CLASSIFICATION OF ASSOCIATION SCHEMES WITH AT MOST TEN VERTICES

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1 Introduction

Association schemes provide a fundamental framework for the study of algebraic combinatorics. Especially they are deeply related to codes, designs, groups and finite geometry. Recently, it has been noticed that they are also related to some questions in topology.

In this paper we classify (not necessarily commutative) association schemes with at most ten vertices, where association schemes are not necessary commutative. We would like to classify association schemes in general, but this is obviously too difficult. So we restrict number of vertices.

Our idea is summarized as follows. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ denote any association scheme with intersection numbers p_{ij}^k ($0 \leq i, j, k \leq d$) and $n = |X|$. Let $A_0, A_1, \dots, A_d$ denote the adjacency matrices with the valencies $k_0, k_1, \dots, k_d$ respectively. We know already A_0 is the identity matrix. So we would like to determine the other adjacency matrices. If more than $n/2$ of the valencies are 1, then the scheme is determined by a group of order n . Therefore we have only to discuss the cases where $\mathcal{X}$ has at most $n/2$ adjacency matrices of valency 1.

We know that the adjacency matrices depend on the ordering of elements of X . Therefore when we construct the remaining d adjacency matrices $A_1, \dots, A_d$ we can assume any convenient ordering of the elements.

We have the useful formulas:

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k \quad (1)$$

$$k_i k_j = \sum_{t=0}^d p_{ij}^t k_t \quad (2).$$

If $A_i A_j$ is symmetric, then we can restrict to p_{ij}^k 's such that $\sum_{k=0}^d p_{ij}^k A_k$ is symmetric. Then we can get the adjacency matrices which have those intersection numbers. We can classify association schemes by these methods. We also give the Hesse diagrams for subschemes, for each number of vertices.

2. Notation

Definition 2.1 (Association scheme) Let X be a set of cardinality n and let R_i ($i=0, 1, \dots, d$) be subsets of $X \times X$ with the property that

- (i) $R_0 = \{(x, x) \mid x \in X\}$
- (ii) $X \times X = R_0 \cup R_1 \cup \dots \cup R_d$ $R_i \cap R_j = \emptyset$ if $i \neq j$
- (iii) ${}^t R_i = R_i$ for some $i' \in \{0, 1, \dots, d\}$

where ${}^t R_i := \{(x, y) \mid (y, x) \in R_i\}$

- (iv) For $i, j, k \in \{0, 1, \dots, d\}$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant whenever $(x, y) \in R_k$. This constant is denoted by p_{ij}^k .

Such a configuration $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called an association scheme of class d on X . The non-negative integers p_{ij}^k are called the intersection numbers of $\mathcal{X}$.

- (v) $p_{ij}^k = p_{ji}^k$ for $\forall i, j, k$

- (vi) ${}^t R_i = R_i$ for $\forall i$

An association schemes with the additional property (v) is called commutative type and an association scheme with the additional property (vi) is called symmetric type.

Let $k_i = p_{ii}^0$. The positive integer k_i is called the valency of R_i .

The i -th adjacency matrix A_i of $\mathcal{X}$ is defined to be the matrix of degree $|X|$ whose rows and columns are indexed by the elements of X and whose (x, y) entries are

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the conditions (i), ..., (vi) are equivalent to the following

- (i)', ..., (vi)' respectively.
- (i)' $A_0 = I$, where I is the identity matrix.
- (ii)' $A_0 + A_1 + \dots + A_d = J$, where J is the matrix whose entries are all 1.
- (iii)' ${}^t A_i = A_{i'}$ for some $i' \in \{0, 1, \dots, d\}$

where tA_i denotes the transpose of A_i .

$$(iv)' \quad A_i A_j = \sum_{k=0}^d p_{ij}^k A_k \text{ for all } i, j \in \{0, 1, \dots, d\}$$

$$(v)' \quad p_{ij}^k = p_{ji}^k \text{ for all } i, j, k$$

$$(vi)' \quad {}^tA_i = A_i \text{ for all } i$$

$\sum_{k=0}^d k A_k$ is called the relation matrix of $\mathcal{X}$.

The relation matrix can be assumed to have the following form.

$$\begin{pmatrix} 0 & 1 \cdots 1 & \cdots & k \cdots k & \cdots & d \cdots d \\ 1' & & & & & \\ \vdots & & & & & \\ 1' & & & & & \\ \vdots & & & & & \\ i' & & & B & & \\ \vdots & & & & & \\ i' & & & & & \\ \vdots & & & & & \\ d' & & & & & \\ \vdots & & & & & \\ d' & & & & & \end{pmatrix}$$

If $p_{ij}^k = t$, then each column of B has tj 's.

Definition 2.2 Let A and B be $n \times n$ matrices. If there exists some permutation matrix P such that $P^{-1}P = B$, then we say that A is isomorphic to B , and write $A \cong B$.

Definition 2.3 Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ and $\mathcal{Y} = (Y, \{S_i\}_{0 \leq i \leq d})$ be association schemes. We say that $\mathcal{X}$ is isomorphic to $\mathcal{Y}$ if there exists the following pair (σ, φ)

$$\begin{aligned} \sigma: \{0, 1, \dots, d\} &\rightarrow \{0, 1, \dots, e\} \\ \varphi: X &\rightarrow Y \end{aligned}$$

where σ and φ are bijections such that

$$(\varphi(x), \varphi(y)) \in S_{\sigma(i)} \text{ if } (x, y) \in R_i \text{ for any } x, y \in X$$

Definition 2.4 (Subscheme) Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme. Let Λ_i ($i=0, 1, \dots, D$) be subsets of $\{0, 1, \dots, d\}$ with the property that $\Lambda_0 = \{0\}$, $\Lambda_0 \cup \Lambda_1 \cup \dots \cup \Lambda_D = \{0, 1, \dots, d\}$ and $\Lambda_i \cap \Lambda_j = \emptyset$ if $i \neq j$. Moreover define R_{Λ_i} to be $\bigcup_{k \in \Lambda_i} R_k$ for each $i \in \{0, 1, \dots, D\}$. If $\mathcal{Y} = (X, \{R_{\Lambda_i}\}_{0 \leq i \leq D})$ is an association scheme, then $\mathcal{Y}$ is called a subscheme of $\mathcal{X}$. We denote this by

$$\mathcal{Y} : \mathcal{X}(0, 1_1 + \dots + 1_{s_1}, \dots, D_1 + \dots + D_{s_d})$$

where $\{k_1, k_2, \dots, k_{s_k}\} = \Lambda_k$

3 Facts

Proposition 3.1 Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme such that each valency is 1.

Then $\mathcal{X}$ is isomorphic to an association scheme whose adjacency matrices are given by a regular representation of some group G of order $|X|$.

Proposition 3.2 Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq 2})$ be an association scheme with two class. If $\mathcal{X}$ is not symmetric, then $|X| \equiv 3 \pmod{4}$.

Definition 3.1 Let A be a $n \times n$ matrix whose entries are 1 or 0.

If A satisfies $AJ = JA = kJ$, then A is called a valency k matrix of degree n , or simply a valency k matrix.

Moreover if $(A)_{ij} = 1$ implies $(A)_{ji} = 0$, then A is call directed. If $(A)_{ij} = (A)_{ji}$ and $(A)_{ii} = 0$, then A is call non-directed.

Remark: It is easy to see that each of the adjacency matrices except A_0 are either directed or non-directed.

Proposition 3.3 If A is a directed valency 1 matrix, then A is isomorphic to some $C_{k_1, \dots, k_d}$.

$$C_{k_1, \dots, k_d} = \begin{pmatrix} C_{k_1} & & & \\ & C_{k_2} & & \\ & & \ddots & \\ & & & C_{k_d} \end{pmatrix}$$

where C_{k_r} is the $k_r \times k_r$ matrix ($k_r \geq 3$) which is defined by

$$(C_{k_r})_{ij} = \begin{cases} 1 & \text{if } j \equiv i+1 \pmod{k_r} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.4 *There do not exist non-directed valency k matrices of degree n if n and k are odd.*

(proof) Let A be such a matrix. We count the number of 1's in A in two ways. As each row has k 1's, A has nk 1's. On the other hand, if the upper triangular part of A has t 1's, then A must have $2t$ 1's. This contradicts that nk is odd.

Proposition 3.5 *Let A be a non-directed valency 2 matrix. Then there exists some directed valency 1 matrix such that A is the sum of it and its transpose.*

4 Classification

Let X be a finite set of cardinality n and $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme. Let A_i be the adjacency matrix for R_i , and let the relation matrix be A . As we know A_0 is the identity matrix, we would like to find A_i ($i=1, 2, \dots, d$).

We will use the symbols that kn_i , kc_i and ks_i to represent non-commutative, commutative and symmetric association schemes on k vertices respectively.

The cases $n=1, 2, 3$ and 4 are well-known. We will consider each $n \geq 4$ separately.

Now we define new notation for valencies. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme and $\{k_0, k_1, \dots, k_d\}$ be the set of valencies. If $R_i \neq R_{i'}$, then we describe $(k_i, k_{i'})$ as a pair of valencies. In this way we denote valencies as the following set.

$$\{k_0, k_{s_1}, \dots, k_{s_l}, (k_{n_1}, k_{n'_1}), \dots, (k_{n_m}, k_{n'_m})\}$$

where R_{s_i} is symmetric for any $1 \leq i \leq l$ and R_{n_i} is non-symmetric for any $1 \leq i \leq m$.

4.1 The case of $n=5$

Theorem 4.1 *Suppose $\mathcal{X}$ is an association scheme with five vertices. Then $\mathcal{X}$ is isomorphic to one of the following association schemes.*

$$\begin{aligned} 5c_1 : Z_5 & \qquad 5s_2 : 5c_1(0, 1+4, 2+3) \\ 5s_3 : 5c_1(0, 1+2+3+4) \end{aligned}$$

Where $5c_1$ is the association scheme with relation matrix given by

$$5c_1: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

(proof) Let $\mathcal{X}$ be an association scheme on 5 vertices. We know there are no non-directed matrices whose valency is odd by proposition 3.4. If the association scheme has a directed adjacency matrix of valency 1, then it is isomorphic to C_5 and we see that the association scheme is isomorphic to $5c_1$.

We can divide into the following cases for the other adjacency matrices.

- (1) $\{1, (2, 2)\}$ (2) $\{1, 2, 2\}$
 (3) $\{1, 4\}$

We discuss each case separately.

- (1) Let A_1 and $A_2 (= {}^t A_1)$ be directed valency 2 matrices.

$$A_1 A_2 = 2A_0 + \sum_{k=1}^2 p_{12}^k A_k.$$

As the left-hand side is symmetric, $\sum_{k=1}^2 p_{12}^k A_k$ must be symmetric. But there do not exist p_{12}^k 's which satisfy that condition.

- (2) Let A_1 be a non-directed matrix of valency 2. Then we may assume A_1 is $C_5 + {}^t C_5$ by proposition 3.5. Hence this association scheme is isomorphic to $5s_2$.

- (3) It is clear that this is isomorphic to $5s_3$.

4.2 The case of $n=6$

Theorem 4.2 Suppose $\mathcal{X}$ is an association scheme with six vertices. Then $\mathcal{X}$ is isomorphic to one of the following association schemes.

$$\begin{array}{ll} 6c_1: Z_6 & 6n_2: S_3 \\ 6c_3: 6c_1(0, 1+3+5, 2, 4) & 6c_4: 6c_1(0, 1+4, 2+5, 3) \\ 6s_5: 6c_1(0, 1+5, 2+4, 3) & 6s_6: 6c_1(0, 1+4+2+5, 3) \\ 6s_7: 6c_1(0, 1+3+5, 2+4) & 6s_8: 6c_1(0, 1+2+3+4+5) \end{array}$$

where $6c_1$ and $6n_2$ are the association schemes with relation matrices given by

$$6c_1: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix} \quad 6n_2: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 1 & 4 & 5 & 3 \\ 1 & 2 & 0 & 5 & 3 & 4 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 3 & 2 & 0 & 1 \\ 5 & 3 & 4 & 1 & 2 & 0 \end{pmatrix}$$

Lemma 4.2.1 *Let $\mathcal{X}$ be an association scheme on 6 vertices. If $\mathcal{X}$ has a directed adjacency matrix of valency 1, then $\mathcal{X}$ is isomorphic to $6c_1$, $6n_2$ or $6c_3$.*

Now we assume that A_1 is $C_{3,3}$. Because A_1^2 is a directed valency 1 matrix, A_1^2 is an adjacency matrix. If $\mathcal{X}$ has another adjacency matrix of valency 1, then all of the adjacency matrices are valency 1 matrices, and $\mathcal{X}$ is isomorphic to $6c_1$ or $6n_2$. If not, $\mathcal{X}$ is isomorphic to $6c_3$.

(proof) Let A_1 be a non-directed adjacency matrix of valency 1. We assume that $\mathcal{X}$ has another non-directed valency 1 adjacency matrix, which is defined as A_2 . As $A_1 A_2$ is also a valency 1 matrix and is neither A_1 or A_2 , it is an adjacency matrix. The order of a maximal subgroup of a group of order 6 is at most 3. Therefore an association scheme with two non-directed valency 1 matrices is isomorphic to $6n_2$.

(1) $\{1, 1, 2, 2\}$ (2) $\{1, 1, (2, 2)\}$
 (3) $\{1, 1, 4\}$

Now we describe the relation matrix follows

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & B_{12} & B_{13} \\ & B_{21} & 0 & 1 \\ & & 1 & 0 \\ B_{31} & B_{32} & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

(1) Let A_2 and A_3 be the non-directed valency 2 matrices. Then

$$A_1 A_2 (= A_2 A_1) = \begin{cases} A_2 & (1-1) \\ A_3 & (1-2) \end{cases}$$

If formula (1-1) is satisfied, then B_{ij} 's must be $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$. Then we can not construct A_2 as a non-directed valency 2 matrix.

If formula (1-2) is satisfied, then B_{ij} 's must be $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Then $\mathcal{X}$ is isomorphic to $6s_5$.

(2) We get $\mathcal{X}$ is isomorphic to $6c_4$ by an argument similar to (1).

(3) It is clear that $\mathcal{X}$ is isomorphic to $6s_6$.

Lemma 4.2.3 *Let $\mathcal{X}$ be an association scheme on 6 vertices. If $\mathcal{X}$ has no adjacency matrices of valency 1 except A_0 , then it is isomorphic to $6s_7$ or $6s_8$.*

As $\mathcal{X}$ has no adjacency matrices of valency 1, we can divide into the following cases for the valencies.

(4) $\{1, 2, 3\}$

(5) $\{1, 5\}$

We discuss each case separately.

(4) Let A_2 be the non-directed valency 2 matrix. Then we may assume A_2 is $C_6 + {}^t C_6$ or $C_{3,3} + {}^t C_{3,3}$.

Condition (4) can not be satisfied if A_2 is $C_6 + {}^t C_6$.

$\mathcal{X}$ is isomorphic to $6s_7$ if A_2 is $C_{3,3} + {}^t C_{3,3}$.

(5) It is clear that $\mathcal{X}$ is $6s_8$

4.3 The case of $n=7$

Theorem 4.3 *Suppose $\mathcal{X}$ is an association scheme with seven vertices. Then $\mathcal{X}$ is isomorphic to one of the following association schemes.*

$$\begin{array}{ll}
7c_1: Z_7 & 7s_2: 7c_1(0, 1+6, 2+5, 3+4) \\
7c_3: 7c_1(0, 1+2+4, 3+5+6) & 7s_4: 7s_2(0, 1+2+3+4+5+6)
\end{array}$$

where $7c_1$ is the association scheme with relation matrix given by

$$7c_1: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \end{pmatrix}$$

(proof) Let $\mathcal{X}$ be an association scheme on 7 vertices. $\mathcal{X}$ is isomorphic to $7c_1$ if it has a directed adjacency matrix of valency 1.

Now assume that $\mathcal{X}$ has no valency 1 matrices except A_0 . Then we can divide into the following cases for the valencies.

(1) $\{1, 2, 2, 2\}$	(2) $\{1, (2, 2), 2\}$
(3) $\{1, 3, 3\}$	(4) $\{1, (3, 3)\}$
(5) $\{1, 6\}$	(6) $\{1, 2, 4\}$

We show each case.

(1) Let A_1, A_2 and A_3 be the non-directed valency-2 matrices. Then they are isomorphic to $C_7 + {}^t C_7$. Therefore $\mathcal{X}$ is isomorphic to $7s_2$.

(2) If $\mathcal{X}$ has a non-directed valency 2 matrix, then it is isomorphic to $7s_2$ because the valencies except k_0 are at least 2. Hence condition (2) can not be satisfied.

(3) Let A_1 and A_2 be the non-directed valency 3 matrices. Then

$$A_1^2 = 3A_0 + \begin{cases} A_1 + A_2 & (3-1) \\ 2A_1 & (3-2) \\ 2A_2 & (3-3) \end{cases}$$

We describe the relation matrix as follows.

$$\left(\begin{array}{c|ccc|ccc} 0 & & & & 2 & 2 & 2 \\ \hline 1 & & & & & & \\ 1 & & B_{11} & & & B_{12} & \\ 1 & & & & & & \\ \hline 2 & & & & & & \\ 2 & & B_{21} & & & B_{22} & \\ 2 & & & & & & \end{array} \right)$$

If formula (3-1) is satisfied, then each column of B_{11} has one 1 by $p_{11}^1 = 1$. But it is impossible to construct B_{11} because A_1 is a non-directed matrix.

If formula (3-2) is satisfied, then $A_2^2 = (J - A_0 - A_1)^2 = 3A_0 + 3A_1 - A_2$. This contradicts $p_{22}^2 \geq 0$.

The case of (3-3) is by an argument similar to that of (3-1) about A_2 .

(4) Let A_1 and A_2 ($= {}^t A_1$) be the directed valency 3 matrices. As $A_1 A_2$ is symmetric, we can get

$$A_1 A_2 = 3A_0 + A_1 + A_2.$$

Then $A_1^2 = A_1 + 2A_2$ and $A_2^2 = 2A_1 + A_2$. We get that $\mathcal{X}$ is isomorphic to $7c_3$ by considering the relation matrix.

(5) It is clear that $\mathcal{X}$ is isomorphic to $7c_4$.

(6) There are no association schemes which satisfy the condition (6) by an argument similar to (2).

4.4 The case of $n=8$

Theorem 4.4 Suppose $\mathcal{X}$ is an association scheme with eight vertices. Then $\mathcal{X}$ is isomorphic to one of the following association schemes.

$8n_1: D_4$	$8n_2: Q_8$
$8c_3: Z_8$	$8c_4: Z_2 \times Z_4$
$8s_5: Z_2 \times Z_2 \times Z_2$	$8c_6: 8c_3(0, 1+5, 2, 3+7, 4, 6)$
$8c_7: 8n_1(0, 1+4, 5, 2+7, 6, 3)$	$8c_8: 8c_4(0, 1+5, 3+7, 2, 4, 6)$
$8s_9: 8n_1(0, 1+4, 5, 2, 3+6, 7)$	$8s_{10}: 8n_1(0, 1+7, 3+6, 2+4, 5)$
$8c_{11}: 8c_3(0, 1+3, 2+6, 4, 5+7)$	$8c_{12}: 8n_1(0, 1+4+2+7, 5, 6, 3)$
$8c_{13}: 8c_4(0, 1+5, 3+7, 2+6, 4)$	$8s_{14}: 8n_1(0, 1+4, 5, 2+7, 3+6)$
$8s_{15}: 8c_4(0, 1+3+5+7, 2, 4, 6)$	$8s_{16}: 8c_4(0, 2+4, 6, 1+3+5+7)$
$8s_{17}: 8n_1(0, 1+4+5, 2, 3+6+7)$	$8s_{18}: 8n_1(0, 1+4+5, 2+3+6+7)$
$8s_{19}: 8n_1(0, 1+3+4+5+6+7, 2)$	$8s_{20}: 8n_1(0, 1+2+3+4+5+6+7)$

where $8n_1$, $8n_2$, $8c_3$, $8c_4$ and $8s_5$ are the association schemes with relation matrices given by

$$\begin{array}{l}
 8n_1: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 6 & 7 & 5 & 4 & 2 & 3 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 6 & 7 & 1 & 0 & 2 & 3 & 5 & 4 \\ 4 & 5 & 3 & 2 & 0 & 1 & 7 & 6 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 3 & 2 & 4 & 5 & 7 & 6 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} \\
 8n_2: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 0 & 1 & 2 & 5 & 6 & 7 & 4 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 1 & 2 & 3 & 0 & 7 & 4 & 5 & 6 \\ 6 & 7 & 4 & 5 & 0 & 1 & 2 & 3 \\ 7 & 4 & 5 & 6 & 3 & 0 & 1 & 2 \\ 4 & 5 & 6 & 7 & 2 & 3 & 0 & 1 \\ 5 & 6 & 7 & 4 & 1 & 2 & 3 & 0 \end{pmatrix} \\
 8c_3: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{pmatrix} \\
 8c_4: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 0 & 1 & 2 & 7 & 4 & 5 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 1 & 2 & 3 & 0 & 5 & 6 & 7 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 7 & 4 & 5 & 6 & 3 & 0 & 1 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 5 & 6 & 7 & 4 & 1 & 2 & 3 & 0 \end{pmatrix} \\
 8s_5: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}
 \end{array}$$

We show the above theorem by the following lemmas.

Lemma 4.4.1 *Let $\mathcal{K}$ be an association scheme on 8 vertices. If $\mathcal{K}$ has a directed adjacency matrix of valency 1, then $\mathcal{K}$ is isomorphic to one of $8n_1$, $8n_2$, $8c_3$, $8c_4$, $8c_6$, $8c_7$ or $8c_{12}$.*

(proof) Let A_1 be a directed adjacency matrix of valency 1. Then A_1 is isomorphic to C_8 or $C_{4,4}$. If A_1 is isomorphic to C_8 , then $\mathcal{X}$ is isomorphic to $8c_3$. If A_1 is isomorphic to $C_{4,4}$, then let A_1 be $C_{4,4}$ without loss of generality and we can get A_2 and A_3 as A_1^2 and A_1^3 respectively. Then we can divide into the following cases for the valencies.

(1) at least five valency 1 matrices (2) $\{1, 1, (1, 1), 2, 2\}$

(3) $\{1, 1, (1, 1), (2, 2)\}$ (4) $\{1, 1, (1, 1), 4\}$

We discuss each case separately.

(1) Let G be a group of order 8. The order of a maximal subgroup of a group of order 8 is at most 4. As an association scheme which satisfies condition (1) has at least five valency 1 matrices, all of valencies must be 1. Hence $\mathcal{X}$ is isomorphic to $8n_1$, $8n_2$, $8c_3$ or $8c_4$.

(2) Let A_4 and A_5 be the non-directed valency 2 matrices. Then

$$A_1 A_4 = \begin{cases} A_4 & (2-1) \\ A_5 & (2-2) \end{cases}$$

If formula (2-1) is satisfied, then $A_2 A_4 = A_3 A_4 = A_4$. But we can not construct A_4 which satisfy those formulas. Now we assume the formula (2-2). Then we have

$$\begin{aligned} A_4 A_1 &= A_5 & A_2 A_4 &= A_4 A_2 = A_4 & A_3 A_4 &= A_4 A_3 = A_5 \\ A_1 A_5 &= A_5 A_1 = A_4 & A_2 A_5 &= A_5 A_2 = A_5 & A_3 A_5 &= A_5 A_3 = A_4 \end{aligned}$$

We get that $\mathcal{X}$ is isomorphic to $8c_7$ by considering the relation matrix.

(3) $\mathcal{X}$ is isomorphic to $8c_6$ by an argument similar to the case (2).

(4) it is clear that $\mathcal{X}$ is isomorphic to $8c_{12}$.

Lemma 4.4.2 *Let $\mathcal{X}$ be an association scheme of 8 vertices. If $\mathcal{X}$ has at least two non-directed adjacency matrices of valency 1 and no directed adjacency matrices of valency 1, then $\mathcal{X}$ is isomorphic to $8s_5$, $8c_8$, $8s_9$ or $8s_{15}$.*

(proof) Let A_1 and A_2 be the non-directed adjacency matrices of valency 1. Let $A_1 + A_2$ be isomorphic to $C_8 + {}^t C_8$, $C_{4,4} + {}^t C_{4,4}$ or $C_{3,5} + {}^t C_{3,5}$ because $A_1 + A_2$ is a non-directed valency 2 matrix.

In the case of $A_1 + A_2 \cong C_8 + {}^t C_8$, we can partition $C_8 + {}^t C_8$ among A_1 and A_2 up to isomorphism as follows.

$$A_1 + 2A_2 \cong \begin{pmatrix} 0 & 1 & & & & & & 2 \\ & 1 & 0 & 2 & & & & \\ & & 2 & 0 & 1 & & & \\ & & & 1 & 0 & 2 & & \\ & & & & 2 & 0 & 1 & \\ & & & & & 1 & 0 & 2 \\ & & & & & & 2 & 0 & 1 \\ & 2 & & & & & & 1 & 0 \end{pmatrix}$$

$\mathcal{X}$ must have a directed adjacency matrix of valency 1 because $A_1 A_2$ is a directed valency 1 matrix. This is in contradiction to our assumption.

In the case of $A_1 + A_2 \cong C_{4,4} + {}^t C_{4,4}$, we can partition $C_{4,4} + {}^t C_{4,4}$ among A_1 and A_2 up to isomorphism as follows.

$$A_1 + 2A_2 \cong \begin{pmatrix} 0 & 1 & & 2 & & & & \\ & 1 & 0 & 2 & & & & \\ & & 2 & 0 & 1 & & & \\ & 2 & & 1 & 0 & & & \\ & & & & & 0 & 1 & 2 \\ & & & & & 1 & 0 & 2 \\ & & & & & & 2 & 0 & 1 \\ & & & & & & & 2 & 1 & 0 \end{pmatrix}$$

Now we define A_1 and A_2 as above without loss of generality. $A_1 A_2$ is also a non-directed valency 1 matrix. Hence it must be an adjacency matrix A_3 . Then we can divide into the following cases for the valencies.

(5) $\{1, 1, 1, 1, 1, 1, 1, 1\}$	(6) $\{1, 1, 1, 1, 1, 1, 2\}$
(7) $\{1, 1, 1, 1, 2, 2\}$	(8) $\{1, 1, 1, 1, (2, 2)\}$
(9) $\{1, 1, 1, 1, 1, 3\}$	(10) $\{1, 1, 1, 1, 4\}$

We discuss each case separately.

- (5) It is clear that $\mathcal{X}$ is isomorphic to $8S_5$.

The order of a maximal subgroup of a group of order 8 is at most 4. Therefore there do not exist association schemes which satisfy conditions (6) or (9).

- (7) Let A_4 and A_5 be the non-directed valency 2 matrices.

$$(A_0 + A_1 + A_2 + A_3)A_4 = 2A_4 + 2A_5 \quad (7-0)$$

$$A_1 A_4 = \begin{cases} A_4 & (7-1) \\ A_5 & (7-2) \end{cases}$$

$$A_2 A_4 = \begin{cases} A_4 & (7-3) \\ A_5 & (7-4) \end{cases}$$

If formulas (7-1) and (7-3) are satisfied, then

$$A_3 A_4 = (A_1 A_2) A_4 = A_1 (A_2 A_4) = A_1 A_4 = A_4.$$

This contradicts the formula (7-0).

If formulas (7-1) and (7-4) are satisfied, then $\mathcal{X}$ is isomorphic to $8s_9$.

If formulas (7-2) and (7-3) are satisfied, then $\mathcal{X}$ is isomorphic to $8s_9$.

If $\mathcal{X}$ satisfies the formulas (7-2) and (7-4), then $\mathcal{X}$ is isomorphic to $8s_9$.

(8) We get that $\mathcal{X}$ is isomorphic to $8c_8$ by an argument similar to (7).

(10) It is clear that $\mathcal{X}$ is isomorphic to $8s_{15}$.

In the case of $A_1 + A_2 \cong C_{3,5} + {}^t C_{3,5}$, we cannot partition $C_{3,5} + {}^t C_{3,5}$ among A_1 and A_2 .

Lemma 4.4.3 *Let $\mathcal{X}$ be an association scheme on 8 vertices. If $\mathcal{X}$ has only one non-directed adjacency matrix of valency 1 and no directed adjacency matrices of valency 1, then $\mathcal{X}$ is isomorphic to $8s_{10}$, $8c_{11}$, $8c_{13}$, $8s_{14}$, $8s_{16}$, $8s_{17}$ or $8s_{19}$.*

(proof) Let A_1 be the non-directed valency 1 matrix. We can divide into the following cases for the valencies.

(11) $\{1, 1, 2, 2, 2\}$	(12) $\{1, 1, 2, (2, 2)\}$
(13) $\{1, 1, 3, 3\}$	(14) $\{1, 1, (3, 3)\}$
(15) $\{1, 1, 2, 4\}$	(16) $\{1, 1, 6\}$

We discuss each case separately.

(11) Let A_2 , A_3 and A_4 be the non-directed valency 2 matrices. A_2 is isomorphic to $C_8 + {}^t C_8$ or $C_{4,4} + {}^t C_{4,4}$.

In the case of $A_2 \cong C_8 + {}^t C_8$ we get A_3 as $A_2^2 - 2A_0$. Here we can get A_1 and A_4 as follows

$$A_1 = \frac{1}{2}A_3^2 - A_0 \quad A_4 = J - A_0 - A_2 - A_3 - A_1$$

Hence $\mathcal{X}$ is isomorphic to $8s_{10}$.

In the case of $A_2 \cong C_{4,4} + {}^t C_{4,4}$ we get A_1 as $\frac{1}{2}A_2^2 - A_0$ and

$$A_3 A_1 = \begin{cases} A_3 & (11-1) \\ A_4 & (11-2) \end{cases}$$

If formula (11-1) is satisfied, then $\mathcal{X}$ is isomorphic to $8s_{14}$.

If formula (11-2) is satisfied, then $\mathcal{X}$ is isomorphic to $8s_{14}$.

(12) $\mathcal{X}$ is isomorphic to $8c_{11}$ or $8c_{13}$ by an argument similar to (11).

(13) Let A_2 and A_3 be the non-directed valency 3 matrices. Then $A_1 A_2$ is A_2 of A_3 . But there exists no A_2 which satisfies $A_1 A_2 = A_2$ because the valency of A_2 is odd. Therefore

$$A_1 A_2 = A_2 A_1 = A_3$$

Similarly

$$A_1 A_3 = A_3 A_1 = A_2$$

Now we can describe the relation matrix as follows:

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ B_{12} & & 1 & 0 & \\ & & & & 0 & 1 \\ B_{13} & & B_{23} & & 1 & 0 \\ & & & & & & 0 & 1 \\ B_{14} & & B_{24} & & B_{34} & & 1 & 0 \end{pmatrix}$$

where $B_{ij} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Moreover we assume $B_{12} = B_{23} = B_{34} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ without loss of generality by using permutations. We can divide into the following cases for the other B_{ij} 's.

$$(13.1) \quad B_{13} = B_{14} = B_{24} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

$$(13.2) \quad B_{13} = B_{14} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad B_{24} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$$(13.3) \quad B_{13} = B_{24} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad B_{14} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$$(13.4) \quad B_{24}=B_{14}=\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \quad B_{13}=\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

In the case of (13.1) $\mathcal{X}$ is isomorphic to $8s_{17}$.

We get that there exist no association schemes with this relation matrix for the cases of (13.2), (13.3) and (13.4) by considering A_2^2 .

(14) Let A_2 and A_3 be two directed valency 3 matrices. Then

$$\begin{aligned} A_1 A_2 &= \begin{cases} A_2 & (14-1) \\ A_3 & (14-2) \end{cases} \\ A_2 A_1 &= \begin{cases} A_2 & (14-3) \\ A_3 & (14-4) \end{cases} \end{aligned}$$

If formulas (14-1) and (14-3) are satisfied, then there exists no A_2 because the valency of A_2 is odd.

If formulas (14-1) and (14-4) are satisfied, then

$$A_3 A_1 = (A_2 A_1) A_1 = A_2 \quad A_3 A_1 = {}^t(A_1 A_2) = {}^t A_2 = A_3.$$

This contradicts $A_2 \neq A_3$.

If formulas (14-2) and (14-3) are satisfied, then this contradicts $A_2 \neq A_3$ by a similar argument.

If formulas (14-2) and (14-4) are satisfied, then there exist no association schemes by an argument similar to (13).

(15) $\mathcal{X}$ is isomorphic to $8s_{16}$ by an argument similar to case of (12).

(16) It is clear that $\mathcal{X}$ is isomorphic to $8s_{19}$.

Lemma 4.4.4 *Let $\mathcal{X}$ be an association scheme on 8 vertices. If $\mathcal{X}$ has no adjacency matrices of valency 1 except A_0 , then $\mathcal{X}$ is isomorphic to $8s_{18}$ or $8s_{20}$.*

(proof) We can divide into the following cases for the valencies.

(17) $\{1, 2, 2, 3\}$	(18) $\{1, 2, 5\}$
(19) $\{1, (2, 2), 3\}$	(20) $\{1, 3, 4\}$
(21) $\{1, 7\}$	

We discuss each case separately.

If $\mathcal{X}$ has a non-directed adjacency matrix of valency 2, then it has to have a non-directed valency 1 matrix. This contradicts the assumption.

Hence the case of (17) or (18) can not be satisfied.

(19) Let A_1 and A_2 ($= {}^t A_1$) be the directed valency 2 matrices. We pay

attention to valencies and then

$$A_1 A_2 = 2A_0 + \begin{cases} A_1 & (19-1) \\ A_2 & (19-2) \end{cases}$$

The left-hand side is symmetric but the each right-hand side is not symmetric, a contradiction.

(20) Let A_1 be the non-directed valency 3 matrix and let A_2 be the non-directed valency 4 matrix. Then

$$A_1^2 = 3A_0 + 2A_1 \quad A_1 A_2 = 3A_2 \quad A_2^2 = 4A_0 + 4A_1$$

Hence $\mathcal{X}$ is isomorphic to $8s_{18}$.

(21) It is clear that $\mathcal{X}$ is isomorphic to $8s_{20}$.

4.5 In the case of $n=9$

Theorem 4.5 *Suppose $\mathcal{X}$ is an association scheme with vertices. Then $\mathcal{X}$ is isomorphic to one of the following association schemes.*

- $9c_1: Z_3 \times Z_3$
- $9c_2: Z_9$
- $9c_3: 9c_1(0, 1+2, 3, 4+5, 6, 7+8)$
- $9s_4: 9c_1(0, 1+2, 3+6, 4+8, 5+7)$
- $9s_5: 9c_2(0, 1+8, 2+7, 3+6, 4+5)$
- $9c_6: 9c_1(0, 1+4+7, 2+5+8, 3, 6)$
- $9s_7: 9c_1(0, 1+2, 3+6, 4+5+7+8)$
- $9c_8: 9c_1(0, 1+2+4+5+7+8, 3, 6)$
- $9c_9: 9c_1(0, 1+4+7, 2+5+8, 3+6)$
- $9s_{10}: 9c_1(0, 1+2+3+6, 4+5+7+8)$
- $9s_{11}: 9c_1(0, 1+2, 3+4+5+6+7+8)$
- $9s_{12}: 9c_1(0, 1+2+3+4+5+6+7+8)$

where $9c_1$ and $9c_2$ are the association schemes with relation matrices given by

$$9c_1: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 0 & 1 & 5 & 3 & 4 & 8 & 6 & 7 \\ 1 & 2 & 0 & 4 & 5 & 3 & 7 & 8 & 6 \\ 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\ 8 & 6 & 7 & 2 & 0 & 1 & 5 & 3 & 4 \\ 7 & 8 & 6 & 1 & 2 & 0 & 4 & 5 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\ 5 & 3 & 4 & 8 & 6 & 7 & 2 & 0 & 1 \\ 4 & 5 & 3 & 7 & 8 & 6 & 1 & 2 & 0 \end{pmatrix} \quad 9c_2: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \end{pmatrix}$$

We show the above theorem by the following lemmas.

Lemma 4.5.1 *Let $\mathcal{K}$ be an association scheme on 9 vertices. If $\mathcal{K}$ has a directed adjacency matrix of valency 1, then $\mathcal{K}$ is isomorphic to one of $9c_1$, $9c_2$, $9c_3$, $9c_6$ or $9c_8$.*

(proof) Let A_1 be the directed adjacency matrix of valency 1. Then A_1 is isomorphic to C_9 or $C_{3,3,3}$.

If A_1 is isomorphic to C_9 , then $\mathcal{K}$ is isomorphic to $9c_2$.

Now assume that A_1 is isomorphic to $C_{3,3,3}$. A_1^2 is also a directed valency 1 matrix. Therefore A_1^2 is an adjacency matrix and let A_2 be A_1^2 . Then we can divide into the following cases for the valencies.

(1) at least four directed valency 1 matrices	(2) $\{1, (1, 1), 2, 2, 2\}$
(3) $\{1, (1, 1), 2, (2, 2)\}$	(4) $\{1, (1, 1), (3, 3)\}$
(5) $\{1, (1, 1), 2, 4\}$	(6) $\{1, (1, 1), 6\}$

We discuss each case separately.

(1) Let G be a group of order 9. The order of a maximal subgroup of G is at most 3. Hence $\mathcal{K}$ is isomorphic to $9c_1$ or $9c_2$.

(2) Let A_3, A_4 and A_5 be the non-directed valency 2 matrices. Then we may assume the following formulas:

$$\begin{aligned} A_1 A_3 &= A_4 & A_1 A_4 &= A_5 & A_1 A_5 &= A_3 \\ A_3 A_1 &= A_5 & A_4 A_1 &= A_3 & A_5 A_1 &= A_4 \end{aligned}$$

We can describe the relation matrix as follows:

$$\begin{pmatrix} 0 & 1 & 2 & & & \\ 2 & 0 & 1 & B_{12} & & B_{13} \\ 1 & 2 & 0 & & & \\ & & & 0 & 1 & 2 \\ & B_{21} & & 2 & 0 & 1 \\ & & & 1 & 2 & 0 \\ & & & & & 0 & 1 & 2 \\ B_{31} & & B_{32} & & & 2 & 0 & 1 \\ & & & & & 1 & 2 & 0 \end{pmatrix}$$

where $B_{ij} = \begin{pmatrix} 3 & 5 & 4 \\ 5 & 4 & 3 \\ 4 & 3 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 5 \\ 3 & 5 & 4 \\ 5 & 4 & 3 \end{pmatrix}$ or $\begin{pmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 3 & 5 & 4 \end{pmatrix}$.

We get that there exist no association schemes with relation matrix for any cases of B_{ij} 's.

(3) We get $\mathcal{X}$ is isomorphic to $9c_3$ by a similar argument as (3) if it is satisfied with the assumption.

(4) Let A_3 and A_4 ($= {}^t A_3$) be the directed valency 3 matrices. Then

$$\begin{aligned} A_1 A_3 &= \begin{cases} A_3 & (4-1) \\ A_4 & (4-2) \end{cases} \\ A_3 A_1 &= \begin{cases} A_3 & (4-3) \\ A_4 & (4-4) \end{cases} \\ A_1 A_4 &= \begin{cases} A_3 & (4-5) \\ A_4 & (4-6) \end{cases} \end{aligned}$$

If formulas (4-1) and (4-3) are satisfied, then

$$A_2 A_3 = A_3 A_2 = A_3 \quad A_1 A_4 = A_4 A_1 = A_2 A_4 = A_4 A_2 = A_4.$$

Hence $\mathcal{X}$ is isomorphic to $9c_6$.

If formulas (4-1) and (4-4) are satisfied, then

$$A_3 = A_1 A_3 = A_1 {}^t A_4 = A_1 {}^t (A_3 A_1) = A_1 (A_2 A_4) = A_4.$$

This contradicts $A_3 \neq A_4$.

If formulas (4-2) and (4-3) are satisfied, then this contradicts $A_3 \neq A_4$ similarly.

If formulas (4-2), (4-4) and (4-5) are satisfied, then this contradicts $A_3 \neq A_4$ similarly.

If formulas (4-2), (4-4) and (4-6) are satisfied, then

$$A_3 A_1 = {}^t A_4 A_1 = {}^2(A_1 A_4) A_1 = (A_3 A_2) A_1 = A_3$$

This contradicts (4-4).

(5) Let A_3 and A_4 be the non-directed valency 2 and 4 matrices respectively. Then we can get following formulas.

$$A_1 A_3 = A_3 A_1 = A_2 A_3 = A_3 A_2 = A_3.$$

The above formula can not be satisfied because the valency of A_3 is 2.

(6) It is clear that $\mathcal{X}$ is isomorphic to $9c_8$.

Lemma 4.5.2 *Let $\mathcal{X}$ be an association scheme on 9 vertices. If $\mathcal{X}$ has no directed adjacency matrices of valency 1 except A_0 , then $\mathcal{X}$ is isomorphic to one of $9s_4$, $9s_5$, $9s_7$, $9c_9$, $9s_{10}$, $9s_{11}$ or $9s_{12}$.*

(proof) We can divide into the following cases for the valencies.

(7) $\{1, 2, 2, 2, 2\}$	(8) $\{1, 2, 2, (2, 2)\}$
(9) $\{1, 2, (3, 3)\}$	(10) $\{1, 2, 2, 4\}$
(11) $\{1, (2, 2), 4\}$	(12) $\{1, (4, 4)\}$
(13) $\{1, 2, 6\}$	(14) $\{1, 8\}$
(15) $\{1, (2, 2), (2, 2)\}$	(16) $\{1, 4, 4\}$

We discuss each case separately.

(7) Let A_1, A_2, A_3 and A_4 be the non-directed valency 2 matrices. A_1 is isomorphic to $C_9 + {}^t C_9$ or $C_{3,3,3} + {}^t C_{3,3,3}$.

If A_1 is isomorphic to $C_9 + {}^t C_9$, then $\mathcal{X}$ is isomorphic to $9s_5$.

We assume that A_1 is isomorphic to $C_{3,3,3} + {}^t C_{3,3,3}$ and that A_2, A_3 and A_4 are also isomorphic to $C_{3,3,3} + {}^t C_{3,3,3}$ similarly. Then

$$A_2^2 = 2A_0 + A_2 \quad A_3^2 = 2A_0 + A_3 \quad A_4^2 = 2A_0 + A_4.$$

Hence $\mathcal{X}$ is isomorphic to $9s_4$.

(8) Let A_1 and A_2 be the non-directed valency 2 matrices and A_3 and A_4 ($= {}^t A_3$) be the directed valency 2 matrices. We can assume that A_1 and A_2 are isomorphic to $C_{3,3,3} + {}^t C_{3,3,3}$.

Then we get the following formulas

$$\begin{aligned}
A_1^2 &= 2A_0 + A_1 & A_2^2 &= 2A_0 + A_2 \\
A_1 A_2 &= A_2 A_1 = A_3 + A_4 & A_1 A_3 &= A_3 A_1 = A_2 + A_4 \\
A_1 A_4 &= A_4 A_1 = A_2 + A_3 & A_2 A_3 &= A_3 A_2 = A_1 + A_4 \\
A_2 A_4 &= A_4 A_2 = A_1 + A_3
\end{aligned}$$

Because $A_3 A_4$ is symmetric,

$$A_3 A_4 = 2A_0 + \begin{cases} A_1 & (8-1) \\ A_2 & (8-2) \end{cases}$$

Assume the formula (8-1). Then

$$\begin{aligned}
A_1 A_3 A_4 A_1 &= (A_1 A_3) (A_4 A_1) \\
&= 2A_0 + 2A_1 + A_2 + A_3 + A_4 + A_4 A_3 \\
A_1 A_3 A_4 A_1 &= A_1 (A_3 A_4) A_1 \\
&= 6A_0 + 5A_1
\end{aligned}$$

Therefore $A_4 A_3 = 4A_0 + 3A_1 - A_2 - A_3 - A_4$, a contradiction.

If we assume formula (8-2), then this gives a contradiction by a similar argument.

(9) Let A_1 be the non-directed valency 2 matrix and let A_2 and $A_3 (= {}^t A_2)$ be the directed valency 3 matrices. We may assume that A_1 is isomorphic to $C_{3,3,3} + {}^t C_{3,3,3}$ as in (7). Then

$$A_1 A_2 = \begin{cases} 2A_2 & (9-1) \\ A_2 + A_3 & (9-2) \\ 2A_3 & (9-3) \end{cases}$$

If formula (9-1) is satisfied, then $\mathcal{X}$ is isomorphic to $9c_9$.

There exist no relation matrices which satisfy the formulas (9-2) or (9-3).

(10) Let A_1 and A_2 be the non-directed valency 2 matrices and let A_3 be the non-directed valency 4 matrix. We may assume that A_1 and A_2 are isomorphic to $C_{3,3,3} + {}^t C_{3,3,3}$ by an argument similar to (7). Then we get that $\mathcal{X}$ is isomorphic to $9s_7$ by considering the relation matrix.

(11) Let A_1 and $A_2 (= {}^t A_1)$ be the directed valency 2 matrices and let A_3 be the non-directed valency 4 matrix. When we pay attention to the valencies, we get the following formulas

$$A_1 A_2 = 2A_0 + \begin{cases} A_1 & (11-1) \\ A_2 & (11-2) \end{cases}$$

In each of the cases, the left-hand side is symmetric but the right-hand side is not symmetric, a contradiction.

(12) There exist no such association schemes by proposition 3.2

(13) Let A_1 be the non-directed valency 2 matrix and let A_2 be the non-directed valency 4 matrix. We may assume that A_1 is isomorphic to $C_{3,3,3} + {}^t C_{3,3,3}$ by an argument similar to (7). Hence $\mathcal{X}$ is isomorphic to $9s_{11}$ by $A_2 := J - A_0 - A_1$

(14) It is clear that $\mathcal{X}$ is isomorphic to $9s_{12}$

(15) There exist no association schemes which satisfy this condition by an argument similar to the case of (11).

(16) Let A_1 and A_2 be the non-directed valency 4 matrices. Then

$$A_1^2 = 4A_0 + \begin{cases} 3A_1 & (16-1) \\ 2A_1 + A_2 & (16-2) \\ 3A_2 & (16-3) \\ A_1 + 2A_2 & (16-4) \end{cases}$$

We get the following results by considering the relation matrix for the each cases.

If formula (16-1), (16-2) or (16-3) is satisfied, then there exist no association schemes.

If formula (16-4) is satisfied, then $\mathcal{X}$ is isomorphic to $9s_{10}$.

4.6 In the case of $n=10$

Theorem 4.6 *Suppose $\mathcal{X}$ is an association scheme with ten vertices. Then $\mathcal{X}$ is isomorphic to one of the following association schemes.*

- $10c_1: \mathbf{Z}_{10}$
- $10n_2: \mathbf{D}_5$
- $10c_3: 10c_1(0, 1+6, 4+9, 2+7, 3+8, 5)$
- $10s_4: 10c_1(0, 1+9, 2+8, 3+7, 4+6, 5)$
- $10c_5: 10c_1(0, 1+3+5+7+9, 2, 4, 6, 8)$
- $10s_6: 10n_2(0, 1+4, 2+3, 6+9, 7+8, 5)$
- $10n_7$
- $10s_8: 10c_1(0, 1+6+4+9, 2+3+7+8, 5)$
- $10s_9: 10c_1(0, 5, 2+8+4+6, 1+3+7+9)$
- $10s_{10}: 10n_2(0, 1+4, 2+3, 5+6+7+8+9)$

$$\begin{aligned}
10s_{11}: & 10c_1(0, 5, 1+2+3+4+6+7+8+9) \\
10s_{12}: & 10n_2(0, 1+2+3+4, 5+6+7+8+9) \\
10s_{13}: & 10n_7(0, 1+3, 2+4+5) \\
10s_{14}: & 10c_1(0, 1+2+3+4+5+6+7+8+9)
\end{aligned}$$

where $10c_1$, $10n_2$ and $10n_7$ are the association schemes with relation matrices given by

$$\begin{aligned}
10c_1: & \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \end{pmatrix} \\
10n_2: & \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 0 & 1 & 2 & 3 & 6 & 7 & 8 & 9 & 5 \\ 3 & 4 & 0 & 1 & 2 & 7 & 8 & 9 & 5 & 6 \\ 2 & 3 & 4 & 0 & 1 & 8 & 9 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 0 & 9 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 & 5 & 4 & 0 & 1 & 2 & 3 \\ 7 & 8 & 9 & 5 & 6 & 3 & 4 & 0 & 1 & 2 \\ 8 & 9 & 5 & 6 & 7 & 2 & 3 & 4 & 0 & 1 \\ 9 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 0 \end{pmatrix} \\
10n_7: & \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 3 & 4 & 5 & 5 & 4 \\ 1 & 0 & 1 & 2 & 2 & 5 & 5 & 4 & 3 & 4 \\ 2 & 1 & 0 & 1 & 2 & 4 & 3 & 4 & 5 & 5 \\ 2 & 2 & 1 & 0 & 1 & 4 & 5 & 5 & 4 & 3 \\ 1 & 2 & 2 & 1 & 0 & 5 & 4 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 & 4 & 0 & 1 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 & 5 & 1 & 0 & 1 & 2 & 2 \\ 4 & 5 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ 4 & 3 & 4 & 5 & 5 & 2 & 2 & 1 & 0 & 1 \\ 5 & 5 & 4 & 3 & 4 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}
\end{aligned}$$

We show the above theorem by the following lemmas.

Lemma 4.6.1 *Let $\mathcal{X}$ be an association scheme on 10 vertices. If $\mathcal{X}$ has a directed adjacency matrix of valency 1, then $\mathcal{X}$ is isomorphic to one of $10c_1$, $10n_2$ or $10c_5$.*

(proof) Let A_1 be the directed adjacency matrix of valency 1. Then A_1 is isomorphic to C_{10} or $C_{5,5}$. If $A_1 \cong C_{10}$, then $\mathcal{X}$ is isomorphic to $10c_1$. If $A_1 \cong C_{5,5}$,

then let A_1 be $C_{5,5}$ without loss of generality and we can get A_2, A_3 and A_4 as A_1^2, A_1^3 and A_1^4 respectively. Then we can divide into the following cases for the valencies.

- (1) at least six valency 1 matrices (2) $\{1, (1, 1), (1, 1), 2, 3\}$
 (3) $\{1, (1, 1), (1, 1), 5\}$

We discuss each case separately.

(1) Let G be a group of order 10. The order of a maximal subgroup of G is at most 5. Hence $\mathcal{K}$ is isomorphic to $10c_1$ or $10n_2$.

(2) Let A_5 be the non-directed valency 2 matrix. As $A_1 A_5$ is also valency 2 matrix, $A_1 A_5 = A_5$. But this formula can not be satisfied because the valency of A_5 is two.

(3) It is clear that $\mathcal{K}$ is isomorphic to $10c_5$.

Lemma 4.6.2 *Let $\mathcal{K}$ be an association scheme on 10 vertices. If $\mathcal{K}$ has one non-directed adjacency matrix of valency 1 and no directed adjacency matrices of valency 1, then $\mathcal{K}$ is isomorphic to $10c_3, 10s_4, 10s_6, 10s_8, 10s_9$ or $10s_{11}$.*

(proof) Let A_1 be the non-directed valency matrix. Then we can describe the relation matrix follows:

$$\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & B_{12} & B_{13} & B_{14} & B_{15} \\ & B_{21} & 0 & 1 & B_{23} & B_{24} & B_{25} \\ & & 1 & 0 & & & \\ & B_{31} & B_{32} & 0 & 1 & B_{34} & B_{35} \\ & & & 1 & 0 & & \\ & B_{41} & B_{42} & B_{43} & 0 & 1 & B_{45} \\ & & & & 1 & 0 & \\ & B_{51} & B_{52} & B_{53} & B_{54} & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}$$

We can divide into the following cases for the valencies.

- (4) at least three non-directed valency 1 matrices (5) $\{1, 1, 2, 2, 2, 2\}$
 (6) $\{1, 1, 2, 2, (2, 2)\}$ (7) $\{1, 1, (2, 2), (2, 2)\}$
 (8) $\{1, 1, 3, 5\}$ (9) $\{1, 1, 2, 3, 3\}$
 (10) $\{1, 1, 2, (3, 3)\}$ (11) $\{1, 1, 2, 2, 4\}$
 (12) $\{1, 1, (2, 2), 4\}$ (13) $\{1, 1, (4, 4)\}$

$$(14) \{1, 1, 2, 6\}$$

$$(15) \{1, 1, 8\}$$

$$(16) \{1, 1, 4, 4\}$$

We discuss each case separately.

(4) Let A_2 be a non-directed valency 1 matrix. As $A_1 + A_2$ is a non-directed valency 2 matrix, it is isomorphic to $C_{10} + {}^t C_{10}$, $C_{5,5} + {}^t C_{5,5}$, $C_{3,7} + {}^t C_{3,7}$, $C_{3,3,4} + {}^t C_{3,3,4}$ or $C_{4,6} + {}^t C_{4,6}$.

If $A_1 + A_2$ is isomorphic to $C_{5,5} + {}^t C_{5,5}$, $C_{3,7} + {}^t C_{3,7}$ or $C_{3,3,4} + {}^t C_{3,3,4}$, then we cannot partition it into A_1 and A_2 . If $A_1 + A_2$ is isomorphic to $C_{4,6} + {}^t C_{4,6}$, then the valency 1 matrix $A_1 A_2$ is neither a directed matrix nor a non-directed matrix, a contradiction.

If $A_1 + A_2$ is isomorphic to $C_{10} + {}^t C_{10}$, then $A_1 A_2$ is a directed valency 1 matrix and it is an adjacency matrix. This contradicts the hypotheses.

(5) Let A_2, A_3, A_4 and A_5 be the non-directed valency 2 matrices. As $A_1 A_2$ is also a non-directed valency 2 matrix, it is an adjacency matrix.

If $A_1 A_2$ is A_2 , then the relation matrix must have $B_{ij} = B_{ji} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$'s. But we can not construct A_2 in this case. Therefore let A_3 be $A_1 A_2$.

We can assume $B_{12} = B_{21} = B_{23} = B_{32} = B_{34} = B_{43} = B_{45} = B_{54} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ and $B_{15} = B_{51} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

If B_{15} is $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$, then $\mathcal{X}$ is isomorphic to $10s_6$.

If B_{15} is $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then $\mathcal{X}$ is isomorphic to $10s_4$.

(6) We get $\mathcal{X}$ is isomorphic to $10n_7$ by an argument similar to (5).

(7) Let $A_2, A_3 (= {}^t A_2), A_4$ and $A_5 (= {}^t A_4)$ be the directed valency 2 matrices.

Then

$$A_1 A_2 = \begin{cases} A_2 & (7-1) \\ A_3 & (7-2) \\ A_4 & (7-3) \\ A_5 & (7-4) \end{cases} \quad A_2 A_1 = \begin{cases} A_2 & (7-5) \\ A_3 & (7-6) \\ A_4 & (7-7) \\ A_5 & (7-8) \end{cases}$$

As $A_i A_{i'}$ is symmetric,

$$A_2 A_3 = A_3 A_2 = A_4 A_5 = A_5 A_4 = 2A_0 + 2A_1.$$

If formulas (7-1) and (7-5) are satisfied, then we get that $\mathcal{X}$ is isomorphic to $10c_3$ by an argument similar to (5).

If formulas (7-1) and (7-6) are satisfied, then this contradicts $A_2 \neq A_3$.

If formulas (7-1) and (7-7) are satisfied, then this contradicts $p_{52}^0 = 0$.

Formulas (7-1) and (7-8) cannot be satisfied in an association scheme.

If formula (7-2) is satisfied, then this contradicts $p_{33}^0 = 0$.

If formula (7-3) is satisfied, then this contradicts $p_{43}^0 = 0$.

If formula (7-4) is satisfied, then a similar argument gives a contradiction.

(8) Let A_2 be the non-directed valency 3 matrix. Then we get $A_1 A_2 = A_2 A_1 = A_2$. But there exists no A_2 which satisfies this formula because the valency of A_2 is odd.

In the cases of (9), (10) and (14), if $\mathcal{X}$ has one non-directed valency 2 matrix, then it must have another valency 2 matrix. Hence there exist no association schemes in these cases.

(11) If $\mathcal{X}$ has two non-directed adjacency matrices of valency 2 and all of the adjacency matrices are non-directed, then it is isomorphic to $10s_4$ or $10s_{14}$ by the case of (4). Hence there exist no association schemes in this case.

(12) Let A_2 and $A_3 (= {}^t A_2)$ be the directed valency 2 matrices and let A_4 be the non-directed valency 4 matrix. Then we proceed as case (7).

$$A_2 A_3 = A_3 A_2 = 2A_0 + 2A_1 \quad A_1 A_2 = A_2 A_1 = A_2$$

We can describe $B_{12} = B_{23} = B_{34} = B_{45} = B_{51} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $B_{21} = B_{32} = B_{43} = B_{54} =$

$B_{15} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$: We get that there exist no association schemes in this case by considering A_2^2 .

(13) Let A_2 and $A_3 (= {}^t A_2)$ be the directed valency 4 matrices. We consider the valencies. As $A_2 A_3$ is symmetric,

$$A_2 A_3 = 4A_0 + \begin{cases} 12A_1 & (13-1) \\ 4A_1 + A_2 + A_3 & (13-1) \end{cases}$$

It is clear that we can not construct A_2 and A_3 which satisfy formula (13-1). If formula (13-2) is satisfied, then we get that there exist no association schemes by an argument similar to (12).

(15) It is clear that $\mathcal{X}$ is isomorphic to $10s_{11}$.

(16) Let A_2 and A_3 be the non-directed valency 4 matrices. Then

$$A_1 A_2 = \begin{cases} A_2 & (16-1) \\ A_3 & (16-2) \end{cases}$$

If formula (16-1) is satisfied, then the B_{ij} 's are $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$.

Therefore $\mathcal{X}$ is isomorphic to $10s_8$.

If formula (16-2) is satisfied, then the B_{ij} 's are $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

As $B_{ij} = B_{ji}$,

$$A_2^2 (= A_3^2) = 4A_0 + \begin{cases} 3A_2 & (16-3) \\ 3A_3 & (16-4) \\ 2A_2 + 3A_3 & (16-5) \\ A_2 + 2A_3 & (16-6) \end{cases}$$

We assume $B_{12} = B_{21} = B_{13} = B_{31} = B_{14} = B_{41} = B_{15} = B_{51} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.

If formula (16-3) is satisfied, then B_{ij} is $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ for any i, j . Hence $\mathcal{X}$ is isomorphic to $10s_9$.

If formula (16-4) is satisfied, then $\mathcal{X}$ is also isomorphic to $10s_9$ by replacing A_2 with A_3 .

If formula (16-5) is satisfied, then one of B_{2i} , B_{3i} , B_{4i} and B_{5i} for each $i \neq 1$ is $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ by $p_{22}^2 = 2$. But we can not construct adjacency matrices which satisfy formula (16-5).

If formula (16-6) is satisfied, then there exist no such association schemes by an argument similar to (16-5).

Lemma 4.6.3 *Let $\mathcal{X}$ be an association scheme on 10 vertices. If $\mathcal{X}$ has no adjacency*

matrices of valency 1 except A_0 , then $\mathcal{X}$ is isomorphic to one of $10s_{10}$, $10s_{13}$, $10s_{12}$ or $10s_{14}$.

(proof) We can divide into the following cases for the valencies.

(17) $\{1, 2, 2, 2, 3\}$	(18) $\{1, 2, (2, 2), 3\}$
(19) $\{1, 2, 2, 5\}$	(20) $\{1, 2, 3, 4\}$
(21) $\{1, 2, 7\}$	(22) $\{1, (2, 2), 5\}$
(23) $\{1, 3, 6\}$	(24) $\{1, 3, 3, 3\}$
(25) $\{1, 3, (3, 3)\}$	(26) $\{1, 4, 5\}$
(27) $\{1, 9\}$	

If $\mathcal{X}$ has a non-directed valency 2 matrix, then its matrix is isomorphic to $C_{5,5} + {}^t C_{5,5}$ because valencies are at least 2. Then $\mathcal{X}$ has an even number of non-directed valency 2 matrices. Therefore there exist no association schemes in cases (17), (18), (20) or (21).

(19) Let A_1 and A_2 be the non-directed valency 2 matrices and let A_3 be the non-directed valency 5 matrix. As A_1 is isomorphic to $C_{5,5} + {}^t C_{5,5}$, we get $A_2 = A_1^2 - 2A_0$. Hence $\mathcal{X}$ is isomorphic to $10s_{10}$.

(22) Let A_1 and $A_2 (= {}^t A_1)$ be the directed valency 2 matrices and let A_3 be the non-directed valency 5 matrix. When we consider the valency, we get the following formulas.

$$A_1 A_2 = 2A_0 + \begin{cases} A_1 \\ A_2 \end{cases}$$

The left-hand side is symmetric but the right-hand side is not symmetric in either case, a contradiction.

(23) Let A_1 and A_2 be the non-directed valency 3 matrix and the non-directed valency 6 matrix respectively. Then

$$A_1^2 = 3A_0 + \begin{cases} 2A_1 & (23-1) \\ A_2 & (23-2) \end{cases}$$

Now we describe the relation matrix as follows.

$$\left(\begin{array}{c|ccc|cccc} 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & & & & & & & & & \\ 1 & & B_{11} & & & & B_{12} & & & \\ 1 & & & & & & & & & \\ \hline 2 & & & & & & & & & \\ 2 & & & & & & & & & \\ 2 & & & & & & & & & \\ 2 & & & & & & & & & \\ 2 & & & & & & & & & \\ 2 & & & & & & & & & \end{array} \right)$$

If formula (23-1) is satisfied, then $B_{11} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Therefore $A_1^2 = 3A_0 +$

$2A_1$ in the restricted B_{22} . But there exists no B_{22} to satisfy that formula.

If formula (23-2) is satisfied, then

$$A_2^2 = 6A_0 + 4A_1 + 3A_2 \quad A_1 A_2 = 2A_1 + 2A_2.$$

We get that $\mathcal{X}$ is isomorphic to $10s_{13}$ by considering the relation matrix.

(24) Let A_1, A_2 and A_3 be the non-directed valency 3 matrices. We describe the relation matrix as follows

$$\left(\begin{array}{c|ccc|ccc|ccc} 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ \hline 1 & & & & & & & & & \\ 1 & & B_{11} & & & B_{12} & & & B_{13} & \\ 1 & & & & & & & & & \\ \hline 2 & & & & & & & & & \\ 2 & & & & & B_{22} & & & B_{23} & \\ 2 & & & & & & & & & \\ \hline 3 & & & & & & & & & \\ 3 & & & & & & & & B_{33} & \\ 3 & & & & & & & & & \end{array} \right)$$

As each of the adjacency matrices is symmetric, B_{ii} is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ or

$$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix} \text{ for any } i.$$

If $B_{11} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, then we must have $A_1^2 = 3A_0 + 2A_1$. But there exist no

association schemes which satisfy that formula by an argument similar to (23).

Similarly $B_{22} \neq \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$, $B_{33} \neq \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$. Therefore we get $p_{ii}^i = 0$. This

contradicts $B_{ii} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$.

(25) Similar to (24).

(26) Let A_1 be the non-directed valency 4 matrix and let A_2 be the non-directed valency 5 matrix. Then we get $A_1^2 = 4A_0 + 3A_1$.

Hence $\mathcal{X}$ is isomorphic to $10s_{12}$.

(27) It is clear that $\mathcal{X}$ is $10s_{14}$.

We can get all association schemes on at most ten vertices as subschemes of group cases except $10n_7$ and $10s_{13}$. We would like to make $10n_7$ from a group.

Let F_{20} be the Frobenius group of order 20 and H be a subgroup of F_{20} of order 2. Let C be the set of right cosets of H in F_{20} . Let A be the permutation representation of F_{20} on C . Let X denote the set $\{1, 2, \dots, 10\}$. A acts on X naturally. The association scheme with respect to the orbits of $AX \times X$ is isomorphic to $10n_7$.

5 Hesse diagrams

For each n , we show the Hesse diagrams for subschemes. Let $\mathcal{X}$ and $\mathcal{Y}$ be association schemes. If $\mathcal{Y}$ is subscheme of $\mathcal{X}$, then we denote this as follows.

$$\mathcal{X} \text{ ————— } \mathcal{Y}.$$

$|X|=5$

$|X|=6$

$|X|=7$

$|X|=8$

 $|X|=9$

$|X|=10$

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