AN INEQUALITY FOR INJECTIVITY RADII OF RIEMANNIAN MANIFOLDS WITH POSITIVE CURVATURE

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Dedicated to Professor Tominosuke Otsuki on his 77th birthday

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1. Introduction

In this paper we give an inequality for injectivity radii of compact, connected, non-simply connected Riemannian manifolds with positive curvature. We also characterize certain homogeneous spherical space forms by using this inequality.

Let M be an m -dimensional complete connected Riemannian manifold and N an n -dimensional ($0 \leq n \leq m-1$) compact connected submanifold embedded in M . Let $\text{Exp}_N: \nu(N) \rightarrow M$ denote the normal exponential map. Here $\nu(N)$ is the total space of the normal bundle of N in M . In the case where N is a point x , $\nu(N)$ stands for the tangent space to M at x . We denote by $i(N)$ the injectivity radius of Exp_N . It is defined as the supremum of the set of all $t > 0$ for which $\text{Exp}_N: \nu_t(N) \rightarrow M$ is an embedding, where $\nu_t(N)$ is the set of all normal vectors to N whose length is less than t . The injectivity radius $\text{Inj}(M)$ of M is defined as $\text{Inj}(M) = \inf\{i(x); x \in M\}$.

From now on let M denote an m -dimensional ($m \geq 3$) compact, connected, non-simply connected Riemannian manifold whose every sectional curvature K_M satisfies $K_M \geq 1$. By the theorem of Bonnet-Myers, the fundamental group $\pi_1(M)$ of M is a finite group. Since the diameter of M is not greater than $\pi/2$ ([9], [14], [16]), we have the inequality $\text{Inj}(M) \leq \pi/2$. By using the maximal diameter theorem due to Toponogov, we can show that $\text{Inj}(M) = \pi/2$ holds if and only if M is isometric to the m -dimensional real projective space $P^m(\mathbf{R})$ with constant curvature 1. Shiohama showed in [14] the following result. If $\pi_1(M) \cong \mathbf{Z}_3$, then the inequality $\text{Inj}(M) \leq \pi/3$ holds. Here the equality holds if and only if M is isometric to the lens space $L^m(3)$ of constant

curvature 1. The definition of $L^m(3)$ will be given in Section 3.

From what stated above we can expect that if the order of $\pi_1(M)$ is greater than three, then the inequality $\text{Inj}(M) \leq \pi/4$ holds and that if the equality holds, then M is isometric to a spherical space form of constant curvature 1. This conjecture is in fact true if the order of $\pi_1(M)$ is not a prime. Our main theorem in this paper is stated as follows.

Main Theorem. *Let M be an m -dimensional ($m \geq 3$) compact, connected, non-simply connected Riemannian manifold whose every sectional curvature K_M satisfies $K_M \geq 1$. Suppose that the order of $\pi_1(M)$ is not a prime. Then we have the inequality $\text{Inj}(M) \leq \pi/4$. Here the equality holds if and only if M is a connected homogeneous Riemannian manifold of constant curvature 1 and $\pi_1(M)$ is isomorphic to $\mathbf{Z}_4$ or $\mathbf{Q8}$ where $\mathbf{Q8}$ denotes the quaternion group.*

The proof will be given in Section 3. This result shows that spherical space forms M of constant curvature 1 with $\text{Inj}(M) = \pi/4$ are of two types, which are homogeneous. We can also give a characterization of a homogeneous spherical space form whose fundamental group is a binary dihedral group by means of the injectivity radius of the normal exponential map for a closed geodesic. Let γ be a simple closed geodesic in M which is homotopically non trivial. By making use of the same method as in the proof of Main Theorem, we show in Section 4 the following result. If M is homogeneous and $\pi_1(M)$ is not a cyclic group, then the inequality $i(\gamma) \leq \pi/4$ holds. Here if the equality holds, then M is of constant curvature 1 and $\pi_1(M)$ is isomorphic to the binary dihedral group $\mathbf{D}_k^*$. Here k is a positive integer connected with the order of $[\gamma] \in \pi_1(M)$. The definition of $\mathbf{D}_k^*$ will be given in Section 3.

2. Preliminaries

We refer to [1] and [3] for tools and results in Riemannian geometry that will be used. We always assume in this paper that manifolds and all apparatus on them are of class C^∞ and that all geodesics have unit speed, unless otherwise stated.

Throughout this section let V denote an m -dimensional ($m \geq 3$) compact, connected, simply connected Riemannian manifold whose every sectional curvature K_V satisfies $K_V \geq 1$. We denote by d the distance function on V induced from the Riemannian metric of V . A nonempty subset C of V is called *totally π -convex* in V if every geodesic in V of length $< \pi$ such that its end points

belong to C is contained in C . The interior of a connected totally π -convex subset of V is a totally geodesic submanifold embedded in V .

We shall state several lemmas which will be used in Sections 3 and 4. Since we have shown these lemmas in [11], the proofs are omitted here.

Lemma 2.1. ([11]) *Let C be a compact totally π -convex proper subset in V .*

- (1) *If C is not connected, then V is isometric to the Euclidean unit m -sphere S^m and C consists of exactly two points x and y with $d(x, y) = \pi$.*
- (2) *If C is connected and $\dim C \geq 1$, then any two points of C can be connected by a minimizing geodesic in V which is contained in C .*

Let A be a compact proper subset of V . We set $B = \{x \in V; d(x, A) \geq \pi/2\}$. Making use of the comparison theorem of Toponogov, we have the following.

Lemma 2.2. ([11]) *Let A and B be as above. Suppose $B \neq \emptyset$. Then the following holds.*

- (1) *B is totally π -convex in V .*
- (2) *If A is invariant under a fixed point free isometry φ of V and $\dim B \geq 1$, then B is a compact connected totally geodesic submanifold embedded in V without boundary and $d(x, A) = \pi/2$ for all $x \in B$.*

Let C_1 and C_2 be compact connected proper subsets in V such that $C_1 = \{x \in V; d(x, C_2) \geq \pi/2\}$ and $C_2 = \{x \in V; d(x, C_1) \geq \pi/2\}$. By Lemma 2.2, these subsets are totally π -convex in V .

Lemma 2.3. ([11]) *Let C_1 and C_2 be as above.*

- (1) *If $\partial C_1 = \emptyset$, then $d(x, C_2) = \pi/2$ for all $x \in C_1$.*
- (2) *If $\partial C_1 = \partial C_2 = \emptyset$, then $d(x, y) = \pi/2$ for all $x \in C_1$ and for all $y \in C_2$.*

The next lemma plays an important role to obtain our results in this paper.

Lemma 2.4. ([11]) *Suppose that V contains compact connected totally geodesic embedded submanifolds C_1 and C_2 with the properties that $\dim C_i \geq 1$, $\partial C_i = \emptyset$, $C_i = \{x \in V; d(x, C_{i+1}) \geq \pi/2\}$ where $i = 1, 2$ and $C_3 = C_1$. If m is odd, then V is isometric to S^m and $\dim C_1 + \dim C_2 = m - 1$.*

3. The proof of Main Theorem

We first explain several notations which will be used in the proofs of our results in this paper. We refer to [17] for notations and results on spherical

space forms. A Clifford translation of a metric space (V, d) is an isometry φ of (V, d) such that the displacement function $d(x, \varphi(x))$ on V is constant.

Let Γ be a properly discontinuous subgroup of the orthogonal group $\mathbf{O}(m+1)$ which acts freely on the Euclidean unit m -sphere S^m . The quotient space S^m/Γ is a homogeneous Riemannian manifold of constant curvature 1 if and only if Γ is a group of Clifford translations of S^m ([17]).

The binary dihedral group $\mathbf{D}_k^*$ is a finite group generated by two elements a and b with relations $a^{2k}=e$, $a^k=b^2$ and $aba=b$ where $2k$ is the order of a ($k \geq 2$) and e denotes the unit element. The order of $\mathbf{D}_k^*$ is equal to $4k$. In case of $k=2$, $\mathbf{D}_2^*$ is the quaternion group $\mathbf{Q}_8$.

Let k be a positive integer ≥ 3 , and let m be of the form $m=2n+1$. Let Λ_k be a cyclic subgroup of $\mathbf{O}(m+1)$ with a generator φ , which is expressed in terms of an orthonormal basis of $\mathbf{R}^{m+1}$ as follows:

$$\varphi = \begin{pmatrix} R(1/k) & & & \\ & R(1/k) & & \\ & & \ddots & \\ & & & R(1/k) \end{pmatrix} =: \underbrace{R(1/k) \oplus \cdots \oplus R(1/k)}_{n+1}$$

where $R(\theta) = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$. Each element of Λ_k is a Clifford translation of S^m . The quotient space $L^m(k) := S^m/\Lambda_k$ is called a lens space. This space is a homogeneous Riemannian manifold of constant curvature 1 with injectivity radius π/k .

Let k be a positive integer ≥ 2 , and let m be of the form $m=4n-1$. Let Σ_{2k} be a subgroup of $\mathbf{O}(m+1)$ with generators φ_1 and φ_2 , which are expressed in terms of an orthonormal basis in $\mathbf{R}^{m+1}$ as follows: $\varphi_1 = R(1/2k) \oplus \cdots \oplus R(1/2k)$ ($2n$ times) and $\varphi_2 = P \oplus \cdots \oplus P$ (n times). Here $P \in \mathbf{O}(4)$ is defined by $P = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then Σ_{2k} is isomorphic to $\mathbf{D}_k^*$ and acts freely on S^m . Each element of Σ_{2k} is a Clifford translation of S^m . The quotient space $M^m(2k) := S^m/\Sigma_{2k}$ is a homogeneous Riemannian manifold of constant curvature 1. Let γ be a simple closed geodesic in $M^m(2k)$ such that $[\gamma] \in \pi_1(M^m(2k))$ corresponds to φ_1 . Then we have that $i(\gamma) = \pi/4$ and $\text{Inj}(M^m(2k)) = \pi/2k$. We note that $L^m(2k)$ is a double covering space of $M^m(2k)$.

From now on let M be an m -dimensional ($m \geq 3$) compact, connected, non-simply connected Riemannian manifold whose every sectional curvature K_M satisfies $K_M \geq 1$. Let $p: V \rightarrow M$ be the universal Riemannian covering. By the theorem of Bonnet-Myers, V is compact and hence $\pi_1(M)$ is a finite group. We denote by d the distance function on V . Let Γ be the deck transformation group on V corresponding to $\pi_1(M)$ which is a finite subgroup of the isometry group of V . We note that if M is homogeneous, then each element of Γ is a Clifford translation of V ([17]). Let $\varphi_1, \dots, \varphi_n$ be elements of $\Gamma \setminus \{I_V\}$ where I_V is the identity transformation of V . We denote by $\Gamma(\varphi_1, \dots, \varphi_n)$ the group generated by $\varphi_1, \dots, \varphi_n$. For a subgroup G of Γ we denote by $\#G$ its order.

Main Theorem follows from the lemmas below. The proof proceeds as follows. Under the assumption that $\# \Gamma$ is not a prime and $\text{Inj}(M) \geq \pi/4$, we first show that the equality $\text{Inj}(M) = \pi/4$ holds and M is of constant curvature 1. We next show that if $\# \Gamma$ is not a prime and the equality $\text{Inj}(M) = \pi/4$ holds, then Γ is isomorphic to A_4 or Σ_4 . The converse will be proved in Lemma 3.10 below.

Under the situation stated above, we shall show several lemmas. In the following lemmas we always assume that $\# \Gamma$ is not a prime unless otherwise states. Then m is odd by Synge's theorem.

Lemma 3.1. *We have the inequality $\text{Inj}(M) \leq \pi/4$. Here if the equality holds, then*

- (1) *For any proper subgroup G of Γ we have $d(x, \varphi(x)) = x/2$ for all $x \in V$ and for all $\varphi \in \Gamma \setminus G$*
- (2) *V is isometric to S^m .*

PROOF. Suppose $\text{Inj}(M) \geq \pi/4$. Then we shall show that the equality $\text{Inj}(M) = \pi/4$ holds and that the assertions (1) and (2) hold. Let G and z be a proper subgroup of Γ and a point of V respectively. The assumption $\text{Inj}(M) \geq \pi/4$ implies that the inequality $d(x, \varphi(x)) \geq \pi/2$ holds for any $\varphi \in \Gamma \setminus \{I_V\}$ and for any $x \in V$. We set $A = \{\varphi(z); \varphi \in G\}$, $B = \{x \in V; d(x, A) \geq \pi/2\}$ and $C = \{x \in V; d(x, B) \geq \pi/2\}$. Clearly G leaves A invariant. By this fact and the assumption $\text{Inj}(M) \geq \pi/4$, we have $\varphi(x) \in B$ for any $x \in A$ and for any $\varphi \in \Gamma \setminus G$. Hence B is not empty and we have $A \subset C$. It follows from Lemma 2.2 (1) that B is totally π -convex in V . Since A is invariant under G , so is B and C . We shall show that B is connected and $\dim B \geq 1$. We now suppose that B is not connected. By Lemma 2.1 (1), V is isometric to S^m and B consists of just two points whose distance is equal to π . Then C is isometric to a great $(m-1)$ -

sphere in S^m . Since G acts freely on V , we have $\#G=2$. This implies that A consists of just two points. Since $A \subset C$, there exists $w \in C$ such that $d(w, A) \geq \pi/2$. This shows $w \in B$, which is a contradiction. Thus we have proved that B is connected. Since G leaves B invariant, we have $\dim B \geq 1$. By Lemma 2.2 (2), B is a compact connected totally geodesic submanifold embedded in V without boundary and we have $d(x, A) = \pi/2$ for all $x \in B$. From Lemma 2.2 (1), C is totally π -convex in V . We first consider the case that C is not connected. By Lemma 2.1 (1), V is isometric to S^m and C consists of just two points whose distance is equal to π . Then we have $A=C$. From this, B is isometric to a great $(m-1)$ -sphere in S^m . In this case we have $d(z, \varphi(z)) = \pi/2$ for any $\varphi \in \Gamma \setminus G$. Next we consider the case that C is connected. Since G leaves C invariant, we have $\dim C \geq 1$. By Lemma 2.2 (2), C is a compact connected totally geodesic submanifold embedded in V without boundary and $d(y, B) = \pi/2$ for all $y \in C$. The subset $B' = \{x \in V; d(x, C) \geq \pi/2\}$ is a connected totally π -convex subset of V because $B \subset B'$ and $\dim B \geq 1$. Since B' is invariant under G , from Lemma 2.2 (2) B' is a compact connected totally geodesic submanifold embedded in V without boundary. It follows from Lemma 2.3 that $d(x, y) = \pi/2$ for all $x \in B'$ and for all $y \in C$. This implies $B=B'$. Hence we have $d(z, \varphi(z)) = \pi/2$ for all $\varphi \in \Gamma \setminus G$. Since m is odd, we can now apply Lemma 2.4 to the present situation. Hence V is isometric to S^m . We complete the proof.

In the following lemmas we always assume $\text{Inj}(M) = \pi/4$ unless otherwise stated. Then we have $d(x, \varphi(x)) \geq \pi/2$ for all $x \in V$ and for all $\varphi \in \Gamma \setminus \{I_V\}$. By Lemma 3.1 V is isometric to S^m . From now on we identify V with S^m . Then Γ can be regarded as a properly discontinuous subgroup of $O(m+1)$ which acts freely on S^m .

Lemma 3.2. *Let G be a proper subgroup of Γ . Let φ be an element of $\Gamma \setminus G$. Then*

- (1) *For each $z \in S^m$ there exists a unique simple closed geodesic $\tau_z: [0, 2\pi] \rightarrow S^m$ with $\tau_z(0) = \tau_z(2\pi) = z$ whose image is invariant under φ .*
- (2) *$\Gamma(\varphi)$ is isomorphic to $\mathbf{Z}_4$.*
- (3) *$\varphi^2 = -I$ where I denotes the unit matrix.*

PROOF. By Lemma 3.1 (1) we have $d(x, \varphi(x)) = \pi/2$ for any $x \in S^m$. There exists for each $x \in S^m$ a unique minimal geodesic segment $\sigma_x: [0, \pi/2] \rightarrow S^m$ from x to $\varphi(x)$. For each $x \in S^m$ we denote by $P(x, \varphi)$ the geodesic polygon with edges $\sigma_x, \varphi \circ \sigma_x, \dots, \varphi^{a-1} \circ \sigma_x$ where a is the order of φ . For each $z \in S^m$ the length

of $P(z, \varphi)$ is equal to $a\pi/2$ and $P(z, \varphi)$ is invariant under φ . This shows that for each $z \in S^m$ $P(z, \varphi)$ is a simple closed geodesic of S^m . Hence we have $a=4$. Thus $\Gamma(\varphi)$ is a cyclic group of order 4. The assertion (3) is now clear.

Lemmas 3.1 and 3.2 yield the following.

Lemma 3.3. *Suppose that Γ is a cyclic group with a generator φ . Then*

- (1) $d(x, \varphi(x)) = \pi/2$ for all $x \in S^m$.
- (2) Γ is isomorphic to $\mathbf{Z}_4$.
- (3) $\varphi^2 = -I$.

PROOF. Since $\#\Gamma$ is not a prime, by Lemma 3.2 we may put $\#\Gamma = 2k$ where k is a positive integer such that $k \geq 2$. Let G be a cyclic group generated by φ^k . It is a proper subgroup of Γ . By Lemma 3.1 we have $d(x, \varphi(x)) = \pi/2$ for all $x \in S^m$. The rest follows from Lemma 3.2.

Using Lemma 3.3 we shall show the following.

Lemma 3.4. *Suppose that Γ is a cyclic group with a generator φ . Then φ can be expressed in terms of an orthonormal basis of $\mathbf{R}^{m+1}$ as $\varphi = R(1/4) \oplus \dots \oplus R(1/4)$ (n times) where $m = 2n + 1$.*

PROOF. By Lemma 3.3 there exists for each $x \in S^m$ a unique simple closed geodesic τ_x in S^m with $\tau_x(0) = \tau_x(2\pi) = x$ whose image is invariant under φ . We now take a simple closed geodesic γ_1 with the property just stated above. We put $N_1 = \gamma_1([0, 2\pi])$ and $V_1 = \{x \in S^m; d(x, N_1) = \pi/2\}$. V_1 is a great $(m-2)$ -sphere in S^m . Since φ leaves V_1 invariant, there exists a unique simple closed geodesic γ_2 in V_1 with the same property as that of γ_1 . We put $N_2 = \gamma_2([0, 2\pi])$. Let W be the great 3-sphere in S^m containing N_1 and N_2 . Then the restriction of φ to W can be expressed in terms of an orthonormal basis as $\varphi = R(1/4) \oplus R(1/4)$. Suppose $m > 3$. We put $V_2 = \{x \in S^m; d(x, W) = \pi/2\}$. It is a great $(m-4)$ -sphere in S^m . Since φ leaves V_2 invariant, we can repeat there the same argument as above. By induction on dimension there exists an orthonormal basis in $\mathbf{R}^{m+1}$ such that φ can be expressed in terms of this basis as $\varphi = R(1/4) \oplus \dots \oplus R(1/4)$ (n times) where $m = 2n + 1$.

From now on let us consider the case that Γ is not a cyclic group.

Lemma 3.3. *Let $\varphi_1 \in \Gamma$ satisfy the condition that $d(x, \varphi_1(x)) = \pi/2$ for all $x \in S^m$. If Γ is not a cyclic group, then we have $\varphi^2 = \varphi_1^2 = -I$ for any $\varphi \in \Gamma \setminus \Gamma(\varphi_1)$.*

PROOF. By the assumption for φ_1 , for each $x \in S^m$ there exists a unique simple closed geodesic τ_x in S^m with $\tau_x(0) = \tau_x(2\pi) = x$ whose image is invariant under φ_1 . From this we have that $\Gamma(\varphi_1)$ is isomorphic to $\mathbf{Z}_4$ and $\varphi_1^2 = -I$. Then the claim follows from Lemma 3.2.

Using Lemmas 3.2 and 3.5 we shall show the following.

Lemma 3.6. *Let $\varphi_1 \in \Gamma$ satisfy the same condition as in Lemma 3.5. If Γ is not a cyclic group, then $\Gamma(\varphi_1, \varphi)$ is isomorphic to the quaternion group for any $\varphi \in \Gamma \setminus \Gamma(\varphi_1)$.*

PROOF. Let φ be an element of $\Gamma \setminus \Gamma(\varphi_1)$. By Lemma 3.2 $\Gamma(\varphi)$ is of order 4. We note $\varphi\varphi_1 \notin \Gamma(\varphi_1)$. Then by Lemma 3.5 we have $(\varphi\varphi_1)^2 = \varphi^2 = \varphi_1^2$. This shows that $\varphi\varphi_1 \neq \varphi_1\varphi$, $\varphi\varphi_1\varphi = \varphi_1$ and $\varphi_1\varphi\varphi_1 = \varphi$. It follows from these relations that $\Gamma(\varphi_1, \varphi)$ is a non-commutative group of order 8. Hence $\Gamma(\varphi_1, \varphi)$ is isomorphic to the quaternion group.

Lemma 3.7. *Suppose that Γ is not a cyclic group. Then we have*

- (1) *Γ is generated by two fixed point free isometries φ_1 and φ_2 of S^m with $d(x, \varphi_1(x)) = d(x, \varphi_2(x)) = \pi/2$ for all $x \in S^m$.*
- (2) *Γ is isomorphic to the quaternion group.*

PROOF. By assumption Γ has a proper subgroup. By Lemma 3.1 there exists $\varphi_1 \in \Gamma$ with the property that $d(x, \varphi_1(x)) = \pi/2$ for all $x \in S^m$. It follows from Lemma 3.2 that $\Gamma(\varphi_1) \cong \mathbf{Z}_4$ and $\varphi_1^2 = -I$. Let φ_2 be an element of $\Gamma \setminus \Gamma(\varphi_1)$. We shall show $\Gamma = \Gamma(\varphi_1, \varphi_2)$. To do that, we now suppose $\Gamma \neq \Gamma(\varphi_1, \varphi_2)$. Pick an element $\psi \in \Gamma \setminus \Gamma(\varphi_1, \varphi_2)$. By Lemma 3.2 $\Gamma(\psi)$ is of order 4. From Lemma 3.5 we have that $\psi^2 = (\varphi_1\psi)^2 = (\varphi_2\psi)^2 = (\varphi_1\varphi_2\psi)^2 = -I$. The relations $\psi^2 = (\varphi_1\psi)^2 = (\varphi_2\psi)^2$ imply $\varphi_1\psi\varphi_1 = \varphi_2\psi\varphi_2 = \psi$. As showed in the proof of Lemma 3.6, we have $\varphi_1\varphi_2\varphi_1 = \varphi_2$. By combining these relations with $\psi^2 = -I$, we obtain $(\varphi_1\varphi_2\psi)^2 = -\varphi_1^2$. This is a contradiction. Thus we have proved $\Gamma = \Gamma(\varphi_1, \varphi_2)$. By Lemma 3.6 Γ is isomorphic to the quaternion group. By Lemma 3.1 we have that $d(x, \varphi_2(x)) = \pi/2$ for all $x \in S^m$.

Lemma 3.8. *Suppose that Γ is not a cyclic group. Let φ_1 and φ_2 be generators of Γ as in Lemma 3.7 (1). Let z be a point of S^m and τ_z a unique simple closed geodesic in S^m with $\tau_z(0) = \tau_z(2\pi) = z$ whose image is invariant under φ_1 . Then there exists a unique Γ -invariant great 3-sphere $V^3(z)$ of S^m containing $\tau_z([0, 2\pi])$ and $\varphi_2(\tau_z([0, 2\pi]))$.*

PROOF. Let z be a point of S^m . We put $A = \{\varphi_1^j(z); 0 \leq j \leq 3\}$ and $A_1 = \tau_z([0, 2\pi])$. As in the proof of Lemma 3.1, we set $B = \{x \in S^m; d(x, A) \geq \pi/2\}$ and $C = \{x \in S^m; d(x, B) \geq \pi/2\}$. φ_1 leaves B and C invariant. As showed there, B and C are compact connected totally π -convex submanifolds embedded in S^m without boundary respectively such that $\dim B + \dim C = m - 1$. We note that $d(x, y) = \pi/2$ for any $x \in B$ and for any $y \in C$. Since $A \subset A_1$, $A \subset C$ and C is totally π -convex in S^m , we have $A_1 \subset C$. By the definition of B we then have $A_1 = C$. From the assumption $\text{Inj}(M) = \pi/4$ and the definition of B we have $\varphi(A) \subset B$ for any $\varphi \in \Gamma \setminus \Gamma(\varphi_1)$. Since B is totally π -convex in S^m , we have $\varphi(A_1) \subset B$ for any $\varphi \in \Gamma \setminus \Gamma(\varphi_1)$. Now we put $A_2 = \varphi_2(A_1)$. Let $V^3(z)$ be the great 3-sphere in S^m containing A_1 and A_2 . It follows from the relation $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$ that each $\psi \in \Gamma$ can be expressed as $\psi = \varphi_2^j \varphi_1^k$ where j and k are integers with $0 \leq j, k \leq 3$. Using this fact and the relations $\varphi_1^2 = \varphi_2^2 = -I$, we have that $\varphi_1(A_2) = A_2$ and $\psi(A_1) = A_2$ for any $\psi \in \Gamma \setminus \Gamma(\varphi_1)$. Hence Γ leaves $A_1 \cup A_2$ invariant. We shall show that Γ leaves $V^3(z)$ invariant. Let x be a point of $V^3(z) \setminus (A_1 \cup A_2)$ and ψ an element of Γ . We can take the great circle σ in $V^3(z)$ with the properties that $\sigma(0) \in A_1$, $\sigma(\pi/2) \in A_2$ and $\sigma(t) = x$ for some $t \in (0, \pi/2)$. We note that $\psi(\sigma(0))$ and $\psi(\sigma(\pi/2))$ are contained in $A_1 \cup A_2$. Since $V^3(z)$ is totally π -convex in S^m and $d(\psi(\sigma(0)), \psi(\sigma(\pi/2))) = \pi/2$, $\psi \circ \sigma$ is a great circle in $V^3(z)$. Hence we have $\psi(x) \in V^3(z)$. This proves Lemma 3.8.

Using Lemma 3.8 we shall show the following.

Lemma 3.9. *Suppose that Γ is not a cyclic group. Let φ_1 and φ_2 be the generators of Γ as in Lemma 3.7 (1). Then we have $m = 4n - 1$ where n is a positive integer. Moreover φ_1 and φ_2 are expressed in terms of an orthonormal basis of $\mathbf{R}^{m+1}$ as follows: $\varphi_1 = R(1/4) \oplus \cdots \oplus R(1/4)$ ($2n$ times) and $\varphi_2 = R \oplus \cdots \oplus P$ (n times).*

PROOF. For each $x \in S^m$ there exists a unique simple closed geodesic τ_x with $\tau_x(0) = \tau_x(2\pi) = x$ whose image is invariant under φ_1 . We now fix a point $z \in S^m$. As in the proof of Lemma 3.8, we put $A_1 = \tau_z([0, 2\pi])$ and $A_2 = \varphi_2(A_1)$. As proved there, we have $d(x, y) = \pi/2$ for any $x \in A_1$ and for any $y \in A_2$. By Lemma 3.8 there exists a unique Γ -invariant great 3-sphere $V^3(z)$ of S^m containing A_1 and A_2 . Then the restrictions of φ_1 and φ_2 to $V^3(z)$ can be expressed in terms of an orthonormal basis as $\varphi_1 = R(1/4) \oplus R(1/4)$ and $\varphi_2 = P$. Suppose that $m \geq 5$. We put $V_1 = \{x \in S^m; d(x, V^3(z)) = \pi/2\}$. It is a great $(m-4)$ -sphere in S^m . Since $V^3(z)$ is invariant under Γ , so is V_1 . We note $\dim V_1 \geq 3$. We can repeat on V_1 the same argument as above. The claim can be

shown by induction on dimension.

Lemma 3.10. *Let L be an m -dimensional ($m \geq 3$) connected homogeneous Riemannian manifold of constant curvature 1 whose fundamental group is isomorphic to $\mathbf{Z}_4$ or $\mathbf{Q}8$. Then we have $\text{Inj}(L) = \pi/4$.*

PROOF. By Lemma 3.1 we have $\text{Inj}(L) \leq \pi/4$. We now suppose $\text{Inj}(L) < \pi/4$. Let $q: S^m \rightarrow L$ be the universal Riemannian covering. There exists a non trivial simple closed geodesic γ in L whose length is less than $\pi/2$. Here we used the fact that for any geodesic $\tau: [0, \pi/2] \rightarrow L$ there is no conjugate point to $\tau(0)$ along τ . Since a lift of γ is a simple closed geodesic in S^m , γ is homotopically non trivial. Let φ be the deck transformation on S^m corresponding to $[\gamma] \in \pi_1(L)$. Since φ is a Clifford translation of S^m , there exists for each $x \in S^m$ a unique simple closed geodesic σ_x with $\sigma_x(0) = \sigma_x(2\pi) = x$ whose image is invariant under φ . From the assumption that $\pi_1(L) \cong \mathbf{Z}_4$ or $\mathbf{Q}8$, we obtain $d(x, \varphi(x)) \geq \pi/2$ for all $x \in S^m$. This shows that the length of γ is not less than $\pi/2$. This is a contradiction. Thus we have $\text{Inj}(L) = \pi/4$.

PROOF OF MAIN THEOREM. We use the same notations as above. By Lemma 3.1 we have the inequality $\text{Inj}(M) \leq \pi/4$. Suppose that $\text{Inj}(M) = \pi/4$. Then by Lemma 3.1 M is of constant curvature 1. We identify V with S^m . In the case where Γ is a cyclic group, Lemma 3.4 shows that M is isometric to $L^m(4)$. If Γ is not a cyclic group, then by Lemma 3.9 M is isometric to $M^m(4)$. Therefore M is a connected homogeneous Riemannian manifold of constant curvature 1 with $\pi_1(M) \cong \mathbf{Z}_4$ or $\mathbf{Q}8$. The converse follows from Lemma 3.10.

4. Spherical space forms M with $\pi_1(M) = D_k^*$

The purpose of this section is to give a characterization of a spherical space form whose fundamental group is a binary dihedral group.

Theorem A. *Let M be an m -dimensional ($m \geq 3$) compact, connected, non-simply connected homogeneous Riemannian manifold whose every sectional curvature K_M satisfies $K_M \geq 1$. Let γ be a simple closed geodesic in M which is homotopically non trivial. Suppose that $\pi_1(M)$ is not a cyclic group. Then we have the inequality $i(\gamma) \leq \pi/4$. Here if the equality holds, then M is isometric to $M^m(2k)$ and the order of $[\gamma] \in \pi_1(M)$ is even.*

Remark. In Theorem A, if the order of $[\gamma]$ is equal to two, then M is

isometric to $M^m(4)$. In the case where the order of $[\gamma]$ is greater than or equal to four, M is isometric to $M^m(2k)$. Here $2k$ is the order of $[\gamma]$ and $k \geq 2$.

From now on let M be as in Theorem A. Let $p: V \rightarrow M$ and Γ be as in Section 3. Let $\gamma: [0, \ell] \rightarrow M$ be a simple closed geodesic which is homotopically non trivial. We set $N = \gamma([0, \ell])$. Let φ_1 denote the deck transformation on V corresponding to $[\gamma] \in \pi_1(M)$. We denote by Γ_1 the cyclic group generated by φ_1 .

Under the situation stated above, we shall show several lemmas. In the following we use the same notations as in Section 3.

Lemma 4.1. *For each $x \in M \setminus \{\gamma(0)\}$ there exists a simple closed geodesic $\gamma_x: [0, \ell] \rightarrow M$ satisfying the conditions: (1) $\gamma_x(0) = \gamma_x(\ell) = x$; (2) γ_x is homotopic to γ ; (3) $i(\gamma_x) = i(\gamma)$.*

PROOF. Let x be a point of $M \setminus \{\gamma(0)\}$. By the assumption for M the isometry group $I(M)$ of M is a compact Lie group which acts on M transitively. We note that the connected component $I_0(M)$ of the identity transformation I_M in $I(M)$ also acts on M transitively. There is a continuous curve $f: [0, 1] \rightarrow I_0(M)$ such that $f(0) = I_M$ and $f(1)(\gamma(0)) = x$. We define a continuous deformation $\Phi: [0, \ell] \times [0, 1] \rightarrow M$ of γ by $\Phi(s, t) = f(t)(\gamma(s))$. If we fix a parameter value $t \in [0, 1]$, then the curve $c_t: [0, \ell] \rightarrow M$ defined $c_t(s) = \Phi(s, t)$ is a simple closed geodesic because $f(t)$ is an isometry of M . Clearly we have $i(c_t) = i(\gamma)$ for any $t \in [0, 1]$. Then c_1 is the required simple closed geodesic.

Let γ_x be a simple closed geodesic in M satisfying the conditions stated in the above lemma. Then a lift of γ_x in V is a simple closed geodesic whose image is invariant under φ_1 .

Lemma 4.1 and the fact just stated above will be used in the proof of Theorem A.

We shall show the following lemma without using the assumption that M is homogeneous.

Lemma 4.2. *Suppose that Γ is not a cyclic group. Then we have $i(\gamma) \leq \pi/4$. Here if the equality holds, then M is of constant curvature 1.*

PROOF. We shall show that the assumption $i(\gamma) \geq \pi/4$ implies that $i(\gamma) = \pi/4$ and M is of constant curvature 1. Suppose $i(\gamma) \geq \pi/4$. By hypothesis Γ_1 is a proper subgroup of Γ . Hence $p^{-1}(N)$ has at least two connected components.

Let $N_1, \dots, N_s$ ($s \geq 2$) be the connected components of $p^{-1}(N)$. Each N_j is a totally geodesic 1-dimensional closed submanifold embedded in V . Γ_1 leaves each N_j invariant. By exchanging the indices if necessary, we let $d(N_1, N_2) = \min\{d(N_i, N_j); 1 \leq i < j \leq s\}$. We now set $B = \{x \in V; d(x, N_1) \geq \pi/2\}$ and $C = \{x \in V; d(x, B) \geq \pi/2\}$. We note that the inequality $d(N_1, N_2) \geq 2i(\gamma)$ holds. By the assumption $i(\gamma) \geq \pi/4$ we get $d(N_i, N_j) \geq \pi/2$ ($1 \leq i < j \leq s$). Hence we have $N_j \subset B$ ($2 \leq j \leq s$) and $N_1 \subset C$. By Lemma 2.2 (1) B and C are connected totally π -convex subsets in V such that $\dim B \geq 1$ and $\dim C \geq 1$. Since N_1 is invariant under Γ_1 , so is B and C . By Lemma 2.2 (2) B and C are compact connected totally geodesic submanifolds embedded in V without boundary. Moreover by Lemma 2.3 we have $d(x, N_1) = \pi/2$ for all $x \in B$ and $d(y, B) = \pi/2$ for all $y \in C$. Using Lemmas 2.2 and 2.3 we can easily show $B = \{x \in V; d(x, C) \geq \pi/2\}$. From this we obtain $d(x, y) = \pi/2$ for all $x \in B$ and for all $y \in C$. Since $\# \Gamma \geq 4, m$ is odd by Synge's theorem. Then we can apply Lemma 2.4 to the present situation. Hence V is isometric to S^m . It is now clear $d(N_i, N_j) = \pi/2$ ($1 \leq i < j \leq s$). Thus we have $i(\gamma) = \pi/4$. We complete the proof.

From now on we always assume that Γ is not a cyclic group and $i(\gamma) = \pi/4$ unless otherwise stated. By Lemma 4.2 V is isometric to S^m . In what follows we identify V with S^m . Then Γ can be regarded as a properly discontinuous subgroup of $O(m+1)$ which acts freely on S^m . Let $N_1, \dots, N_s$ ($s \geq 2$) be the connected components of $p^{-1}(N)$. We have that $d(x, y) = \pi/2$ for all $x \in N_i$ and for all $y \in N_j$ ($i \neq j$). We set $B = \{x \in S^m; d(x, N_1) \geq \pi/2\}$ and $C = \{x \in S^m; d(x, B) \geq \pi/2\}$. As showed in the proof of Lemma 4.2, we get that $B = \{x \in S^m; d(x, N_1) = \pi/2\}$ and $C = \{x \in S^m; d(x, B) = \pi/2\}$. By the definition of B we have $N_1 = C$. Hence B is a great $(m-2)$ -sphere in S^m .

Under the situation stated above, we have the following.

Lemma 4.3 *Let φ be an element of $\Gamma \setminus \Gamma_1$. Then*

- (1) $\varphi(N_1) \subset B$ and $\varphi(N_1)$ is a connected component of $p^{-1}(N)$.
- (2) $d(x, \varphi(x)) = \pi/2$ for any $x \in S^m$.

PROOF. We shall show $\varphi(N_1) \cap N_1 = \emptyset$. Suppose $\varphi(N_1) \cap N_1 \neq \emptyset$. There exists a point $x \in N_1$ such that $\varphi(x) \in N_1$. Since γ is a simple closed geodesic in M , we have $\varphi(x) = \varphi_1^k(x)$ for some positive integer k . This is a contradiction because Γ acts freely on S^m and $\varphi \notin \Gamma_1$. Thus we have proved $\varphi(N_1) \cap N_1 = \emptyset$. Then the assumption $i(\gamma) = \pi/4$ implies $d(N_1, \varphi(N_1)) \geq \pi/2$. Hence we have $\varphi(N_1) \subset B$. This shows that $d(x, \varphi(x)) = \pi/2$ for any $x \in N_1$. Since φ is a

Clifford translation of S^m , (2) holds.

Let φ be an element of $\Gamma \setminus \Gamma_1$. By Lemma 4.3 (2) there exists for each $x \in N_1$ a unique minimal geodesic segment $\sigma_x: [0, \pi/2] \rightarrow S^m$ with $\sigma_x(0) = x$ and $\sigma_x(\pi/2) = \varphi(x)$. For each $x \in N_1$ let $P(x, \varphi)$ denote the geodesic polygon with edges $\sigma_x, \varphi \circ \sigma_x, \dots, \varphi^{a-1} \circ \sigma_x$, where $a = \# \Gamma(\varphi)$.

Lemma 4.4. *We have*

- (1) $\Gamma(\varphi) \cong \mathbf{Z}_4$ and $\varphi^2 = -I$ for any $\varphi \in \Gamma \setminus \Gamma_1$.
- (2) The order of φ_1 is even.
- (3) For any $\varphi \in \Gamma \setminus \Gamma_1$ we have $\varphi^2 = \varphi_1^j$ where $2j$ is the order of φ_1 .

PROOF. Let φ be an element of $\Gamma \setminus \Gamma_1$. Since by Lemma 4.3 (2) we have $d(x, \varphi(x)) = \pi/2$ for all $x \in S^m$, for each $x \in N_1$ the geodesic polygon $P(x, \varphi)$ is a simple closed geodesic $\tau_x: [0, 2\pi] \rightarrow S^m$ with $\tau_x(0) = \tau_x(2\pi) = x$ whose image is invariant under φ . This shows that $\# \Gamma(\varphi) = 4$ and $\varphi^2 = -I$. For each $x \in N_1$ we have $\varphi^2(x) \in N_1$. Since γ is a simple closed geodesic in M , we have $\varphi^2 = \varphi_1^j$ for some positive integer j . Since φ^2 is an involution on S^m , the order of φ_1 is equal to $2j$.

Lemma 4.5. *If $\# \Gamma_1 = 2$, then we have $\Gamma \cong \mathbf{D}_2^*$.*

PROOF. As proved in Lemma 4.4, we have $\varphi^2 = \varphi_1 = -I$ for any $\varphi \in \Gamma \setminus \Gamma_1$. We fix an element $\varphi_2 \in \Gamma \setminus \Gamma_1$. By the assumption for Γ there exists an element $\varphi_3 \in \Gamma \setminus \Gamma(\varphi_2)$. By Lemma 4.4 we have that $\varphi_2^2 = \varphi_3^2 = (\varphi_2 \varphi_3)^2 = -I$. From this we obtain the relation $\varphi_2 \varphi_3 \varphi_2 = \varphi_3$. Hence we have $\Gamma(\varphi_2, \varphi_3) \cong \mathbf{D}_2^*$. By using the same way as in the proof of Lemma 3.7, we can conclude that $\Gamma = \Gamma(\varphi_2, \varphi_3)$.

Lemma 4.6. *If $\# \Gamma_1 \geq 4$, then we have $\Gamma \cong \mathbf{D}_k^*$ where $2k$ is the order of φ_1 and $k \geq 2$.*

PROOF. Let $2k$ be the order of φ_1 where $k \geq 2$. We fix an element $\varphi_2 \in \Gamma \setminus \Gamma_1$. Since M is homogeneous, φ_1 and φ_2 are Clifford translations of S^m . As proved in Lemma 4.4, we have that $\varphi_1^k = \varphi_2^2 = (\varphi_1 \varphi_2)^2 = -I$. From the relation $(\varphi_1 \varphi_2)^2 = \varphi_2^2$, we obtain $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$. Then $\Gamma_2 := \Gamma(\varphi_1, \varphi_2)$ is isomorphic to $\mathbf{D}_k^*$. We shall show $\Gamma = \Gamma_2$. For this we now suppose $\Gamma \neq \Gamma_2$. Let ϕ be an element of $\Gamma \setminus \Gamma_2$. Since $\varphi_1 \varphi_2 \phi \notin \Gamma_2$, we have $\phi^2 = (\phi \varphi_1)^2 = (\phi \varphi_2)^2 = (\varphi_1 \varphi_2 \phi)^2 = -I$. From the relations $\phi^2 = (\phi \varphi_1)^2 = (\phi \varphi_2)^2$, we get $\varphi_1 \phi \varphi_1 = \varphi_2 \phi \varphi_2 = \phi$. By using the relations showed above, we obtain $(\varphi_1 \varphi_2 \phi)^2 = -\varphi_1^2$. This is a contradic-

tion. Hence we have proved $\Gamma = \Gamma_2$.

PROOF OF THEOREM A. We use the same notations as above. By Lemma 4.2 we have the inequality $i(\gamma) \leq \pi/4$. Here if the equality holds, then V is isometric to S^m . From now on we assume $i(\gamma) = \pi/4$. We identify V with S^m . We first suppose $\# \Gamma_1 = 2$. By Lemma 4.5 Γ is isomorphic to $\mathbf{D}_2^*$. Then Main Theorem proves that M is isometric to $M^m(4)$. In the following we assume $\# \Gamma_1 \geq 4$. By Lemma 4.6 Γ is isomorphic to $\mathbf{D}_k^*$ where $2k$ is the order of φ_1 and $k \geq 2$. Let φ_2 be an element of $\Gamma \setminus \Gamma_1$. As proved in the proof of Lemma 4.6, we have $\Gamma = \Gamma(\varphi_1, \varphi_2)$. We shall show that φ_1 and φ_2 can be expressed in terms of an orthonormal basis of $\mathbf{R}^m$ as $\varphi_1 = R(1/2k) \oplus \cdots \oplus R(1/2k)$ ($2n$ times) and $\varphi_2 = P \oplus \cdots \oplus P$ (n times) where $m = 4n - 1$. We now put $N_2 = \varphi_2(N_1)$. Since $\varphi_1(N_1) = N_1$, the relation $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$ shows that φ_1 leaves N_2 invariant. We note here that by Lemma 4.3 we have $d(x, y) = \pi/2$ for all $x \in N_1$ and for all $y \in N_2$. There exists a unique great 3-sphere V_1 in S^m containing N_1 and N_2 . It follows from the relation $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$ that $\phi \in \Gamma$ can be expressed as $\phi = \varphi_2^a \varphi_1^b$ where a and b are integers such that $0 \leq a \leq 3$ and $0 \leq b \leq 2k - 1$. This fact and the relations $\varphi_2^2 = \varphi_1^k = -I$ imply that Γ leaves V_1 invariant. This claim can be proved by the same way as in the proof of Lemma 3.8. Hence $p^{-1}(N)$ has just two connected components N_1 and N_2 . The restrictions of φ_1 and φ_2 to V_1 can be expressed in terms of an orthonormal basis as $\varphi_1 = R(1/2k) \oplus R(1/2k)$ and $\varphi_2 = P$. We now suppose that $m > 3$. We put $V_2 = \{x \in S^m; d(x, V_1) = \pi/2\}$. It is a great $(m-4)$ -sphere in S^m . Since Γ leaves V_2 invariant, we have $\dim V_2 \geq 3$. We now fix a point $z \in V_2$ and put $x = p(z)$. Let γ_x be a simple closed geodesic in M satisfying the conditions stated in Lemma 4.1. The lift of γ_x passing through z is a simple closed geodesic in V_2 whose image is invariant under φ_1 . This can be proved by using the following facts: (1) φ_1 leaves V_2 invariant; (2) $d(y, \varphi_1(y)) = x/k \leq \pi/2$ for any $y \in V_2$; (3) V_2 is totally π -convex in S^m . Then we can repeat on V_2 the same argument as above. The claim can be shown by induction on dimension. Thus M is isometric to $M^m(2k)$. We complete the proof.

We can characterize $M^m(4)$ by means of the injectivity radii of the normal exponential maps of simple closed geodesics.

Theorem B. *Let M be an m -dimensional ($m \geq 3$) compact, connected, non-simply connected homogeneous Riemannian manifold such that every sectional curvature K_M satisfies $K_M \geq 1$ and $\pi_1(M)$ is not a cyclic group. Let γ_1 and γ_2 be homotopically non trivial simple closed geodesics in M such that $[\gamma_2]$ is not an element of the cyclic group*

generated by $[\gamma_1]$. If γ_1 and γ_2 satisfy the condition that $i(\gamma_1) = i(\gamma_2) = \pi/4$, then M is isometric to $M^m(4)$.

PROOF. We use the same notations as above. By Lemma 4.2 M is of constant curvature 1. Let $p: S^m \rightarrow M$ be the universal Riemannian covering. We shall show $\Gamma \cong \mathbf{D}_2^*$. Let φ_1 and φ_2 be deck transformations on S^m corresponding to $[\gamma_1]$ and $[\gamma_2]$ respectively. If the order of φ_1 or φ_2 is equal to two, then by Lemma 4.5 we have that $\Gamma \cong \mathbf{D}_2^*$. In the case where $\# \Gamma(\varphi_1) \geq 4$ and $\# \Gamma(\varphi_2) \geq 4$, the assumption $\varphi_2 \notin \Gamma(\varphi_1)$ and Lemma 4.4 imply that $\Gamma(\varphi_2) \cong \mathbf{Z}_4$. This shows that $\varphi_1 \notin \Gamma(\varphi_2)$. Hence by Lemma 4.4 we have $\Gamma(\varphi_1) \cong \mathbf{Z}_4$. Then Lemma 4.6 shows $\Gamma \cong \mathbf{D}_2^*$. Thus we have proved the claim. Theorem A proves that M is isomorphic to $M^m(4)$.

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