A HYPERGEOMETRIC INTEGRAL ATTACHED TO THE CONFIGURATION OF THE MIRRORS OF THE REFLECTION GROUP S_{n+2} ACTING ON $\mathbf{P}_n$

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Dedicated to Professor Katsumi Shiratani

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Introduction

In this paper we present functions defined by the following integral of hypergeometric type :

$$\int \prod_{i=1}^n t_i^{\alpha_i} (1-t_i)^{\beta_i} \prod_{1 \leq i < j \leq n} (t_i - t_j)^{\gamma_{ij}} (\sum x^i t_i)^{\lambda_0} dt_1 \wedge \cdots \wedge dt_n,$$

which is, when $n=1$, the Euler integral representation of the hypergeometric function. Geometric and analytic properties of the functions are studied. Symmetry with respect to the permutation group S_{n+2} is emphasized.

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1. Hyperplane arrangements in $\mathbf{P}^n$ and in $\mathbf{P}_n$

Let A be an arrangement of $n+2$ hyperplanes in general position in the n -dimensional complex projective space $\mathbf{P}^n$, which is known to be unique up to projective motions. We study in this section the arrangement A' in the dual

projective space $\mathbf{P}^n$ dual to A , and the arrangement A'' in $\mathbf{P}^n$ dual to A' . Figure 1 illustrates these when $n=2$.

Let us fix a homogeneous coordinate system $(x^0 : \cdots : x^n)$ on $\mathbf{P}^n$ and normalize A by a projective transformation as follows:

$$A: \begin{aligned} H^i &:= \{x^i = 0\} \quad (i=0, \dots, n) \\ H &= H^{01\dots n} := \left\{ \sum_{i=0}^n x^i = 0 \right\}. \end{aligned}$$

Any n hyperplanes meet at a point; there are $\binom{n+2}{2}$ such points:

$$p^i := (0 : \cdots : 0 : \overset{i}{1} : \cdots : 0) \quad (i=0, \dots, n)$$

the intersection of $H^0, \dots, H^{i-1}, H^{i+1}, \dots$, and H^n .

$$p^{ij} := (0 : \cdots : 0 : \overset{i}{1} : 0 : \cdots : 0 : -\overset{j}{1} : \cdots : 0) \quad (0 \leq i < j \leq n)$$

the intersection of $H^0, \dots, H^{i-1}, H^{i+1}, \dots, H^{j-1}, H^{j+1}, \dots, H^n$, and H .

Thus the dual arrangement A' to A is given by:

$$A': \begin{aligned} H_i &:= \{t_i = 0\} \quad (i=0, \dots, n) \\ H_{ij} &:= \{t_i - t_j = 0\} \quad (0 \leq i < j \leq n), \end{aligned}$$

where $(t_0 : \cdots : t_n)$ is the homogeneous coordinate system on $\mathbf{P}_n$ dual to $(x^0 : \cdots : x^n)$.

Any projectively independent n hyperplanes in A' meet at a point; there are $2^{n+1} - 1$ such. Indeed they are given by:

$$p_{i_1 \dots i_k} := (0 : \cdots : 0 : \overset{i_1}{1} : 0 : \cdots : 0 : \overset{i_2}{1} : 0 : \cdots : 0) \quad (0 \leq i_1 < \cdots < i_k \leq n)$$

the intersection of H_j ($j \notin \{i_1, \dots, i_k\}$) and $H_{i_l i_{l+1}}$ ($l=1, \dots, k-1$).

Thus the dual arrangement A'' of A' is given by:

$$A'': \quad H^{i_1 \dots i_k} := \{x^{i_1} + \cdots + x^{i_k} = 0\} \quad (0 \leq i_1 < \cdots < i_k \leq n).$$

Figure 1

Of course A'' is nothing but the arrangement of hyperplanes spanned by n projectively independent intersection points of A . The permutation group S_{n+2} acts on $(x^0 : \cdots : x^n) \in \mathbf{P}^n$ as follows:

$$\begin{aligned}
 S_{n+2} \ni (i, j) &\mapsto \begin{pmatrix} & & & & i & j & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} \in PGL(n+1) \\
 &\hspace{15em} 0 \leq i, j \leq n, \\
 S_{n+2} \ni (i, n+1) &\mapsto \begin{pmatrix} & & & & i & & & \\ & -1 & & & 1 & & & \\ & & \ddots & & \vdots & & & \\ & & & -1 & \vdots & & & \\ & & & & 1 & & & \\ & & & & \vdots & -1 & & \\ & & & & \vdots & & \ddots & \\ & & & & 1 & & & -1 \end{pmatrix} \in PGL(n+1),
 \end{aligned}$$

where S_{n+2} is regarded as the permutation group of $\{0, 1, \dots, n+1\}$. This action preserves A and A'' , and so the dual action preserves A' . The actions are transitive on A and A' , but not on A'' . In fact we have

Lemma 1.1. *The orbit decomposition of A'' under the S_{n+2} action is given by:*

$$\begin{aligned}
 A'' &= \prod_{k=1}^{\lfloor n/2 \rfloor + 1} O_k, \\
 O_k &:= \{H^{i_1 \dots i_k}, H^{j_1 \dots j_{n+2-k}}\}_{0 \leq i_1 < \dots < i_k \leq n, 0 \leq j_1 < \dots < j_{n+2-k} \leq n}.
 \end{aligned}$$

Proof. The permutation $\sigma := (i, j)$ ($0 \leq i, j \leq n$) acts on A'' by

$$H^{i_1, \dots, i_k} \mapsto H^{\sigma(i_1), \dots, \sigma(i_k)},$$

and $(i, n+1)$ acts on A'' by

$$\begin{cases} H^{i_1, \dots, i_k} \mapsto H^{i_1, \dots, i_k} & \text{if } i \notin \{i_1, \dots, i_k\}, \\ H^{i_1, \dots, i_k} \mapsto H^{j_1, \dots, j_{n+2-k}} & \text{if } i \in \{i_1, \dots, i_k\}, \end{cases}$$

where $\{0, 1, \dots, n\} = \{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{n+2-k}\}$ and $i \in \{i_1, \dots, i_k\} \cap \{j_1, \dots, j_{n+2-k}\}$. Now the assertion easily follows. $\square$

Notice that the orbit O_k consists of $\binom{n+2}{k}$ hyperplanes.

2. Twisted (co)homologies

For complex numbers λ_j such that

$$\sum_{j=0}^N \lambda_j = -n-1, \quad N := \binom{n+2}{2}$$

consider the integral

$$\int \prod_{j=0}^N l_j(t)^{\lambda_j} dt,$$

where

$$dt = \sum (-1)^i t_i dt_0 \wedge \dots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \dots \wedge dt_n,$$

and $l_j(t)$ are linear forms in $t = (t_0, \dots, t_n)$ such that $\prod_{j=1}^N l_j$ is a defining equation of the arrangement A' in $\mathbf{P}_n$, and $l_0(t)$ is a generic one.

We state here three conditions (NPI), (NNI) and (NI) for the exponents λ_j ($0 \leq j \leq N$). Let us rename the members of A' as follows:

$$h_j := \{t \mid l_j = 0\} \quad j=1, \dots, N$$

and put $h_0 := \{t \mid l_0 = 0\}$.

(NPI)(resp. (NNI), (NI)). For any k -dimensional ($0 \leq k \leq n-2$) linear subspace $L \subset \mathbf{P}_n$ such that the number p of hyperplanes h_j passing through L , say $h_{i_1}, \dots, h_{i_p}$, exceeds $n-k$,

$$\sum_{j=1}^p \lambda_{i_j} \notin \mathbf{N} - \{0\} \quad (\text{resp. } \notin -\mathbf{N}, \notin \mathbf{Z})$$

Furthermore for any j ($0 \leq j \leq N$),

$$\lambda_j \notin \mathbf{N} - \{0\} \quad (\text{resp. } \notin -\mathbf{N}, \notin \mathbf{Z}),$$

where $\mathbf{N} = \{0, 1, 2, \dots\}$.

Notice that (NI) is equivalent to (NPI) together with (NNI).

Let us express the linear form l_0 as follows :

$$l_0 = l_0(x, t) = - \sum_{i=0}^n x^i t_i,$$

where $x \in \mathbf{P}^n - A''$. After normalizing A' as in Section 1 and putting $t_0 = 1$ and $x_0 = -1$, the integral takes the form

$$\int \Phi dt_1 \wedge \dots \wedge dt_n$$

where

$$\Phi = \prod_{i=1}^n t_i^{\alpha_i} (1-t_i)^{\beta_i} \prod_{1 \leq i < j \leq n} (t_i - t_j)^{\gamma_{ij}} (1 - \sum x^i t_i)^{\lambda_0},$$

where the exponents α_i , β_i , γ_{ij} are related to λ_j accordingly.

Remark 2.1 : The integral can be thought of as an iterated integral of the Appell-Lauricella function F_D (cf. Section 4):

$$\begin{aligned} \int dt_1 t_1^{\alpha_1} (1-t_1)^{\beta_1} \int \dots \int dt_{n-1} t_{n-1}^{\alpha_{n-1}} (1-t_{n-1})^{\beta_{n-1}} \prod_{j < n-1} (t_j - t_{n-1})^{\gamma_{j, n-1}} \\ \int dt_n t_n^{\alpha_n} (1-t_n)^{\beta_n} \prod_{j < n} (t_j - t_n)^{\gamma_{j, n}} (1 - \sum x^i t_i)^{\lambda_0} \end{aligned}$$

Remark 2.2 : When $n=2$ and $x_2=0$ the integral as a function of x_1 gives a solution of the ordinary Fuchsian differential equation of degree 4 studied in [ST], [Har] and [Mim].

Remark 2.3 : When $n=2$ and $\gamma_{12}=1$, the integral gives Appell's hypergeometric function F_2 . (cf. [Yos, Section 6.4])

For $x \notin A''$, let H_x be the hyperplane defined by $l_0(x, t)$; put

$$B := A' \cup H_x, \quad T := \mathbf{P}_n - B.$$

Let us define a covariant derivation $\nabla : \mathcal{O}_T \rightarrow \mathcal{Q}_T^1$ (the sheaves of holomorphic functions and holomorphic 1-forms on T are denoted by $\mathcal{O}_T$ and $\mathcal{Q}_T^1$, respectively)

by

$$\nabla := d + \omega \wedge, \quad \omega := d\Phi/\Phi;$$

the kernel, which is a locally constant sheaf on T , is denoted by $\mathcal{S}$; this is quasi-isomorphic to $(\Omega_T^\bullet, \nabla)$. The following isomorphisms and dualities are known ([Aom1,2], [Del], [ESV], [KN]). The p -th twisted cohomology group

$$H^p(\nabla) := H^p(\Gamma(\Omega_T^\bullet), \nabla)$$

is (since T is affine) isomorphic to $H^p(T, \mathcal{S})$.

(1) Under the condition (NPI), $H^p(\nabla)$ is isomorphic to $H^p(\Gamma(\mathbf{P}_n, \Omega^\bullet(\log B)), \nabla)$ and is dual to the p -th twisted homology group $H_p(T, \mathcal{S})$. Here $\Omega^p(\log B)$ denotes the sheaf of p -forms on $\mathbf{P}_n$ such that under a modification $(\tilde{\mathbf{P}}_n, \tilde{B}) \rightarrow (\mathbf{P}_n, B)$, with center in B and normally crossing $\tilde{B}$, they have at most logarithmic poles along $\tilde{B}$.

(2) Under (NNI), the p -th locally finite twisted homology group $H_p^{lf}(T, \mathcal{S}^\vee)$ is dual to

$$H^p(\Gamma(\mathbf{P}_n, \Omega^\bullet(\log B)(-B)), \nabla) \cong H^p(\Gamma(\mathbf{P}_n, \Omega^\bullet(\log B)), \nabla),$$

where $\mathcal{S}^\vee$ is a locally constant sheaf dual to $\mathcal{S}$.

(3) Under (NI), $H^\bullet(\nabla)$ is purely n -dimensional, so $H_\bullet(T, \mathcal{S})$ and $H_\bullet^{lf}(T, \mathcal{S})$ are purely n -dimensional; moreover there is an isomorphism

$$reg: H_n^{lf}(T, \mathcal{S}) \rightarrow H_n(T, \mathcal{S})$$

called the regularization.

2.1. A basis of twisted homologies

Proposition 2.1.1. *Under the assumption (NI), we have*

$$\dim H^n(T, \mathcal{S}) = (n+1)!$$

Proof due to Professor Toshio Sumi. Note that

$$\begin{aligned} (-1)^n \dim H^n(T, \mathcal{S}) &= \chi(T, \mathcal{S}) \\ &= \chi(T), \end{aligned}$$

where χ stands for Euler characteristic. We shall prove

$$\chi(T) = (-1)^n (n+1)!$$

Let

$$e_n := \chi(T), \quad f_n := \chi(A' \cap \mathbf{C}^n).$$

Since the boundary of the tubular neighborhood $V_{A'}$ of $A' \cap \mathbf{C}^n$ in $\mathbf{C}^n$ is an S^{2q-1} -bundle, $\chi(T \cap V_{A'}) = 0$. Thus, $T = \mathbf{C}^n - A'$ implies $e_n = 1 - f_n$. By the same reason,

$$A' \cap \mathbf{C}^n = (\cup H_i \cap \mathbf{C}^n) \cup (\cup H_{ij} \cap \mathbf{C}^n)$$

leads to

$$\begin{aligned} f_n &= \chi(\cup H_i \cap \mathbf{C}^n) + \chi(\cup H_{ij} \cap \mathbf{C}^n - \cup H_i) \\ &= 1 + (n+1)e_n, \end{aligned}$$

because $\cup H_i \cap \mathbf{C}^n$ is contractible. This completes the proof.

Conjecture. (a basis of the twisted homology group) The simplex

$$\Delta = \{t_0, \dots, t_n \in \mathbf{RP}_n \mid t_i > 0, i=0, \dots, n\}$$

is barycentrically subdivided by H_{ij} into $(n+1)!$ chambers. Under the assumption (NI), $(n+1)!$ twisted cycles with supports on the chambers form a basis of the n -dimensional locally finite twisted homology group $H^{lf}(T, \mathcal{S})$.

We shall see in Section 4 that this conjecture is true when $n=2$.

2.2. A basis of twisted cohomologies

We are going to present an explicit basis of $H^n(\nabla)$. For notational simplicity, we express Φ as follows:

$$\Phi = \prod_{0 \leq i < j \leq n+1} (t_i - t_j)^{\lambda_{ij}} (1 - x^1 t_1 - \dots - x^n t_n)^{\lambda_0},$$

where

$$t_0 = 1, \quad t_{n+1} = 0.$$

Proposition 2.2.1. *If the exponents satisfy (NPI) and if $\lambda_0 \neq 0$ then the following $(n+1)!$ monomials generate the twisted cohomology group $H^n(\nabla)$:*

$$\begin{aligned} &\langle s_0, s_1, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \\ &:= d \log l_{s_0, s_1} \wedge \dots \wedge d \log l_{s_{a-1}, s_a} \wedge d \log l_{s_{a+1}, s_{a+2}} \wedge \dots \wedge d \log l_{s_n, s_{n+1}} \\ &\in \Gamma(\mathbf{P}_n, \Omega^n(\log B)), \quad 0 \leq a \leq n \end{aligned}$$

where

$$l_{ij} := t_i - t_j, \quad s_0 = 0, \quad s_{n+1} = n+1,$$

and $\{s_1, \dots, s_n\}$ are permutations of $\{1, \dots, n\}$.

Proof. Under the condition (NPI), the space $H^n(\nabla)$ is generated by wedge products of n 1-forms among $d \log l_0$ and $d \log l_{ij}$ ([ESV]). Computing the covariant differential of an $(n-1)$ -wedge products of the $d \log l_{ij}$, if $\lambda_0 \neq 0$ we know that terms involving $d \log l_0$ can be eliminated, that is, $H^n(\nabla)$ can be generated by logarithmic n -forms

$$d \log l_{1,i_1} \wedge \dots \wedge d \log l_{n,i_n}, \quad 0 \leq i_j \leq n+1,$$

which will be called *monomials*. Let us attach to the monomial the graph with vertices $0, \dots, n+1$ and with edges joining j and i_j for $j=1, \dots, n$. Length of each edge is supposed to be $1km$. We identify a monomial and the corresponding graph. The graph has two connected components, one with vertex 0 and the other with vertex $n+1$. Figure 2 shows the graphs of the monomials presented in the Proposition 2.2.1.

Figure 2

Such a graph consisting of the two linear components will be said to be *reduced*. For any monomial $d \log l_{1,i_1} \wedge \dots \wedge d \log l_{n,i_n}$, we shall express it as a $\mathbf{Z}$ -linear combination of reduced ones.

(Step 1) If there are more than one edges joining the vertex 0 , by virtue of the identity

$$\frac{1}{(t_0 - t_1)(t_0 - t_2)} = \frac{1}{(t_0 - t_1)(t_1 - t_2)} + \frac{1}{(t_0 - t_2)(t_2 - t_1)},$$

one can express the graph as the sum of graphs with at least one edge starting 0 (see Figure 3).

(Step 2) If there is a graph, among the ones obtained above, that has more than one vertices with distance $2km$ from the vertex 0 , then by the same reasoning we can express it as the sum of graphs that has at least one vertex with distance $2km$

from the vertex 0.

(Step 3) Repeat this process until the original graph may be expressed as a sum of graphs each of which the connected component with vertex 0 is a linear graph.

(Step 4) Do the same for the vertex $n+1$.

Eventually our graph is expressed as a sum of graphs with linear graphs only. This completes the proof. $\square$

Figure 3

Notice that $\lambda_0 \neq 0$ is only a sufficient condition so that $H^n(\nabla)$ is generated by $(n+1)!$ forms; this condition makes the story less complicated. Anyhow we get

Corollary 2.2.2. *(a basis of the twisted cohomology group) If the exponents satisfy (NI), then the twisted cohomology is purely n -codimensional, and the $(n+1)!$ monomials above form a basis of $H^n(\nabla)$.*

Proof. Only to use $(n+1)! = \dim H_n^{lf}(T, \mathcal{S}) = \dim H^n(T, \mathcal{S})$. $\square$

Example 2.2.1. Following is a basis when $n=2$:

$$\begin{aligned} \varphi_1 &= \frac{dt_1 \wedge dt_2}{t_1(t_2-1)}, \quad \varphi_2 = \frac{dt_1 \wedge dt_2}{(t_1-1)t_2}, \quad \varphi_3 = \frac{dt_1 \wedge dt_2}{t_1(t_2-t_1)}, \\ \varphi_4 &= \frac{dt_1 \wedge dt_2}{(t_1-t_2)t_2}, \quad \varphi_5 = \frac{dt_1 \wedge dt_2}{(t_1-1)(t_2-t_1)}, \quad \varphi_6 = \frac{dt_1 \wedge dt_2}{(t_1-t_2)(t_2-1)} \end{aligned}$$

3. System of differential equations

By the geometric consideration made in Section 1, the $(n+1)!$ functions f_j defined by the integral

$$f_j = \int \Phi \varphi_j, \quad \{\varphi_j\}: \text{a basis of } H^n(\nabla)$$

satisfy a Fuchsian system of differential equations in the form

$$d^t(f_1, \dots, f_{(n+1)!}) = \Omega^t(f_1, \dots, f_{(n+1)!})$$

of rank $(n+1)!$ with regular singularities at least along the arrangement A'' . In fact the set of $x \in \mathbf{P}^n$ such that the arrangement $B = A' \cup H_x$ changes its intersection pattern is nothing but A'' , by the definition of duality. Notice that we do not know a priori the singularities of the matrix 1-form Ω . What we prove in the following proposition is that by virtue of the clever choice of the basis, the matrix Ω has logarithmic poles exactly along A'' and the residues are matrices with linear forms in the exponents λ_{ij} and λ_0 as entries. The system admits symmetry under the group S_{n+2} , of course. The 1-form Ω , when $n=2$, is given separately.

Set

$$\langle \langle s_0, s_1, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \rangle := \int \Phi \langle s_0, s_1, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle.$$

We are going to find a system of differential equations for them. According to

$$d_x \int \Phi \varphi = \int ((d_x \Phi) \varphi + \Phi d_x \varphi) = \int \Phi ((d_x \log \Phi) \varphi + d_x \varphi),$$

we define derivations

$$\nabla_j := \frac{\partial}{\partial x^j} + \frac{\partial \log \Phi}{\partial x^j}, \quad \nabla_x := \sum_{j=1}^n \nabla_j dx^j.$$

Hereafter d_t denotes the differentiation with respect to t and d_x the differentiation with respect to x . By the Lemma 3.2 below, for each $1 \leq j \leq n$, we have

$$\begin{aligned} \nabla_{s_j} \langle s_0, s_1, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle &= -\frac{\lambda_0 t_{s_j}}{l_0} \langle s_0, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \\ &= \sum_{k=1}^a \frac{\lambda_0 (-1)^{k-1} a_{s_k}^{(s_j)}}{x^{s_k} + \dots + x^{s_a}} d_t \log l_0 \wedge \langle s_0, \dots, s_{k-1}; s_k, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \\ &\quad + \sum_{k=a+1}^n \frac{\lambda_0 (-1)^{k-1} a_{s_k}^{(s_j)}}{x^{s_{a+1}} + \dots + x^{s_k}} d_t \log l_0 \wedge \langle s_0, \dots, s_a; s_{a+1}, \dots, s_k; s_{k+1}, \dots, s_{n+1} \rangle \\ &\quad - \lambda_0 a_{s_{n+1}}^{(s_j)} \langle s_0, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle, \end{aligned}$$

where $\langle s_0, \dots, s_a; s_{a+1}, \dots, s_k; s_{k+1}, \dots, s_{n+1} \rangle$ denotes the $(n-1)$ -form

$$\begin{aligned} & \langle\langle s_0, s_1 \rangle\rangle \wedge \langle\langle s_1, s_2 \rangle\rangle \wedge \cdots \wedge \langle\langle s_{a-1}, s_a \rangle\rangle \wedge \langle\langle s_{a+1}, s_{a+2} \rangle\rangle \wedge \\ & \cdots \wedge \langle\langle s_{k-1}, s_k \rangle\rangle \wedge \langle\langle s_{k+1}, s_{k+2} \rangle\rangle \wedge \cdots \wedge \langle\langle s_n, s_{n+1} \rangle\rangle, \end{aligned}$$

with

$$\langle\langle i, j \rangle\rangle := d_i \log l_{ij};$$

and where the $a_{s_k}^{(s_j)}$'s, all with the denominator

$$d(a) := 1 - t_0 \sum_{r=1}^a x^{s_r} - t_{n+1} \sum_{r=a+1}^n x^{s_r},$$

are given as follows (summations are taken over r):
when $1 \leq k \leq a$

$$a_{s_k}^{(s_j)} = \begin{cases} t_0 \sum_k^a x^{s_r} / d(a), & 1 \leq j \leq k-1, \\ (1 - t_0 \sum_{a+1}^n x^{s_r} - t_{n+1} \sum_{a+1}^n x^{s_r}) / d(a), & k \leq j \leq a, \\ t_{n+1} \sum_k^a x^{s_r} / d(a), & a+1 \leq j \leq n, \end{cases}$$

when $a+1 \leq k \leq n$,

$$a_{s_k}^{(s_j)} = \begin{cases} t_0 \sum_{a+1}^k x^{s_r} / d(a), & 1 \leq j \leq a, \\ (1 - t_0 \sum_1^a x^{s_r} - t_{n+1} \sum_{k+1}^n x^{s_r}) / d(a), & a+1 \leq j \leq k, \\ t_{n+1} \sum_{a+1}^k x^{s_r} / d(a), & k+1 \leq j \leq n, \end{cases}$$

when $k = n+1$,

$$a_{s_k}^{(s_j)} = \begin{cases} t_0 / d(a), & 1 \leq j \leq a, \\ t_{n+1} / d(a), & a+1 \leq j \leq n. \end{cases}$$

Recall that $t_0 = 1$ and $t_{n+1} = 0$. Let us see that the numerators of the coefficients of $d_i \log l_0 \wedge \langle \dots; \dots; \dots \rangle$ can be expressed as sums of terms whose numerators do not include x .

When $1 \leq k \leq a$,

$$a_{s_k}^{(s_j)} / \sum_k^a x^{s_r} = \begin{cases} t_0 / d(a), & 1 \leq j \leq k-1, \\ t_0 / d(a) + 1 / \sum_k^a x^{s_r}, & k \leq j \leq a, \\ t_{n+1} / d(a), & a+1 \leq j \leq n. \end{cases}$$

When $a+1 \leq k \leq n$,

$$a_{s_k}^{(s_j)} / \sum_{a+1}^k x^{s_r} = \begin{cases} t_0/d(a), & 1 \leq j \leq a, \\ t_{n+1}/d(a) + 1/\sum_{a+1}^k x^{s_r}, & a+1 \leq j \leq k, \\ t_{n+1}/d(a), & k+1 \leq j \leq n. \end{cases}$$

Thus we know

$$\begin{aligned} & \nabla_x \langle s_0, s_1, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \\ &= \lambda_0 \sum_{k=1}^a (-1)^{k-1} \{ -d_x \log(d(a)) + d_x \log(\sum_k^a x^{s_r}) \} \\ & \quad \times d_t \log l_0 \wedge \langle s_0, \dots, s_{k-1}; s_k, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \\ &+ \lambda_0 \sum_{k=a+1}^n (-1)^{k-1} \{ -d_x \log(d(a)) + d_x \log(\sum_{a+1}^k x^{s_r}) \} \\ & \quad \times d_t \log l_0 \wedge \langle s_0, \dots, s_a; s_{a+1}, \dots, s_k; s_{k+1}, \dots, s_{n+1} \rangle \\ &+ \lambda_0 d_x \log(d(a)) \langle s_0, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle, \end{aligned}$$

Our next task is to express $d_t \log l_0 \wedge \langle s_0, \dots; \dots; \dots, s_{n+1} \rangle$ as a $\mathbf{Z}[\lambda_{ij}]$ -linear combination of reduced monomials in $H^n(\nabla)$. In order to avoid notation with clumsy indices, we introduce the following graphs:

$$\begin{aligned} & \langle \begin{array}{ccccccc} \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\ s_0 & & s_l & & s_a & & s_{a+1} & & s_r & & s_k & & s_{k+1} & & s_{n+1} \end{array} \rangle \\ &= \langle (s_0, s_1) \rangle \wedge \langle (s_1, s_2) \rangle \wedge \cdots \wedge \langle (s_{a-1}, s_a) \rangle \wedge \langle (s_{a+1}, s_{a+2}) \rangle \wedge \\ & \quad \cdots \wedge \langle (s_{r-1}, s_r) \rangle \wedge \langle (s_l, s_r) \rangle \wedge \langle (s_r, s_{r+1}) \rangle \wedge \\ & \quad \cdots \wedge \langle (s_{k-1}, s_k) \rangle \wedge \langle (s_{k+1}, s_{k+2}) \rangle \wedge \cdots \wedge \langle (s_n, s_{n+1}) \rangle, \\ & \langle \begin{array}{ccccccc} \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\ s_0 & & s_a & & s_{a+1} & & s_l & & s_k & & s_{k+1} & & s_r & & s_{n+1} \end{array} \rangle \\ &= \langle (s_0, s_1) \rangle \wedge \langle (s_1, s_2) \rangle \wedge \cdots \wedge \langle (s_{a-1}, s_a) \rangle \wedge \langle (s_{a+1}, s_{a+2}) \rangle \wedge \\ & \quad \cdots \wedge \langle (s_{l-1}, s_l) \rangle \wedge \langle (s_l, s_r) \rangle \wedge \langle (s_l, s_{l+1}) \rangle \wedge \\ & \quad \cdots \wedge \langle (s_{k-1}, s_k) \rangle \wedge \langle (s_{k+1}, s_{k+2}) \rangle \wedge \cdots \wedge \langle (s_n, s_{n+1}) \rangle. \end{aligned}$$

For a d_t -closed form ϕ , the form

$$\begin{aligned} \nabla_\omega \phi &= \omega \wedge \phi \\ &= \{\lambda_0 d_t \log l_0 + \sum_{i < j} \lambda_{ij} \langle (i, j) \rangle\} \wedge \phi \end{aligned}$$

is cohomologous to zero; thus we have

$$\begin{aligned}
& - \lambda_0 d_i \log l_0 \wedge \langle s_0, \dots, s_a ; s_{a+1}, \dots, s_k ; s_{k+1}, \dots, s_{n+1} \rangle \\
& \equiv \sum_{1 \leq i < j \leq n} \lambda_{ij} \langle (i, j) \rangle \wedge \langle s_0, \dots, s_a ; s_{a+1}, \dots, s_k ; s_{k+1}, \dots, s_{n+1} \rangle \\
& = \left\{ \sum_{\substack{0 \leq l \leq a \\ a+1 \leq r \leq k}} + \sum_{\substack{a+1 \leq l \leq k \\ k+1 \leq r \leq n+1}} \right\} \lambda_{s_l s_r} \langle (s_l, s_r) \rangle \wedge \langle s_0, \dots, s_a ; s_{a+1}, \dots, s_k ; s_{k+1}, \dots, s_{n+1} \rangle \\
& = \sum_{\substack{0 \leq l \leq a \\ a+1 \leq r \leq k}} \lambda_{s_l s_r} (-1)^{r-1} \langle \text{Diagram 1} ; \text{Diagram 2} ; \text{Diagram 3} \rangle \\
& \quad + \sum_{\substack{a+1 \leq l \leq k \\ k+1 \leq r \leq n+1}} \lambda_{s_l s_r} (-1)^{l-1} \langle \text{Diagram 4} ; \text{Diagram 5} ; \text{Diagram 6} \rangle ;
\end{aligned}$$

in the sequel, we call (I) and (II) the first and the second terms, respectively.

The following computation, which is a recurrent use of the formula in the proof of Proposition 2.2.1 in Section 2.2, gives the reader an idea of expressing a branched graph in terms of linear graphs:

$$\begin{aligned}
& \langle \text{Diagram 1} \rangle \\
&= \langle \text{Diagram 2} \rangle + \langle \text{Diagram 3} \rangle + \dots \\
&= \sum_{c: a,b} sgn \left(\begin{matrix} a_p, \dots, a_1, b_1, \dots, b_q \\ c_1, c_2, \dots && \dots, c_{p+q} \end{matrix} \right) \langle \text{Diagram 4} \rangle,
\end{aligned}$$

where $\sum_{c,a,b}$ stands for the sum over all reorderings $c_1, c_2, \dots$ of the a 's and the b 's such that a_i is the left side of a_j , and b_i is the left side of b_j for $i < j$; we shall call

$$\{a_p \succ \cdots \succ a_1\} \quad \text{and} \quad \{b_1 \prec \cdots \prec b_q\};$$

Applying the above formula, we have

where $\sum_{\langle \mathbf{I} \rangle}$ stands for summation over all shuffles $s'_{m+1}, \dots, s'_k$ of the three ordered sets:

Since we can change the order of the sums

we eventually have

and

$$\begin{aligned}
(\text{II}) &= \sum_{\substack{a+1 \leq l \leq k \\ k \leq m \leq n}} (\lambda_{s_l, s_{m+1}} + \dots + \lambda_{s_l, s_{n+1}}) (-1)^{l-1} \\
&\times \sum_{\langle \text{II} \rangle} \text{sgn} \left(\begin{array}{c} s_{a+1}, \dots, s_{m-1}, s_m \\ s'_{a+1}, \dots, s'_{m-1}, s_l \end{array} \right) \langle s_0, \dots, s_a; s'_{a+1}, \dots, s'_{m-1}, s_l, s_{m+1}, \dots, s_{n+1} \rangle,
\end{aligned}$$

where $\sum_{\langle \Pi \rangle}$ stands for summation over all shuffles $s'_{a+1}, \dots, s'_{m-1}$ of the three ordered sets:

$$\{s_{k+1} < \dots < s_{r-1}\}, \quad \{s_{a+1} < \dots < s_{l-1}\}, \quad \{s_{l+1} > \dots > s_k\}.$$

Therefore we obtain

Propositon 3.1. *The functions $\langle \langle s_0, s_1, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \rangle$ satisfy the following Fuchsian system of differential equations:*

$$\begin{aligned} d_x \langle \langle s_0, s_1, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \rangle &= - \sum_{k=1}^a (-1)^{k-1} \{ -d_x \log(d(a)) + d_x \log(\sum_k^a x^{s_r}) \} \\ &\quad \times \sum_{\substack{a+1 \leq r \leq k \\ 1 \leq m \leq a+1}} (\lambda_{s_0, s_r} \dots + \lambda_{s_{m-1}, s_r}) (-1)^{r-1} \\ &\quad \times \sum_{\langle I \rangle} \operatorname{sgn} \begin{pmatrix} s_m, s_{m+1}, \dots, s_k \\ s_r, s'_{m+1}, \dots, s'_k \end{pmatrix} \langle \langle s_0, \dots, s_{m-1}, s_r, s'_{m+1}, \dots, s'_k; s_{k+1}, \dots, s_{n+1} \rangle \rangle \\ &- \sum_{k=a+1}^n (-1)^{k-1} \{ -d_x \log(d(a)) + d_x \log(\sum_{a+1}^k x^{s_r}) \} \\ &\quad \times \sum_{\substack{a+1 \leq l \leq k \\ k \leq m \leq n}} (\lambda_{s_l, s_{m+1}} + \dots + \lambda_{s_l, s_{n+1}}) (-1)^{l-1} \\ &\quad \times \sum_{\langle II \rangle} \operatorname{sgn} \begin{pmatrix} s_a, \dots, s_{m-1}, s_m \\ s'_a, \dots, s'_{m-1}, s_l \end{pmatrix} \langle \langle s_0, \dots, s_{a-1}, s'_a, \dots, s'_{m-1}, s_l, s_{m+1}, \dots, s_{n+1} \rangle \rangle \\ &+ \lambda_0 d_x \log(d(a)) \langle \langle s_0, \dots, s_a; s_{a+1}, \dots, s_{n+1} \rangle \rangle. \end{aligned}$$

Lemma 3.2.

$$\begin{aligned} \frac{t_{s_l}}{l_0} \prod_{i=1}^a \frac{1}{t_{s_i} - t_{s_{i-1}}} \prod_{i=a+1}^n \frac{1}{t_{s_i} - t_{s_{i+1}}} &= \sum_{k=1}^a \frac{a_{s_k}^{(s_j)}}{l_0 \prod_{\substack{1 \leq i \leq a \\ i \neq k}} (t_{s_i} - t_{s_{i-1}}) \prod_{i=a+1}^n (t_{s_i} - t_{s_{i+1}})} \\ &+ \sum_{k=a+1}^n \frac{a_{s_k}^{(s_j)}}{l_0 \prod_{1 \leq i \leq a} (t_{s_i} - t_{s_{i-1}}) \prod_{\substack{a+1 \leq i \leq n \\ i \neq k}} (t_{s_i} - t_{s_{i+1}})} \\ &+ a_{s_k}^{(s_j)} \prod_{i=1}^a \frac{1}{t_{s_i} - t_{s_{i-1}}} \prod_{i=a+1}^n \frac{1}{t_{s_i} - t_{s_{i+1}}}. \end{aligned}$$

Proof. Multiplying the denominators, we have

$$t_{s_j} = \sum_{k=1}^a a_{s_k}^{(s_j)} (t_{s_k} - t_{s_{k-1}}) + \sum_{k=a+1}^n a_{s_k}^{(s_j)} (t_{s_k} - t_{s_{k+1}}) + a_{s_{n+1}}^{(s_j)} l_0.$$

When $1 \leq k \leq a$, by substituting

$$t_{s_i} = \begin{cases} t_0, & 1 \leq i \leq k-1 \\ (1 - t_0 \sum_1^{k-1} x^{s_r} - t_{n+1} \sum_{a+1}^n x^{s_r}) / \sum_k^a x^{s_r}, & k \leq i \leq a \\ t_{n+1}, & a+1 \leq i \leq n, \end{cases}$$

one gets $a_{s_k}^{(s_j)}$. When $a+1 \leq k \leq n$, substitute

$$t_{s_i} = \begin{cases} t_0, & 1 \leq i \leq a \\ (1 - t_0 \sum_1^a x^{s_r} - t_{n+1} \sum_{k+1}^n x^{s_r}) / \sum_{a+1}^k x^{s_r}, & a+1 \leq i \leq k \\ t_{n+1}, & k+1 \leq i \leq n. \end{cases}$$

When $k=n+1$, substitute

$$t_{s_i} = \begin{cases} t_0, & 1 \leq i \leq a \\ t_{n+1}, & a+1 \leq i \leq n, \end{cases}$$

Proposition 3.3. (System of differential equations when $n=2$) The six functions f_j defined by the integral $f_j = \int \varphi_j \Phi$ satisfy the following system :

$$d^t(f_1, \dots, f_6) = \Omega^t(f_1, \dots, f_6),$$

where Ω is a logarithmic 1-form, with poles exactly on A'' , given as follows :

$$\Omega = A_0 \frac{dx}{x} + a_1 \frac{d(x-1)}{x-1} + B_0 \frac{dy}{y} + B_1 \frac{d(y-1)}{y-1} + C_0 \frac{d(x+y)}{x+y} + C_1 \frac{d(x+y-1)}{x+y-1}$$

and

$$A_0 = \begin{pmatrix} -\alpha_1 & 0 & 0 & 0 & -\beta_1 & -\beta_1 - \gamma \\ 0 & -\beta_1 & -\alpha_1 & -\alpha_1 - \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_1 & -\alpha_1 & -\alpha_1 - \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 & -\beta_1 & -\beta_1 - \gamma \end{pmatrix},$$

$$\begin{aligned}
A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 + \beta_1 + \lambda & \alpha_1 & \alpha_1 + \gamma & \beta_2 + \gamma & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
B_0 &= \begin{pmatrix} -\beta_2 & 0 & -\alpha_2 - \gamma & -\alpha_2 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 & -\beta_2 - \gamma & -\beta_2 \\ -\beta_2 & 0 & -\alpha_2 - \gamma & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 & -\beta_2 - \gamma & -\beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
B_1 &= \begin{pmatrix} \beta_2 + \alpha_1 + \lambda & 0 & \alpha_2 + \gamma & \alpha_2 & \beta_1 & \beta_1 + \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
C_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 & \alpha_2 & -\beta_1 & \beta_2 \\ 0 & 0 & \alpha_1 & -\alpha_2 & \beta_1 & -\beta_2 \\ 0 & 0 & -\alpha_1 & \alpha_2 & -\beta_1 & \beta_2 \\ 0 & 0 & \alpha_1 & -\alpha_2 & \beta_1 & -\beta_2 \end{pmatrix}, \\
C_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & -\alpha_2 & \beta_1 + \beta_2 + \gamma + \lambda & 0 \\ \alpha_1 & 0 & -\alpha_1 & \alpha_2 & 0 & \beta_1 + \beta_2 + \gamma + \lambda \end{pmatrix},
\end{aligned}$$

where $\gamma = \gamma_{12}$, $\lambda = \lambda_0$, $x = x^1$, $y = x^2$.

Remark 3.1: One can enjoy the S_4 -symmetry, e.g. the set of eigenvalues of the residues along each divisor. Notice that the divisors defined by $x, y, x+y-1, 1$ are in one orbit and those defined by $x-1, y-1, x+y$ are in one orbit.

4. Intersection numbers

It is known under the condition (NI) that there is a perfect pairing called the intersection form:

$$H_n(X, \mathcal{S}) \times H_n^{lf}(X, \mathcal{S}^\vee) \rightarrow \mathbf{C}.$$

Basic definitions and facts of twisted homology and its intersection are collected in [KY]. We give in this section when $n=2$ the intersection numbers with respect to the following basis. Let $D_1, \dots, D_6 \in H_2^{lf}(T, \mathcal{S})$ be 3! twisted cycles illustrated in Figure 4, the arguments of the $\binom{2+2}{2}$ linear forms $l_1, \dots, l_6$ are 0 or $-\pi$ according to their signs.

Figure 4

Let $D_j^\vee \in H_2^{lf}(T, \mathcal{S}^\vee)$ be cycles analogously defined using Φ^{-1} instead, and $\text{reg } D_j \in H_2(T, \mathcal{S})$ the regularization of D_j . Put

$$c_j := \exp \alpha_j, \quad d_j := c_j - 1, \quad d_{jk\dots} := c_j c_k \cdots - 1.$$

Then the intersection numbers $X(i, j) := (\text{reg } D_j \cdot D_k^\vee)$ are given as follows:

$$\begin{aligned}
X(1, 1) &:= 1 + \frac{1}{d_3} + \frac{1}{d_6} + \frac{1}{d_4} + \frac{1}{d_3 \cdot d_6} + \frac{1}{d_{234}} \cdot \frac{d_{34}}{d_3 \cdot d_4} + \frac{1}{d_{456}} \cdot \frac{d_{46}}{d_4 \cdot d_6}, \\
X(2, 2) &:= 1 + \frac{1}{d_3} + \frac{1}{d_6} + \frac{1}{d_5} + \frac{1}{d_3 \cdot d_6} + \frac{1}{d_{135}} \cdot \frac{d_{35}}{d_3 \cdot d_5} + \frac{1}{d_{456}} \cdot \frac{d_{56}}{d_5 \cdot d_6}, \\
X(3, 3) &:= 1 + \frac{1}{d_1} + \frac{1}{d_4} + \frac{1}{d_5} + \frac{1}{d_1 \cdot d_4} + \frac{1}{d_{135}} \cdot \frac{d_{15}}{d_1 \cdot d_5} + \frac{1}{d_{456}} \cdot \frac{d_{45}}{d_4 \cdot d_5}, \\
X(4, 4) &:= 1 + \frac{1}{d_1} + \frac{1}{d_4} + \frac{1}{d_6} + \frac{1}{d_1 \cdot d_4} + \frac{1}{d_{126}} \cdot \frac{d_{16}}{d_1 \cdot d_6} + \frac{1}{d_{456}} \cdot \frac{d_{46}}{d_4 \cdot d_6}, \\
X(5, 5) &:= 1 + \frac{1}{d_2} + \frac{1}{d_5} + \frac{1}{d_6} + \frac{1}{d_2 \cdot d_5} + \frac{1}{d_{126}} \cdot \frac{d_{26}}{d_2 \cdot d_6} + \frac{1}{d_{456}} \cdot \frac{d_{56}}{d_5 \cdot d_6}, \\
X(6, 6) &:= 1 + \frac{1}{d_2} + \frac{1}{d_5} + \frac{1}{d_4} + \frac{1}{d_2 \cdot d_5} + \frac{1}{d_{234}} \cdot \frac{d_{24}}{d_2 \cdot d_4} + \frac{1}{d_{456}} \cdot \frac{d_{45}}{d_4 \cdot d_5}, \\
X(1, 6) &:= -\frac{c_4}{d_4} \cdot \left(\frac{1}{d_{234}} + 1 + \frac{1}{d_{456}} \right), \quad X(6, 1) := -\frac{1}{d_4} \cdot \left(\frac{1}{d_{234}} + 1 + \frac{1}{d_{456}} \right), \\
X(5, 4) &:= -\frac{c_6}{d_6} \cdot \left(\frac{1}{d_{126}} + 1 + \frac{1}{d_{456}} \right), \quad X(4, 5) := -\frac{1}{d_6} \cdot \left(\frac{1}{d_{126}} + 1 + \frac{1}{d_{456}} \right), \\
X(3, 2) &:= -\frac{c_5}{d_5} \cdot \left(\frac{1}{d_{135}} + 1 + \frac{1}{d_{456}} \right), \quad X(2, 3) := -\frac{1}{d_5} \cdot \left(\frac{1}{d_{135}} + 1 + \frac{1}{d_{456}} \right), \\
X(1, 2) &:= -\frac{c_6}{d_6} \cdot \left(\frac{1}{d_3} + 1 + \frac{1}{d_{456}} \right), \quad X(2, 1) := -\frac{1}{d_6} \cdot \left(\frac{1}{d_3} + 1 + \frac{1}{d_{456}} \right), \\
X(5, 6) &:= -\frac{c_5}{d_5} \cdot \left(\frac{1}{d_2} + 1 + \frac{1}{d_{456}} \right), \quad X(6, 5) := -\frac{1}{d_5} \cdot \left(\frac{1}{d_2} + 1 + \frac{1}{d_{456}} \right), \\
X(3, 4) &:= -\frac{c_4}{d_4} \cdot \left(\frac{1}{d_1} + 1 + \frac{1}{d_{456}} \right), \quad X(4, 3) := -\frac{1}{d_4} \cdot \left(\frac{1}{d_1} + 1 + \frac{1}{d_{456}} \right), \\
X(1, 4) &:= \frac{c_4 \cdot c_6}{d_{456}} \cdot \left(\frac{1}{d_4} + 1 + \frac{1}{d_6} \right), \quad X(4, 1) := \frac{c_5}{d_{456}} \cdot \left(\frac{1}{d_4} + 1 + \frac{1}{d_6} \right), \\
X(5, 2) &:= \frac{c_5 \cdot c_6}{d_{456}} \cdot \left(\frac{1}{d_5} + 1 + \frac{1}{d_6} \right), \quad X(2, 5) := \frac{c_4}{d_{456}} \cdot \left(\frac{1}{d_5} + 1 + \frac{1}{d_6} \right), \\
X(3, 6) &:= \frac{c_4 \cdot c_5}{d_{456}} \cdot \left(\frac{1}{d_4} + 1 + \frac{1}{d_5} \right), \quad X(6, 3) := \frac{c_6}{d_{456}} \cdot \left(\frac{1}{d_4} + 1 + \frac{1}{d_5} \right), \\
X(1, 3) &:= -\frac{c_6 \cdot c_4}{d_4 \cdot d_{456}}, \quad X(3, 1) := -\frac{c_5 \cdot c_4}{d_4 \cdot d_{456}}, \\
X(1, 5) &:= -\frac{c_4 \cdot c_6}{d_6 \cdot d_{456}}, \quad X(5, 1) := -\frac{c_5 \cdot c_6}{d_6 \cdot d_{456}}, \\
X(5, 3) &:= -\frac{c_6 \cdot c_5}{d_5 \cdot d_{456}}, \quad X(3, 5) := -\frac{c_4 \cdot c_5}{d_5 \cdot d_{456}}, \\
X(6, 2) &:= -\frac{c_6 \cdot c_5}{d_5 \cdot d_{456}}, \quad X(2, 6) := -\frac{c_4 \cdot c_5}{d_5 \cdot d_{456}},
\end{aligned}$$

$$X(2, 4) := -\frac{c_4 \cdot c_6}{d_6 \cdot d_{456}}, \quad X(4, 2) := -\frac{c_5 \cdot c_6}{d_6 \cdot d_{456}},$$

$$X(4, 6) := -\frac{c_5 \cdot c_4}{d_4 \cdot d_{456}}, \quad X(6, 4) := -\frac{c_6 \cdot c_4}{d_4 \cdot d_{456}}.$$

The monodromy group of the system obtained above leaves the matrix $X = (X(i, j))$ invariant; the following is the precise meaning. For an arbitrary $0 \neq \varphi \in H^2$, let $g_i(x)$ be functions around a point, say $b \in \mathbf{P}^2 - A''$ defined by

$$g_i(x) := \int_{D_i} \Phi \varphi \quad 1 \leq i \leq 6.$$

For any loop ρ in $\mathbf{P}^2 - A''$, there is an $M(\rho) \in GL(6, \mathbf{Q}(c_1, \dots, c_6))$ such that the resulting function g_i^ρ after the analytic continuation along ρ can be expressed as

$$(g_1^\rho, \dots, g_6^\rho) = (g_1, \dots, g_6) M(\rho).$$

Let $\vee$ be the automorphism on $\mathbf{Q}(c_1, \dots, c_6)$ over $\mathbf{Q}$ defined by $c_j \mapsto c_j^{-1}$. Then we have

$$X = {}^t M^\vee(\rho) X M(\rho), \quad \rho \in \pi_1(\mathbf{P}^2 - A'', b).$$

By the way, a computation shows that the determinant of the intersection matrix is given by

$$\frac{(c_1 c_2 c_3 c_4 c_5 c_6 - 1)^4}{(c_1 c_2 c_6 - 1)(c_1 c_3 c_5 - 1)(c_2 c_3 c_4 - 1)(c_4 c_5 c_6 - 1) \prod_{j=1}^6 (c_j - 1)^2}.$$

One can easily check that the determinant is non-zero finite if and only if the exponents satisfy the condition (NI); this implies that the cycles $D_1, \dots, D_6$ form a basis of $H_2^{lf}(T, \mathcal{A})$, proving the Conjecture in 2.1 when $n=2$.

5. Solutions given by power series

Let us consider a function $f_D(x)$ on $\mathbf{P}^n$ defined by the integral stated in the introduction over a cycle $D \in H_n^{lf}(T, \mathcal{A})$; generically it is singular along A'' . Around each point of A'' where n hyperplanes in A meet, there are $n!$ linearly independent cycles D_j such that $f_{D_j}(x)$ are holomorphic at the point. Indeed this can be led by the fact that for each vertex v of the n -simplex $\Delta \subset \mathbf{RP}_n$ there are $n!$

chambers (among the $(n+1)!$) with v as a vertex (see Section 2.1). In this section, when $n=2$, we present them in power series. For notational simplicity we write (s, t) and (x, y) in place of (t_1, t_2) and (x^1, x^2) . Let

$$D_1 := \{(s, t) \in \mathbf{R}^2 \mid s > 0, 1-s > 0, s-t > 0\}$$

(see Figure 5) and put

$$f(x, y) := \int_{D_1} s^{\alpha_1} t^{\alpha_2} (1-s)^{\beta_1} (1-t)^{\beta_2} (s-t)^{\gamma} (1-xs-yt)^{\lambda} dt \wedge ds,$$

where (x, y) is supposed to be sufficiently near the origin and the arguments of the linear forms in the integrand are 0.

Figure 5

Substituting the binomial formula

$$(1-xs-yt)^{\lambda} = \sum_{m,n=0}^{\infty} \frac{(-\lambda, m+n)}{(1, m)(1, n)} s^m t^n x^m y^n,$$

where $(\lambda, m) := \lambda(\lambda+1)\cdots(\lambda+m-1)$, we have

$$f = \sum_{m,n} B_{m,n} \frac{(-\lambda, m+n)}{(1, m)(1, n)} x^m y^n,$$

where

$$\begin{aligned}
B_{m,n} &= \int_{D_1} s^{\alpha_1+m} t^{\alpha_2+n} (1-s)^{\beta_1} (1-t)^{\beta_2} (s-t)^\tau dt \wedge ds \\
&= \int_0^1 s^{\alpha_1+m} (1-s)^{\beta_1} \left(\int_0^s t^{\alpha_2+n} (1-t)^{\beta_2} (s-t)^\tau dt \right) ds.
\end{aligned}$$

Recall the formulae (cf. [IKSY, pp 38, 99])

$$\begin{aligned}
&\int_0^s t^{a-c} (1-t)^{c-b-1} (s-t)^{-a} dt \\
&= \frac{\Gamma(a-c+1)\Gamma(1-a)}{\Gamma(2-c)} s^{1-c} F(a-c+1, b-c+1, 2-c; s),
\end{aligned}$$

where

$$F(a, b, c; s) := \sum_{p=0}^{\infty} \frac{(a, p)(b, p)}{(c, p)(1, p)} s^p.$$

Putting these formulae into the expression of $B_{m,n}$ and using the Beta-formula, we have

$$\begin{aligned}
B_{m,n} &= \sum_p \frac{\Gamma(\alpha_2+1+n)\Gamma(\gamma+1)}{\Gamma(\alpha_2+\gamma+2+n)} \frac{(\alpha_2+1+n, p)(\beta_2, p)}{(\alpha_2+\gamma+2+n, p)(1, p)} \\
&\quad \times \frac{\Gamma(\alpha_1+\alpha_2+\gamma+2+m+n+p)\Gamma(\beta_1)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\gamma+3+m+n+p)};
\end{aligned}$$

eventually we get

Proposition 5.1.

$$f = C \sum_{m,n,p} \frac{(\alpha_2+1, n+p)(\gamma', m+n+p)(-\beta_1, p)(-\lambda, m+n)}{(\alpha_2+\gamma+2, n+p)(\beta_1+\gamma'+1, m+n+p)(1, p)} \frac{x^m y^n}{m! n!},$$

where

$$C = \frac{\Gamma(\alpha_2+1)\Gamma(\beta_1+1)\Gamma(\gamma+1)\Gamma(\gamma')}{\Gamma(\alpha_2+\gamma+2)\Gamma(\beta_1+\gamma'+1)}, \quad \gamma' := \alpha_1 + \alpha_2 + \gamma + 2.$$

Notice that each sum over p gives a special value of the generalized hypergeometric series ${}_3F_2$ at 1 (cf. [SU]). Taking

$$D_2 = \{(s, t) \in \mathbf{R}^2 \mid t > 0, 1-t > 0, t-s > 0\}$$

instead of D_1 , we get a power series, obtained from the above by the interchange of the indices $1 \leftrightarrow 2$.

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