ON THE POLES OF RIEMANNIAN MANIFOLDS AND ALEXANDROV SPACES OF CURVATURE BOUNDED BELOW

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1. INTRODUCTION

A point p of a complete Riemannian manifold M is said to be a pole if the exponential mapping $\exp_p: T_p M \rightarrow M$ is a diffeomorphism. Maeda [6] defined a constant that measures how a Riemannian manifold expands at infinity, and showed that the constant restricts the size of the set of poles in case of nonnegative curvature. The main purpose of this paper is to give a similar estimate for Riemannian manifolds and Alexandrov spaces of curvature bounded below.

The Alexandrov space is a locally compact and complete length space with Toponogov's comparison property. The length space is a metric space such that the distance $d(p, q)$ of any two points p and q is realized by the length of a minimizing curve that connects them. Such a minimizing curve is called a geodesic as in Riemannian geometry. In this paper, we assume that all geodesics are parametrized by arc length. Let k be a real number. An Alexandrov space M is said to be of curvature bounded below by k if it has the following

Toponogov's Comparison Property. *Let Δpqr be a geodesic triangle in M and $\Delta \bar{p}\bar{q}\bar{r}$ a geodesic triangle in the simply connected complete 2-dimensional Riemannian manifold of constant curvature k with $pq = \bar{p}\bar{q}$, $pr = \bar{p}\bar{r}$ and $\angle rpq = \angle \bar{r}\bar{p}\bar{q}$. Then $qr \leq \bar{q}\bar{r}$.*

All complete Riemannian manifolds of curvature bounded below are Alexandrov spaces. For the precise definition and properties of the Alexandrov space, please refer to [1] or [7].

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A curve of an Alexandrov space defined on the half line $[0, \infty)$ is called a ray if its any subarc is a geodesic. A point of an Alexandrov space is called a pole if every geodesic emanating from the point can be extended to a ray.

Let M denote a Riemannian manifold or an Alexandrov space. For a point $p \in M$, let $S_t(p) = \{q \in M; d(p, q) = t\}$ denote the metric sphere of radius t centered at p . Maeda defined the constant by

$$\mu_0(p) = \limsup_{t \rightarrow \infty} \frac{(\text{diam}_M S_t(p))^2}{t},$$

where $\text{diam}_M S_t(p) = \sup\{d(q_1, q_2); q_1, q_2 \in S_t(p)\}$. Since $\mu_0(p)$ is independent of p , we denote it by $\mu_0 = \mu_0(M)$. Then we have

Theorem 1.1. *Let M be a complete noncompact Riemannian manifold or a noncompact Alexandrov space. If M is of nonnegative curvature, then for any two poles p_1, p_2 of M*

$$d(p_1, p_2) \leq \mu_0/8.$$

This theorem was proved in [8] for the Riemannian manifold. The proof is valid without any change for the Alexandrov space. In this paper we shall give a new and shorter proof of this theorem in section 2. In section 4, we shall show that the equality holds for some Alexandrov spaces.

For a Riemannian manifold or an Alexandrov space of curvature bounded below by some negative constant, we need to modify the definition of the constant as follows

$$\begin{aligned} \mu_-(p) &= \limsup_{t \rightarrow \infty} \text{diam}_M S_t(p) \\ \mu_- &= \mu_-(M) = \sup_{p \in M} \mu_-(p). \end{aligned}$$

Then our main result is the following

Theorem 1.2. *Let M be a complete noncompact Riemannian manifold or a noncompact Alexandrov space. If M is of curvature bounded below by a negative constant k , then for any two poles p_1, p_2 of M*

$$d(p_1, p_2) \leq \frac{\log \cosh^2(\sqrt{-k} \mu_-/2)}{\sqrt{-k}}.$$

The proof shall be given in section 3. In section 4 we shall give some examples of Alexandrov spaces with two poles to show our estimate is asymptotically sharp.

2. PROOF OF THEOREM 1.1

Throughout this section let M denote a complete noncompact Riemannian manifold or a noncompact Alexandrov space of nonnegative curvature.

First we prove a lemma that plays a key role in the proof.

Lemma 2.1 ([8]). *Let $\gamma: [0, \infty) \rightarrow M$ be a ray emanating from a point $p \in M$. For a point $q \in M$ and a positive number t_0 , let θ denote the angle at $\gamma(t_0)$ of a geodesic triangle $\Delta p \gamma(t_0) q$. Then we have*

$$d(p, q) \geq d(p, \gamma(t_0)) - d(\gamma(t_0), q) \cos \theta.$$

Proof. For $t > t_0$, we consider a geodesic triangle $\Delta \gamma(t_0) \gamma(t) q$. It follows from the cosine formula and Toponogov's Comparison Property that

$$d(q, \gamma(t))^2 \leq d(q, \gamma(t_0))^2 + (t - t_0)^2 + 2d(q, \gamma(t_0))(t - t_0) \cos \theta.$$

Since γ is a ray, $t \leq d(p, q) + d(q, \gamma(t))$. Hence we get

$$\begin{aligned} d(p, q) &\geq t - d(q, \gamma(t)) \\ &\geq t - \sqrt{d(q, \gamma(t_0))^2 + (t - t_0)^2 + 2d(q, \gamma(t_0))(t - t_0) \cos \theta}. \end{aligned}$$

Letting $t \rightarrow \infty$, we get the lemma. $\square$

Let $a = d(p_1, p_2)$. Let $\gamma_1(t)$ (resp. $\gamma_2(t)$) ($0 \leq t$) denote the ray that emanates from p_1 (resp. p_2) and passes p_2 (resp. p_1). Let $q_i = \gamma_i(a + t)$ ($i = 1, 2$), and let q denote the middle point of a minimizing geodesic that connects q_1 and q_2 . Let $\theta_i = \angle p_i q_i q$ ($i = 1, 2$), and $\tau = d(q_1, q_2)$. It follows from the cosine formula and Toponogov's Comparison Property for a geodesic triangle $\Delta p_1 q_2 q$ that

$$d(p_1, q)^2 \leq t^2 + \tau^2/4 - t\tau \cos \theta_2.$$

Applying Lemma 2.1 to the ray γ_1 and the point q , we see

$$d(p_1, q) \geq a + t - (\tau/2) \cos \theta_1.$$

Hence we get

$$t^2 + \tau^2/4 - t\tau \cos \theta_2 \geq (a + t - \tau/2 \cos \theta_1)^2,$$

which yields

$$\tau^2(1 - \cos^2 \theta_1)/4 \geq a^2 + 2at - a\tau \cos \theta_1 + t\tau(\cos \theta_2 - \cos \theta_1).$$

Since we may assume that $\cos \theta_2 \geq \cos \theta_1$, we get

$$\frac{\tau^2}{4t}(1 - \cos^2 \theta_1) \geq \frac{a^2}{t} + 2a - \frac{a\tau \cos \theta_1}{t}.$$

Letting $t \rightarrow \infty$, we get $\mu_0/4 \geq 2a$.

3. PROOF OF THEOREM 1.2

Let M denote a complete noncompact Riemannian manifold or an Alexandrov space of curvature bounded below by -1 .

First we prove a hyperbolic version of Lemma 2.1.

Lemma 3.1. *Let $\gamma: [0, \infty) \rightarrow M$ be a ray emanating from a point $p \in M$. For a point $q \in M$ and a positive number t_0 , let θ denote the angle at $\gamma(t_0)$ of a geodesic triangle $\Delta p\gamma(t_0)q$. Then we have*

$$d(p, q) \geq d(p, \gamma(t_0)) - \log(\cosh d(\gamma(t_0), q) + \sinh d(\gamma(t_0), q) \cos \theta).$$

Proof. For $t > t_0$, we consider a geodesic triangle $\Delta \gamma(t_0)\gamma(t)q$. Let $t_1 = t - t_0$, $\tau_0 = d(q, \gamma(t_0))$ and $\tau = d(q, \gamma(t))$. It follows from the cosine formula in hyperbolic geometry and from Toponogov's Comparison Property that

$$\cosh \tau \leq \cosh \tau_0 \cosh t_1 + \sinh \tau_0 \sinh t_1 \cos \theta.$$

Since γ is a ray, we have $\cosh(t_0 + t_1) \leq \cosh(d(p, q) + \tau)$. Hence we get

$$\begin{aligned} \cosh t_0 + \sinh t_0 \frac{\sinh t_1}{\cosh t_1} \\ \leq \left(\cosh d(p, q) + \sinh d(p, q) \frac{\sinh \tau}{\cosh \tau} \right) \left(\cosh \tau_0 + \sinh \tau_0 \frac{\sinh t_1}{\cosh t_1} \cos \theta \right). \end{aligned}$$

When t_1 goes to infinity, so does τ . Hence, letting $t_1 \rightarrow \infty$, we get the lemma. $\square$

Let $a = d(p_1, p_2)$. Let $\gamma_1(t)$ (resp. $\gamma_2(t)$) ($0 \leq t$) denote the ray that emanates from p_1 (resp. p_2) and passes p_2 (resp. p_1). Let $q_i = \gamma_i(a + t)$ ($i = 1, 2$). Let q denote the middle point of a minimizing geodesic that connects q_1 and q_2 . Let $\theta_i = \angle p_i q_i q$ ($i = 1, 2$), and $\tau = d(q_1, q_2)$. It follows from the cosine formula in hyperbolic geometry and from Toponogov's Comparison Property for a geodesic triangle $\Delta p_1 q_2 q$ that

$$\cosh d(p_1, q) \leq \cosh t \cosh \frac{\tau}{2} - \sinh t \sinh \frac{\tau}{2} \cos \theta_2.$$

Applying Lemma 3.1 to the ray γ_1 and the point q , we see

$$\begin{aligned} \cosh(a+t) + \sinh(a+t) &\leq (\cosh d(p_1, q) + \sinh d(p_1, q)) \left(\cosh \frac{\tau}{2} + \sinh \frac{\tau}{2} \cos \theta_1 \right) \\ &\leq 2 \left(\cosh t \cosh \frac{\tau}{2} - \sinh t \sinh \frac{\tau}{2} \cos \theta_2 \right) \left(\cosh \frac{\tau}{2} + \sinh \frac{\tau}{2} \cos \theta_1 \right) \\ &\leq 2 \cosh t \left(\cosh \frac{\tau}{2} + \frac{1}{2} \left(\cos \theta_1 - \frac{\sinh t}{\cosh t} \cos \theta_2 \right) \sinh \frac{\tau}{2} \right)^2. \end{aligned}$$

Since

$$\lim_{t \rightarrow \infty} \left\{ \left(\cos \theta_2 - \cos \theta_1 \frac{\sinh t}{\cosh t} \right) + \left(\cos \theta_1 - \cos \theta_2 \frac{\sinh t}{\cosh t} \right) \right\} = 0,$$

(if necessary exchanging the roles of θ_1 and θ_2) we may assume there is a divergent sequence $t_1 < t_2 < \dots$ with

$$-2 \leq \lim_{i \rightarrow \infty} \left(\cos \theta_1 - \cos \theta_2 \frac{\sinh t_i}{\cosh t_i} \right) \leq 0.$$

Therefore we get

$$e^a = \lim_{i \rightarrow \infty} \frac{\cosh(a+t_i) + \sinh(a+t_i)}{2 \cosh t_i} \leq \limsup_{i \rightarrow \infty} \cosh^2 \frac{\tau(t_i)}{2} \leq \cosh^2 \frac{\mu_-}{2}.$$

4. SOME ALEXANDROV SPACES OF CURVATURE BOUNDED BELOW

In this section we shall give some examples of Riemannian and Alexandrov surfaces with two poles and of curvature bounded below.

Example 4.1 (cf. [8]). Let (x_0, x_1, x_2) denote the canonical coordinates of $\mathbf{R}^3$. For two real numbers $0 < a_0 < a_1$, let $M(a_0, a_1)$ denote the elliptic paraboloid defined by an equation

$$\frac{x_0^2}{a_0} + \frac{x_1^2}{a_1} = 2x_2.$$

Since it is convex, it follows from Gauss' equation that its sectional curvature is

nonnegative. The two umbilic points

$$\left(\pm \sqrt{a_0(a_1 - a_0)}, 0, \frac{a_1 - a_0}{2} \right)$$

are the only poles and $\mu_0(M(a_0, a_1)) < \infty$.

Example 4.2. The limit

$$X(a_1) = \lim_{a_0 \rightarrow 0} M(a_0, a_1)$$

is the double of the convex domain in $\mathbf{R}^2$ bounded by a parabola

$$\frac{x_1^2}{a_1} = 2x_2$$

is an Alexandrov space of nonnegative curvature with

$$\mu_0(X(a_1)) = \lim_{a_0 \rightarrow 0} \mu_0(M(a_0, a_1)) < \infty.$$

The two foci of the parabola are the only poles and their distance is equal to $\mu_0(X(a_1))/8$.

Example 4.3. For three real numbers $a_0 < a_1 < 0 < a_2$, we consider the elliptic hyperboloid of two sheets defined by an equation

$$\frac{x_0^2}{a_0} + \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} = 1.$$

Let $M = M(a_0, a_1, a_2)$ denote the upper sheet of the hyperboloid. We consider it as a submanifold of the Poincaré upper half space $\mathbf{H}^3 = \{(x_0, x_1, x_2) ; x_2 > 0\}$ with the metric $(dx_0^2 + dx_1^2 + dx_2^2)/x_2^2$. Since M is convex in $\mathbf{H}^3$, it follows from Gauss' equation that its sectional curvature is not less than -1 .

Theorem 4.4.

(1) *The two umbilic points*

$$\left(0, \pm \sqrt{a_1(a_0 - a_1)/(a_2 - a_1)}, \sqrt{a_2(a_0 - a_2)/(a_1 - a_2)} \right)$$

of $M(a_0, a_1, a_2)$ are the only poles.

(2) $\mu_-(M(a_0, a_1, a_2)) < \infty$

Proof. (1) We first note that our surface has a special coordinate system $(u_0, u_1) \in (-\infty, a_0] \times [a_0, a_1]$ defined by

$$\begin{aligned} x_0^2 &= \frac{a_0(u_0 - a_0)(u_1 - a_0)}{(a_1 - a_0)(a_2 - a_0)} \\ x_1^2 &= \frac{a_1(u_0 - a_1)(u_1 - a_1)}{(a_0 - a_1)(a_2 - a_1)} \\ x_2^2 &= \frac{a_2(u_0 - a_2)(u_1 - a_2)}{(a_0 - a_2)(a_1 - a_2)}. \end{aligned}$$

In this coordinate system, the first fundamental form has the expression

$$ds^2 = \left(\frac{1}{u_0 - a_2} - \frac{1}{u_1 - a_2} \right) (U_0 du_0^2 + U_1 du_1^2),$$

where

$$U_i = \frac{(-1)^{i+1}(a_0 - a_2)(a_1 - a_2)u_i}{4a_2(a_0 - u_i)(a_1 - u_i)(a_2 - u_i)} \quad (i=0, 1).$$

Changing coordinates by

$$\begin{aligned} du'_0 &= \sqrt{\frac{1}{u_0 - a_2} - \frac{1}{\gamma - a_2}} \sqrt{U_0} du_0 \pm \sqrt{-\frac{1}{u_1 - a_2} + \frac{1}{\gamma - a_2}} \sqrt{U_1} du_1 \\ du'_1 &= \sqrt{U_0} du_0 / \sqrt{\frac{1}{u_0 - a_2} - \frac{1}{\gamma - a_2}} \mp \sqrt{U_1} du_1 / \sqrt{-\frac{1}{u_1 - a_2} + \frac{1}{\gamma - a_2}}. \end{aligned}$$

for a real number $\gamma \in (-\infty, a_0)$ or (a_0, a_1) , we get

$$ds^2 = du_0'^2 + \left(\frac{1}{u_0 - a_2} - \frac{1}{\gamma - a_2} \right) \left(-\frac{1}{u_1 - a_2} + \frac{1}{\gamma - a_2} \right) du_1'^2.$$

Then by a similar argument as in [8], we can investigate the behavior of geodesics on M and conclude that those two points are the only poles.

(2) Let C denote the cone defined by an equation

$$\frac{x_0^2}{a_0} + \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} = 0.$$

For $\alpha > 0$, define $M(\alpha) = \{q \in M; x_2(q) = \alpha\}$ and $C(\alpha) = \{q \in C; x_2(q) = \alpha\}$. Since both $C(\alpha)$ and $M(\alpha)$ are convex in $\{q \in \mathbf{H}^3; x_2(q) = \alpha\}$ that is isometric to

the Euclidean plane and since $M(\alpha)$ is inside $C(\alpha)$, $\text{length } M(\alpha) \leq \text{length } C(\alpha) = \text{const}$. Let p be a point of M and $q_1, q_2 \in S_t(p)$. For large $t > 0$, we may assume $x_2(p) < x_2(q_1) \leq x_2(q_2)$. Let γ denote a minimizing geodesic on M from p to q_2 . Let q denote the point on γ with $x_2(q) = x_2(q_1)$. Since $d(p, q_2) = d(p, q_1)$, we see

$$d(q, q_2) = d(p, q_1) - d(p, q) \leq d(q, q_1).$$

Hence

$$\begin{aligned} d(q_1, q_2) &\leq d(q_1, q) + d(q, q_2) \leq 2d(q_1, q) \leq \text{length } M(x_2(q_1)) \\ &\leq \text{length } C(x_2(q_1)). \end{aligned} \quad \square$$

Example 4.5. We take two copies of the Poincaré upper half plane

$$\mathbf{H}_{\pm}^2 = \{(x_0, x_2) \in \mathbf{R}^2; 0 < x_2\}, (dx_0^2 + dx_2^2)/x_2^2\}$$

of constant sectional curvature -1 . For $a_0 < 0 < a_1$, let $X_{\pm} = X_{\pm}(a_0, a_1)$ denote the domains defined by

$$X_{\pm} = \{(x_0, x_2) \in \mathbf{H}_{\pm}^2; 1 - x_0^2/a_0 \leq x_2^2/a_2\}.$$

Identifying their boundaries ∂X_+ and ∂X_- , we get an Alexandrov space

$$X(a_0, a_2) = X_+ \cup_{\partial X_+ = \partial X_-} X_-$$

of curvature bounded below by -1 , which is isometric to $\lim_{a_1 \rightarrow 0} M(a_0, a_1, a_2)$.

Let $p_{\pm} = (0, \sqrt{-a_0 + a_2}) \in X_{\pm}$.

Proposition 4.6. *The two points $p_{\pm}$ are the only poles of $X(a_0, a_2)$.*

Proof. Let $p = (p_0, p_2)$ be a point on the hyperbola ∂X_+ with $p_0 \neq 0$. Let S be the circle in $\mathbf{R}^2$ that passes three points p_+ , $-p_+$ and p . Let $q = (q_0, 0)$ be a point of intersection of S and x_0 -axis with $p_0 \cdot q_0 < 0$. Since p_+ and $-p_+$ are the foci of the hyperbola

$$\frac{x_0^2}{a_0} + \frac{x_2^2}{a_2} = 1,$$

the segment qp tangents the hyperbola at p . Since the length of two arcs $\text{arc}(qp_+p)$ and $\text{arc}(q(-p_+)(-p))$ are equal, we get $\angle(\text{arc}(p_+p), pq) = \angle(-p)pq$. Hence $\text{arc}(p_+p)$ and a half-line $\{(x_0, x_2) \in X_-; x_2 \geq p_2\}$ form a

geodesic in $X(a_0, a_2)$. Since no such geodesics intersect except at p_+ , the point p_+ is a pole. Similarly we see that p_- is also a pole. It is clear that there are no other poles in $X(a_0, a_2)$. $\square$

Let $p = (p_0, p_2)$ be a point in X_+ . For a positive real number t , let $S_t(p, \mathbf{H}_+^2)$ denote the metric circle with respect to the Poincaré metric centered at p . Then $S_t(p, \mathbf{H}_+^2)$ is a Euclidean circle. Let $(\chi_0(t), \chi_2(t))$ denote its Euclidean center. For the isometries

$$\frac{1}{\chi_2(t)} : (x_0, x_2) \mapsto \frac{(x_0, x_2)}{\chi_2(t)}$$

of $\mathbf{H}_\pm^2$, let

$$V_\pm = \mathbf{H}_\pm^2 \cap \lim_{t \rightarrow \infty} \frac{X_\pm}{\chi_2(t)}.$$

Then

$$V_\pm = \{(x_0, x_2) \in \mathbf{H}_\pm^2; x_2/\sqrt{a_2} \geq |x_0|/\sqrt{-a_0}\}.$$

Lemma 4.7.

$$(1) \quad \lim_{t \rightarrow \infty} \frac{S_t(p, X(a_0, a_2)) \cap X_+}{\chi_2(t)} = \{(x_0, x_2) \in V_+; x_0^2 + (x_2 - 1)^2 = 1\}.$$

$$(2) \quad \lim_{t \rightarrow \infty} \frac{S_t(p, X(a_0, a_2)) \cap X_-}{\chi_2(t)} \subset \left\{ (x_0, x_2) \in V_-; x_0^2 + (x_2 - 1)^2 \leq 1 \text{ and } x_2 \geq \frac{2a_2}{a_2 - a_0} \right\}.$$

If $p = p_+$, then

$$(3) \quad \lim_{t \rightarrow \infty} \frac{S_t(p, X(a_0, a_2)) \cap X_-}{\chi_2(t)} = \left\{ (x_0, x_2) \in V_-; x_2 = \frac{2a_2}{a_2 - a_0} \right\}.$$

Proof. (1) Since $S_t(p, \mathbf{H}^2)$ is a Euclidean circle in the upper half space whose inside contains p and since $\lim_{t \rightarrow \infty} p/\chi_2(t) = (0, 0)$, $\lim_{t \rightarrow \infty} S_t(p, \mathbf{H}^2)/\chi_2(t)$ is the Euclidean circle of radius 1 and centered at $(0, 1)$.

(2) For each point $x = (x_0, x_2) \in X_+$, let x' denote the point in X_- with the same coordinates (x_0, x_2) . Then

$$d(p', x') \leq \inf_{q \in \partial X_+} \{d(p', q) + d(q', x')\}$$

$$\begin{aligned}
&= \inf_{q \in \partial X_+} \{d(p, q) + d(q', x')\} \\
&= d(p, x').
\end{aligned}$$

Hence it follows that

$$S_t(p, X(a_0, a_2)) \cap X_- \subset \{x' \in X_-; d(p', x') \leq t\}.$$

On the other hand we have

$$S_t(p, X(a_0, a_2)) \cap X_- \subset X_- \setminus \bigcup_{q \in \partial X_+} \text{int } B_{t-d(p,q)}(q', \mathbf{H}_-^2),$$

where $B_{t-d(p,q)}(q', \mathbf{H}_-^2)$ denotes the metric ball in $\mathbf{H}_-^2$ of radius $t-d(p, q)$ and centered at $q' \in \partial X_-$. If $\lim_{t \rightarrow \infty} q/\chi_2(t) = q_\infty \in \partial X_+ \setminus \{(0, 0)\}$, then $\lim_{t \rightarrow \infty} S_{t-d(p,q)}(q', \mathbf{H}_-^2)/\chi_2(t)$ is a Euclidean circle that tangents to the circle

$$\lim_{t \rightarrow \infty} S_t(p_-, \mathbf{H}_-^2)/\chi_2(t) = \{(x_0, x_2); x_0^2 + (x_2 - 1)^2 = 1\}$$

from inside and tangents to the horizontal line $x_2 = 2a_2/(-a_0 + a_2)$ from below. Hence, for any $x' \in V_-$ under the line $x_2 = 2a_2/(-a_0 + a_2)$, there are $q \in \partial X_+$ and t_0 such that

$$x' \in \text{int } B_{t-d(p,q)}(q', \mathbf{H}_-^2)$$

for any $t > t_0$, which proves (2).

(3) Since $S_t(p_+, X(a_0, a_2))$ is perpendicular to the rays emanating from p_+ , which are vertical lines in X_- , the assertion is obvious. $\square$

Let S_1 (resp. S_2) be the circle in $\mathbf{R}^2$ of radius $2a_2/(-a_0 + a_2)$ (resp. 2) and centered at the origin. Let

$$L^\pm = \{(x_0, x_2); x_2 > 0, x_2/\sqrt{a_2} = \pm x_0/\sqrt{-a_0}\}.$$

Let $Q_i^\pm$ denote the points of intersection of S_i and $L^\pm$ ($i=1, 2$). Then we easily see that the diameter of the union of

$$\{(x_0, x_2) \in V_+; x_0^2 + (x_2 - 1)^2 = 1\}$$

and

$$\left\{ (x_0, x_2) \in V_-; x_0^2 + (x_2 - 1)^2 \leq 1 \text{ and } x_2 \geq \frac{2a_2}{a_2 - a_0} \right\}$$

in $V_+ \cup_{\partial V_+ = \partial V_-} V_-$ is less than one half the length of

$$Q_1^+ Q_2^+ \cup \text{arc}(Q_2^+ Q_2^-) \cup Q_2^- Q_1^- \cup \text{arc}(Q_1^- Q_1^+).$$

Hence we get

Lemma 4.8.

$$\lim_{t \rightarrow \infty} \text{diam } S_t(p, X(a_0, a_2)) \leq \log \frac{\sqrt{-a_0 + a_2} - \sqrt{-a_0}}{\sqrt{-a_0 + a_2} + \sqrt{-a_0}} + \sqrt{\frac{-a_0 + a_2}{a_2}} \log \frac{-a_0 + a_2}{a_2}.$$

From Lemma 4.8 we get

Theorem 4.9.

$$\lim_{a_0 \rightarrow 0} \frac{\log \cosh^2(\mu_-(X(a_0, a_2))/2)}{d(p_-, p_+)} = 1.$$

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