The Subconstituent Algebra of a Strongly Regular Graph

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(Received May 9, 1994)

1 Introduction

In the following let Γ be a strongly regular graph on a vertex set X of cardinality n . For a given $x \in X$, let $\Gamma_1(x)$ denote the set of vertices in X adjacent to x in Γ and $\Gamma_2(x)$ the set of vertices which neither coincide nor are adjacent to x . Let A_1 denote the adjacency matrix of Γ and A_2 the adjacency matrix of Γ^c , the complement of Γ .

Let ∂ denote the ordinary distance relation between two vertices and, for $i=0, 1, 2$, we write

$$R_i = \{(x, y) \in X \times X \mid \partial(x, y) = i\}.$$

We know that $Y = (X, \{R_0, R_1, R_2\})$ is a (symmetric) P -polynomial association scheme with 2 classes, which is defined in section 2.

We write $\lambda = p_{11}^1$, $\mu = p_{11}^2$, and $k_i = p_{ii}^0$ for $i=0, 1, 2$, where p_{ij}^l is defined in Definition 2.1.

Pick any $x \in \Gamma$. The induced subgraphs of Γ with vertex sets $\Gamma_1(x)$, $\Gamma_2(x)$ are called the subconstituents of Γ with respect to x . We note that they are regular graphs with valencies λ and $k_1 - \mu$, respectively.

It is the important properties of strongly regular graphs but a union of cliques that A_1 has just three eigenvalues k_1, r, s with $k > r \geq 0$ and $s < -1$. See Theorem 1.3.1 (i), (iii), (iv) in [2].

Terwilliger introduced in [4, 5, 6] the subconstituent algebras of commutative association schemes. In particular, he considered the P - and Q -polynomial association scheme with D classes ($D \geq 3$). In this paper we consider the subconstituent algebra of an association scheme constructed from a strongly regular graph, i.e., the symmetric P - and Q -polynomial association scheme with 2 classes.

The main theorem in this paper is as follows.

Theorem 1.1 *Let Γ be a strongly regular graph and let $Y = (X, \{R_0, R_1, R_2\})$ be the*

symmetric association scheme constructed from Γ . Let A_1 be the adjacency matrix of Γ and k_1, r, s be eigenvalues of A_1 . Pick any $x \in X$ and let $T = T(x)$ be the subconstituent algebra of Y with respect to x . Let B_1 and B_2 be the adjacency matrices of the subconstituents of Γ with respect to x .

Then

$$\begin{aligned} \dim T &= m_1 + m'_1 + 4m_2 + 9 \\ &= m_1 + m'_1 + 4m'_2 + 9, \end{aligned}$$

where m_1 (resp. m_2) is the number of different eigenvalues of B_1 contained in (resp. not contained in) $\{r, s\}$ with respect to the eigenvectors which is orthogonal to an all 1's vector of size k_1 , and where m'_1 (resp. m'_2) is the number of different eigenvalues of B_2 contained in (resp. not contained in) $\{r, s\}$ with respect to the eigenvectors which is orthogonal to an all 1's vector of size k_2 .

Remark. It is clear that B_1 (resp. B_2) has an eigenvalue λ (resp. $k_1 - \mu$) with respect to an all 1's vector. We know that, if Γ is not a complete multipartite graph, then $\Gamma_2(x)$ is connected, so that the eigenspace with respect to the eigenvalue $k_1 - \mu$ is spanned by the all 1's vector. See [2]. But $\Gamma_1(x)$ may be disconnected even if $\lambda \neq 0$. In this case the eigenvalue λ has multiplicity at least 2. So we must count λ as an eigenvalue of B_1 with respect to the eigenvectors which is orthogonal to an all 1's vector.

2 The subconstituent algebras of association schemes

Definition 2.1 Let X be a nonempty finite set and, for a non-negative integer D , let $R_0, R_1, \dots, R_D$ be a family of nonempty subsets of the Cartesian product $X \times X$.

$Y = (X, \{R_i\}_{0 \leq i \leq D})$ is called a (symmetric) association scheme with D classes if the following hold.

- (i) $R_0 = \{(x, x) \mid x \in X\}$.
- (ii) $(x, y) \in R_i$ for exactly one i ($0 \leq i \leq D$), $(x, y \in X)$.
- (iii) $R_i = R_i^t = \{(y, x) \mid (x, y) \in R_i\}$.
- (iv) For all integers i, j, l with $0 \leq i, j, l \leq D$ and all $x, y \in X$ with $(x, y) \in R_l$,

$$p_{ij}^l \neq \{z \in X \mid (x, y) \in R_i, (z, y) \in R_j\}$$

depends only on i, j, l .

For each integer i with $0 \leq i \leq D$, the i -th associate matrix $A_i \in \text{Mat}_X(\mathbf{C})$ satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{if } (x, y) \notin R_i. \end{cases}$$

Let M be the linear span of $\{A_i\}_{0 \leq i \leq D}$ over $\mathbf{C}$. Then we immediately see that M is a commutative subalgebra of $\text{Mat}_X(\mathbf{C})$. We call M the Bose-Mesner algebra of Y .

Now fix any $x \in X$. For each integer i with $0 \leq i \leq D$, let $E_i = E_i(x)$ denote the diagonal matrix of order n satisfying

$$(E_i^*)_{uv} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{if } (x, y) \notin R_i. \end{cases}$$

Let M^* be the linear span of $\{E_i^*\}_{0 \leq i \leq D}$ over $\mathbf{C}$. Then we immediately see that M^* is a commutative subalgebra of $\text{Mat}_X(\mathbf{C})$. We call M^* the dual Bose-Mesner algebra of Y with respect to x . Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbf{C})$ generated by M and M^* . We call T the subconstituent algebra of Y with respect to x .

Let $V = \mathbf{C}^{|X|}$ (column vectors) and let $\langle, \rangle$ denote the Hermitean form

$$\langle u, v \rangle = u^t \bar{v}$$

for $u, v \in V$ where t denotes transpose and $\bar{}$ complex conjugate. Then $\text{Mat}_X(\mathbf{C})$ acts on V by left multiplication. V is called the standard module. Since T is closed under the conjugate transpose map, T is semisimple. Hence V can be expressed as an orthogonal direct sum of irreducible T -modules. By representation theory, we have the following.

Proposition 2.2 *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be an association scheme with D -classes. Pick any $x \in X$ and let $T = T(x)$ be the subconstituent algebra of Y .*

Suppose that the standard module V can be expressed as

$$V = \oplus_{j,l} W_{jl} \quad (\text{direct sum})$$

where $\{W_{jl}\}_{j,l}$ are irreducible T -modules, W_{jl} and $W_{j'l'}$ are T -isomorphic if and only if $j=j'$, and where $\dim W_{jl} = m_j$. Then

$$\dim T = \sum_j m_j^2.$$

□

Definition 2.3 Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be an association scheme. Then Y is said to be P -polynomial (with respect to the given ordering $A_0, A_1, \dots, A_D$ of the associate matrices), if for all integers i, j, k ($0 \leq i, j, k \leq D$), $p_{ij}^k = 0$ (resp. $p_{ij}^k \neq 0$) whenever one of i, j, k is greater than (resp. equal to) the sum of the other two.

The following holds.

Lemma 2.4 [4, Lemma 3.8 (2)] Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a P -polynomial association scheme with respect to the ordering $A_0, A_1, \dots, A_D$ of the associate matrices. Then there exists a polynomial $\varphi_i \in \mathbf{R}[\lambda]$ ($0 \leq i \leq D$) such that

$$\deg \varphi_i = i \quad (0 \leq i \leq D),$$

$$A_i = \varphi_i(A_1) \quad (0 \leq i \leq D).$$

□

By the previous lemma, we easily see the following.

Lemma 2.5 Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a P -polynomial association scheme with respect to the ordering $A_0, A_1, \dots, A_D$ of the associate matrices and M the Bose-Mesner algebra. Let W be a subspace of the standard module over $\mathbf{C}$.

Then W is M -invariant by left multiplication if and only if W is A_1 -invariant. □

In the following, we will mainly consider the following modules.

Definition 2.6 Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be an association scheme with D -classes. Pick any $x \in X$ and write $E_i^* = E_i^*(x)$, $T = T(x)$. Let W denote an irreducible T -module. Then W is said to be thin whenever

$$\dim E_i^* W \leq 1$$

for all i with $0 \leq i \leq D$.

Y is said to be thin with respect to x if each irreducible $T(x)$ -module is thin. Y is said to be thin if Y is thin with respect to each vertex $x \in X$.

We can firstly find one thin irreducible module of each subconstituent algebra.

Lemma 2.7 [4, Lemma 3.6] *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a symmetric association scheme with D -classes. Pick any $x \in X$ and write $E_i^* = E_i^*(x)$ ($0 \leq i \leq D$), $T = T(x)$. Let δ denote the all 1's vector in the standard module.*

Then the vector space over $\mathbb{C}$ which is spanned by $\{E_i^ \delta\}_{0 \leq i \leq D}$ is a thin irreducible T -module of dimension $D+1$. $\square$*

3 Proof of Theorem 1.1

In the following let $Y = (X, \{R_0, R_1, R_2\})$ be a symmetric association scheme with 2 classes. Pick any $x \in X$ and let $T = T(x)$ be the subconstituent algebra of Y .

Let the associate matrix A_1 of Y be decomposed according to the partition of $X = \{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)$ into

$$A_1 = \begin{bmatrix} 0 & \mathbf{1}' & \mathbf{0}' \\ \mathbf{1} & B_1 & N \\ 0 & N' & B_2 \end{bmatrix},$$

where $\mathbf{1}$ is an all 1's column vector of size k_1 , $\mathbf{0}$ is a zero column vector of size k_2 , N is a $k_1 \times k_2$ matrix, and where B_i is a $k_i \times k_i$ matrix for $i=1, 2$. Note that B_1 and B_2 are the adjacency matrices of the subconstituents of Γ with respect to x .

Similarly we decompose any vector u in the standard module V into three parts.

Let us call an eigenvector of B_i ($i=1, 2$) restricted whenever it is orthogonal to the all 1's vector of size k_i .

Proposition 3.1 [3, Theorem 5.1] *Let Γ be a strongly regular graph with eigenvalues k, r, s .*

The following hold.

(1) *Suppose that u is a restricted eigenvector of B_1 with an eigenvalue θ . Then u is the eigenvector of NN' with the eigenvalue $(r-\theta)(\theta-s)$, and $N'u$ is zero vector or the restricted eigenvector of B_2 with the eigenvalue $r+s-\theta$.*

In particular, $N'u$ is zero vector if and only if $\theta \in \{r, s\}$.

(2) *Suppose that v is a restricted eigenvector of B_2 with an eigenvalue θ' . Then v is the eigenvector of $N'N$ with the eigenvalue $(r-\theta')(\theta'-s)$, and Nv is zero vector or the restricted eigenvector of B_1 with the eigenvalue $r+s-\theta'$.*

In particular, Nv is zero vector if and only if $\theta' \in \{r, s\}$. $\square$

Lemma 3.2 *Pick any $x \in X$ and write $T = T(x)$.*

Then the following hold.

- (1) *Let u be a restricted eigenvector of B_1 with an eigenvalue θ .*

Then the vector space W over $\mathbf{C}$ which is spanned by $(0, u^t, \mathbf{0}^t)^t$ and $(0, \mathbf{0}^t, (N^t u)^t)^t$ is a thin irreducible T -module over $\mathbf{C}$.

In particular,

$$\dim W = \begin{cases} 1 & \text{if } \theta \in \{r, s\}, \\ 2 & \text{if } \theta \notin \{r, s\}. \end{cases}$$

- (2) *Let v be a restricted eigenvector of B_2 with an eigenvalue θ' .*

Then the vector space W' over $\mathbf{C}$ which is spanned by $(0, \mathbf{0}^t, v^t)^t$ and $(0, (Nv)^t, \mathbf{0}^t)^t$ is a thin irreducible T -module over $\mathbf{C}$.

In particular,

$$\dim W' = \begin{cases} 1 & \text{if } \theta' \in \{r, s\}, \\ 2 & \text{if } \theta' \notin \{r, s\}. \end{cases}$$

PROOF: (1) It follows from Proposition 3.1 that

$$\begin{aligned} (0, (B_1 u)^t, \mathbf{0}^t)^t, (0, (NN^t u)^t, \mathbf{0}^t)^t &\in \text{Span}\{(0, u^t, \mathbf{0}^t)^t\}, \\ (\mathbf{0}, \mathbf{0}^t, (B_2 N^t u)^t)^t &\in \text{Span}\{(0, \mathbf{0}^t, (N^t u)^t)^t\}. \end{aligned}$$

Hence

$$\begin{aligned} &A_1(0, u^t, \mathbf{0}^t)^t \\ &= (0, (B_1 u)^t, \mathbf{0}^t)^t + (0, \mathbf{0}^t, (N^t u)^t)^t \\ &\in W, \end{aligned}$$

$$\begin{aligned} &A_1(0, \mathbf{0}^t, (N^t u)^t)^t \\ &= (0, (NN^t u)^t, \mathbf{0}^t)^t + (0, \mathbf{0}^t, (B^2 N^t u)^t)^t \\ &\in W. \end{aligned}$$

Therefore W is A_1 -invariant, i.e., M -invariant by Lemma 2.5. Since it is clear that W is M^* -invariant, we have the claim of (1).

- (2) Similar to (1). □

Since B_1 is a real symmetric matrix, there exist eigenvectors

$$u_1, u_2, \dots, u_f, u_{f+1}, \dots, u_{k_1}$$

such that $\{u_i\}_{0 \leq i \leq k_1}$ span $E_i^* V \simeq \mathbf{C}^{k_1}$, u_1 is an all 1's vector, $\langle u_i, u_j \rangle = 0$ for $i \neq j$. We may assume that $\theta_i \notin \{r, s\}$ for $2 \leq i \leq f$ and that $\theta_i \in \{r, s\}$ for $f+1 \leq i \leq k$, where θ_i is the eigenvalue with respect to u_i .

It follows from Lemma 3.2 that, if we write $v_i = N^t u_i$ for $1 \leq i \leq f$, then $v_1, \dots, v_f$ are eigenvectors of B_2 . Let W_i denote the linear span of $(\langle u_i, u_1 \rangle, \mathbf{0}^t, \mathbf{0}^t)^t$, $(0, u_i^t, \mathbf{0}^t)^t$, $(0, \mathbf{0}^t, (N^t u_i)^t)^t$ over $\mathbf{C}$. Then we see that W_i is a thin irreducible T -module and that $W_i \cap W_j = \{0\}$ for $i \neq j$. In particular,

$$\dim W_i = \begin{cases} 3 & \text{if } i=1, \\ 2 & \text{if } 2 \leq i \leq f, \\ 1 & \text{if } f+1 \leq i \leq k_1. \end{cases}$$

Now we can put the rest of eigenvectors of B_2

$$v_{f+1}, v_{f+2}, \dots, v_{k_2}$$

such that $v_1, \dots, v_{k_2}$ span $E_2^* V \simeq \mathbf{C}^{k_2}$ and that $\langle v_i, v_j \rangle = 0$ for $1 \leq i \neq j \leq k_2$. Note that, for $f+1 \leq i \leq k_2$, the eigenvalue with respect to v_i is contained in $\{r, s\}$ and that, if we let W'_i be the linear span of $(0, \mathbf{0}^t, v_i^t)^t$ over $\mathbf{C}$, W'_i is a thin irreducible T -module of dimension 1.

Thus we have the following.

Lemma 3.3 *Let V , $\{W_i\}_{1 \leq i \leq k_1}$, $\{W'_j\}_{f+1 \leq j \leq k_2}$ be as above. Then*

$$V = (\oplus_{1 \leq i \leq k_1} W_i) \oplus (\oplus_{f+1 \leq j \leq k_2} W'_j) \quad (\text{orthogonal direct sum}).$$

In particular, $T = T(x)$ is thin for any $x \in X$, i.e., Y is thin. $\square$

Now we need the following lemma in order to calculate the dimension of T .

Lemma 3.4 *Let $\{\theta_i\}_{1 \leq i \leq k_1}$, $\{\theta'_j\}_{f+1 \leq j \leq k_2}$, $\{W_i\}_{1 \leq i \leq k_1}$ and $\{W'_j\}_{f+1 \leq j \leq k_2}$ be as above.*

Then the following hold.

- (1) *For all i with $2 \leq i \leq k_1$, W_1 and W_i are not T -isomorphic.*
- (2) *For all i and j with $1 \leq i \leq k_1$ and $f+1 \leq j \leq k_2$, W_i and W'_j are not T -isomorphic.*
- (3) *For i and j with $2 \leq i, j \leq k_1$, W_i and W_j are T -isomorphic if and only if $\theta_i = \theta_j$.*
- (4) *For i and j with $f+1 \leq i, j \leq k_2$, W'_i and W'_j are T -isomorphic if and only if $\theta'_i = \theta'_j$.*

PROOF: (1) Immediate.

(2) It is clear for $1 \leq i \leq f$.

Now we suppose that W_i and W_j' are T -isomorphic for some i and j with $f+1 \leq i \leq k_1$ and $f+1 \leq j \leq k_2$. Then there exists a vector space isomorphism $\sigma: W_i \rightarrow W_j'$ such that

$$(\sigma a - a\sigma)W_i = 0 \quad \text{for all } a \in T.$$

However, as $E_1^* W_i = W_i$ and $E_1^* W_j' = 0$,

$$\begin{aligned} & (\sigma E_1^* - E_1^* \sigma)W_i \\ &= (\sigma - E_1^* \sigma)W_i \\ &= W_j', \end{aligned}$$

which is a contradiction. Therefore W_i and W_j' are not T -isomorphic.

(3) It is clear for $2 \leq i \leq f$ and $f+1 \leq j \leq k_1$.

Let $2 \leq i, j \leq f$. Then, by Proposition 3.1, we see that

$$[A_1]_{w_i} = \begin{bmatrix} \theta_i & (r - \theta_i)(\theta_i - s) \\ 1 & s + r - \theta_i \end{bmatrix},$$

where $[A_1]_{w_i}$ denotes the matrix representing A_1 with respect to the basis

$$\{(0, u_i^t, \mathbf{0}^t)^t, (0, \mathbf{0}^t, (N^t u_i)^t)^t\}.$$

Hence, if $\theta_i = \theta_j$, then $[A_1]_{w_i} = [A_1]_{w_j}$. Now we assume that $\sigma: W_i \rightarrow W_j$ is a vector space isomorphism such that

$$\begin{aligned} \sigma(0, u_i^t, \mathbf{0}^t)^t &= (0, u_j^t, \mathbf{0}^t)^t, \\ \sigma(0, \mathbf{0}^t, (N^t u_i)^t)^t &= (0, \mathbf{0}^t, (N^t u_j)^t)^t. \end{aligned}$$

Then, as $[A_1]_{w_i} = [A_1]_{w_j}$, we easily have

$$(\sigma A_1 - A_1 \sigma)W_i = 0.$$

By Lemma 2.4,

$$(\sigma a - a\sigma)W_i = 0 \quad \text{for all } a \in M.$$

It is clear that

$$(\sigma a - a\sigma)W_i = 0 \quad \text{for all } a \in M^*.$$

Therefore we have that W_i and W_j are T -isomorphic if $2 \leq i, j \leq f$ and $\theta_i = \theta_j$.

Similarly, if $f+1 \leq i, j \leq k_1$ and $\theta_i = \theta_j$, then there exists a vector space isomorphism $\sigma: W_i \rightarrow W_j$ such that

$$\begin{aligned}\sigma(0, u_i^t, \mathbf{0}^t) &= (0, u_j^t, \mathbf{0}^t)^t, \\ (\sigma a - a\sigma)W_i &= 0 \quad \text{for all } a \in T.\end{aligned}$$

Thus W_i and W_j are T -isomorphic if $2 \leq i, j \leq k_1$ and $\theta_i = \theta_j$.

Suppose that $2 \leq i, j \leq k_1$, $\theta_i \neq \theta_j$ and that W_i and W_j are T -isomorphic. Let $\sigma: W_i \rightarrow W_j$ be a T -isomorphism. Note that $\sigma(0, u_i^t, \mathbf{0}^t)^t \neq 0$ and that, by Proposition 3.1, $(0, u_i^t, \mathbf{0}^t)^t \in E_1^* W_i$ and $E_1^* \sigma(0, u_i^t, \mathbf{0}^t)^t \in E_1^* W_i$ are the eigenvectors of $E_1^* A_1 E_1^*$ with eigenvalues θ_i and θ_j , respectively. However,

$$\begin{aligned}(\sigma E_1^* A_1 E_1^* - E_1^* A_1 E_1^* \sigma)(0, u_i^t, \mathbf{0}^t)^t \\ = (\theta_i - \theta_j) \sigma(0, u_i^t, \mathbf{0}^t)^t \\ \neq 0,\end{aligned}$$

which contradicts that σ is a T -isomorphism. Therefore, if $2 \leq i \leq k_1$ and $\theta_i \neq \theta_j$, then W_i and W_j are not T -isomorphic.

(4) Similar to (3). □

By Proposition 2.2, Lemma 3.3 and the previous lemma, we conclude the proof of Theorem 1.1.

4 Examples

In this section we determine the dimension of Γ for the Johnson graph $J(8, 2)$ and three Chang graphs. They have the same parameters $(v, k_1, \lambda, \mu) = (28, 12, 6, 4)$, and the same eigenvalues $k_1 = 12$, $r = 4$, $s = -2$. (See [2].)

We denote by $\text{Aut } \Gamma$ the automorphism group of a graph Γ . Take x and y in X . Suppose that there is some $\pi \in \text{Aut } \Gamma$ such that $y = \pi(x)$. Then we easily see that the subconstituents of Γ with respect to x is isomorphic to those with respect to y . Therefore in this case we have $\dim T(x) = \dim T(y)$.

Let $\Omega = \{1, 2, \dots, 8\}$ and $X = \binom{\Omega}{2}$, i.e., the collection of all 2-subsets of Ω .

Example 4.1 Let Γ be a Johnson graph $J(8, 2)$. $\text{Aut } \Gamma (= S_8)$ acts on X transitively.

Take any $x \in X$. Then the spectrums of B_1 and B_2 are

$$\begin{aligned} (6^1) \quad & 0^5 \quad 4^1 \quad -2^5, \\ (8^1) \quad & 2^5 \quad \quad \quad -2^9, \end{aligned}$$

where $(\quad)$ is associated with all 1's eigenvector. Note that both subconstituents are connected. Now we have $(m_1, m_2, m'_1, m'_2) = (2, 1, 1, 1)$ and

$$\dim T(x) = 3 \cdot 1 + 1 \cdot 4 + 9 = 16.$$

Example 4.2 The Chang graphs are constructed from $J(8, 2)$ by switching with respect to one of the sets. (See [2].)

- (1) $\{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\},$
- (2) $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{1, 8\}\},$
- (3) $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{4, 8\}\}.$

By using a computer, we have the following.

Case (1). In this case $\text{Aut } \Gamma$ is of order 384 and the orbits of $\text{Aut } \Gamma$ on X are U_1, U_2 , where

$$U_1 = \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}, \quad U_2 = X \setminus U_1.$$

So we have only to check the case $x \in \{\{1, 5\}, \{1, 2\}\}.$

If $x = \{1, 5\},$

$$\begin{aligned} B_1: \quad & (6^1) \quad 2^3 \quad 0^2 \quad -2^6 \\ B_2: \quad & (8^1) \quad 0^3 \quad 2^2 \quad \quad 4^1 \quad -2^8, \end{aligned}$$

so that $(m_1, m_2, m'_1, m'_2) = (1, 2, 2, 2).$

If $x = \{1, 2\},$

$$\begin{aligned} B_1: \quad & (6^1) \quad 1+\sqrt{5}^1 \quad 2^1 \quad 0^3 \quad 1-\sqrt{5}^1 \quad -2^6 \\ B_2: \quad & (8^1) \quad 1-\sqrt{5}^1 \quad 0^1 \quad 2^3 \quad 1+\sqrt{5}^1 \quad \quad -2^8, \end{aligned}$$

so that $(m_1, m_2, m'_1, m'_2) = (1, 4, 1, 4).$

Hence we have

$$\begin{aligned}\dim T(x) &= 20 & \text{if } x \in U_1, \\ \dim T(x) &= 27 & \text{if } x \in U_2.\end{aligned}$$

Case (2). In this case $\text{Aut } \Gamma$ is of order 96 and the orbits of $\text{Aut } \Gamma$ are U_1, U_2 , where

$$U_1 = \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}, \quad U_2 = X \setminus U_1.$$

So we check $x \in \{\{1, 5\}, \{1, 2\}\}$.

If $x = \{1, 5\}$,

$$\begin{aligned}B_1: & (6^1) \quad 1 + \sqrt{3}^2 \quad 0^2 \quad 1 - \sqrt{3}^2 \quad -2^6 \\ B_2: & (8^1) \quad 1 - \sqrt{3}^2 \quad 2^2 \quad 1 + \sqrt{3}^2 \quad -2^8,\end{aligned}$$

so that $(m_1, m_2, m'_1, m'_2) = (1, 3, 1, 3)$.

If $x = \{1, 2\}$,

$$\begin{aligned}B_1: & (6^1) \quad 1 + \sqrt{3}^1 \quad 2^1 \quad \sqrt{2}^1 \quad 0^1 \quad 1 - \sqrt{3}^1 \quad -\sqrt{2}^1 \quad -2^5 \\ B_2: & (8^1) \quad 1 - \sqrt{3}^1 \quad 0^1 \quad 2 - \sqrt{2}^1 \quad 2^1 \quad 1 + \sqrt{3}^1 \quad 2 + \sqrt{2}^1 \quad -2^8,\end{aligned}$$

so that $(m_1, m_2, m'_1, m'_2) = (1, 6, 1, 6)$.

Hence we have

$$\begin{aligned}\dim T(x) &= 23 & \text{if } x \in U_1, \\ \dim T(x) &= 35 & \text{if } x \in U_2.\end{aligned}$$

Case (3). In this case $\text{Aut } \Gamma$ is of order 360 and the orbits of $\text{Aut } \Gamma$ are U_1, U_2 , where

$$U_1 = \{\{i, j\} \mid 4 \leq i < j \leq 8\}, \quad U_2 = X \setminus U_1.$$

So we check $x \in \{\{4, 5\}, \{1, 2\}\}$.

If $x = \{4, 5\}$,

$$\begin{aligned}B_1: & (6^1) \quad \frac{1 + \sqrt{13}^2}{2} \quad 1^2 \quad \frac{1 - \sqrt{13}^2}{2} \quad -2^5 \\ B_2: & (8^1) \quad \frac{3 - \sqrt{13}^2}{2} \quad 1^2 \quad \frac{3 + \sqrt{13}^2}{2} \quad -2^8,\end{aligned}$$

so that $(m_1, m_2, m'_1, m'_2) = (1, 3, 1, 3)$.

If $x = \{1, 2\}$,

$$B_1: (6^1) \quad 3^1 \quad \frac{1+\sqrt{5}^2}{2} \quad \frac{1-\sqrt{5}^2}{2} \quad -1^1 \quad -2^5$$

$$B_2: (8^1) \quad -1^1 \quad \frac{3-\sqrt{5}^2}{2} \quad \frac{3+\sqrt{5}^2}{2} \quad 3^1 \quad -2^8,$$

so that $(m_1, m_2, m'_1, m'_2) = (1, 4, 1, 4)$.

Hence we have

$$\begin{aligned} \dim T(x) &= 23 \quad \text{if } x \in U_1, \\ \dim T(x) &= 27 \quad \text{if } x \in U_2. \end{aligned}$$

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