q -CONFORMALLY FLAT MANIFOLDS

Masatoshi KOJIMA

(Received May 6, 1994)

(Revised June 13, 1994)

Introduction

Let (M, g) be an n -dimensional Riemannian manifold with fundamental metric tensor g . Let R , C , C_0 and K denote the curvature tensor, Weyl conformal curvature tensor, Weyl 3-index curvature tensor and the sectional curvature, respectively. A well-known theorem says that (M, g) is conformally flat if and only if $C=0$ for $n \geq 4$ and $C_0=0$ for $n=3$. The tensor g , R and C are typical examples of curvature structures of order one and two.

Research on curvature structures of higher order, e.g. the q -th Gauss-Kronecker curvature tensor R^q , have been made by many scholars [4], [5], [6], [7], [8], [9], [11], [12], [13] and [15].

For a double form ω , R.S. Kulkarni [9] and T. Nasu [11] have introduced important double forms $\mathbf{con} \omega$ and $\mathbf{con}_0 \omega$ such as $\mathbf{con} R = C$ and $(n-2)\mathbf{con}_0 R = C_0$. They also proved that $\mathbf{con} \omega$ and $\mathbf{con}_0 \omega$ have the same algebraic properties as the tensor C and C_0 , respectively.

In this paper, an n -dimensional Riemannian manifold is said to be q -conformally flat if $\mathbf{con} R^q = 0$ for $n \geq 4q$ and $\mathbf{con}_0 R^q = 0$ for $n = 4q - 1$.

A characterization of a conformally flat manifold is given by the following.

A manifold (M, g) is conformally flat if and only if for every quadruple of orthogonal vectors $\{e_1, e_2, e_3, e_4\}$ at each point of M

$$K_{12} + K_{34} = K_{13} + K_{24}$$

where K_{ij} denotes the sectional curvature with respect to the plane spanned by $\{e_i, e_j\}$ (see Theorem 4.1 (Kulkarni [8])).

Let $\{e_1, \dots, e_n\}$ be locally defined orthonormal vector fields. Then we have the following (see Theorem 4.2 (Nasu & Kojima [13])) which is a generalized version of Theorem 4.1.

Let (M, g) be a Riemannian manifold of dimension $n \geq 4q$. (M, g) is q -conformally flat if and only if for all distinct indices $i_1, \dots, i_{2q}, k_1, \dots, k_{2q}$,

$$\sum_{\substack{\bar{t}=t, k_t \\ t=1, \dots, 2q}} (-1)^\tau R^q(e_1 \cdots e_{\bar{2}q} \otimes e_1 \cdots e_{\bar{2}q}) = 0$$

where τ is the number of index t such that $\bar{t}=k_t$.

Let M be immersed in an $(n+1)$ -dimensional Euclidean space and g be the induced metric and h be the second fundamental form of M .

We define the notion of *k-umbilical* as follows.

If a k -th exterior power $(h - \rho g)^k = 0$ for some function ρ , then M is said to be *k-umbilical* (umbilical for $k=1$) (see Definition 5.1).

The following are well known.

Let M be a hypersurface of an $(n+1)$ -dimensional Euclidean space and $n \geq 4$. Then M is conformally flat if and only if M is 2-umbilical (see Theorem 5.1 (Cartan [1], Schouten [14])).

Let M be a hypersurface of a 4-dimensional Euclidean space. If M is 2-umbilical then M is conformally flat. (see Theorem 5.2 (Chen & Yano [2])).

Applying Theorem 4.2 we obtain generalized theorems of the above theorems.

Let M be a hypersurface of an $(n+1)$ -dimensional Euclidean space and $n \geq 4q$. Then M is q -conformally flat if and only if M is $2q$ -umbilical (see Theorem 5.3 (cf. Theorem 1 in [6])).

Let M be a hypersurface of a $4q$ -dimensional Euclidean space. If M is $2q$ -umbilical then M is q -conformally flat (see Theorem 5.4).

Applying Theorem 5.3 and Theorem 5.4 we describe, for each $q \geq 2$, first examples of q -conformally flat Riemannian manifolds which are not $(q-1)$ -conformally flat. These are certain hyperquadrics of $(n+1)$ -dimensional Euclidean spaces for $n \geq 4q-1$ (see Theorem 6.1 and 6.2).

Now, we consider (M, g) to be a product Riemannian manifold of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with dimensions n_1 and n_2 , respectively. Then we have the following.

Let (M, g) be a product Riemannian manifold of constant curvature spaces M_1 and M_2 . If $\dim M_1 \geq 2q$ and $\dim M_2 \geq 2q$, then M is q -conformally flat if and only if M is conformally flat in the usual sense (see Theorem 7.1 ([12])).

We define the notion of *k-concircularly flat* as follows.

If a k -th exterior power $(R - \rho g^2)^k = 0$ for some function ρ , then M is said to be *k-concircularly flat* (see Definition 1.2).

As a result we have the following.

Let (M, g) be a product Riemannian manifold of an n_1 -dimensional space M_1

and an n_2 -dimensional constant curvature space M_2 . If $n_1 \geq 4q - 1$ and $n_2 \geq 2q$, then M is q -conformally flat if and only if M_1 is q -concentrically flat (see Theorem 7.2).

1. Preliminaries

In this section, let us recall the main facts on the calculus of double forms described by Gray [4], Kowalski [7], Kulkarni [9] and Nasu [11] for later use.

Let (M, g) be an n -dimensional C^∞ Riemannian manifold, $\mathcal{F}(M)$ the ring of C^∞ functions on M and $T(M)$ the Lie algebra of vector fields on M . Let Λ^{*p} be the bundle of p -forms on M and $\Lambda^* = \sum_{0 \leq p} \Lambda^{*p}$. We put.

$$\Xi^{p,q} = \Lambda^{*p} \otimes \Lambda^{*q}, \quad \Xi = \sum_{0 \leq p, q \leq n} \Xi^{p,q}$$

where the tensor products are taken over $\mathcal{F}(M)$. We call an element ω of $\Xi^{p,q}$ a *double form of type (p, q)* on M . It is an $\mathcal{F}(M)$ -multilinear map

$$\omega: T(M)^p \times T(M)^q \rightarrow \mathcal{F}(M)$$

which is skew-symmetric in the first p variables and also in the last q variables. We shall use the notation $\omega(u_1 \cdots u_p \otimes v_1 \cdots v_q)$ to denote the value of ω on the vector fields $u_1, \dots, u_p$ and $v_1, \dots, v_q$. For convenience, we identify Λ^{*p} with $\Xi^{p,0}$ unless stated otherwise. Furthermore, let Υ^p denote symmetric elements of $\Xi^{p,p}$. We shall call an element of Υ^p a *curvature structure of order p* . Set $\Upsilon = \sum_{0 \leq p} \Upsilon^p$.

The metric tensor field $g \in \Upsilon^1$. The Riemannian curvature tensor $R \in \Upsilon^2$ which is defined by, for any vector fields $u, v, w, y \in T(M)$,

$$R(uv \otimes wy) = g(\nabla_{[u,v]} w - \nabla_u \nabla_v w + \nabla_v \nabla_u w, y)$$

where ∇ is the Riemannian connection by g .

As de Rham has noted, it is possible to define the exterior product $\omega \wedge \theta$ of two double forms $\omega \in \Xi^{p,q}$ and $\theta \in \Xi^{r,s}$ by the formula

$$(\omega \wedge \theta)(u_1 \cdots u_{p+r} \otimes v_1 \cdots v_{q+s}) = \sum_{\rho \in Sh(p,r)} \sum_{\sigma \in Sh(q,s)} \varepsilon_\rho \varepsilon_\sigma \omega(u_{\rho(1)} \cdots u_{\rho(p)} \otimes v_{\sigma(1)} \cdots v_{\sigma(q)}) \cdot \theta(u_{\rho(p+1)} \cdots u_{\rho(p+r)} \otimes v_{\sigma(q+1)} \cdots v_{\sigma(q+s)})$$

for $u_1, \dots, u_{p+r}, v_1, \dots, v_{q+s} \in T(M)$. Here, ε_ρ denotes the sign of ρ and $Sh(p, r)$ denotes the set of all (p, r) -shuffles;

$$Sh(p, r) = \{\rho \in S_{p+r} : \rho(1) < \cdots < \rho(p) \text{ and } \rho(p+1) < \cdots < \rho(p+r)\}$$

where S_{p+r} is the symmetric group of degree $p+r$. It is not difficult to show that the multiplication $\wedge$ is associative and that we have

$$(1.1) \quad \omega \wedge \theta = (-1)^{pr+qs} \theta \wedge \omega$$

for $\omega \in \mathcal{E}^{p,q}$ and $\theta \in \mathcal{E}^{r,s}$. Λ^* , $\mathcal{E}$ and $\mathcal{I}$ are graded associative rings. In particular $\mathcal{I}$ is a commutative ring.

Let ω^k denote the k -th exterior power of $\omega \in \mathcal{I}^p$. Then we see

$$g^k(u_1 \cdots u_k \otimes v_1 \cdots v_k) = k! \cdot \det(g(u_i \otimes v_j)).$$

Let G_p denote the Grassmann bundle of p -planes on M and $\omega \in \mathcal{I}^p$. We define the corresponding *curvature function* $K_\omega : G_p \rightarrow \mathfrak{R}$ as follows. For $\phi \in G_p$, $K_\omega(\phi) = \omega(\phi, \phi) / \|\phi\|$ where $\|\cdot\|$ denotes the norm of a p -vector induced by g .

Let us remind some further notations and results of the theory of the ring $\mathcal{E}$.

The *first Bianchi sum* $\mathcal{S}$ is a map of $\mathcal{E}^{p,q}$ into $\mathcal{E}^{p+1,q-1}$ defined as follows.

For $\omega \in \mathcal{E}^{p,q}$, if $q=0$, we put $\mathcal{S}\omega=0$ and $q \geq 1$, we put

$$\mathcal{S}\omega(x_1 \cdots x_{p+1} \otimes y_1 \cdots y_{q-1}) = \sum_{j=1}^{p+1} (-1)^j \omega(x_1 \cdots \hat{x}_j \cdots x_{p+1} \otimes x_j y_1 \cdots y_{q-1})$$

where $x_1, \dots, x_{p+1}, y_1, \dots, y_{q-1} \in T(M)$ and the symbol $\hat{}$ denotes omission.

The *second Bianchi sum* $\mathcal{D}$ is a map of $\mathcal{E}^{p,q}$ into $\mathcal{E}^{p+1,q}$ defined as follows.

For $\omega \in \mathcal{E}^{p,q}$, we put

$$\mathcal{D}\omega(x_1 \cdots x_{p+1} \otimes y_1 \cdots y_q) = \sum_{j=1}^{p+1} (-1)^j (\nabla_{x_j} \omega)(x_1 \cdots \hat{x}_j \cdots x_{p+1} \otimes y_1 \cdots y_q)$$

where ∇ denotes the covariants differentiation with respect to the metric g .

The *contraction* $\mathbf{c}$ is a map of $\mathcal{E}^{p,q}$ into $\mathcal{E}^{p-1,q-1}$ defined as follows. For $\omega \in \mathcal{E}^{p,q}$ with $p=0$ or $q=0$, we put $\mathbf{c}\omega=0$. If both p and $q \geq 1$, then we put

$$(\mathbf{c}\omega)(e_{i_1} \cdots e_{i_{p-1}} \otimes e_{j_1} \cdots e_{j_{q-1}}) = \sum_{\alpha=1}^n \omega(e_\alpha e_{i_1} \cdots e_{i_{p-1}} \otimes e_\alpha e_{j_1} \cdots e_{j_{q-1}})$$

where $\{e_1, \dots, e_n\}$ are locally defined orthonormal vector fields. We shall say ω to be *effective* if $\mathbf{c}\omega=0$. Let $\mathcal{E}^{p,q}$ denote the set of effective element of $\mathcal{E}^{p,q}$.

Definition 1.1. $\mathcal{R}_1^p = \mathcal{R}^p \cap \ker \mathcal{S}$. $\mathcal{R}_2^p = \mathcal{R}^p \cap \ker \mathcal{D}$. $\mathcal{R}_0^p = \mathcal{R}_1^p \cap \mathcal{R}_2^p$.

Definition 1.2. We call the tensor $(R - \rho g^2)^q$, q -th concircular curvature tensor. And if $(R - \rho g^2)^q = 0$ for some function ρ , then M is said to be q -concircularly flat. 1-concircularly flat is concircularly flat which is constant curvature.

2. Elementary properties

Let $\{e_1, \dots, e_n\}$ be locally defined orthonormal vector fields. In this section, we denote the local components of the tensor $\omega \in \Xi^{p,q}$ by

$$\omega_{i_1 \dots i_p j_1 \dots j_q} = \omega(e_{i_1} \cdots e_{i_p} \otimes e_{j_1} \cdots e_{j_q}).$$

Concerning operators in section 1, the following propositions are well-known ([4], [9], [11]).

Proposition 2.1. Let $\omega \in \Xi^{p,q}$ and $\theta \in \Xi^{r,s}$. Then we have

- (a) $\mathcal{S} \cdot \mathbf{c} = \mathbf{c} \cdot \mathcal{S}$ on Ξ ,
- (b) $\mathcal{S}(\omega \wedge \theta) = (\mathcal{S}\omega) \wedge \theta + (-1)^{p+q} \omega \wedge (\mathcal{S}\theta)$,
- (c) $\mathcal{D}(\omega \wedge \theta) = (\mathcal{D}\omega) \wedge \theta + (-1)^p \omega \wedge (\mathcal{D}\theta)$.

Proposition 2.2. $\ker \mathcal{S}$ and $\ker \mathcal{D}$ are closed under multiplication in Ξ .

Proposition 2.3. Let $\omega, \theta \in \mathcal{R}_1^p$ such that $K_\omega = K_\theta$. Then $\omega = \theta$.

Proposition 2.4. For any $\omega \in \Xi^{p,q}$, $\mathbf{c}(g \wedge \omega) = g \wedge \mathbf{c}\omega + (n - p - q)\omega$.

Corollary 2.5. Let $\omega \in \Xi^{p,q}$. For any positive integer r ,

$$\mathbf{c}^r(g \wedge \omega) = g \wedge \mathbf{c}^r \omega + r(n - p - q + r - 1)\mathbf{c}^{r-1} \omega.$$

Proposition 2.6. Multiplication with g is injective on $\sum_{p+q < n} \Xi^{p,q}$.

Remark. When $n = p + q$, we have $\mathbf{c}(g \wedge \omega) = g \wedge \mathbf{c}\omega$. Hence multiplication with g is not injective.

By Proposition 2.4, the next proposition follows by induction.

Proposition 2.7. For any $\omega \in \Xi^{p,q}$ and any positive integer r we have

$$\mathbf{c}(g^r \wedge \omega) = g^r \wedge \mathbf{c}\omega + r(n - p - q - r + 1)g^{r-1} \wedge \omega.$$

It is well known that the binomial coefficients are defined by, for any number x and for any positive integer r ,

$$\binom{x}{0} = 1, \quad \binom{x}{r} = \frac{1}{r!} \left(\prod_{\alpha=1}^r (x-r+\alpha) \right).$$

By Proposition 2.7, the next proposition follows by induction.

Proposition 2.8. *Let $r \leq n-p$ and k be a non-negative integer. For any $\omega \in Y^p$, we have*

$$\mathbf{c}^k(g^r \wedge \omega) = \sum_{\substack{i=0 \\ i \leq r}}^k \binom{k}{i} \binom{r}{i} \binom{n-2p-r+k}{i} (i!)^2 g^{r-i} \wedge \mathbf{c}^{k-i} \omega.$$

Proposition 2.9. *For any $\omega \in \Xi^{p,q}$ we have*

$$\mathbf{c}(\mathcal{D} \cdot \mathbf{c} + \mathbf{c} \cdot \mathcal{D}) + (\mathcal{D} \cdot \mathbf{c} + \mathbf{c} \cdot \mathcal{D}) \mathbf{c} = 0.$$

Proof. For any $\omega \in \Xi^{p,q}$ and for any indices $i_1, \dots, i_p, j_1, \dots, j_{q-1}$ we have

$$((\mathcal{D} \cdot \mathbf{c} + \mathbf{c} \cdot \mathcal{D}) \omega)_{i_1 \dots i_p j_1 \dots j_{q-1}} = - \sum_{\alpha=1}^n (\nabla_{\alpha} \omega)_{i_1 \dots i_p, \alpha j_1 \dots j_{q-1}}.$$

Hence, the proposition follows immediately.

Proposition 2.10. *We have the following relation*

$$(2.1) \quad \mathbf{c} \cdot \mathcal{D} \cdot \mathbf{c}^r = -\frac{1}{r+1} \{r \mathcal{D} \cdot \mathbf{c}^{r+1} + (-1)^{r+1} \mathbf{c}^{r+1} \cdot \mathcal{D}\}.$$

Proof. If $r=1$, (2.1) is trivial by Proposition 2.9. Suppose that we get (2.1) for $r=1, \dots, t$. Then we have

$$(2.2) \quad \begin{aligned} \mathbf{c} \cdot \mathcal{D} \cdot \mathbf{c}^{t+1} &= (\mathbf{c} \cdot \mathcal{D} \cdot \mathbf{c}^t) \mathbf{c} = -\frac{1}{t+1} \{t \mathcal{D} \cdot \mathbf{c}^{t+2} + (-1)^{t+1} \mathbf{c}^t \cdot \mathcal{D} \mathbf{c}\} \\ &= -\frac{1}{t+1} \left\{ t \mathcal{D} \cdot \mathbf{c}^{t+2} + \frac{(-1)^t}{2} (\mathbf{c}^t \cdot \mathcal{D} \mathbf{c}^2 + \mathbf{c}^{t+2} \cdot \mathcal{D}) \right\}. \end{aligned}$$

On the other hand, by (2.1) for $r=t-1$, we get

$$\mathbf{c} \cdot \mathcal{D} \cdot \mathbf{c}^{t+1} = \mathbf{c} \cdot \mathcal{D} \cdot \mathbf{c}^{t-1} \mathbf{c}^2 = -\frac{1}{t} \{ (t-1) \mathcal{D} \cdot \mathbf{c}^{t+2} + (-1)^t \mathbf{c}^t \cdot \mathcal{D} \mathbf{c}^2 \}.$$

Substituting this into (2.2) we find that (2.1) is true when $t+1$. By induction we obtain the proposition.

3. q -th conformal curvature tensor

Weyl conformal curvature tensor C is a double form of type (2,2) given by

$$C = R - \frac{1}{n-2} g \wedge \mathbf{c}R + \frac{1}{2(n-2)(n-1)} g^2 \cdot \mathbf{c}^2 R.$$

C is an effective and vanishes identically if $n=3$. Weyl 3-index tensor C_0 is defined by the formula

$$C_0 = \mathcal{D}(\mathbf{c}R - \frac{1}{2(n-1)} g \cdot \mathbf{c}^2 R).$$

C is a conformal invariant for $n \geq 3$ and C_0 is a conformal invariant for $n=3$.

As a generalization of Weyl conformal curvature tensor C , we define the following ([9]).

Definition 3.1. Let $p+q+1 \leq n$ and $\mu = \min(p, q)$. Then, the conformal map $\mathbf{con}$ is by definition a map of $\Xi^{p+1, q+1}$ into itself such that

$$(3.1) \quad \mathbf{con} \omega = \omega + \sum_{r=1}^{\mu+1} \frac{(-1)^r g^r \wedge \mathbf{c}^r \omega}{r! \Pi_{j=0}^{r-1} (n-p-q+j)}$$

for any double form $\omega \in \Xi^{p+1, q+1}$.

Especially, $\mathbf{con} R^q$ is called the q -th conformal curvature tensor. $\mathbf{con} R$ coincides with C . The following proposition due to Kulkarni [9] is important.

Proposition 3.1. Let $p+q+1 \leq n$ ($= \dim M$) and $\omega \in \Xi^{p+1, q+1}$. Then

- (a) The map $\mathbf{con}$ is a projection $\Xi^{p+1, q+1}$ onto $\mathcal{E}^{p+1, q+1}$.
- (b) $\mathbf{con} \omega = 0$ if and only if $\omega = g \wedge \phi$ for a unique $\phi \in \Xi^{p, q}$.
- (c) If $n = p+q+1$ then $\mathbf{con} \omega = 0$.
- (d) There are unique elements $\phi \in \Xi^{p, q}$ such that

$$(3.2) \quad \omega = \mathbf{con} \omega + g \wedge \phi.$$

Following Kowalski [7], we define the deviation map.

Definition 3.2. We call the correspondence $\omega \rightarrow \phi$ given by the formula (3.2), the deviation map and we write $\phi = \text{dev } \omega$.

From (3.1), we can get an explicit expression for the map $\text{dev}: \Xi^{p+1, q+1} \rightarrow \Xi^{p, q}$ such that for $\mu = \min(p, q)$,

$$\text{dev } \omega = \sum_{r=1}^{\mu+1} \frac{(-1)^{r-1} g^{r-1} \wedge \mathbf{c}^r \omega}{r! \Pi_{j=0}^{r-1} (n-p-q+j)}.$$

As a generalization of Weyl 3-index tensor C_0 , we define the following ([11]).

Definition 3.3. The map $\mathbf{con}_0$ is defined by $\mathbf{con}_0 = \mathcal{D} \cdot \text{dev}$.

Especially, $(n-2)\mathbf{con}_0 R$ coincides with Weyl 3-index tensor C_0 .

$\mathbf{con} R^q$ is a conformal invariant (by Kurkarni [9]) and $\mathbf{con}_0 R^q$ is a conformal invariant for $n=4q-1$ (by Nasu [11]).

Now, we have the following propositions (by Nasu [11]).

Proposition 3.2. Let $p+q+1 \leq n (= \dim M)$ and $\omega \in \Xi^{p+1, q+1}$. Then we have $(\mathcal{D} \cdot \mathbf{c} + \mathbf{c} \cdot \mathcal{D}) \mathbf{con} \omega = (n-p-q-1)(\mathbf{con}_0 \omega + \text{dev} \cdot \mathcal{D} \omega)$.

Remark. Suppose that $n \geq 2q+2$, $\omega \in \Gamma_2^{q+1}$ and $\mathbf{con} \omega = 0$ then $\mathbf{con}_0 \omega = 0$. Moreover, we have the following proposition.

Proposition 3.3. Let $p+q+1 \leq n (= \dim M)$, $\mu = \min(p, q)$ and $\omega \in \Xi^{p+1, q+1}$. We have

$$\mathbf{c} \cdot \mathbf{con}_0 \omega = \sum_{r=1}^{\mu+1} \frac{(-1)^r g^{r-1} \wedge \mathbf{c}^{r+1} \cdot \mathcal{D} \omega}{(r+1)! \Pi_{j=0}^{r-1} (n-p-q+j)}.$$

Proof. Since $\mathcal{D}g=0$ and $\mathbf{c}^{\mu+2} \omega = 0$, by making use of (c) in Proposition 2.1, Proposition 2.7 and Proposition 2.10, we have

$$\begin{aligned} \mathbf{c} \cdot \mathbf{con}_0 \omega &= \mathbf{c} \cdot \mathcal{D} \cdot \text{dev } \omega = \mathbf{c} \left(\sum_{r=1}^{\mu+1} \frac{g^{r-1} \wedge \mathcal{D} \cdot \mathbf{c}^r \omega}{r! \Pi_{j=0}^{r-1} (n-p-q+j)} \right) \\ &= - \sum_{r=1}^{\mu+1} \frac{g^{r-1} \wedge (r \mathcal{D} \cdot \mathbf{c}^{r+1} \omega + (-1)^{r+1} \mathbf{c}^{r+1} \cdot \mathcal{D} \omega)}{(r+1)! \Pi_{j=0}^{r-1} (n-p-q+j)} \\ &\quad + \sum_{r=2}^{\mu+1} \frac{(r-1)(n-p-q+r-1) g^{r-2} \wedge \mathcal{D} \cdot \mathbf{c}^r \omega}{r! \Pi_{j=0}^{r-1} (n-p-q+j)} \\ &= \sum_{r=1}^{\mu+1} \frac{(-1)^r g^{r-1} \wedge \mathbf{c}^{r+1} \cdot \mathcal{D} \omega}{(r+1)! \Pi_{j=0}^{r-1} (n-p-q+j)}. \end{aligned}$$

Hence we have the proposition.

Corollary 3.4. *Under the hypothesis of Proposition 3.3, if $\mathcal{D}\omega=0$ then we have $\mathbf{c} \cdot \mathbf{con}_0 \omega = 0$.*

Now we have the algebraic propositions with combinations which is useful to compute the q -th conformal curvature tensors.

Proposition 3.5. *Let integer $s \geq 1$ and r be non-negative integer. Then we have, for any integer t ,*

$$(3.3) \quad 1 + \sum_{k=1}^s (-1)^k \frac{\binom{s}{k} \binom{t+k}{k}}{\binom{r+k}{r}} = \prod_{\alpha=1}^s \frac{r-t-1+\alpha}{r+\alpha}.$$

Proof. Let r be a non-negative integer and t be an integer.

It is clear that the equation (3.3) is true for $s=1$. Now, suppose that the equation (3.3) is true for $s=1, 2, \dots, m-1$. When $s=m$, we have

$$\begin{aligned} \text{(the left-hand side of (3.3))} &= 1 + \sum_{k=1}^m (-1)^k \frac{\left(1 + \frac{m-k}{k}\right) \binom{m-1}{k-1} \binom{t+k}{k}}{\binom{r+k}{r}} \\ &= \frac{r-t}{r+1} + \sum_{k=1}^{m-1} (-1)^k \binom{m-1}{k} \left\{ -\frac{\binom{t+k+1}{k+1}}{\binom{r+k+1}{r}} + \frac{\binom{t+k}{k}}{\binom{r+k}{r}} \right\} \\ &= \frac{r-t}{r+1} \left\{ 1 + \sum_{k=1}^{m-1} (-1)^k \frac{\binom{m-1}{k} \binom{t+k}{k}}{\binom{r+1+k}{r+1}} \right\}. \end{aligned}$$

From the assumption, we find the equation (3.3) is true for $s=m$.

By induction we see that the equation (3.3) is true for any $s \geq 1$.

Proposition 3.6. *Let $r > k$ and k be non-negative integers. Then we have, for a positive integer a ,*

$$(3.4) \quad \sum_{i=0}^k \frac{(-1)^{r-i}(a+2i)}{i!(r-i)!\Pi_{j=i}^{r+i}(a+j)} = \frac{(-1)^{r-k}}{r \cdot k!(r-k-1)!\Pi_{j=k+1}^{k+r}(a+j)}.$$

Proof. If $k=0$ then for every positive integer r , the equation (3.4) is true.

Now, suppose that the equation (3.4) is true for $k=0, 1, \dots, m-1$ and for every integer $r > m-1$.

When $k=m$, we have for every integer $r > m$

$$\begin{aligned} \sum_{i=0}^m \frac{(-1)^{r-i}(a+2i)}{i!(r-i)!\Pi_{j=i}^{r+i}(a+j)} &= \frac{(-1)^{r-m+1}}{r \cdot (m-1)!(r-m)!\Pi_{j=m}^{m+r-1}(a+j)} \\ &\quad + \frac{(-1)^{r-m}(a+2m)}{m!(r-m)!\Pi_{j=m}^{m+r}(a+j)} \\ &= \frac{(-1)^{r-m}}{r \cdot m!(r-m-1)!\Pi_{j=m+1}^{m+r}(a+j)}. \end{aligned}$$

From the assumption, we find the equation (3.4) is true for $k=m$.

By induction we see that the equation (3.4) is true for any $k \geq 0$ and every non-negative integer $r \geq k$.

From the (d) of Proposition 3.1 we have a decomposition

$$\Xi^{p+1,q+1} = \mathcal{E}^{p+1,q+1} \oplus g \wedge \mathcal{E}^{p,q} \oplus g^2 \wedge \mathcal{E}^{p-1,q-1} \oplus \dots$$

where the sum is direct. We give an explicit expression of this decomposition.

Proposition 3.7. *Let $n \geq 4q-1$. We have*

$$R^q = \mathbf{con} R^q + \sum_{r=1}^{2q} \frac{g^r \wedge \mathbf{con}(\mathbf{c}^r R^q)}{r! \Pi_{\alpha=0}^{r-1}(n-4q+r+1+\alpha)}.$$

Proof. By using Proposition 3.6 we have the proposition.

4. q -Conformally flat Riemannian spaces

Let $\{e_1, \dots, e_n\}$ be locally defined orthonormal vector fields. In this section, we denote local components of the tensor $\omega \in \Xi^{p,q}$ by

$$\omega_{i_1 \dots i_p, j_1 \dots j_q} = \omega(e_{i_1} \cdots e_{i_p} \otimes e_{j_1} \cdots e_{j_q}).$$

A Riemannian manifold is *conformally flat* if and only if $\mathbf{con} R = 0$ for $n \geq 4$

and $\text{con}_0 R = 0$ for $n = 3$.

A characterization of a conformally flat Riemannian manifold is given by the following theorem.

Theorem 4.1 (Kulkarni [8]). *A manifold M is conformally flat if and only if for every quadruple of orthogonal vectors $\{e_1, e_2, e_3, e_4\}$ at each point of M*

$$K_{12} + K_{34} = K_{13} + K_{24}$$

where K_{ij} denotes the sectional curvature with respect to the plane spanned by $\{e_i, e_j\}$.

In this paper, we define q -conformally flat the following ([9], [11]).

Definition 4.1. *An n -dimensional Riemannian manifold is said to be q -conformally flat if $\text{con} R^q = 0$ for $n \geq 4q$ and $\text{con}_0 R^q = 0$ for $n = 4q - 1$.*

It is clear that q -conformally flat is $(q+1)$ -conformally flat, but $(q+1)$ -conformally flat is not q -conformally flat. One of main purpose of this paper is to obtain the examples of q -conformally flat for $n \geq 4q - 1$ which are not $(q-1)$ -conformally flat (in sections 5, 6).

Proposition 4.1. *Let $n \geq 2p$, $\theta \in Y^{p-1}$. For all distinct indices $i_1, \dots, i_p, k_1, \dots, k_p$, we have*

$$\sum_{\substack{\bar{t}=i_t, k_t \\ t=1, \dots, p}} (-1)^\tau ((g \wedge \theta)_{\bar{i} \dots \bar{p}, \bar{i} \dots \bar{p}}) = 0$$

where τ is the number of index t such that $\bar{t} = k_t$.

Proof.

$$\begin{aligned} \sum_{\substack{\bar{t}=i_t, k_t \\ t=1, \dots, p}} (-1)^\tau ((g \wedge \theta)_{\bar{i} \dots \bar{p}, \bar{i} \dots \bar{p}}) &= \sum_{\substack{\bar{t}=i_t, k_t \\ t=1, \dots, p}} (-1)^\tau \sum_{j=1}^p \theta_{\bar{i} \dots \bar{j} \dots \bar{p}, \bar{i} \dots \bar{j} \dots \bar{p}} \\ &= \sum_{j=1}^p \sum_{\substack{\bar{t}=i_t, k_t \\ t \neq j}} \theta_{\bar{i} \dots \bar{j} \dots \bar{p}, \bar{i} \dots \bar{j} \dots \bar{p}} ((-1)^\tau + (-1)^{\tau-1}) = 0 \end{aligned}$$

where the symbol $\widehat{}$ denotes omission.

Now, we are going to prove the following theorem which is a generalized version of Theorem 4.1.

Theorem 4.2([13]). *Let (M, g) be a Riemannian manifolds of dimension $n \geq 4q$. (M, g) is q -conformally flat if and only if for all distinct indices $i_1, \dots, i_{2q}, k_1, \dots, k_{2q}$,*

$$\sum_{\substack{\bar{t}=i_t, k_t \\ t=1, \dots, 2q}} (-1)^\tau (R^q_{i_1 \dots \bar{2q}, \bar{1} \dots \bar{2q}}) = 0$$

where τ is the number of index t such that $\bar{t}=k_t$.

Proof. Necessity follows from Proposition 4.1.

Conversely, taking the summation with distinct indices $k_1, \dots, k_{2q}$ on the range $1, \dots, n$, we have

$$\begin{aligned} 0 &= \sum_{\substack{k_1, \dots, k_{2q}=1 \\ \neq i_1, \dots, i_{2q}}}^n \left(\sum_{\substack{\bar{t}=i_t, k_t \\ t=1, \dots, 2q}} (-1)^\tau R^q_{i_1 \dots \bar{2q}, \bar{1} \dots \bar{2q}} \right) \\ &= (\text{const.}) \text{ con } R^q_{i_1 \dots i_{2q}, i_1 \dots i_{2q}} \end{aligned}$$

Using Proposition 2.3, we see

$$\text{con } R^q = 0.$$

Therefore, we obtain the theorem.

5. Hypersurfaces in a Euclidean space

Let M be a hypersurface in an $(n+1)$ -dimensional Euclidean space E^{n+1} and g be the induced metric and h be the second fundamental form of M for a unit normal vector. By the equation of Gauss we have $R = h^2/2$. Now we define the notion of k -umbilical as follows.

Definition 5.1. If a k -th exterior power $(h - \rho g)^k = 0$ for some function ρ , then M is said to be k -umbilical (umbilical for $k=1$).

The following are well known.

Theorem 5.1 (Cartan [1], Schouten [14]). Let M be a hypersurface of an $(n+1)$ -dimensional Euclidean space and $n \geq 4$. Then M is conformally flat if and only if M is 2-umbilical.

Theorem 5.2 (Chen & Yano [2]). Let M be a hypersurface of a 4-dimensional Euclidean space. If M is 2-umbilical then M is conformally flat.

Since the second fundamental tensor is symmetric, there exists an orthonormal basis $e_1, \dots, e_n$ of M at x which are the eigenvectors of h , that is

$$h(e_i, e_j) = \eta_i \delta_{ij}, \quad g_{ij} = \delta_{ij}, \quad g^{ij} = \delta^{ij}.$$

Moreover, in this section, we write local components of the tensor $\omega \in \Xi^{p,q}$ by

$$\omega_{i_1 \dots i_p, j_1 \dots j_q} = \omega(e_{i_1} \cdots e_{i_p} \otimes e_{j_1} \cdots e_{j_q}).$$

We put a matrix $H = (h_{ij})$ and the unit matrix $I = (\delta_{ij})$. Let P be an $m \times n$ matrix (p_{ij}) . Take r -tuples $(i_1, \dots, i_r)$ and $(j_1, \dots, j_r)$, where $1 \leq i_a \leq m$ and $1 \leq j_b \leq n$. We denote a minor matrix of degree r of matrix P by $P_{(j_1, \dots, j_r)}^{(i_1, \dots, i_r)}$ whose (s, t) -component is the (i_s, j_t) -component of P for each s and t . Then we have

$$(R^q)_{i_1 \dots i_{2q}, j_1 \dots j_{2q}} = \frac{(2q)!}{2^q} \left| H_{(j_1, \dots, j_{2q})}^{(i_1, \dots, i_{2q})} \right| = \frac{(2q)!}{2^q} \eta_{i_1} \cdots \eta_{i_{2q}} \left| I_{(j_1, \dots, j_{2q})}^{(i_1, \dots, i_{2q})} \right|.$$

Now, we have the following theorem which is a generalization of Theorem 5.1.

Theorem 5.3. *Let M be a hypersurface of an $(n+1)$ -dimensional Euclidean space and $n \geq 4q$. Then M is q -coformally flat if and only if M is $2q$ -umbilical.*

Proof. If M is a q -conformally flat, by Theorem 4.2, we have

$$0 = \sum_{\substack{\bar{i}=i_t, k_t \\ t=1, \dots, 2q}} (-1)^\tau (R^q_{\bar{i} \dots \bar{2q}, \bar{i} \dots \bar{2q}}) = \frac{(2q)!}{2^q} (\eta_{i_1} - \eta_{k_1}) \cdots (\eta_{i_{2q}} - \eta_{k_{2q}})$$

where τ is the number of index t such that $\bar{i} = k_t$. By replacing indices, we can assume without loss of generality that

$$\eta_{2q} = \cdots = \eta_n = \eta_0.$$

Hence, by setting $\rho = \eta_0$ we see M is $2q$ -umbilical.

Conversely, if $(h - \rho g)^{2q} = 0$ for some function ρ then we have

$$R^q = h^{2q} / 2^q = g \wedge \{ \cdots \}.$$

By the (b) of Proposition 3.1 we see $\text{con } R^q = 0$.

Moreover, we have the following theorem which is a generalized version of Theorem 5.2.

Theorem 5.4. *Let M be a hypersurface of a $4q$ -dimensional Euclidean space. If M is $2q$ -umbilical then M is q -conformally flat.*

Proof. Suppose that M is $2q$ -umbilical then we have $(h - \rho g)^{2q} = 0$ for some function ρ . Then we have

$$R^q = \frac{1}{2^q} h^{2q} = \frac{1}{2^q} g \wedge \left\{ \sum_{r=0}^{2q-1} \binom{2q}{r} \rho^{2q-r} (-g)^{2q-r-1} \wedge h^r \right\}.$$

Since $n=4q-1$, by Proposition 3.1 (c), we see $\mathbf{con} R^q=0$. And from Definition 3.2 we have

$$\mathit{dev} R^q = \frac{1}{2^q} \left\{ \sum_{r=0}^{2q-1} \binom{2q}{r} \rho^{2q-r} (-g)^{2q-r-1} \wedge h^r \right\}.$$

By Definition 3.3, (c) of Proposition 2.1, $\mathcal{D}g=0$ and $\mathcal{D}h=0$ we have

$$\begin{aligned} \mathbf{con}_0 R^q &= \frac{1}{2^q} \left\{ \sum_{r=0}^{2q-1} \binom{2q}{r} (2q-r) \rho^{2q-r-1} \mathcal{D}(\rho) \wedge (-g)^{2q-r-1} \wedge h^r \right\} \\ &= \frac{2q}{2^q} \mathcal{D}(\rho) \wedge (h - \rho g)^{2q-1}. \end{aligned}$$

Let $u=h-\rho g$, $u_{ij}=u(f_i \otimes f_j)$ and the matrix $U=(u_{ij})$ where $\{f_1, \dots, f_n\}$ are locally defined orthonormal vector fields. Since we have $u^{2q}=0$, without loss of generality, we may assume that

$$u_{ij} = \nu_i \delta_{ij}, \quad \nu_{2q} = \dots = \nu_n = 0.$$

If $\sum_{t=1}^{2q-1} \sum_{s=1}^{2q} \delta_{ti_s} < 2q-1$ or $\sum_{t=1}^{2q-1} \sum_{s=1}^{2q} \delta_{ti_s} < 2q-1$, then we see

$$\begin{aligned} \mathbf{con}_0 R^q(f_{i_1} \cdots f_{i_{2q}} \otimes f_{j_1} \cdots f_{j_{2q-1}}) &= \frac{2q}{2^q} \left\{ \sum_{l=1}^{2q} (-1)^l \nabla_{f_{i_l}}(\rho) | U_{(j_1, \dots, j_{2q-1})}^{(i_1, \dots, i_{2q})} | \right\} \\ &= 0. \end{aligned}$$

And from Corollary 3.4, we see, for $k=2q, \dots, n$,

$$0 = (\mathbf{c} \cdot \mathbf{con}_0 R^q)(f_k f_1 \cdots f_{2q-2} \otimes f_1 \cdots f_{2q-2}) = \frac{2q}{2^q} \nabla_{f_k}(\rho) \cdot \nu_1 \cdots \nu_{2q-1}.$$

Consequently, we see $\mathbf{con}_0 R^q=0$. Hence the theorem follows.

6. Quadric hypersurface

In this section, we shall give some examples of q-conformally flat quadric hypersurfaces.

Let us recall well known facts on quadric hypersurfaces. Let M be a properly n -dimensional quadric hypersurface in E^{n+1} , and b_{ij} be the coefficient of $x^i x^j$. We shall omit singular points in M . We call the symmetric matrix $B=(b_{ij})$ the matrix of M . Then the equation of M may be written by

$$(6.1) \quad z = x^{n+1} = \sum_{i=1}^n \frac{a_i (x^i)^2}{2} \quad a_i = -\lambda_i \neq 0 \quad (i=1, \dots, n)$$

or

$$(6.2) \quad z = x^{n+1} = \sqrt{a - \sum_{i=1}^n a_i (x^i)^2} \quad a_i = \frac{\lambda_i}{\lambda_{n+1}} \neq 0 \quad (i=1, \dots, n)$$

where λ_i 's are proportional to the eigenvalues of the matrix B . And we put

$$z_i = \frac{\partial z}{\partial x^i}, \quad z_{ij} = \frac{\partial^2 z}{\partial x^i \partial x^j}.$$

Then, the local components of the induced metric g and second fundamental form h on M , with respect to the basis vectors $f_i = \partial/\partial x^i$ are given the following.

$$g_{ij} = \delta_{ij} + z_i z_j, \quad g^{ij} = \delta^{ij} - \frac{z_i z_j}{g} \quad \text{and} \quad h_{ij} = \frac{z_{ij}}{\sqrt{g}} \quad \text{where} \quad g = 1 + \sum_{i=1}^n (z_i)^2.$$

From (6.1) and (6.2) we have

$$z_i = a_i x^i, \quad z_{ij} = \delta_{ij} a_i \quad \text{and} \quad z_i = -\frac{a_i x^i}{z}, \quad z_{ij} = -\frac{a_i \delta_{ij} + z_i z_j}{z}, \quad (z \neq 0),$$

respectively. When $n \geq 4q$, from Theorem 5.3, we have the following theorem (cf. Theorem 2 and 3 in [6]).

Theorem 6.1. *Let $n \geq 4q$. A properly n -dimensional quadric hypersurface M in E^{n+1} is q -conformally flat if and only if at least $(n-2q+2)$ eigenvalues of the matrix of M are equal.*

Proof. Suppose that $n \geq 4q$ and M is q -conformally flat, then by Theorem 5.3, we see M is $2q$ -umbilical. Hence we set, for some function ρ ,

$$u_{ij} = h_{ij} - \rho g_{ij} = \xi_i \delta_{ij} + \zeta z_i z_j$$

$$\text{where} \quad \begin{cases} \xi_i = a_i/\sqrt{g} - \rho, \quad \zeta = -\rho & \text{for (6.1)} \\ \xi_i = -(a_i/(z\sqrt{g}) + \rho), \quad \zeta = -(1/(z\sqrt{g}) + \rho) & \text{for (6.2).} \end{cases}$$

We put matrix $U = (u_{ij})$. Hence we have for all distinct indices $i_1, \dots, i_{2q}, j$,

$$0 = (u^{2q})_{i_1 \dots i_{2q}, i_1 \dots i_{2q-1} j} = (2q)! |U_{(i_1, \dots, i_{2q-1}, j)}^{(i_1, \dots, i_{2q})}| = (2q)! \zeta z_{i_{2q}} z_j \prod_{k=1}^{2q-1} \xi_{i_k}.$$

The first, if the function $\zeta \neq 0$, then by replacing indices, we can assume without loss of generality that

$$a_{2q-1} = \cdots = a_n.$$

Next, if the function $\zeta = 0$ for 6.2, then we have, for all distinct indices $i_1, \dots, i_{2q}$,

$$0 = (u^{2q})_{i_1 \dots i_{2q}, i_1 \dots i_{2q}} = (2q)! \rho \prod_{k=1}^{2q} (a_{i_k} - 1).$$

By replacing indices, we can assume without loss of generality that

$$a_{2q} = \cdots = a_n = 1.$$

Consequently, we find at least $(n-2q+2)$ eigenvalues of the matrix of M are equal.

Conversely, in this case, we see that M is $2q$ -umbilical. Therefore, we obtain the theorem.

Next, If $n=4q-1$, from Theorem 5.4, we have the following theorem.

Theorem 6.2. *Let $n=4q-1$ and M be a properly n -dimensional quadric hypersurface in E^{n+1} . Suppose that at least $(n-2q+2)$ eigenvalues of the matrix of M are equal, then M is q -conformally flat.*

Proof. From assumption of the theorem, we may set

$$a_{2q-1} = \cdots = a_n = a_0 \text{ or } a_{2q} = \cdots = a_n = a_0 = 1.$$

When we set $\rho = a_0/\sqrt{g}$, we see $(h - \rho g)^{2q} = 0$. Therefore, from Theorem 5.4 we obtain the theorem.

7. Product Riemannian manifolds

Let V be a product vector space of two vector spaces V_1 and V_2 with dimensions n_1 and n_2 , respectively. Let us set $V = V_1 \oplus V_2$ and identify V_1, V_2 with the subspaces of V . An element $\xi \in \Lambda^p(V^*)$ is called of *type* (p_1, p_2) if $p = p_1 + p_2$ and, for vectors $x_i \in V_1$ or V_2 , $\xi(x_1, \dots, x_p) = 0$ except for the case when the p_1 vectors x_i belong to V_1 and the other p_2 vectors x_j belong to V_2 . The set of such elements is indicated by $\Lambda^{p_1, p_2}(V^*)$. Then we have $\Lambda^p(V^*) = \sum_{p_1+p_2=p} \Lambda^{p_1, p_2}(V^*)$ (direct sum).

Now, we consider the spaces $\Xi^{(p_1, q_1; p_2, q_2)}(V) = \Lambda^{p_1, p_2}(V^*) \otimes \Lambda^{q_1, q_2}(V^*)$ and we

call an element $\omega \in \mathcal{E}^{(p_1, q_1; p_2, q_2)}(V)$ the double form of type $(p_1, q_1; p_2, q_2)$ on V . Also, we can identify

$$\mathcal{E}^{(p_1, q_1)}(V_1) = \mathcal{E}^{(p_1, q_1; 0, 0)}(V), \quad \mathcal{E}^{(p_2, q_2)}(V_2) = \mathcal{E}^{(0, 0; p_2, q_2)}(V_2).$$

Then we have the following proposition.

Proposition 7.1.

(a) For any $\omega \in \mathcal{E}^{(p_1, q_1; p_2, q_2)}(V)$ and $\theta \in \mathcal{E}^{(r_1, s_1; r_2, s_2)}(V)$,

$$\omega \wedge \theta \in \mathcal{E}^{(p_1+r_1, q_1+s_1; p_2+r_2, q_2+s_2)}(V).$$

(b) $\mathcal{E}^{(p, q)}(V) = \sum_{p_1+p_2=p} \sum_{q_1+q_2=q} \mathcal{E}^{(p_1, q_1; p_2, q_2)}(V)$ (direct sum).

Let g_1, g_2 be metrics on the vector spaces V_1, V_2 , respectively. We introduce a metric g on V by the formula

$$g(u \otimes v) = g_1(u_1 \otimes v_1) + g_2(u_2 \otimes v_2)$$

where u_ι, v_ι are V_ι -components and $u, v \in V, \iota = 1, 2$. Also, we define the two mappings $\mathbf{c}_\iota: \mathcal{E}^{(p, q)}(V) \rightarrow \mathcal{E}^{(p-1, q-1)}(V)$ as follow.

For $\omega \in \mathcal{E}^{p, q}$ with $p=0$ or $q=0$, we put $\mathbf{c}_\iota \omega = 0, \iota = 1, 2$. If both p and $q \geq 1$, then we put

$$\begin{aligned} (\mathbf{c}_1 \omega)(x_{i_1} \cdots x_{i_{p-1}} \otimes x_{j_1} \cdots x_{j_{q-1}}) &= \sum_{\alpha=1}^{n_1} \omega(e_\alpha x_{i_1} \cdots x_{i_{p-1}} \otimes e_\alpha x_{j_1} \cdots x_{j_{q-1}}), \\ (\mathbf{c}_2 \omega)(x_{i_1} \cdots x_{i_{p-1}} \otimes x_{j_1} \cdots x_{j_{q-1}}) &= \sum_{\beta=n_1+1}^n \omega(e_\beta x_{i_1} \cdots x_{i_{p-1}} \otimes e_\beta x_{j_1} \cdots x_{j_{q-1}}) \end{aligned}$$

where $e_1, \dots, e_{n_1}$ and $e_{n_1+1}, \dots, e_n$ are orthonormal bases for V_1 and V_2 , respectively. It is easy to see that

$$(7.1) \quad \mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 \quad \text{on} \quad \mathcal{E}(V)$$

and for any double forms $\omega \in \mathcal{E}^{p_1, q_1; 0, 0}$ and $\theta \in \mathcal{E}^{0, 0; p_2, q_2}$, $\mathbf{c}_2 \omega = 0$ and $\mathbf{c}_1 \theta = 0$.

Proposition 7.2. For any double forms ω, θ of types $(p_1, q_1; 0, 0), (0, 0; p_2, q_2)$, respectively, we have $\mathbf{c}(\omega \wedge \theta) = \mathbf{c}_1 \omega \wedge \theta + (-1)^{p_1+q_1} \omega \wedge \mathbf{c}_2 \theta$.

Proof. From assumptions of Proposition 7.2 and the definition of $\mathbf{c}_1$, we find, for $x_2, \dots, x_{p_1+p_2}, y_2, \dots, y_{q_1+q_2} \in V$,

$$\begin{aligned}
& \mathbf{c}_1(\omega \wedge \theta)(x_2 \cdots x_{p_1+p_2} \otimes y_2 \cdots y_{q_1+q_2}) \\
&= \sum_{i=1}^{n_1} \sum_{\rho \in Sh(p_1-1, p_2)} \sum_{\sigma \in Sh(q_1-1, q_2)} \varepsilon_\rho \varepsilon_\sigma \omega(\ell_i x_{\rho(2)} \cdots x_{\rho(p_1)} \otimes \ell_i y_{\sigma(2)} \cdots y_{\sigma(q_1)}) \\
&\quad \cdot \theta(x_{\rho(p_1+1)} \cdots x_{\rho(p_1+p_2)} \otimes y_{\sigma(q_1+1)} \cdots y_{\sigma(q_1+q_2)}) \\
&= (\mathbf{c}_1 \omega \wedge \theta)(x_2 \cdots x_{p_1+p_2} \otimes y_2 \cdots y_{q_1+q_2}).
\end{aligned}$$

Similarly, we see that $\mathbf{c}_2(\theta \wedge \omega) = \mathbf{c}_2 \theta \wedge \omega$. Thus, the proposition follows from equations (1.1) and (7.1).

Corollary 7.3. *For any curvature structures $\omega \in \mathcal{R}^p(V_1)$ and $\theta \in \mathcal{R}^p(V_2)$, we have $\mathbf{c}^r(\omega \wedge \theta) = \sum_{k=0}^r \binom{r}{k} \mathbf{c}_1^{r-k} \omega \wedge \mathbf{c}_2^k \theta$.*

In this section, let (M, g) be a product Riemannian manifold of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with dimensions n_1 and n_2 , respectively. Then the tangent space $T_m(M)$ at each point $m = (m_1, m_2)$ ($m_1 \in M_1$, $m_2 \in M_2$), is isomorphic in a natural way to the direct sum $T_{m_1}(M_1) \oplus T_{m_2}(M_2)$, so we identify $T_m(M) = T_{m_1}(M_1) \oplus T_{m_2}(M_2)$. Thus, all algebraic operations and rules mentioned in the previous section can be now reformulated for the manifolds M , M_1 and M_2 .

Now, let R , R_1 and R_2 denote the curvature tensors of M , M_1 and M_2 , respectively. And let $\mathbf{c}$, $\mathbf{c}_1$ and $\mathbf{c}_2$ denote the contractions of M , M_1 and M_2 , respectively. Then we have $g = g_1 + g_2$, $R = R_1 + R_2$, $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$. Here we set $n_1, n_2 \geq 2q$. From Corollary 7.3 we have

$$\mathbf{c}^r R^q = \sum_{i=0}^r \sum_{j=0}^q \binom{r}{i} \binom{q}{j} \mathbf{c}_1^{r-i} R_1^{q-j} \wedge \mathbf{c}_2^i R_2^j.$$

Let γ_λ denote the component of type $(2q-\lambda, 2q-\lambda, \lambda, \lambda)$ of $\mathbf{con} R^q$. We are going to compute γ_0 .

We introduce the symbol $[x]$ which is an integral part of x . Then we have

$$\gamma_0 = R_1^q + \sum_{r=1}^{2q} \frac{(-1)^r g_1^r \wedge \left(\sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2j} \binom{q}{j} \mathbf{c}_1^{r-2j} R_1^{q-j} \wedge \mathbf{c}_2^{2j} R_2^j \right)}{r! \prod_{\alpha=0}^{r-1} (n-4q+2+\alpha)}.$$

Next using Proposition 2.8 we have, for $\mu \leq 2q$

$$\mathbf{c}_1^\mu \gamma_0 = \mathbf{c}_1^\mu R_1^q + \sum_{r=1}^{2q} \frac{(-1)^r \sum_{\substack{i=0 \\ i \leq r}}^\mu \binom{\mu}{i} \binom{r}{i} \binom{n_1-4q+r+\mu}{i} (i!)^2 g_1^{r-i}}{r! \prod_{\alpha=0}^{r-1} (n-4q+2+\alpha)}$$

$$\begin{aligned}
 & \wedge \left(\sum_{j=0}^{\left[\frac{r}{2}\right]} \binom{r}{2j} \binom{q}{j} \mathbf{c}_1^{r-2j+\mu-i} R_1^{q-j} \wedge \mathbf{c}_2^{2j} R_2^j \right) \\
 &= \sum_{j=0}^{[\mu/2]} \mathbf{c}_1^{\mu-2j} R_1^{q-j} \wedge \mathbf{c}_2^{2j} R_2^j \cdot \frac{\binom{q}{j} \binom{\mu}{2j} \binom{n_1-4q+\mu+2j}{2j}}{\binom{n-4q+1+2j}{2j}} \\
 & \quad \left\{ \sum_{r=0}^{\mu-2j} \frac{(-1)^{r+2j} \binom{\mu-2j}{\mu-2j-r} \binom{n_1-4q+r+\mu+2j}{n_1-4q+\mu+2j}}{\binom{n-4q+1+r+2j}{n-4q+1+2j}} \right\} + g_1 \wedge \{\dots\}
 \end{aligned}$$

(using Proposition 3.5)

$$= \sum_{j=0}^{[\mu/2]} \mathbf{c}_1^{\mu-2j} R_1^{q-j} \wedge \mathbf{c}_2^{2j} R_2^j \cdot \frac{\binom{q}{j} \binom{n_2-2j}{\mu-2j} \binom{n_1-4q+\mu+2j}{2j}}{\binom{n-4q+1+\mu}{\mu}} + g_1 \wedge \{\dots\}.$$

When $\mu=2q$, we have

$$\mathbf{c}_1^{2q} \gamma_0 = \sum_{j=0}^q \mathbf{c}_1^{2q-2j} R_1^{q-j} \cdot \mathbf{c}_2^{2j} R_2^j \cdot \frac{\binom{q}{j} \binom{n_2-2j}{2q-2j} \binom{n_1-2q+2j}{2j}}{\binom{n-2q+1}{2q}}.$$

Suppose that M is q -conformally flat then we have $\gamma_0=0$. Therefore we have the following proposition.

Proposition 7.4. *Let (M, g) be an n -dimensional product Riemannian manifold of an n_1 -dimensional Riemannian manifold (M_1, g_1) and an n_2 -dimensional Riemannian manifold (M_2, g_2) , and R, R_1 and R_2 be curvature tensors of M, M_1 and M_2 respectively.*

If (M, g) is q -conformally flat and $n_1 \geq 2q$ and $n_2 \geq 2q$, then we have

$$\sum_{j=0}^q \mathbf{c}_1^{2q-2j} R_1^{q-j} \cdot \mathbf{c}_2^{2j} R_2^j \cdot \binom{q}{j} \binom{n_2-2j}{2q-2j} \binom{n_1-2q+2j}{2j} = 0.$$

Moreover if M_1 and M_2 are constant curvature spaces then the following theorem ([12]).

Theorem 7.1. *Let (M, g) be a product Riemannian manifold of constant curvature spaces M_1 and M_2 . If $\dim M_1 \geq 2q$ and $\dim M_2 \geq 2q$, then M is q -conformally flat if and only if M is conformally flat in the usual sense.*

Proof. By the assumption, we have for $\iota = 1, 2$, $R_\iota = \kappa_\iota g_\iota^2$. From Proposition 2.8 we see, for $k \leq q$,

$$\mathbf{c}_\iota^{2k} R_\iota^k = \mathbf{c}_\iota^{2k} (\kappa_\iota^k g_\iota^{2k}) = \kappa_\iota^k \frac{(2k)! n_\iota!}{(n_\iota - 2k)!}.$$

Substituting these equations to the formula of Proposition 7.4, we see $(\kappa_1 + \kappa_2)^q = 0$. Hence we have $\kappa_1 + \kappa_2 = 0$. This corollary follows.

The other hand we see, for $n_1 \geq 4q - 1$,

$$\mathbf{con}_1 \gamma_0 = \mathbf{con}_1 R_1^q = 0, \quad \mathbf{con}_1 (\mathbf{c}_1 \gamma_0) = \mathbf{con}_1 (\mathbf{c}_1 R_1^q) = 0.$$

And from $\mathbf{con}_1 (\mathbf{c}_1^\mu \gamma_0) = 0$, we have, for $\mu \geq 2$,

$$\sum_{j=0}^{\lfloor \frac{\mu}{2} \rfloor} \frac{\binom{q}{j} \binom{\mu}{2j} \binom{n_1 - 4q + \mu + 2j}{2j}}{\binom{n_2}{2j}} \mathbf{con}_1 (\mathbf{c}_1^{\mu-2j} R_1^{q-j}) \wedge \mathbf{c}_2^{2j} R_2^j = 0.$$

Applying these equations to Proposition 3.7, we have

$$\begin{aligned} R_1^q &= - \sum_{r=2}^{2q} g_1^r \wedge \\ &\quad \sum_{j=1}^{\lfloor r/2 \rfloor} \frac{\binom{q}{j} \binom{r}{2j} \binom{n_1 - 4q + r + 2j}{2j}}{r! \binom{n_2}{2j} \prod_{\alpha=0}^{r-1} (n_1 - 4q + 1 + r + \alpha)} \mathbf{con}_1 (\mathbf{c}_1^{r-2j} R_1^{q-j}) \wedge \mathbf{c}_2^{2j} R_2^j \\ &= - \sum_{k=1}^q (\mathbf{c}_2^{2k} R_2^k) \wedge g_1^{2k} \wedge \\ &\quad \frac{\binom{q}{k}}{((2k)!)^2 \binom{n_2}{2k}} \left(\mathbf{con}_1 R_1^{q-k} + \sum_{s=1}^{2q-2k} \frac{g_1^s \wedge \mathbf{con}_1 (\mathbf{c}_1^s R_1^{q-k})}{s! \prod_{\alpha=0}^{s-1} (n_1 - 4q + 1 + s + 4k + \alpha)} \right) \end{aligned}$$

(from Proposition 3.7)

$$= - \sum_{k=1}^q (\mathbf{c}_2^{2k} R_2^k) \wedge g_1^{2k} \wedge \frac{\binom{q}{k}}{((2k)!)^2 \binom{n_2}{2k}} R_1^{q-k}.$$

Therefore, we obtain the following proposition.

Proposition 7.5. *Let (M, g) be an n -dimensional product Riemannian manifold of an n_1 -dimensional Riemannian manifold (M_1, g_1) and an n_2 -dimensional Riemannian manifold (M_2, g_2) and R , R_1 and R_2 be curvature tensors of M , M_1 and M_2 , respectively.*

If (M, g) is q -conformally flat and $n_1 \geq 4q - 1$ and $n_2 \geq 2q$, then we have

$$R_1^q + \sum_{k=1}^q g_1^{2k} \wedge \frac{\binom{q}{k}}{((2k)!)^2 \binom{n_2}{2k}} R_1^{q-k} \wedge \mathbf{c}_2^{2k} R_2^k = 0.$$

If M_2 is a constant curvature space then we have the following theorem (cf. [5]).

Theorem 7.2. *Let (M, g) be a product Riemannian manifold of an n_1 -dimensional space M_1 and an n_2 -dimensional constant curvature space M_2 . If $n_1 \geq 4q - 1$ and $n_2 \geq 2q$, then M is q -conformally flat if and only if M_1 be q -concentrically flat.*

Proof. By the assumption, we have $R_2 = \kappa_2 g_2^2$. From Proposition 2.8 we see $\mathbf{c}_2^{2k} R_2^k = \mathbf{c}_2^{2k} (\kappa_2^k g_2^{2k}) = \kappa_2^k (2k)! n_2! / (n_2 - 2k)!$. Substituting this to the formula of Proposition 7.5, this theorem follows.

Acknowledgment

The author would like to thank Professors Katuhiro Shiohama, Toshio Nasu, Keisaku Kumahara and Masato Wakayama for their valuable comments in preparing this paper.

REFERENCES

- [1] E. Cartan, *La déformation des hypersurfaces dans l'espace conforme réel a $n \geq 5$ dimensions*, Bull. Soc. Math. France **45**(1917), 57–121.
- [2] B.-y. Chen and K. Yano, *Conformally flat Submanifolds*, Mimeographic Notes, Michigan State University (1972).
- [3] L.P. Eisenhart, *Riemannian Geometry*, Princeton Univ. Press (1949).
- [4] A. Gray, *Some relations between curvature and characteristic classes*, Math. Ann. **184** (1970), 257–267.
- [5] M. Kojima, *A theorem on 2-conformally flat product Riemannian manifolds*, J. of the Faculty of General Education of Tottori Univ. **8** (1975), 177–183.
- [6] ———, *q-Conformally flat hypersurfaces*, Differential Geometry and its Applications (to appear).
- [7] O. Kowalski, *On the Gauss-Kronecker curvature tensors*, Math. Ann. **203** (1973), 335–343.
- [8] R.S. Kulkarni, *Curvature structures and conformal transformations*, J. Differential Geometry **4** (1969), 425–451.
- [9] ———, *On the Bianchi identities*, Math. Ann. **199** (1972), 175–204.
- [10] ———, *Conformally flat manifolds*, Proc. Nat. Sci. USA **69**, No.9 (1972), 2675–2676.
- [11] T. Nasu, *On conformal invariants of higher order*, Hiroshima Math. J. **5** (1975), 43–60.
- [12] T. Nasu and M. Kojima, *A remark on q-conformally flat product Riemannian manifolds*, Proc. Japan Acad. **50** (1974), 793–796.
- [13] ———, *Characterizations of Riemannian manifolds with vanishing conformal curvature tensor of higher order*, J. of the Faculty of General Education of Tottori U. **16** (1982), 1–6.
- [14] J.A. Schouten, *Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Massbestimmung auf eine Mannigfaltigkeit mit euklidischer Massbestimmung*, Math. Zeit **11** (1921), 58–88.
- [15] J.A. Thorpe, *Sectional curvatures and characteristic classes*, Ann. of Math. **80** (1964), 429–443.

Faculty of General Education
Tottori University
Koyama, Tottori 680
Japan