THE LENGTH OF A CLOSED GEODESIC ON A COMPACT SURFACE

Masao MAEDA

Dedicated to Professor Tominosuke Otsuki
on his 75th birthday

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0. Introduction.

In [4], B. C. Croke showed the following interesting fact

FACT ([4; THEOREM 4.1]). Let M be a compact surface which is homeomorphic to a 2-dimensional sphere S^2 . Then in M there exists a non-trivial closed geodesic c whose length $L=L(c)$ satisfies

$$L \leq 9d(M)$$

where $d(M)$ denotes the diameter of M .

Furthermore in the remark following Theorem 4.1, Croke is asserting that one could improve this estimate $L \leq 9d(M)$ to

$$L \leq 6d(M)$$

by improving Lemma 3.1 which is the main part of the estimate in Theorem 4.1.

Now the purpose of this note is to give a more sharp estimate than Croke which is stated as follows;

THEOREM. Let M be a compact surface which is homeomorphic to a sphere S^2 . Then in M there exists a non-trivial closed geodesic c whose length $L=L(c)$ satisfies

$$L \leq 5d(M).$$

This can be proved by improving Lemma 3.1 in [4] following the same idea used by Croke.

2. Decomposition by convex domains

In the following, let M always denotes a compact surface which is homeomorphic to a sphere S^2 in a 3-dimensional Euclidean space E^3 . Let $x, y \in M$ be the points such that

$$d(x, y) = d(M) \quad (:= \sup_{p, q \in M} d(p, q))$$

where d is the distance function on M induced from the Riemannian metric on M and put

$$G(x, y) := \{\gamma; \gamma: [0, d(x, y)] \rightarrow M \text{ is a shortest geodesic from } x \text{ to } y\}.$$

In this note, we assume that all geodesics are parametrized by arc-length.

Following lemma is well known.

Lemma (M. Berger). Let M' be a compact manifold and $x, y \in M'$ be the points such that $d(x, y) = d(M')$. Then for any non-zero tangent vector $v \in T_x(M')$, there exists a geodesic $\gamma \in G(x, y)$ satisfying

$$\angle(v, \dot{\gamma}(0)) \leq \pi/2.$$

Here $\angle(v, \dot{\gamma}(0))$ denotes the angle between the vector v and the tangent vector $\dot{\gamma}(0)$.

Using this lemma, we can show the following;

Lemma 1. There exists $\{\gamma_1, \gamma_2, \dots, \gamma_m\} \subset G(x, y)$ and mutually disjoint domains $\{D_i\}_{1 \leq i \leq m}$ in M such that

- (i) $\partial D_i = \gamma_i([0, d(x, y)]) \cup \gamma_{i+1}([0, d(x, y)])$ for $i=1, 2, \dots, m$ (γ_{m+1} denotes γ_1)
- (ii) each D_i is a convex subset of M and $\bigcup_{i=1}^m \bar{D}_i = M$.

Here a domain $D \subset M$ is called to be convex if for each point $p \in M$, there exists a number $r = r(p) > 0$ such that for any points $q, q' \in B_r(p) \cap D$ and any shortest geodesic $\sigma \in G(q, q')$,

$$\sigma([0, d(q, q')]) \subset B_r(p) \cap D$$

where $B_r(p) \subset M$ denotes the geodesic ball with radius r centered at p .

Since each boundary ∂D_i of the domain D_i in Lemma 1 is the images of two shortest geodesics $\gamma_i \in G(x, y)$ and $\gamma_{i+1} \in G(x, y)$, we see that D_i is convex if and only if $\angle_{D_i}(\dot{\gamma}_i(0), \dot{\gamma}_{i+1}(0)) \leq \pi$ and $\angle_{D_i}(-\dot{\gamma}_i(d(x, y)), -\dot{\gamma}_{i+1}(d(x, y))) \leq \pi$

where $\angle_{D_i}(\dot{\gamma}_i(0), \dot{\gamma}_{i+1}(0))$ denotes the angle between the vectors $\dot{\gamma}_i(0)$ and $\dot{\gamma}_{i+1}(0)$ measured on $\bar{D}_i$.

For each fixed $i, 1 \leq i \leq m$, put

$$\delta_i := \sup_{p \in D_i} d(p, \partial D_i).$$

Then we have the following ;

PROPOSITION 2. Following (i) or (ii) occurs

- (i) $\delta_i \leq d(M)/2$ for all $i=1, 2, \dots, m$
- (ii) $\delta_{i_0} > d(M)/2$ for some number i_0 and $\delta_i \leq d(M)/2$ for all $i, i \neq i_0$.

Proof. Assume that case (i) does not occur. Then there exists some number $i_0, 1 \leq i_0 \leq m$ such that

$$\delta_{i_0} > d(M)/2.$$

So it suffices to show that

$$\delta_{i_0} \leq d(M)/2 \text{ for all } i, i \neq i_0.$$

Let $D'_{i_0} := \text{interior of } \bigcup_{k \neq i_0} \bar{D}_k$. Then D_{i_0} and D'_{i_0} are mutually disjoint and satisfy the following

- (i) $\bar{D}_{i_0} \cup \bar{D}'_{i_0} = M$
- (ii) $\partial D_{i_0} = \partial D'_{i_0} = \gamma_{i_0}([0, d(x, y)]) \cup \gamma_{i_0+1}([0, d(x, y)])$
- (iii) $\gamma_i((0, d(x, y))) \subset D'_i$ for all $i, i \neq i_0, i_0+1$.

Let put $\delta'_{i_0} := \sup_{p \in D'_{i_0}} d(p, \partial D'_{i_0})$.

Then there exist a point $p_o \in D_{i_0}$ and $p'_o \in D'_{i_0}$ such that

$$\delta_{i_0} = d(p_o, \partial D_{i_0})$$

and

$$\delta'_{i_0} = d(p'_o, \partial D'_{i_0}).$$

Let $\tau : [0, d(p_o, p'_o)] \rightarrow M$ be a shortest geodesic from p_o to p'_o . Then by Jordan's Theorem, there exists s_o such that $\tau(s_o) \in \partial D_{i_0}$. So

$$\delta_{i_0} = d(p_o, \partial D_{i_0}) \leq d(p_o, \tau(s_o))$$

and

$$\delta'_{i_0} = d(p'_o, \partial D_{i_0+1}) \leq d(p'_o, \tau(s_o)).$$

Thus

$$\delta_{i_0} + \delta'_{i_0} \leq d(p_o, \tau(s_o)) + d(p'_o, \tau(s_o)) = d(p_o, p'_o) \leq d(M)$$

because τ is a shortest geodesic. So we have

$$\delta'_{i_0} \leq d(M)/2$$

and Proposition 1 follows from following lemma.

Lemma 3. For any compact two domains $D, D' \subset M$ such that $D \subset D'$, it holds

$$\sup_{p \in D} d(p, \partial D) \leq \sup_{p \in D'} d(p, \partial D').$$

Proof. Let $p_1 \in D$ be a point such that

$$d(p_1, \partial D) = \sup_{p \in D} d(p, \partial D).$$

For this point p_1 , we can find a point $q_1 \in \partial D'$ such that

$$d(p_1, q_1) = d(p_1, \partial D').$$

Let $\sigma : [0, d(p_1, q_1)] \rightarrow M$ be a shortest geodesic from p_1 to q_1 . Then by assumption $D \subset D'$, we can find a point $\sigma(s_1) \in \partial D$. Thus

$$d(p_1, \partial D) \leq d(p_1, \sigma(s_1)) \leq d(p_1, q_1) = d(p_1, \partial D') \leq \sup_{p \in D'} d(p, \partial D').$$

q.e.d.

2. Birkhoff's curve shortenning process.

Let D be a compact convex domain in M with piecewise smooth boundary ∂D . For this ∂D , we will apply the Birkhoff's curve shortenning process. Then we can get the following fact which is detailed in [4 ; p3–p5] and there is no necessity to explain Birkhoff's curve shortening process for our purpose.

PROPOSITION 4. There exists a free homotopy $\{\sigma_t\}_{0 \leq t \leq 1}$ of a simply closed curve $\sigma_0([0, 1]) = \partial D$ satisfying the following properties ;

- (i) each $\sigma_t : [0, 1] \rightarrow \bar{D}$, $0 \leq t \leq 1$ is piecewise smooth. In particular, if σ_0 is a broken geodesic curve, then each σ_t is also a broken geodesic curve.
- (ii) $\Omega_t \subset \Omega_{t'}$, if $t > t'$ where Ω_t is the convex subdomain of D such that $\partial\Omega_t = \sigma_t([0, 1])$ and $L(\sigma_t) \leq L(\sigma_{t'})$ if $t > t'$.
- (iii) σ_1 is a simple closed geodesic or a point curve i.e. $\sigma_1([0, 1]) = p_1$ for some point $p_1 \in \bar{D}$.

Now let $\{\gamma_1, \gamma_2, \dots, \gamma_m\} \subset G(x, y)$ and D_i , $1 \leq i \leq m$ are the geodesics and domains obtained in § 1. Then renumbering if necessary, we may assume that following case (i) or case (ii) occur

case (i) $\delta_i \leq d(M)/2$ for all i , $1 \leq i \leq m$

case (ii) $\delta_1 > d(M)/2$ and $\delta_i \leq d(M)/2$ for all i , $2 \leq i \leq m$.

Note that case (i) and case (ii) do not occur simultaneously.

PROPOSITION 5. For each fixed D_i , $i=2, \dots, m$, there exists a non-trivial simple closed geodesic c with length $L(c) \leq 2d(M)$ or there exists a piecewise smooth homotopy $\{b_s\}_{0 \leq s \leq 1}$ from γ_i to γ_{i+1} whose images are in $\bar{D}_i$ and satisfying

$$L(b_s) \leq 4d(M) \text{ for all } s$$

where γ_i and γ_{i+1} are the shortest geodesics from x to y such that

$$\gamma_i([0, d(x, y)]) \cup \gamma_{i+1}([0, d(x, y)]) = \partial D_i.$$

Proof. Fix a domain D_i and denote it as D for the simplicity. Let $\{\sigma_t\}_{0 \leq t \leq 1}$ be the homotopy of ∂D obtained by an application of Proposition 4. Then $\sigma_1 : [0, 1] \rightarrow \bar{D}$ is a simple closed geodesic with length $L(\sigma_1) \leq L(\sigma_0) = L(\partial D) = 2d(M)$ or a point curve i.e. $\sigma_1(s) = p_1$ for all s , $0 \leq s \leq 1$ where $p_1 \in \bar{D}$. If the first case occur, then there is nothing to prove. So assuming the second case occur, we will prove our assertion. If $p_1 \in \partial D$, then the following argument can be done more easily comparison with the case $p_1 \in D$. Thus we assume $p_1 \in D$. Let $p_o \in \partial D$ be a point such that $d(p_1, p_o) = d(p_1, \partial D)$. Then by definition

$$d(p_1, p_o) \leq \sup_{p \in \partial D} d(p, \partial D) \leq d(M)/2.$$

We consider the case $p_o \in \gamma_i([0, d(x, y)])$. The other case $p_o \in \gamma_{i+1}([0, d(x, y)])$ will be considered similarly. Let $\tau : [0, d(p_o, p_1)] \rightarrow M$ be a shortest geodesic from p_o to p_1 . Then since $L(\tau) = d(p_o, p_1) = d(p_1, \partial D)$,

$$\tau((0, d(p_o, p_1))) \subset D.$$

For each t , let $\Omega_t \subset D$ be the convex domain whose boundary $\partial\Omega_t$ is the image of the curve σ_t . Put

$$t_o := \min\{t; p_1 \in \partial\Omega_t\}.$$

Then from property (ii) in Proposition 4 and the definition of t_o ,

$$p_1 \in \Omega_t \text{ for all } t, t < t_o$$

and

$$p_1 \in \partial\Omega_t \text{ for all } t, t \geq t_o.$$

Note that $t_o > 0$ by our assumption for p_1 . So for $t, t < t_o$, $p_1 \in \Omega_t$. Then we have

Lemma 6. For all $t, t < t_o$, it holds

$$\tau([0, d(p_o, p_1)]) \cap \partial\Omega_t = \{p_t\}$$

for some $p_t \in \partial\Omega_t$.

Proof. Since $p_1 \in \Omega_t$ and $p_o \in \Omega_t^c$, $\tau([0, d(p_o, p_1)]) \cap \partial\Omega_t \neq \emptyset$ by Jordan's Theorem. So there exists some s_o such that $\tau([0, s_o]) \subset \Omega_t$ and $\tau(s_o) \in \partial\Omega_t$. Then from a property of convex set, there exists a positive number $\varepsilon > 0$ such that $\tau((s_o, s_o + \varepsilon)) \cap \bar{\Omega}_t = \emptyset$. Thus if $\# \tau([0, d(p_o, p_1)]) \cap \partial\Omega_{t_2} \geq 2$ for some $t_2 < t_o$, then there exists s_1 such that $\tau((s_o, s_1)) \cap \bar{\Omega}_{t_2} = \emptyset$ and $\tau(s_1) \in \partial\Omega_{t_2}$ where $\#$ denotes the number of elements of a set. Put $t' := \inf\{t; t < t_2 \text{ and } \# \tau((s_o, s_1)) \cap \partial\Omega_t \geq 2\}$. Then by continuity of the curve $\sigma_t = \partial\Omega_t$ for the parameter t , we see $t' < t_o$ and $\# \tau((s_o, s_1)) \cap \partial\Omega_{t'} = 1$. Thus there exists s_2 such that $\tau((s_o, s_2)) \subset \Omega_{t'}$, $\tau((s_2, s_1)) \subset \Omega_{t'}$ and $\tau(s_2) \in \partial\Omega_{t'}$. But since $\tau((s_o, s_2)) \subset \Omega_{t'}$ and $\tau(s_2) \in \partial\Omega_{t'}$, also from the property of convex set $\Omega_{t'}$, there exists $\varepsilon' > 0$ such that $\tau((s_2, s_2 + \varepsilon')) \cap \bar{\Omega}_{t'} = \emptyset$. This is a contradiction. q.e.d.

For $\tau([0, d(p_o, p_1)]) \cap \partial\Omega_{t_o}$, there are two possibilities (i) or (ii);

- (i) $\tau([0, d(p_o, p_1)]) \cap \partial\Omega_{t_o} = \{p_1\}$
- (ii) $\tau([0, d(p_o, p_1)]) \cap \partial\Omega_{t_o} \supsetneq \{p_1\}$.

If case (i) occur, then

$$\tau([0, d(p_o, p_1)]) \cap \partial\Omega_t = \{p_1\} \text{ for all } t > t_o$$

because $\Omega_{t_0} \supset \Omega_{t'}$ for $t', t' > t_0$.

If case (ii) occur, we put

$$t_1 := \sup\{t; \tau([0, d(p_o, p_1)]) \cap \partial\Omega_t \neq \emptyset\}.$$

Then from the proof of Lemma 6, we see

$$\tau([0, d(p_o, p_1)]) \cap \partial\Omega_t = \{p_t, p_1\} \text{ for all } t, t < t_1.$$

Also from the definition of t_1 , we have

$$\tau([0, d(p_o, p_1)]) \cap \partial\Omega_t = \{p_1\} \text{ for all } t, t > t_1.$$

For $t = t_1$, $\tau([0, d(p_o, p_1)]) \cap \partial\Omega_{t_1} = \{p_1\}$. This follows from the continuity of the homotopy $\{\sigma_t\}$. Summarizing above, we have

Lemma 7. (i) For all $t, t_0 \leq t < t_1$, it holds

$$\tau([0, d(p_o, p_1)]) \cap \partial\Omega_t = \{p_t, p_1\}$$

for some point $p_t (\neq p_1) \in \partial\Omega_t$ and the map $p: [t_0, t_1] \rightarrow \tau([t_0, t_1])$ defined by $p(t) := p_t$ is continuous.

(ii) For all $t, t_1 \leq t \leq 1$, it holds

$$\tau([0, d(p_o, p_1)]) \cap \partial\Omega_t = \{p_1\}.$$

By Lemma 7, in the following we can assume that each homotopy curve σ_t is parametrized as

$$\sigma_t(0) = \begin{cases} p_t & \text{for } t, 0 \leq t \leq t_1 \\ p_1 & \text{for } t, t_1 \leq t \leq 1. \end{cases}$$

Now we will define a homotopy $\{b_s\}_{0 \leq s \leq 1}$. This homotopy is constructed by four homotopies $\{b_s\}_{0 \leq s \leq 1/4}$, $\{b_s\}_{1/4 \leq s \leq 1/2}$, $\{b_s\}_{1/2 \leq s \leq 3/4}$, $\{b_s\}_{3/4 \leq s \leq 1}$. First homotopy $\{b_s\}_{0 \leq s \leq 1/4}$ is a homotopy from the curve $\gamma_i = b_o$ to the curve

$$b_{1/4} = \gamma_i \mid [d(x, p_o), d(x, y)] \circ \tau^{-1} \circ \sigma_1^{-1} \circ \tau \circ \gamma_i \mid [0, d(x, p_o)]$$

through the curve

$$b_s = \begin{cases} \gamma_i \mid [d(x, p_o), d(x, y)] \circ (\tau \mid [0, 8sd(p_o, p_1)])^{-1} \circ \tau \mid [0, 8sd(p_o, p_1)] \circ \\ \quad \gamma_i \mid [0, d(x, p_o)] & \text{for } 0 \leq s \leq 1/8 \\ \gamma_i \mid [d(x, p_o), d(x, y)] \circ (\tau \mid [0, t_1])^{-1} \circ (\sigma_1 \mid [0, 8s-1])^{-1} \circ \tau \mid [0, t_1] \circ \\ \quad \gamma_i \mid [0, d(x, p_o)] & \text{for } 1/8 \leq s \leq 1/4. \end{cases}$$

Second homotopy $\{b_s\}_{1/4 \leq s \leq 1/2}$ is a homotopy from the curve $b_{1/4}$ to

$$b_{1/2} = \gamma_i \left| [d(x, p_o), d(x, y)] \circ \tau^{-1} \circ \sigma_{t_1}^{-1} \circ \tau \circ \gamma_i \right| [0, d(x, p_o)]$$

through the curve

$$b_s = \gamma_i \left| [d(x, p_o), d(x, y)] \circ \tau^{-1} \circ \sigma_{4(t_1-1)s+2-t_1}^{-1} \circ \tau \circ \gamma_i \right| [0, d(x, p_o)].$$

Third homotopy $\{b_s\}_{1/2 \leq s \leq 3/4}$ is a homotopy from the curve $b_{1/2}$ to

$$b_{3/4} = \gamma_i \left| [d(x, p_o), d(x, y)] \circ (\gamma_i \left| [d(p_o, y), d(x, y)] \right|)^{-1} \circ \gamma_{i+1} \circ \right. \\ \left. (\gamma_i \left| [0, d(x, p_o)] \right|)^{-1} \circ \gamma_i \right| [0, d(x, p_o)]$$

through the curve

$$b_s = \gamma_i \left| [d(x, p_o), d(x, y)] \circ (\tau \left| [0, d(p_o, p_{(3-4s)t_1})] \right|)^{-1} \circ \sigma_{(3-4s)t_1}^{-1} \circ \right. \\ \left. \tau \left| [0, d(p_o, p_{(3-4s)t_1})] \right| \circ \gamma_i \right| [0, d(x, p_o)].$$

Last homotopy $\{b_s\}_{3/4 \leq s \leq 1}$ is a homotopy from the curve $b_{3/4}$ to γ_{i+1} through the curve

$$b_s = \gamma_i \left| [(d(x, y) - d(x, p_o))(4s-3) + d(x, p_o), d(x, y)] \circ \right. \\ \left. (\gamma_i \left| [(d(x, y) - d(x, p_o))(4s-3) + d(x, p_o), d(x, y)] \right|)^{-1} \circ \gamma_{i+1} \circ \right. \\ \left. (\gamma_i \left| [0, (4-4s)d(x, p_o)] \right|)^{-1} \circ \gamma_i \right| [0, (4-4s)d(x, p_o)].$$

Each parameter of b_s is chosen so as the homotopy $\{b_s\}_{0 \leq s \leq 1}$ is continuous.
q.e.d.

Proof of the Theorem. Let $x, y \in M$ be the points such that $d(x, y) = d(M)$ and $\{\gamma_i\}_{i=1,2,\dots,m}$ be the shortest geodesics from x to y such that the domains $\{D_i\}$ bounded by $\gamma_{i+1}^{-1} \circ \gamma_i$, $i=1, 2, \dots, m$ are mutually disjoint and each domain D_i , $i=1, 2, \dots, m$ is convex. These $\{\gamma_i\}_{i=1,2,\dots,m}$ exist by Lemma 1. Then exchanging the number if necessary, we can assume that

$$\delta_i (:= \sup_{p \in \bar{D}_i} d(p, \partial D_i)) \leq d(M)/2$$

for $i=2, 3, \dots, m$ by Proposition 2. For D_1 , we will apply Proposition 4. Then we get a homotopy $\{b_i\}_{0 \leq i \leq 1}$ of the closed curve $\sigma_o = \partial D_1 = \gamma_2^{-2} \circ \gamma_1$. If σ_1 is a closed geodesic, then σ_1 is simple by construction and satisfies

$$L(\sigma_1) \leq L(\sigma_o) = 2d(M).$$

Thus in this case, there is nothing to prove. So it suffices to consider the case that σ_1 is a point curve i.e. for some point $p_1 \in D$, $\sigma_1(t) = p_1$ for all t .

We will also apply Proposition 4 to the domain D_m . Then by the same reason stated as above, the last curve $\tilde{\sigma}_1$ of the homotopy $\{\tilde{\sigma}_t\}$ from the curve $\tilde{\sigma}_o = \gamma_1^{-1} \circ \gamma_m = \partial D_m$ which is obtained by an application of Proposition 4 is also a point curve i.e. for some point $p_2 \in D_m$

$$\tilde{\sigma}_1(t) = p_2 \text{ for all } t.$$

Now we will define a free homotopy $\{\mu_s\}$ from the point curve $\sigma_1(t) = p_1$ to the point curve $\tilde{\sigma}_1(t) = p_2$. First homotopy $\{\mu_s\}_{0 \leq s \leq 1/m}$ is from the curve σ_1 to the curve $\sigma_o = \gamma_2^{-1} \circ \gamma_1$ defined by

$$\mu_s = \sigma_{1-ms}.$$

Second homotopy $\{\mu_s\}_{1/m \leq s \leq 2/m}$ is from the curve $\mu_{1/m}$ to the curve $\mu_{2/m} = \gamma_3^{-1} \circ \gamma_1$ defined by

$$\mu_s = b_{ms-1}^{-1} \circ \gamma_1$$

where $\{b_s\}$ is the homotopy from γ_2 to γ_3 which is constructed in the proof of Proposition 6. Similar for each $i = 2, 3, \dots, m$ we define a homotopy $\{\mu_s\}_{(i-1)/m \leq s \leq i/m}$ from the curve $\gamma_i^{-1} \circ \gamma_1$ to $\gamma_{i+1}^{-1} \circ \gamma_1$ using the homotopy in Proposition 6.

Finally we define a homotopy $\{\mu_s\}_{(m-1)/m \leq s \leq 1}$ from the curve $\gamma_m^{-1} \circ \gamma_1$ to the point curve $\tilde{\sigma}_1^{-1}$ defined by

$$\mu_s = \tilde{\sigma}_{ms-(m-1)}^{-1}.$$

Thus we have a 1-parameter family of curves $\{\mu_s\}_{0 \leq s \leq 1}$ starting a point curve and ending a point curve. From this family of curves, we can get a map $F : S^2 \rightarrow M$ which has non-zero degree. Then by Birkhoff's argument we know that there exists a non-trivial closed geodesic in M whose length is not greater than the length of the longest curve in this 1-parameter family of curves. Thus there exists a non-trivial closed geodesics in M with length $\leq 5d(M)$. Q.E.D.

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*Department of Mathematics
Faculty of Education
Yokohama National University*