DIMENSIONS IN A SEPARABLE METRIC SPACE

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Abstract. It often occurs that a Borel finite measure plays an important role in estimating the Hausdorff dimension of a Borel set of the Euclidean space, for example [6, 12, 16, 21, 22]. In particular, in the Euclidean space and homogeneous (in the sense of Coifman and Weiss) space, Frostman's theorem [8, 14] and Tricot's theorem [20] hold and those facts imply the equivalence among dimensions and several indices of a finite Borel measure [7]. But it is not known whether Frostman's theorem and Tricot's theorem can be extended to general metric space or not.

The aim of this paper is to analyse systematically the several fractional dimensions of a subset and a finite Borel measure and establish new relations among them in a general separable metric space. As an application, we prove that Frostman's and Tricot's theorems are equivalent.

1. Introduction and the Statements of the Results

1a. Fractional Dimensions Let (T, d) be a separable metric space and $\mathcal{G}$ the family of the Hausdorff functions h , that is, $h: [0, \infty) \rightarrow [0, \infty]$ is *non-decreasing*, $h(0) = \inf_{x>0} h(x) = 0$, $h(x) > 0$ if $x > 0$ and $h(2x) = O(h(x))$ ($x \rightarrow 0$). In this section, based on a Hausdorff function, we define the Hausdorff, packing and box dimensions of a subset of T and then, in Section 2, estimate them by finite Borel measures on it.

Let $\mathcal{B}$ be the Borel field of T and $F \subseteq T$. Then we denote the diameter of F by $d(F)$ and put, for convention, $h(F) = h(d(F))$. Let $\{B(t_j; r_j)\}_j$, $t_j \in T$, $r_j > 0$, be a countable family of *closed* balls. Then, for $\delta > 0$, we call $\{B(t_j; r_j)\}_j$ a δ -*cover* of $F \subseteq T$ if it covers F , $t_j \in F$ and $d(B(t_j; r_j)) \leq \delta$ for each j , and a δ -*packing* of F if $t_j \in F$, $2r_j \leq \delta$ for each j and $\{B(t_j; r_j)\}_j$ are disjoint each other. For each $h \in \mathcal{G}$, $\alpha > 0$, $\delta > 0$ and $F (\neq \emptyset) \subset T$, define

$$h\mathcal{H}_\delta^\alpha(F) = \inf_j \{ \sum h(B(t_j; r_j))^\alpha; \{B(t_j; r_j)\}_j \text{ is the } \delta\text{-cover of } F \},$$

$$h\mathcal{P}_\delta^\alpha(F) = \sup_j \{ \sum h(2r_j)^\alpha; \{B(t_j; r_j)\}_j \text{ is the } \delta\text{-packing of } F \}$$

and $h\mathcal{H}_\delta^\alpha(\emptyset) = h\tilde{\mathcal{P}}_\delta^\alpha(\emptyset) = 0$. Since $h\mathcal{H}_\delta^\alpha(F)$ is non-decreasing as $\delta \searrow 0$, the limit $h\mathcal{H}^\alpha(F) = \lim_{\delta \rightarrow 0} h\mathcal{H}_\delta^\alpha(F) (\leq \infty)$ is well-defined and $h\mathcal{H}^\alpha$ is a Borel measure on T [18], which we call the α -dimensional Hausdorff measure with respect to $h \in \mathcal{G}$. On the other hand, $h\tilde{\mathcal{P}}_\delta^\alpha(F)$ is non-increasing as $\delta \searrow 0$, so that the limit $h\tilde{\mathcal{P}}^\alpha(F) = \lim_{\delta \rightarrow 0} h\tilde{\mathcal{P}}_\delta^\alpha(F) (\leq \infty)$ is well-defined. The α -dimensional packing measure with respect to $h \in \mathcal{G}$ is defined by

$$h\mathcal{P}^\alpha(F) = \inf \left\{ \sum_n h\tilde{\mathcal{P}}^\alpha(F_n) ; F \subseteq \bigcup_n F_n, F_n \in \mathcal{B} \right\},$$

and $h\mathcal{P}^\alpha$ is a Borel measure on T [10]. The Hausdorff and packing dimensions of $F \subseteq T$ with respect to $h \in \mathcal{G}$ are defined by

$$h\text{-dim } F = \sup \{ \alpha > 0 ; h\mathcal{H}^\alpha(F) = \infty \}$$

and

$$h\text{-Dim } F = \sup \{ \alpha > 0 ; h\mathcal{P}^\alpha(F) = \infty \},$$

respectively, with convention $\sup \{ \cdot ; \emptyset \} = 0$.

Let $\mathcal{M}(T)$ be a family of the finite measures on $(T, \mathcal{B})$ and define, for each $F \in \mathcal{B}$, $\mathcal{M}^+(F) = \{ \mu \in \mathcal{M}(T) ; \mu(F) > 0 \}$. Then, for $h \in \mathcal{G}$ and $\mu \in \mathcal{M}(T)$, the lower and upper Lipschitz exponents of μ at $t \in T$ with respect to h are defined by

$$\phi_h(\mu : t) = \liminf_{r \searrow 0} \frac{\log \mu(B(t : r))}{\log h(r)}$$

and

$$\Phi_h(\mu : t) = \limsup_{r \searrow 0} \frac{\log \mu(B(t : r))}{\log h(r)},$$

respectively. In the case where $h(x) = x$ we shall always omit the symbol h (e.g. $\dim$ and $\phi(\mu : t) \cdots$). Also denote the restriction of $\mu \in \mathcal{M}(T)$ to $F \in \mathcal{B}$ by μ_F .

The purpose of Section 2 is to estimate the Hausdorff and packing dimensions of a Borel set by $\phi_h(\mu : t)$ and $\Phi_h(\mu : t)$, respectively. If (T, d) is the Euclidean space, Tricot gave the following theorem.

Theorem 1.1 [20] *Let (T, d) be the Euclidean space and $F \in \mathcal{B}$. Then we have*

$$\begin{aligned} \text{(a)} \quad \dim F &= \sup_{\mu \in \mathcal{M}^+(F)} \inf_{t \in F} \phi(\mu: t), \\ \text{(b)} \quad \text{Dim } F &= \inf_{\mu \in \mathcal{M}^+(F)} \sup_{t \in F} \Phi(\mu: t). \end{aligned}$$

It is not known whether the above theorem can be extended to a general separable metric space or not, but we prove the following theorem in Section 3.

Theorem 1.2 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $F \in \mathcal{B}$. Then we have*

$$\begin{aligned} & \inf_{\mu \in \mathcal{M}^+(F)} \text{ess.inf}_{t \in F} \phi_h(\mu: t) = \inf_{\mu \in \mathcal{M}^+(F)} \text{ess.sup}_{t \in F} \phi_h(\mu: t) \\ & \leq \sup_{\mu \in \mathcal{M}^+(F)} \inf_{t \in F} \phi_h(\mu: t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.inf}_{t \in F} \phi_h(\mu: t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.sup}_{t \in F} \phi_h(\mu: t) \\ & \leq h\text{-dim } F \\ & \leq \inf_{\mu \in \mathcal{M}^+(F)} \sup_{t \in F} \phi_h(\mu: t) \end{aligned}$$

and

$$\begin{aligned} & \inf_{\mu \in \mathcal{M}^+(F)} \text{ess.inf}_{t \in F} \Phi_h(\mu: t) = \inf_{\mu \in \mathcal{M}^+(F)} \text{ess.sup}_{t \in F} \Phi_h(\mu: t) \\ & \leq \sup_{\mu \in \mathcal{M}^+(F)} \inf_{t \in F} \Phi_h(\mu: t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.inf}_{t \in F} \Phi_h(\mu: t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.sup}_{t \in F} \Phi_h(\mu: t) \\ & \leq h\text{-Dim } F \\ & \leq \inf_{\mu \in \mathcal{M}^+(F)} \sup_{t \in F} \Phi_h(\mu: t). \end{aligned}$$

As an immediate corollary of Theorem 1.2 we have the following results.

Corollary 1.3 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$, $\mu \in \mathcal{M}^+(T)$ and $\alpha > 0$. Then we have*

$$\begin{aligned} \text{(a)} \quad \mu(t \in T: \phi_h(\mu: t) \leq \alpha) &= \sup \{ \mu(F) : h\text{-dim } F \leq \alpha, F \in \mathcal{B} \}, \\ \text{(b)} \quad \mu(t \in T: \Phi_h(\mu: t) \leq \alpha) &= \sup \{ \mu(F) : h\text{-Dim } F \leq \alpha, F \in \mathcal{B} \}. \end{aligned}$$

Note that Cutler [2] proved (a) in the case where $h(x) = x$, and she [3] also proved (b) in the case where $h(x) = x$ and $\mu \in \mathcal{M}^+(T)$ possesses the centred Vitali covering property, see [3] for precise definition.

Let $N(F; r)$, $F \subseteq T$, $r > 0$, be the minimal cardinal number of the family of closed balls with centers in F and radius r which cover F and define, for each $h \in \mathcal{G}$ and totally bounded $F \subseteq T$, the lower and upper box dimensions of F with respect to h by

$$h\text{-}\underline{\dim}_B F = \liminf_{r \searrow 0} \frac{\log N(F; r)}{-\log h(r)}$$

and

$$h\text{-}\overline{\dim}_B F = \limsup_{r \searrow 0} \frac{\log N(F; r)}{-\log h(r)},$$

respectively. We estimate the box dimensions of a Borel set as follows.

Theorem 1.4 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $F \in \mathcal{B}$. Assume that F is totally bounded. Then we have*

$$\begin{aligned} \text{(a)} \quad h\text{-}\dim F &\leq h\text{-}\underline{\dim}_B F \leq \inf_{\mu \in \mathcal{M}^+(F)} \liminf_{r \searrow 0} \sup_{t \in F} \frac{\log \mu(B(t; r))}{\log h(r)}, \\ \text{(b)} \quad \text{Dim } F &\leq \overline{\dim}_B F \leq \inf_{\mu \in \mathcal{M}^+(F)} \limsup_{r \searrow 0} \sup_{t \in F} \frac{\log \mu(B(t; r))}{\log r}. \end{aligned}$$

In the case where (T, d) is the Euclidean space and $h(x) = x$ the estimations $\dim \leq \underline{\dim}_B$ and $\text{Dim} \leq \overline{\dim}_B$ were proved by several authors, for example [6].

1b. Capacity Dimension Let $\mathcal{G}'$ be the family of Hausdorff functions h such that there exists $0 < \eta_h \leq \delta_h$, where $\delta_h = \sup \{x > 0; h(x) \leq 1\}$, such that h is continuous on $(0, \eta_h]$. For any $h \in \mathcal{G}'$, $\mu \in \mathcal{M}^+(T)$ and $\alpha \geq 0$, define the α -potential of μ with respect to h by

$$U_h^\alpha(\mu; t) = \int_T \frac{1}{h(d(s, t))^\alpha} \mu(ds), \quad t \in T$$

and the α -energy of μ with respect to h by

$$I_h^\alpha(\mu) = \int_T U_h^\alpha(\mu; t) \mu(dt).$$

For $h \in \mathcal{G}'$, $\alpha \geq 0$ and $F \in \mathcal{B}$, define the α -capacity of F with respect to h by

$$h\text{-Cap}^\alpha F = \left(\inf_{\mu \in \mathcal{M}^+(F)} I_h^\alpha(\mu) \right)^{-1},$$

and the capacity dimension of F with respect to h by

$$h\text{-dim}_c F = \sup \{ \alpha \geq 0; h\text{-Cap}^\alpha(F) > 0 \}.$$

Our definition of the capacity is a little different from the previous one, for

example [13, 7]. But our capacity dimension coincides with the previous one in the case where every finite Borel measure on T is a Radon measure.

Theorem 1.5 [8, 14, Frostman's theorem] *Let (T, d) be the Euclidean space. Then we have $\dim F = \dim_c F$ for each $F \in \mathcal{B}$.*

In Section 2b we prove the following theorem.

Theorem 1.6 *Let (T, d) be a separable metric space, $h \in \mathcal{G}'$ and $F \in \mathcal{B}$. Then*

$$h\text{-dim}_c F = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.inf}_{t \in F} \phi_h(\mu : t) \leq h\text{-dim } F.$$

Corollary 1.7 *Let (T, d) be a separable metric space and $h \in \mathcal{G}'$. Then the following two statements are equivalent.*

- (a) *For each $F \in \mathcal{B}$, we have $h\text{-dim } F = h\text{-dim}_c F$.*
- (b) *For each $F \in \mathcal{B}$, we have $h\text{-dim } F = \sup_{\mu \in \mathcal{M}^+(F)} \inf_{t \in F} \phi_h(\mu : t)$.*

Corollary 1.7 says that Frostman's theorem and Tricot's theorem (a) are equivalent.

1c. Dimensions of a Measure Let $h \in \mathcal{G}$ and $\mu \in \mathcal{M}^+(T)$. The *upper Hausdorff* and *upper packing* dimensions of μ with respect to h are defined by

$$h\text{-dim}^* \mu = \inf \{ h\text{-dim } F ; \mu(F) = \mu(T), F \in \mathcal{B} \}$$

and

$$h\text{-Dim}^* \mu = \inf \{ h\text{-Dim } F ; \mu(F) = \mu(T), F \in \mathcal{B} \},$$

respectively. On the other hand the *lower Hausdorff* and *lower packing* dimensions of μ with respect to h are defined by

$$h\text{-dim}_* \mu = \inf \{ h\text{-dim } F ; \mu(F) > 0, F \in \mathcal{B} \}$$

and

$$h\text{-Dim}_* \mu = \inf \{ h\text{-Dim } F ; \mu(F) > 0, F \in \mathcal{B} \},$$

respectively. The definition of the upper Hausdorff dimension was owed to Young [23], who established the relation between the upper Hausdorff dimension and the box dimensions (see below) in the case where $h(x) = x$ and (T, d) is compact. The lower Hausdorff and packing dimensions were defined by Haase

[11], who estimated these dimensions of a product measure in the case where $h(x) = x$ and (T, d) is the Euclidean space. On the other hand Fan [7] analysed the upper and lower Hausdorff dimensions systematically in the case where $h(x) = x$ and (T, d) is *homogeneous* (in the sense of Coifman and Weiss), that is,

$$\sup_{t \in T, r > 0} N\left(B(t; r) : \frac{r}{2}\right) < \infty.$$

Assume that a set function $\mathcal{J} : \mathcal{B} \rightarrow [0, \infty)$ satisfies $\mathcal{J}(A) \leq \mathcal{J}(B)$ for $A, B (A \subseteq B) \in \mathcal{B}$. Then we say that $\mu \in \mathcal{M}(T)$ is *absolutely continuous* with respect to $\mathcal{J}$ ($\mu \ll \mathcal{J}$) if $\mathcal{J}(F) = 0, F \in \mathcal{B}$, implies $\mu(F) = 0$ and that μ and $\mathcal{J}$ are *singular* ($\mu \perp \mathcal{J}$) if there exists an $F \in \mathcal{B}$ such that $\mu(F) = \mu(T)$ and $\mathcal{J}(F) = 0$. We prove the following theorem in Section 3.

Theorem 1.8 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $\mu \in \mathcal{M}^+(T)$. Then*

- (a) $h\text{-dim}_* \mu = \text{ess. inf}_{t \in T} \phi_h(\mu : t) = \sup \{ \alpha > 0 ; \mu \ll h\text{-}\mathcal{H}^\alpha \},$
- (b) $h\text{-dim}^* \mu = \text{ess. sup}_{t \in T} \phi_h(\mu : t) = \inf \{ \alpha > 0 ; \mu \perp h\text{-}\mathcal{H}^\alpha \},$
- (c) $h\text{-Dim}_* \mu = \text{ess. inf}_{t \in T} \Phi_h(\mu : t) = \sup \{ \alpha > 0 ; \mu \ll h\text{-}\mathcal{P}^\alpha \},$
- (d) $h\text{-Dim}^* \mu = \text{ess. sup}_{t \in T} \Phi_h(\mu : t) = \inf \{ \alpha > 0 ; \mu \perp h\text{-}\mathcal{P}^\alpha \},$

with convention $\inf \{ \cdot ; \emptyset \} = \infty$. If $h \in \mathcal{G}'$ then we have

- (e) $h\text{-dim}_* \mu = \sup \{ \alpha \geq 0 ; U_h^\alpha(\mu : t) < \infty \mu\text{-a.e.} \}$
 $= \sup \{ \alpha \geq 0 ; \mu = \sum_n \mu_n, \mu_n \in \mathcal{M}^+(T) \text{ and } I_h^\alpha(\mu_n) < \infty, \forall n \geq 1 \}$
 $= \sup \{ \alpha \geq 0 ; \mu \ll h\text{-Cap}^\alpha \},$

where $\mu = \sum_n \mu_n$ means the convergence in the total variation, and

- (f) $h\text{-dim}^* \mu = \inf \{ \alpha > 0 ; U_h^\alpha(\mu : t) = \infty \mu\text{-a.e.} \}$
 $= \inf \{ \alpha > 0 ; \text{if } \mu(F) > 0, F \in \mathcal{B}, \text{ then } I_h^\alpha(\mu_F) = \infty \}$
 $= \inf \{ \alpha > 0 ; \mu \perp h\text{-Cap}^\alpha \}.$

Fan [7] proved (a),(b),(e) and (f) in the case where $h(x) = x$ and (T, d) is *homogeneous*. In his arguments, the homogeneous condition plays a central role. Our proofs much depend on Theorem 1.2.

$\mu \in \mathcal{M}^+(T)$ is said to have a *totally bounded support* if there exists a totally bounded set $E \in \mathcal{B}$ such that $\mu(E) = \mu(T)$. Assume that μ has a *totally bounded*

support $E \in \mathcal{B}$ and define the *lower* and *upper* box dimensions of μ with respect to $h \in \mathcal{G}$ by

$$h\text{-}\underline{\dim}_{\mathbf{B}} \mu = \liminf_{\delta \searrow 0} \{h\text{-}\underline{\dim}_{\mathbf{B}} F; \mu(E \setminus F) \leq \delta, F(\subseteq E) \in \mathcal{B}\}$$

and

$$h\text{-}\overline{\dim}_{\mathbf{B}} \mu = \liminf_{\delta \searrow 0} \{h\text{-}\overline{\dim}_{\mathbf{B}} F; \mu(E \setminus F) \leq \delta, F(\subseteq E) \in \mathcal{B}\},$$

respectively. Our definitions of box dimensions coincide with those of Young [23] in the case where (T, d) is compact. In Section 3 we also prove the following theorem.

Theorem 1.9 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and assume that $\mu \in \mathcal{M}(T)$ has a totally bounded support. Then we have*

$$h\text{-}\dim^* \mu \leq h\text{-}\underline{\dim}_{\mathbf{B}} \mu \leq h\text{-}\overline{\dim}_{\mathbf{B}} \mu \leq h\text{-}\text{Dim}^* \mu$$

and, in particular, $\overline{\dim}_{\mathbf{B}} \mu = \text{Dim}^* \mu$.

Young [23] proved $\dim^* \mu \leq \underline{\dim}_{\mathbf{B}} \mu \leq \overline{\dim}_{\mathbf{B}} \mu$ in the case where (T, d) is compact.

In Section 4 we characterize the *exact dimensionality* and the *dimension regularity* (see Section 4 for the definitions) of a finite Borel measure. In Section 5 we estimate the dimensions of a product measure.

2. Estimations of the Dimensions of a Borel Set

2a. Fractional Dimensions The following well known lemma plays an important role in this subsection. For $B = B(t; r)$ denote $\widehat{B} = B(t; 5r)$.

Lemma 2.1 *Let (T, d) be a separable metric space, $\delta > 0$ and $\mathcal{F} \neq \emptyset$ a family of closed balls of T . Denote $F = \{t \in T; B(t; r) \in \mathcal{F}\}$ and assume that $\sup\{2r; B(t; r) \in \mathcal{F}\} \leq \delta$. Then there exists a δ -packing $\{B_j\}_j \subseteq \mathcal{F}$ of F such that $F \subseteq \bigcup_j \widehat{B}_j$.*

PROOF. See page 27 of [4]. They state the result for the Euclidean space but the proof also works in our case.

For an $h \in \mathcal{G}$, there exists a constant $T_h \geq 1$ such that $h(2x) \leq T_h h(x)$ for each $0 \leq x < 2^{-1} \delta_h$, and we have $h\text{-}\dim F = \inf\{\alpha > 0; h\text{-}\mathcal{H}^\alpha(F) = 0\}$ and $h\text{-}\text{Dim} F = \inf\{\alpha > 0; h\text{-}\mathcal{P}^\alpha(F) = 0\}$ for each $F \subseteq T$. Moreover, the Hausdorff and packing

dimensions have important properties as follows.

Proposition 2.2 *Let (T, d) be a separable metric space and $h \in \mathcal{G}$. The Hausdorff and packing dimensions have the following properties.*

(a) (Monotonicity) *If $E \subseteq F (\subseteq T)$, then we have $h\text{-dim } E \leq h\text{-dim } F$, $h\text{-Dim } E \leq h\text{-Dim } F$.*

(b) (Countable stability) *For a countable family $\{F_n\}_n$ of subsets of T , we have $h\text{-dim } (\cup_n F_n) = \sup_n h\text{-dim } F_n$ and $h\text{-Dim } (\cup_n F_n) = \sup_n h\text{-Dim } F_n$.*

(c) *We have $h\text{-dim } F \leq h\text{-Dim } F$ for each $F \subseteq T$.*

PROOF. (a) and (b) are easy to check, for example [6].

(c) Without loss of generality we may assume $h\text{-Dim } F < \infty$. Choose any $h\text{-Dim } F < \alpha < \infty$. Then by definition we have $h\text{-}\mathcal{P}^\alpha(F) = 0$, so that there exist $\{F_n\}_n \subseteq \mathcal{B}$ such that $F \subseteq \cup_n F_n$ and $\sum_n h\text{-}\tilde{\mathcal{P}}^\alpha(F_n) < 1$. Fix any integer $n \geq 1$. Since $h\text{-}\tilde{\mathcal{P}}^\alpha(F_n) < 1$ there is a $0 < \delta_0 < 10^{-1} \delta_h$ such that $h\text{-}\tilde{\mathcal{P}}_\delta^\alpha(F_n) \leq 1$ for each $0 < \delta < \delta_0$. Fix any $0 < \delta < \delta_0$ and denote $\mathcal{F}_\delta = \{B(t; r) : t \in F_n, 2r \leq \delta\}$. By Lemma 2.1 there is a δ -packing $\{B(t_j; r_j)\}_{j \in \mathcal{F}_\delta}$ of F_n such that $F_n \subseteq \cup_j B(t_j; 5r_j)$, therefore we have

$$\sum_j h(B(t_j; 5r_j))^\alpha \leq \sum_j h(10r_j)^\alpha \leq T_h^{3\alpha} \sum_j h(2r_j)^\alpha \leq T_h^{3\alpha} h\text{-}\tilde{\mathcal{P}}_\delta^\alpha(F_n),$$

thus $h\text{-}\mathcal{H}_{5\delta}^\alpha(F_n) \leq T_h^{3\alpha}$. Taking $\delta \searrow 0$ we have $h\text{-}\mathcal{H}^\alpha(F_n) \leq T_h^{3\alpha}$ and $h\text{-dim } F_n \leq \alpha$ by definition of the Hausdorff dimension. The monotonicity and countable stability of the Hausdorff dimension imply $h\text{-dim } F \leq \alpha$. Since $h\text{-Dim } F < \alpha < \infty$ was arbitrary we have $h\text{-dim } F \leq h\text{-Dim } F$. $\square$

Let $h \in \mathcal{G}$ and $\mu \in \mathcal{M}(T)$. Then $\phi_h(\mu; t)$ and $\Phi_h(\mu; t)$ are Borel measurable functions on T since $\mu(B(t; r))$ is upper semi continuous for each $r > 0$.

Proposition 2.3 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $F \in \mathcal{B}$. Then*

- (a) $\sup_{\mu \in \mathcal{M}^+(F)} \inf_{t \in F} \phi_h(\mu; t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.inf}_{t \in F} \phi_h(\mu; t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.sup}_{t \in F} \phi_h(\mu; t),$
(b) $\sup_{\mu \in \mathcal{M}^+(F)} \inf_{t \in F} \Phi_h(\mu; t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.inf}_{t \in F} \Phi_h(\mu; t) = \sup_{\mu \in \mathcal{M}^+(F)} \text{ess.sup}_{t \in F} \Phi_h(\mu; t).$

PROOF. (a) Denote the first and third terms by Δ_1 and Δ_2 , respectively. It is enough to prove $\Delta_1 \geq \Delta_2$ and we may assume $\Delta_2 > 0$. Choose any $0 < \alpha < \Delta_2$. Then there exists $\mu \in \mathcal{M}^+(F)$ such that $\mu(t \in F; \phi_h(\mu; t) > \alpha) > 0$. Choose a closed set $K \subset \{t \in F; \phi_h(\mu; t) > \alpha\}$ such that $\mu(K) > 0$. It is easy to see that we

have $\phi_h(\mu_K: t) > \alpha$ for each $t \in K$ and $\phi_h(\mu_K: t) = \infty$ for each $t \in T \setminus K$, so that $\Delta_1 \geq \alpha$. Then it follows $\Delta_1 \geq \Delta_2$. We omit the proof of (b) since it is the same arguments as that of (a). $\square$

Proposition 2.4 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $F \in \mathcal{B}$. Then*

$$\begin{aligned} \text{(a)} \quad & \sup_{\mu \in \mathcal{M}^+(F)} \operatorname{ess.\,inf}_{t \in F} \phi_h(\mu: t) \leq h\text{-dim } F \leq \inf_{\mu \in \mathcal{M}^+(F)} \sup_{t \in F} \phi_h(\mu: t), \\ \text{(b)} \quad & \sup_{\mu \in \mathcal{M}^+(F)} \operatorname{ess.\,inf}_{t \in F} \Phi_h(\mu: t) \leq h\text{-Dim } F \leq \inf_{\mu \in \mathcal{M}^+(F)} \sup_{t \in F} \Phi_h(\mu: t). \end{aligned}$$

Cutler [2] proved (a) in the case where $h(x) = x$ and she [3] also proved (b) in the case where $h(x) = x$ and each $\mu \in \mathcal{M}^+(T)$ possesses the centred Vitali covering property.

PROOF OF PROPOSITION 2.4

(a) Denote the first and third terms by Δ_1 and Δ_2 , respectively.

First we prove $\Delta_1 \leq h\text{-dim } F$. We may assume $\Delta_1 > 0$. Choose any $0 < \alpha < \Delta_1$. Then we have $\phi_h(\mu: t) > \alpha$ μ a.e. on F for some $\mu \in \mathcal{M}^+(F)$, so that there exists a $0 < \delta_0 < \delta_h$ such that $\mu(A) > 0$, where $A = \{t \in F; \mu(B(t: r)) \leq h(r)^\alpha, 0 < \forall r \leq \delta_0\}$. Fix any $0 < \delta < \delta_0$ and choose any δ -cover $\{B(t_j: r_j)\}_j$ of A . Then we have, by the definition of A ,

$$0 < \mu(A) \leq \sum_j \mu(B(t: d(B(t_j: r_j)))) \leq \sum_j h(B(t_j: r_j))^\alpha,$$

which implies $h\mathcal{H}_\delta^\alpha(A) \geq \mu(A)$. Taking $\delta \searrow 0$ we have $h\mathcal{H}^\alpha(A) \geq \mu(A)$ and $h\text{-dim } F \geq \alpha$ by the definition and the monotonicity of the Hausdorff dimension, thus $h\text{-dim } F \geq \Delta_1$.

Next we prove $\Delta_2 \geq h\text{-dim } F$. We may assume $\Delta_2 < \infty$. Choose any $\Delta_2 < \alpha < \infty$. Then there exists $\mu \in \mathcal{M}^+(F)$ such that $\phi_h(\mu: t) < \alpha$, $\forall t \in F$, so that we may choose, for each $t \in F$, a real sequence $\{r_n(t)\}_n$, $0 < 20r_n(t) < \delta_h$, such that $\lim_{n \rightarrow \infty} r_n(t) = 0$ and

$$(2.1) \quad \mu(B(t: r_n(t))) \geq h(r_n(t))^\alpha.$$

Fix any $0 < \delta < \delta_h$ and put $\mathcal{F}_\delta = \{B(t: r_n(t)); t \in F, 2r_n(t) \leq \delta\}$. By Lemma 2.1 there exists a δ -packing $\{B(t_n: r_n)\}_n \subseteq \mathcal{F}_\delta$ of F such that $F \subseteq \bigcup_n B(t_n: 5r_n)$, thus (2.1) implies

$$\sum_n h(B(t_n: 5r_n))^\alpha \leq T_h^{4\alpha} \sum_n \mu(B(t_n: r_n)) \leq T_h^{4\alpha} \mu(T),$$

and $h\mathcal{H}_{5\delta}^\alpha(F) \leq T_h^{4\alpha} \mu(T)$. Taking $\delta \searrow 0$ we have $h\mathcal{H}^\alpha(F) \leq T_h^{4\alpha} \mu(T)$ and $h\text{-dim } F \leq \alpha$. Thus it follows $\Delta_2 \geq h\text{-dim } F$.

(b) Denote the first and third terms by ∇_1 and ∇_2 , respectively.

First we prove $\nabla_1 \leq h\text{-Dim } F$. We may assume $\nabla_1 > 0$. Choose any $0 < \alpha < \nabla_1$. Then there exists $\mu \in \mathcal{M}^+(F)$ such that $\mu(G) > 0$, where $G = \{t \in F; \Phi_h(\mu: t) > \alpha\}$. We prove $h\mathcal{P}^\alpha(G) > 0$, which implies $h\text{-Dim } F \geq \alpha$. Assume $G \subseteq \bigcup_n G_n$, $G_n \in \mathcal{B}$. Then there exists an integer $N \geq 1$ such that $\mu(R) \geq 2^{-1} \mu(G)$, where $R = G \cap (\bigcup_{j=1}^N G_j)$. For each $t \in R$ we may choose $\{r_n(t)\}_n$, $0 < 2r_n(t) < \delta_h$, such that $\lim_{n \rightarrow \infty} r_n(t) = 0$ and

$$(2.2) \quad \mu(B(t: r_n(t))) \leq h(r_n(t))^\alpha.$$

Fix any $0 < \delta < \delta_h$ and put $\mathcal{F}_\delta = \{B(t: l_n(t)); t \in R, 2l_n(t) \leq \delta\}$, where $l_n(t) = 5^{-1} r_n(t)$. Lemma 2.1 implies that there exists a δ -packing $\{B(t_n: l_n)\}_n \subseteq \mathcal{F}_\delta$ of R such that $R \subseteq \bigcup_n B(t_n: 5l_n) = \bigcup_n B(t_n: r_n)$. Then we have, by (2.2), that

$$0 < \frac{1}{2} \mu(G) \leq \sum_n \mu(B(t_n: r_n)) \leq \sum_n h(5l_n)^\alpha \leq T_h^{2\alpha} \sum_n h(2l_n)^\alpha,$$

thus $h\tilde{\mathcal{P}}_\delta^\alpha(R) \geq (T_h^{2\alpha} 2)^{-1} \mu(G)$. Then we have $h\mathcal{P}^\alpha(G) > 0$ and $h\text{-Dim } F \geq \nabla_1$.

Next we prove $\nabla_2 \geq h\text{-Dim } F$. We may assume $\nabla_2 < \infty$. Choose any $\nabla_2 < \alpha < \infty$. Then we have $\Phi_h(\mu: t) < \alpha$, $\forall t \in F$, for some $\mu \in \mathcal{M}^+(F)$. Denote, for each integer $n > 2\delta_h^{-1}$, $A_n = \{t \in F; \mu(B(t: r)) \geq h(r)^\alpha, 0 < \forall r \leq n^{-1}\}$. Then $F = \bigcup_n A_n$. For an $n > 2\delta_h^{-1}$ and any δ -packing $\{B(t_j: r_j)\}_j$, $0 < \delta < n^{-1}$, of A_n we have

$$\sum_j h(2r_j)^\alpha \leq T_h^\alpha \sum_j h(r_j)^\alpha \leq T_h^\alpha \sum_j \mu(B(t_j: r_j)) \leq T_h^\alpha \mu(T),$$

so that $h\tilde{\mathcal{P}}_\delta^\alpha(A_n) \leq T_h^\alpha \mu(T)$. Then it follows $h\text{-Dim } F \leq \alpha$, and $h\text{-Dim } F \leq \nabla_2$. $\square$

Next we prove Theorem 1.4. Denote, for $F \subseteq T$ and $r > 0$, the maximal cardinal number of the family of the disjoint closed balls with centers in T and radius $r > 0$ by $M(F: r)$. Then we have $N(F: 2r) \leq M(F: r) \leq N(F: r)$ and, for each $h \in \mathcal{G}$,

$$\lim_{x \searrow 0} \frac{\log h(2x)}{\log h(x)} = 1,$$

thus it follows

$$h\text{-}\underline{\dim}_{\mathbf{B}} F = \liminf_{r \searrow 0} \frac{\log M(F; r)}{-\log h(r)} \quad \text{and} \quad h\text{-}\overline{\dim}_{\mathbf{B}} \mu = \limsup_{r \searrow 0} \frac{\log M(F; r)}{-\log h(r)}.$$

PROOF OF THEOREM 1.4

(a) First we prove $h\text{-}\dim F \leq h\text{-}\underline{\dim}_{\mathbf{B}} F$. We may assume $h\text{-}\underline{\dim}_{\mathbf{B}} F < \infty$. Choose any $h\text{-}\underline{\dim}_{\mathbf{B}} F < \alpha < \infty$ and a real sequence $\{r_n\}_n$, $0 < 4r_n < \delta_h$, such that $\lim_n r_n = 0$ and $N(F; r_n) h(r_n)^\alpha \leq 1$, $\forall n$. Fix any integer $n \geq 1$ and let $\{B(t_j; r_n)\}_{j=1}^{N(F; r_n)}$ be the minimal family of the closed balls with centers in T and radius r_n which cover F . Then

$$\sum_{j=1}^{N(F; r_n)} h(B(t_j; r_n))^\alpha \leq T_h^\alpha N(F; r_n) h(r_n)^\alpha \leq T_h^\alpha,$$

so that $h\text{-}\mathcal{H}_{2r_n}^\alpha(F) \leq T_h^\alpha$. Then we have $h\text{-}\dim F \leq \alpha$, thus $h\text{-}\dim F \leq h\text{-}\underline{\dim}_{\mathbf{B}} F$.

Denote the third term by Δ . To show the inequality $h\text{-}\underline{\dim}_{\mathbf{B}} F \leq \Delta$ we may assume $\Delta < \infty$. Choose any $\Delta < \alpha < \infty$. Then there exists $\mu \in \mathcal{M}^+(F)$ and a real sequence $\{r_n\}_n$, $r_n \searrow 0$, such that $\mu(B(t; r_n)) \geq h(r_n)^\alpha$, $\forall t \in F$. For any integer $n \geq 1$, let $\{B(t_j; r_n)\}_{j=1}^{N(F; r_n)}$ be the maximal family of the disjoint closed balls with centers in F and radius r_n . Then we have

$$M(F; r_n) h(r_n)^\alpha \leq \sum_{j=1}^{N(F; r_n)} \mu(B(t_j; r_n)) \leq \mu(T),$$

so that $h\text{-}\underline{\dim}_{\mathbf{B}} F \leq \alpha$, thus $h\text{-}\underline{\dim}_{\mathbf{B}} F \leq \Delta$.

(b) The proof of $\dim F \leq h\text{-}\overline{\dim}_{\mathbf{B}} F$ is the same argument as that of [6, Lemma 3.7]. We also omit the proof of the last inequality, which is the same as that of (a). $\square$

2b. Capacity Dimension In this subsection we always assume $h \in \mathcal{G}'$. It is easy to see that if $0 \leq \alpha \leq \beta < \infty$ then $I_h^\alpha(\mu) = \infty$ implies $I_h^\beta(\mu) = \infty$, thus we have, for each $F \in \mathcal{B}$, $h\text{-}\dim_{\mathbf{C}} F = \inf\{\alpha > 0; h\text{-}\text{Cap}^\alpha(F) = 0\}$.

Lemma 2.5 *Let (T, d) be a separable metric space, $a > 0$ and $F \in \mathcal{B}$. If $h\text{-}\text{Cap}^a(F) > 0$ then there exists $\mu \in \mathcal{M}^+(F)$ and a constant $M > 0$ such that $U_h^a(\mu; t) \leq M$, $\forall t \in T$.*

PROOF. Assume $h\text{-Cap}^\alpha F > 0$. Then, by the definition, there exists $\nu \in \mathcal{M}^+(F)$ such that $I_h^\alpha(\nu) < \infty$, thus we may choose an integer $N \geq 1$ and a closed set $K \subseteq F$ such that $K \subseteq \{t \in F; U_h^\alpha(\nu: t) \leq N\}$ and $\nu(K) > 0$. It is enough to prove that we have

$$(2.3) \quad U_h^\alpha(\mu: t) \leq T_h^{2\alpha} N + h(4^{-1} \delta_h)^{-1} \nu(K), \quad \forall t \in T,$$

where $\mu = \nu_K$. If $t \in K$ then (2.3) is trivial. Assume $t \notin K$ then, since K is closed, there exists a $t_0 \in K$ such that $d(t, t_0) < 2 \inf\{d(s, t); s \in K\}$. If $d(s, t) < 4^{-1} \delta_h$, $s \in K$, then we have $T_h^{-2} h(d(s, t_0)) \leq T_h^{-2} h(3d(s, t)) \leq h(d(s, t))$, and thus

$$\begin{aligned} U_h^\alpha(\mu: t) &\leq \int_{\{d(s, t) < 4^{-1} \delta_h\}} \frac{1}{h(d(s, t))^\alpha} \nu(ds) + h(4^{-1} \delta_h)^{-1} \nu(K) \\ &\leq T_h^{2\alpha} N + h(4^{-1} \delta_h)^{-1} \nu(K), \end{aligned}$$

which proves (2.3). $\square$

Proposition 2.6 *Let (T, d) be a separable metric space, $\alpha > 0$ and $F \in \mathcal{B}$. Then we have $h\text{-Cap}^\alpha(F) = 0$ if and only if $U_h^\alpha(\mu: t) = \infty$ μ a.e. on F for $\mu \in \mathcal{M}^+(F)$.*

PROOF. The “if” part is trivial. Assume $U_h^\alpha(\mu: t) < \infty$ μ a.e. on F for some $\mu \in \mathcal{M}^+(F)$. Then there exists an integer $N \geq 1$ such that $\mu(A) > 0$, where $A = \{t \in F; U_h^\alpha(\mu: t) \leq N\}$. Then it is easy to check $I_h^\alpha(\mu_A) \leq N$ and $\mu_A \in \mathcal{M}^+(F)$, thus $h\text{-Cap}^\alpha F > 0$. Therefore we have the “only if” part. $\square$

Set, for each $\mu \in \mathcal{M}^+(T)$ and $0 \leq \alpha < \infty$, $S_h^\alpha(\mu) = \{t \in T; U_h^\alpha(\mu: t) = \infty\}$. $S_h^\alpha(\mu)$ is called the (h, α) -singular set of μ .

Lemma 2.7 [15] *We have $h\text{-Cap}^\alpha S_h^\alpha(\mu) = 0$.*

PROOF. Use Lemma 2.5 $\square$

Let $\alpha \geq 0$ and $\mu \in \mathcal{M}^+(T)$. We say that μ is (h, α) -regular if there exist $\{\mu_n\}_n \subseteq \mathcal{M}^+(T)$ such that $I_h^\alpha(\mu_n) < \infty \forall n$ and $\mu = \sum_n \mu_n$. It is easy to see that if μ is (h, α) -regular then μ is absolutely continuous with respect to $h\text{-Cap}^\alpha$.

Proposition 2.8 Let (T, d) be a separable metric space, $\alpha > 0$ and $\mu \in \mathcal{M}^+(T)$. Then μ is uniquely decomposed as $\mu = \mu_s + \mu_r$ ($\mu_s, \mu_r \in \mathcal{M}(T)$) where $\mu_s \perp h\text{-Cap}^\alpha$ and μ_r is $(h : \alpha)$ -regular.

Above decomposition is called *Kahane decomposition* or *decomposition with respect to dimension*, which was owed to Kahane. The proof is same as that of [15].

Lemma 2.9 Let (T, d) be a separable metric space, $\alpha \geq 0$, $t \in T$ and $\mu \in \mathcal{M}^+(T)$.

(a) Assume that there exist $\beta > \alpha$ and $0 < \delta < \eta_h$ such that $\mu(B(t : r)) \leq h(r)^\beta$ for each $0 < r < \delta$. Then we have $U_h^\alpha(\mu : t) < \infty$.

(b) We have $\mu(B(t : r))h(2r)^{-\alpha} \leq U_h^\alpha(\mu : t)$ for each $r > 0$.

PROOF. (a) Set $F(r) = \mu(B(t : r))$ and observe that

$$\begin{aligned} U_h^\alpha(\mu : t) &\leq \int_{(0, \infty)} h(r)^{-\alpha} F(dr) \\ &\leq \int_{(0, \delta]} h(r)^{-\alpha} F(dr) + h(\delta)^{-\alpha} \mu(T) \\ &\leq \alpha \int_{(0, \delta]} h(r)^{-\alpha+\beta-1} h(dr) + 2h(\delta)^{-\alpha} \mu(T) < \infty. \end{aligned}$$

(b) was essentially proved by Fan [7]. □

PROOF OF THEOREM 1.6

Denote the second term by λ . First we prove $h\text{-}\underline{\dim}_c F \geq \lambda$. We may assume $\lambda > 0$. Choose any $0 < \alpha < \beta < \lambda$. Then there exists $\mu \in \mathcal{M}^+(F)$ and $0 < \delta < \eta_h$ such that $\mu(A) > 0$, where $A = \{t \in F; \mu(B(t : r)) \leq h(r)^\beta, 0 < r \leq \delta\}$. Set $\eta = \mu_A \in \mathcal{M}^+(F)$. Then we have, by Lemma 2.9 (a) and Proposition 2.6, $h\text{-}\underline{\dim}_c F \geq \alpha$, thus $h\text{-}\underline{\dim}_c F \geq \lambda$. Next we prove $h\text{-}\underline{\dim}_c F \leq \lambda$. We may assume $h\text{-}\underline{\dim}_c F > 0$. Choose any $h\text{-}\underline{\dim}_c F > \alpha > 0$. Then there exists $\mu \in \mathcal{M}^+(F)$ such that $U_h^\alpha(\mu : t) < \infty$ μ a.e. on F . Lemma 2.9 (b) implies $\text{ess.inf}_{t \in F} \phi_h(\mu : t) \geq \alpha$ and $\lambda \geq \alpha$, thus $h\text{-}\underline{\dim}_c F \leq \lambda$. □

Combining Theorem 1.6 and Proposition 2.3 (a) we have Corollary 1.7.

3. Dimensions of a Measure

Proposition 3.1 [7] *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $\mu \in \mathcal{M}^+(F)$. Then we have*

- (a) $h\text{-dim}^* \mu = \min \{h\text{-dim } F; \mu(F) = \mu(T), F \in \mathcal{B}\}.$
- (b) $h\text{-Dim}^* \mu = \min \{h\text{-Dim } F; \mu(F) = \mu(T), F \in \mathcal{B}\}.$

Lemma 3.2 *Let (T, d) be a separable metric space, $h \in \mathcal{G}'$, $F \in \mathcal{B}$, $\mu \in \mathcal{M}^+(F)$ and $\alpha > 0$. Then we have*

- (a) $\text{ess.inf}_{t \in F} \phi_h(\mu; t) > \alpha \Rightarrow U_h^\alpha(\mu; t) < \infty \text{ } \mu\text{-a.e. on } F \Rightarrow \text{ess.inf}_{t \in F} \phi_h(\mu; t) \geq \alpha.$
- (b) $\text{ess.sup}_{t \in F} \phi_h(\mu; t) < \alpha \Rightarrow U_h^\alpha(\mu; t) = \infty \text{ } \mu\text{-a.e. on } F \Rightarrow \text{ess.sup}_{t \in F} \phi_h(\mu; t) \leq \alpha.$

PROOF. (a) Assume $\text{ess.inf}_{t \in F} \phi_h(\mu; t) > \alpha$ and choose any $\text{ess.inf}_{t \in F} \phi_h(\mu; t) > \beta > \alpha$. Set $A_n = \{t \in F; \mu(B(t; r)) \leq h(r)^\beta, 0 < \forall r \leq n^{-1}\}$, $n > \eta_h^{-1}$. Then we have $\mu(F) = \mu(\bigcup_n A_n)$, thus $U_h^\alpha(\mu; t) < \infty \text{ } \mu\text{-a.e. on } F$ by Lemma 2.9 (a). Conversely assume $U_h^\alpha(\mu; t) < \infty \text{ } \mu\text{-a.e. on } F$. Then Lemma 2.9 (b) implies $\text{ess.inf}_{t \in F} \phi_h(\mu; t) \geq \alpha$. (b) is easy to check by (a). $\square$

PROOF OF THEOREM 1.8

- (a) Denote the second and third terms by Δ_1 and Δ_2 , respectively.

First we prove $\Delta_1 \leq h\text{-dim}_* \mu$. We may assume $\Delta_1 > 0$. Choose any $0 < \alpha < \Delta_1$ and assume $\mu(F) > 0$, $F \in \mathcal{B}$. Then $\phi_h(\mu; t) > \alpha \text{ } \mu\text{-a.e. on } F$. Proposition 2.4 (a) implies $h\text{-dim } F \geq \alpha$, thus $h\text{-dim } \mu \geq \Delta_1$.

Next we prove $h\text{-dim}_* \mu \leq \Delta_2$. We may assume $h\text{-dim}_* \mu > 0$. Choose any $0 < \alpha < h\text{-dim}_* \mu$. By the definition $\mu(F) > 0$, $F \in \mathcal{B}$, implies $h\text{-dim } F > \alpha$. Then it is easy to check $\mu \ll h\mathcal{H}^\alpha$, thus $\Delta_2 \geq \alpha$. Therefore we have $h\text{-dim}_* \mu \leq \Delta_2$.

Finally we prove $\Delta_2 \leq \Delta_1$. We may assume $\Delta_1 < \infty$. Choose any $\Delta_1 < \beta < \alpha < \infty$ and denote $F' = \{t \in T; \phi_h(\mu; t) < \beta\}$. Then we have $\mu(F') > 0$ and $h\mathcal{H}^\alpha(F') = 0$ by Proposition 2.4 (a). Thus it follows $\Delta_2 \leq \alpha$ and $\Delta_2 \leq \Delta_1$.

- (b) Denote the second and third terms by ∇_1 and ∇_2 , respectively.

First we prove $\nabla_1 \leq h\text{-dim}^* \mu$. We may assume $\nabla_1 > 0$, thus it suffices to prove that we have $h\text{-dim}^* \mu \geq \alpha$ for each $0 < \alpha < \nabla_1$. Fix any $F \in \mathcal{B}$ such that $\mu(F) = \mu(T)$ and put $F' = F \cap \{t \in T; \phi_h(\mu; t) > \alpha\}$. Then we have $\mu(F') > 0$ and, by Proposition 2.4 (a), $h\text{-dim } F' \geq \alpha$, thus $h\text{-dim}^* \mu \geq \alpha$.

Next we prove $h\text{-dim}^* \mu \leq \nabla_2$. We may assume $\nabla_2 < \infty$. Choose any $\nabla_2 < \alpha < \infty$. By the definition there exists an $F \in \mathcal{B}$ such that $\mu(F) = \mu(T)$ and $h\text{-}\mathcal{H}^\alpha(F) = 0$, so that $h\text{-dim}^* \mu \leq \alpha$. Thus we have $h\text{-dim}^* \mu \leq \nabla_2$.

Finally we prove $\nabla_2 \leq \nabla_1$. We may assume $\nabla_1 < \infty$. Choose any $\nabla_1 < \beta < \alpha < \infty$ and put $F = \{t \in T; \phi_h(\mu: t) < \beta\}$. Then we have $\mu(F) = \mu(T)$ and, by Proposition 2.4 (a), $h\text{-}\mathcal{H}^\alpha(F) = 0$, so that $\nabla_2 \leq \alpha$. Thus it follows $\nabla_2 \leq \nabla_1$.

The proof of (c) and (d) are the same as them of (a) and (b), respectively (use Proposition 2.4 (b) in place of Proposition 2.4 (a)).

(e) Denote the second, third and fourth terms by κ_1 , κ_2 and κ_3 , respectively.

The inequality $h\text{-dim}_* \mu \leq \kappa_1$ is easy to check by Theorem 1.8 (a) and Lemma 3.2 (a). Next we prove $\kappa_1 \leq \kappa_2$. We may assume $\kappa_1 > 0$. Choose any $0 < \alpha < \kappa_1$ and denote, for each integer $n \geq 1$, $A_n = \{t \in T; n-1 \leq U_h^\alpha(\mu: t) < n\}$ and $\mu_n = \mu_{A_n}$. Then we have $\mu = \sum_n \mu_n$ and $I_h^\alpha(\mu_n) < \infty$ for each $n \geq 1$. Thus it follows $\kappa_2 \geq \alpha$ and $\kappa_2 \geq \kappa_1$.

Third we prove $\kappa_2 \leq \kappa_3$. We may assume $\kappa_2 > 0$. Choose any $0 < \alpha < \kappa_2$. Then there exists $\{\mu_n\}_n \subseteq \mathcal{M}(T)$ such that $\mu = \sum_n \mu_n$ and $I_h^\alpha(\mu_n) < \infty$ for each $n \geq 1$. Assume $h\text{-Cap}^\alpha F = 0$. Then it is easy to check that $\mu(F) = 0$ and $\kappa_3 \geq \alpha$. Therefore we have $\kappa_3 \geq \kappa_2$.

Finally we prove $\kappa_3 \leq h\text{-dim}_* \mu$. We may assume $h\text{-dim}_* \mu < \infty$. Choose any $h\text{-dim}_* \mu < \alpha < \infty$ and put $F = \{t \in T; U_h^\alpha(\mu: t) = \infty\}$. Then we have, by Theorem 1.8 (a) and Lemma 3.2 (b), $\mu(F) > 0$. On the other hand, Lemma 2.7 implies $h\text{-Cap}^\alpha F = 0$, and $\kappa_3 \leq \alpha$. Thus it follows $\kappa_3 \leq h\text{-dim}_* \mu$.

(f) Denote the second, third and fourth terms by θ_1 , θ_2 and θ_3 , respectively.

The inequality $h\text{-dim}^* \mu \geq \theta_1$ is easy to check by Theorem 1.8 (b) and Lemma 3.2 (b). On the other hand, the inequality $\theta_1 \geq \theta_2$ is easy to check by Lemma 2.7. Third we prove $\theta_2 \geq \theta_3$. We may assume $\theta_2 < \infty$. Choose any $\theta_2 < \alpha < \infty$. Then $\mu(F) > 0$, $F \in \mathcal{B}$, implies $I_h^\alpha(\mu_F) = \infty$, thus it is easy to see that we have $\mu(t \in T; U_h^\alpha(\mu: t) = \infty) = \mu(T)$. Lemma 2.7 implies $\mu \perp h\text{-Cap}^\alpha$ and $\theta_2 \leq \alpha$. Then it follows $\theta_3 \leq \theta_2$.

Finally we prove $\theta_3 \geq h\text{-dim}^* \mu$. We may assume $\theta_3 < \infty$. Choose any $\theta_3 < \alpha < \infty$. Then there exists an $F \in \mathcal{B}$ such that $\mu(F) = \mu(T)$ and $h\text{-Cap}^\alpha F = 0$. Proposition 2.6 and Lemma 3.2 (b) imply $\phi_h(\mu: t) \leq \alpha$ *a.e.*. Therefore we have $h\text{-dim}^* \mu \leq \alpha$ by Theorem 1.8 (b). Thus $h\text{-dim}^* \mu \leq \theta_3$. $\square$

PROOF OF THEOREM 1.9

Theorem 1.4 implies the inequalities $h\text{-dim}^* \mu \leq h\text{-dim}_B \mu$ and $\text{Dim}^* \mu \leq \overline{\text{dim}}_B \mu$.

Next we prove $h\text{-dim}_B \mu \leq h\text{-Dim}^* \mu$. We may assume $h\text{-Dim}^* \mu < \infty$. Choose any $h\text{-Dim}^* \mu < \alpha < \infty$. Then Theorem 1.8 (d) implies $\Phi_h(\mu: t) < \alpha$ μ a.e.. Denote, for each integer $n \geq 1$, $F_n = \{t \in E; \mu(B(t: r)) \geq h(r)^\alpha, 0 < r \leq n^{-1}\}$. Then $\mu(E) = \mu(\cup_n F_n)$. Also denote $R_n = \cup_{j=1}^n F_j$ and fix any integer $n \geq 1$. Now we prove

$$(3.1) \quad \limsup_{r \rightarrow 0} \frac{\log M(R_n: r)}{-\log h(r)} \leq \alpha,$$

which implies $h\text{-dim}_B \mu \leq \alpha$ and thus $h\text{-dim}_B \mu \leq h\text{-Dim}^* \mu$. Let $\{B(t_k: r)\}_{k=1}^{M(R_n: r)}$, $0 < r \leq n^{-1}$, be the maximal family of the disjoint closed balls with centers in R_n and radius r . Then, by definition of F_j , we have

$$M(R_n: r) h(r)^\alpha \leq \sum_{k=1}^{M(R_n: r)} \mu(B_k(t: r)) \leq \mu(E).$$

which implies (3.1). □

Proposition 3.3 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $F \in \mathcal{B}$. Then*

- (a)
$$\inf_{\mu \in \mathcal{M}^+(F)} \text{ess. sup}_{t \in F} \phi_h(\mu: t) = \inf_{\mu \in \mathcal{M}^+(F)} \text{ess. inf}_{t \in F} \phi_h(\mu: t),$$
- (b)
$$\inf_{\mu \in \mathcal{M}^+(F)} \text{ess. sup}_{t \in F} \Phi_h(\mu: t) = \inf_{\mu \in \mathcal{M}^+(F)} \text{ess. inf}_{t \in F} \Phi_h(\mu: t).$$

PROOF. (a) Denote the left and right hand side by Δ and ∇ , respectively. It is enough to prove $\Delta \leq \nabla$ and we may assume $\nabla < \infty$. Choose any $\nabla < \alpha < \infty$. Then there exists $\mu \in \mathcal{M}^+(F)$ such that $\mu(A) > 0$, where $A = \{t \in F; \phi_h(\mu: t) < \alpha\}$. By Proposition 2.4 (a) we have $h\text{-dim} A \leq \alpha$, thus Theorem 1.8 (b) implies $\phi_h(\mu_A: t) \leq \alpha$ μ a.e. on F , therefore we have $\Delta \leq \alpha$ and $\Delta \leq \nabla$. The proof of (b) are the same argument. □

Combining Proposition 2.3, 2.4 and Proposition 3.3 we have Theorem 1.2.

4. Dimension Regularity

Let $h \in \mathcal{G}$, $\mu \in \mathcal{M}^+(T)$ and $\alpha \geq 0$. We say that μ has the *exact* Hausdorff (resp. packing) *dimension* [2, 3] α with respect to h if $h\text{-dim}_*\mu = h\text{-dim}^*\mu = \alpha$ (resp. $h\text{-Dim}_*\mu = h\text{-Dim}^*\mu = \alpha$) and that μ has the exact dimension if μ has the exact Hausdorff and packing dimension. On the other hand, μ is called *dimension regular* [2, 7] with respect to h if $\phi_h(\mu: t) = \Phi_h(\mu: t)$ μ a.e., which is equivalent to the condition that the indicated limit exists μ a.e.;

$$h\text{-}\bar{\mu}(t) = \lim_{r \searrow 0} \frac{\log \mu(B(t: r))}{h(r)}.$$

A finite Borel measure which has the exact Hausdorff dimension is also called *unidimensional* [15], and dimension regular measure μ is also called a *fractal measure* [19].

By Theorem 1.8 and Proposition 3.1 we may characterize the exact dimensionality and dimension regularity of a finite Borel measure as follows.

Theorem 4.1 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$, $\mu \in \mathcal{M}^+(T)$ and $\alpha > 0$.*

- (i) *The following four statements are equivalent.*
 - (a) μ has the exact Hausdorff dimension α with respect to h .
 - (b) $\phi_h(\mu: t) = \alpha$ μ -a.e..
 - (c) We have $\mu \ll h\text{-}\mathcal{H}^\beta$ for each $0 < \beta < \alpha$ and $h\text{-}\mathcal{H}^\gamma \perp \mu$ for each $\alpha < \gamma < \infty$.
 - (d) There exists an $F \in \mathcal{B}$ such that $\mu(F) = \mu(T)$ and $h\text{-dim } F = \alpha$, and we have $\mu(G) = 0$ for each $G \in \mathcal{B}$ with $h\text{-dim } G < \alpha$.
- (ii) *The following four statements are equivalent.*
 - (a) μ has the exact packing dimension α with respect to h .
 - (b) $\Phi_h(\mu: t) = \alpha$ μ -a.e..
 - (c) We have $\mu \ll h\text{-}\mathcal{P}^\beta$ for each $0 < \beta < \alpha$ and $h\text{-}\mathcal{P}^\gamma \perp \mu$ for each $\alpha < \gamma < \infty$.
 - (d) There exists an $F \in \mathcal{B}$ such that $\mu(F) = \mu(T)$ and $h\text{-Dim } F = \alpha$, and we have $\mu(G) = 0$ for each $G \in \mathcal{B}$ with $h\text{-Dim } G < \alpha$.
- (iii) *The following four statements are equivalent.*
 - (a) μ is dimension regular with respect to h and $h\text{-}\bar{\mu}(t) = \alpha$ μ a.e..
 - (b) $h\text{-dim}_*\mu = h\text{-dim}^*\mu = h\text{-Dim}_*\mu = h\text{-Dim}^*\mu = \alpha$.
 - (c) We have $\mu \ll h\text{-}\mathcal{H}^\beta$ for each $0 < \beta < \alpha$ and $h\text{-}\mathcal{P}^\gamma \perp \mu$ for each $\alpha < \gamma < \infty$.
 - (d) There exists an $F \in \mathcal{B}$ such that $\mu(F) = \mu(T)$ and $h\text{-Dim } F = \alpha$, and we have $\mu(E) = 0$ for each $E \in \mathcal{B}$ with $h\text{-dim } E < \alpha$.

- (iv) Assume $h \in \mathcal{G}$. Then the following four statements are equivalent.
- (a) μ has the exact Hausdorff dimension α with respect to h .
- (b) We have $U_h^\beta(\mu; t) < \infty$ μ a.e. for each $0 < \beta < \alpha$ and $U_h^\gamma(\mu; t) = \infty$ μ a.e. for each $\alpha < \gamma < \infty$.
- (c) μ is (h, α) -regular for each $0 < \beta < \alpha$ and if $\mu(F) > 0$, $F \in \mathcal{B}$, then $\Gamma_h(\mu_F) = \infty$ for each $\alpha < \gamma < \infty$.
- (d) We have $\mu \ll h\text{-Cap}^\beta$ for each $0 < \beta < \alpha$ and $\mu \perp h\text{-Cap}^\gamma$ for each $\alpha < \gamma < \infty$.

Fan proved Theorem 4.1 (i) and (iv) in the case where $h(x) = x$ and (T, d) is homogeneous.

The following proposition was proved by Taylor [19] in the case where $h(x) = x$ and (T, d) is the Euclidean space, which is induced from Theorem 2.4.

Proposition 4.2 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$ and $\mu \in \mathcal{M}^+(T)$. Assume that μ is dimension regular with respect to h .*

- (a) *Denote $F(\alpha) = \{t \in T; h\text{-}\hat{\mu}(t) = \alpha\}$, $\alpha \geq 0$. If $\mu(F(\alpha)) > 0$ then we have $h\text{-dim } F(\alpha) = h\text{-dim } F(\alpha) = \alpha$.*
- (b) *Denote $\delta = \text{ess. sup}_{t \in T} h\text{-}\hat{\mu}(t)$ and $F = \{t \in T; 0 \leq h\text{-}\hat{\mu}(t) \leq \delta\}$. Then we have $h\text{-dim } F = h\text{-dim } F = \delta$.*

The following result is an immediate corollary of Theorem 1.9, which was proved by Young [23] in the Euclidean space.

Corollary 4.3 *Let (T, d) be a separable metric space, $h \in \mathcal{G}$, $\mu \in \mathcal{M}^+(T)$ and $\alpha \geq 0$. Assume that μ is dimension regular with respect to h , $h\text{-}\hat{\mu}(t) = \alpha$ μ a.e. and has a totally bounded support. Then we have*

$$\dim_* \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = \text{Dim}^* \mu = \alpha.$$

5. Product Measure

In this section we estimate the dimensions of a product measure. Let (T_1, d_1) and (T_2, d_2) be the separable metric spaces. On the product space $T = T_1 \times T_2$ we shall equip a metric d such that

$$d((t_1, t_2), (s_1, s_2)) = \max\{d_1(t_1, s_1), d_2(t_2, s_2)\}, \quad (t_1, t_2), (s_1, s_2) \in T.$$

For each $\mu \in \mathcal{M}^+(T_1)$ and $\nu \in \mathcal{M}^+(T_2)$, denote the product measure by $\mu \times \nu$. Note that we have $B(t; r) = B(t_1; r) \times B(t_2; r)$, $t = (t_1, t_2)$, $r > 0$. Then

Theorem 1.8 and Theorem 4.1 imply the following results.

Theorem 5.1 *Let (T, d) be the product space as above, $h \in \mathcal{G}$, $\mu \in \mathcal{M}^+(T_1)$ and $\nu \in \mathcal{M}^+(T_2)$. Then we have*

- (a)
$$\begin{aligned} h\text{-dim}_* \mu + h\text{-dim}_* \nu &\leq h\text{-dim}_* \mu \times \nu \leq h\text{-dim}_* \mu + h\text{-Dim}_* \nu \\ &\leq h\text{-Dim}_* \mu \times \nu \leq h\text{-Dim}_* \mu + h\text{-Dim}_* \nu, \end{aligned}$$
- (b)
$$\begin{aligned} h\text{-dim}^* \mu + h\text{-dim}^* \nu &\leq h\text{-dim}^* \mu \times \nu \leq h\text{-dim}^* \mu + h\text{-Dim}^* \nu \\ &\leq h\text{-Dim}^* \mu \times \nu \leq h\text{-Dim}^* \mu + h\text{-Dim}^* \nu. \end{aligned}$$

Assume that $\mu \in \mathcal{M}^+(T_1)$ and $\nu \in \mathcal{M}^+(T_2)$ are dimension regular. Then $\mu \times \nu$ is also dimension regular and above all inequalities are replaced by equalities.

Note that Fan [7] established the relation between $h\text{-dim}_* \mu + h\text{-dim}_* \nu$, $h\text{-dim}_* \mu \times \nu$, $h\text{-dim}^* \mu + h\text{-dim}^* \nu$ and $h\text{-dim}^* \mu \times \nu$ in the case where $h(x) = x$ and (T_1, d_1) and (T_2, d_2) are homogeneous. On the other hand Haase [11] proved (a) in the case where $h(x) = x$ and (T_1, d_1) and (T_2, d_2) are the Euclidean spaces.

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