_{1 \leq x \leq fp^n} \chi(x) [x]^{n} q^x$$

$$+ [p]^{n-1} \chi(p) \beta_{n, x}(q^n),$$

that is,
\[ \beta_{n,x}(q) - [p]^{n-1} \chi(p) \beta_{n,x}(q^p) = \lim_{p \to \infty} \frac{1}{[fp^p]} \sum_{1 \leq x \leq fp^p} * \chi(x)[x]^n q^{cx} \]

We choose a rational number \( c \in \mathbb{Z} \) such that \((c, pf) = 1\), \( c \neq \pm 1 \).

Let \( x \) run over the range \( 1 \leq x \leq fp^p \), \( (x, p) = 1 \), \( x_0 \) run over the range \( 1 \leq x_0 \leq fp^p \), \( (x_0, p) = 1 \), and determine a number \( r_\rho(x) \in \mathbb{Z} \) by \( x_0 = cx + r_\rho(x)fp^p \).

Taking the \( n \)-th power of this equality and making sum with the character \( \chi \) we obtain

\[ \frac{1}{[fp^p]} \sum_{1 \leq x \leq fp^p} * \chi(x_0)[x_0]^n q^{cx} = \frac{1}{[fp^p]} \sum_{1 \leq x \leq fp^p} * \chi(cx)[cx]^n q^{cx} \]
\[ + \sum_{1 \leq x \leq fp^p} * \chi(cx)(n[cx]^{n-1}q^{cx} + (q-1)[cx]^n)[r_\rho(x):q^{lpq}]q^{cx} \]

(mod \([fp^p])

If \( p \to \infty \), it holds that

\[ \beta_{n,x}(q) - [p]^{n-1} \chi(p) \beta_{n,x}(q^p) = \chi(c)[c]^n(\beta_{n,x}(q^c) - [p : q^c]^{n-1} \chi(p) \beta_{n,x}(q^{pc})) \]
\[ + \sum_{1 \leq x \leq fp^p} * \chi(cx)(n[cx]^{n-1}q^{cx} + (q-1)[cx]^n)[r_\rho(x):q^{lpq}]q^{cx} \]

Thus we have

\[ \lim_{p \to \infty} \sum_{1 \leq x \leq fp^p} * \chi(cx)(n[cx]^{n-1}q^{cx} + (q-1)[cx]^n)[r_\rho(x):q^{lpq}]q^{cx} \]
\[ = \beta_{n,x}(q) - [p]^{n-1} \chi(p) \beta_{n,x}(q^p) - \chi(c)[c]^n(\beta_{n,x}(q^c) - [p : q^c]^{n-1} \chi(p) \beta_{n,x}(q^{pc})). \]

Now, we easily see that

\[ (1-\chi^p)(1-[c] \chi^c)\beta_{n,x}(q) = \beta_{n,x}(q) - [p]^{n-1} \chi(p) \beta_{n,x}(q^p) - \chi(c)[c]^n(\beta_{n,x}(q^c) - [p : q^c]^{n-1} \chi(p) \beta_{n,x}(q^{pc})). \]

Therefore we obtain the following

**Theorem 2.** For \( n \geq 1 \) we have

\[ (1-\chi^p)(1-[c] \chi^c)\beta_{n,x}(q) = \lim_{p \to \infty} \sum_{1 \leq x \leq fp^p} * \chi(cx)(n[cx]^{n-1}q^{cx} + (q-1)[cx]^n)[r_\rho(x):q^{lpq}]q^{cx}. \]
We set $\chi_k = \chi \omega^{-k}$. By the above theorem
\[
(1 - \chi_k^i) (1 - [c] \chi_k^i) \beta_{n,k}(q) = \lim_{\rho \to \infty} \sum_{1 \leq x < \rho/p^\theta} \ast (n \chi \omega^{-1}(cx) < cx > n^{-1} q^{c^x} + (q-1) \chi (cx) < cx > n) [r_p(x) : q^{fp^\rho}] q^{c^x}.
\]

Let $t = 1 - q$ such that $|t| < p^{-1/\theta}$. Then we easily see that $<x : q> \equiv 1 \pmod{p^\min(e_p(p^\theta), e_p(1))}$. Hence $<x : q> p^\theta \equiv 1 \pmod{p^{n+1/\theta}}$. Therefore we obtain the following

**Theorem 3.** For $k \equiv k_1 \pmod{p^n}$ we have
\[
(1 - \chi_k^i) (1 - [c] \chi_k^i) \frac{\beta_{k,k}(q) - \chi_k^i}{k} = (1 - \chi_k^i)(1 - [c] \chi_k^i) \frac{\beta_{k,k}(q) - \chi_k^i}{k_1} \pmod{p^n},
\]
where $e_n = \min(n + \min(v_p(p^\theta), v_p(1-q)), n - v_p(kk_1) + v_p(1-q))$.

Now we define
\[
\mu_k^{(c)} = (\mu_{k,0}(U) - [c] \mu_{k,0}(\frac{1}{c}, U)),
\]
where $U \subseteq X$ is compact open set.

Note that $\mu_k^{(c)}$ distribution on $X$.

This distribution yields an integral as follows. For $n \geq 1$, we have
\[
\int_{x \in X} \chi(x) d\mu_n^{(c)}(x) = (\beta_{n,0}(q) - \langle p \rangle^{n-1} \langle p \rangle \beta_{n,0}(q^p)) - \chi(c) [c]^{n} (\beta_{n,0}(q^c) - [p : q^c]^{n-1} \chi(p) \beta_{n,0}(q^{p^c}))
\]
\[
= \lim_{\rho \to \infty} \sum_{1 \leq x < \rho/p^\theta} \ast \chi (cx) (n [cx]^{n-1} q^{c^x} + (q-1) [cx]^{n}) [r_p(x) : q^{lp^\rho}] q^{c^x}
\]
\[
= \lim_{\rho \to \infty} \sum_{1 \leq x < \rho/p^\theta} \ast \chi (cx) (n [cx]^{n-1} q^{c^x} + (q-1) [cx]^{n}) \left[\left[ -\frac{cx}{fp^\rho} \right] c : q^{lp^\rho} \right] q^{c^x},
\]
where $[\cdot]_G$ is Gauss' symbol.

**Theorem 4.** For $n \geq 1$ we have
\[
\int_{x \in X} \chi(x) d\mu_n^{(c)}(x)
\]
\[ \lim_{p \to \infty} \sum_{1 \leq x < j_p} * \chi(cx) (n[cx]^{n-1} q^{cx} + (q-1) [cx]^n) \left[ \frac{cx}{j_p} \right]_q q^{cx}. \]

For \( r \in \mathbb{Z}, \ r \neq 1 \), we define
\[ \zeta_{p, d}(r) = \frac{1}{r-1} \lim_{k \to \infty} \frac{1}{[p^k]} \sum_{m=0}^{p^k-1} q^m \left[ \frac{m}{r} \right]^{-1}. \]

Then we have
\[ \zeta_{p, d}(1-k) = \frac{\beta_k(q)}{k}. \]

Here we can also define as follows.
\[ L_{p, d}(r, \chi) = \frac{1}{r-1} \lim_{k \to \infty} \frac{1}{[p^k]} \sum_{1 \leq x < j_p^k} * \chi(n) \omega^{r-1}(n) q^n \]
\[ = \frac{1}{r-1} \lim_{k \to \infty} \frac{1}{[p^k]} \sum_{1 \leq x < j_p^k} * \chi(n) < n >^{1-r} q^n. \]

Thus we find that
\[ L_{p, d}(1-k, \chi \omega^k) = \frac{1}{k} \sum_{m=0} \frac{1}{[p^m]} \sum_{1 \leq x < j_p^m} * \chi(n) [n]^k q^n \]
\[ = \frac{1}{k} \int_{[x]} \chi(x) [x]^k d\mu_q(x) \]
\[ = \frac{1}{k} \left( \int_{[x]} \chi(x) [x]^k d\mu_q(x) - \int_{[px]} \chi(x) [x]^k d\mu_q(x) \right) \]
\[ = \frac{1}{k} \left( \int_{[x]} \chi(x) [x]^k d\mu_q(x) \right) \]
\[ - \chi(p) [p]^{k-1} \int_{[x]} \chi(x) [x : q^p]^k d\mu_{q^p}(x). \]

We easily see that
\[ [px]^k = [p]^k [x : q^p]^k, \ \mu_q(px) = \frac{1}{[p]} \mu_{q^p}(x). \]

Therefore we have
\[ L_{p, d}(1-k, \chi \omega^k) = -\frac{1}{k} (\beta_{k,x}(q) - \chi(p) [p]^{k-1} \beta_{k,x}(q^p)). \]
§ 4. A formula of \( L_{p,q}(s, \chi) \)

In this section we study the value of \( L_{p,q}(s, \chi) \) at \( s=1 \).

In complex case, the another \( q \)-Bernoulli numbers \( B_k(q) \) are defined in [14], [1] by

\[
B_0(q) = \frac{q-1}{\log q} \quad B_k(q) = \begin{cases} 1 & \text{if } k=1 \\ =0 & \text{if } k>1, \end{cases}
\]

with the usual convention about replacing \( B_i(q) \) by \( B_i(q) \).

Here, we see that \( \lim_{q \to 1} B_k(q) = B_k \).

It is known in [1]. [14] that

\[
B_m(q) = \frac{1}{(q-1)^m} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \frac{i}{[i]}.
\]

Let \( q \in C_p \) with \( |1-q|_p < p^{-\frac{1}{p-1}} \). Then the \( p \)-adic \( q \)-Bernoulli numbers have the property as follows:

\[
B_m(q) = \int_{\mathbb{Z}_p} q^{-x} [x]^m d\mu_q(x) = \lim_{N \to -\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} [x]^m.
\]

This is easily proved in [4].

In this section, we use the same notation of Section 3. Let \( B_{n,x}(q) \) denote the \( n \)-th generalized \( q \)-Bernoulli numbers belonging to the character \( \chi \). Then we have \( q \)-analogue of Witt's formula in the \( p \)-adic cyclotomic field \( Q_p(\chi) \) as follows:

\[
B_{n,x}(q) = \lim_{\rho \to -\infty} \frac{1}{[fp^{\rho}]} \sum_{x=1}^{fp^{\rho}} \chi(x) [x]^n.
\]

By the same method of section 3, we have

\[
B_{n,x}(q) = \lim_{\rho \to -\infty} \frac{1}{[fp^{\rho}]} \sum_{1 \leq x \leq fp^{\rho}} \chi(x) [x]^n
\]

\[
+ [p]^{x-1} \chi(p) B_{n,x}(q^p),
\]

that is,

\[
B_{n,x}(q) - [p]^{x-1} \chi(p) B_{n,x}(q^p) = \lim_{\rho \to -\infty} \frac{1}{[fp^{\rho}]} \sum_{1 \leq x \leq fp^{\rho}} \chi(x) [x]^n.
\]
Taking the $n$-th power of this equality and making sum with the character $\chi$ we obtain
\[
\frac{1}{[fp^p]} \sum_{1 \leq x \leq fp^p} \chi(x_\rho)[x_\rho]^n = \frac{1}{[fp^p]} \sum_{1 \leq x \leq fp^p} \chi(cx)[cx]^n + n \sum_{1 \leq x \leq fp^p} \chi(cx)[cx]^{n-1}[r_\rho(x) : q^{fp^p}] q^{cx} \mod [fp^p].
\]

If $\rho \to \infty$, it holds that
\[
B_{n,x}(q) - [p]^{n-1} \chi(p) B_{n,x}(q^p) = \chi(c) [c]^n (B_{n,x}(q^c) - [p : q^c]^{n-1} \chi(p) B_{n,x}(q^{pc})) + n \lim_{p \to \infty} \sum_{1 \leq x \leq fp^p} \chi(cx)[cx]^{n-1}[r_\rho(x) : q^{fp^p}] q^{cx}.
\]

Thus we have
\[
\lim_{p \to \infty} \sum_{1 \leq x \leq fp^p} \chi(cx)[cx]^{n-1}[r_\rho(x) : q^{fp^p}] q^{cx} = \frac{1}{n} (B_{n,x}(q) - [p]^{n-1} \chi(p) B_{n,x}(q^p)) - \frac{1}{n} \chi(c) [c]^n (B_{n,x}(q^c) - [p : q^c]^{n-1} \chi(p) B_{n,x}(q^{pc})) = (1 - \chi^c)(1 - [c] \chi^c) \frac{B_{n,x}(q)}{n}.
\]

We set $\chi_k = \chi k^{-1}$.

For any positive integer $k$, we can define the $p$-adic $q$-$L$-function as follows:
\[
L_{p,q}(1-k, \chi) = - (1 - \chi^c)(1 - [c] \chi^c) \frac{B_{k,x_k}(q)}{k}.
\]

Note that $\lim_{q \to 1} L_{p,q}(1-k, \chi) = L_p(1-k, \chi)$.

Let
\[
Q_{k,q}(\chi) = (1 - \chi^c)(1 - [c] \chi^c) \frac{B_{k,x_k}(q)}{k}.
\]

For any sequence $(a_k)$, we define an operator $\Delta$ by $\Delta a_k = a_{k+1} - a_k$. Since $<cx> \equiv 1 \mod p^e$ with $e = \min(v_p(p^e), v_p(1-q))$, we have
\[ \Delta^n <cx>^n = (cx)^n \equiv 0 \pmod{p^n}, \]
hence
\[ \Delta^n \Omega_{1,q}(\chi) \to 0 \text{ as } n \to \infty. \]

For any positive integer \( k \),
\[
- \sum_{m=0}^{\infty} \binom{-(1-k)}{m} \Delta^m \Omega_{1,q}(\chi) = - \sum_{m=0}^{k-1} \Delta^m \Omega_{1,q}(\chi) \binom{k-1}{m} = -(1+\Delta)^{k-1} \Omega_{1,q}(\chi) = -\frac{1}{k} (1-\chi^{q}) (1-[c]\chi^{k}) B_{k,\chi}(q).
\]

Here, we see that \( p \)-adic \( q \)-L-function \( L_{p,q}(s, \chi) \) is defined as such a continuous function of \( s \) in \( \mathbb{Z}_p \) that takes the value
\[
-\frac{B_{k,\chi}(q)}{k} (1-\chi^{q}) (1-[c]\chi^{k})
\]
at each point \( s = 1-k \) with above integer \( k \), that is,
\[
L_{p,q}(s, \chi) = - \sum_{k=0}^{\infty} \binom{-s}{k} \Delta^k \Omega_{1,q}(\chi).
\]

**Proposition 3.** For \( s \in \mathbb{Z}_p \), we have
\[
L_{p,q}(s, \chi) = - \sum_{k=0}^{\infty} \binom{-s}{k} \Delta^k \Omega_{1,q}(\chi)
\]
if \( \chi \neq \chi_0 \). In particular
\[
L_{p,q}(1, \chi) = - \sum_{k=0}^{\infty} (-1)^k \Delta^k \Omega_{1,q}(\chi)
\]
for \( \chi \neq \chi_0 \). Here \( \chi_0 \) denotes the trivial character.

**References**


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