A QUANTUM ANALOGUE OF THE HYPERGEOMETRIC SYSTEM $E_{3,6}$

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Introduction

It has been studied the relation between the representation of Lie algebra and the several hypergeometric functions (see e.g. [G], [S]). For example, the contiguity relations for the Gauss hypergeometric functions give a representation of Lie algebra $gl(4)$. From this point of view, E. Horikawa [Ho] showed that the contiguity relations for the q -hypergeometric function ${}_2\varphi_1$ that is a q -analogue of the Gauss hypergeometric function, give a representation of $U_q(gl(4))$. Here $U_q(gl(4))$ is the quantum analogue of the universal enveloping algebra of $gl(4)$.

From a different viewpoint, M. Noumi [N] defined a q -hypergeometric series φ_D that is a q -analogue of Lauricella's hypergeometric function F_D of $n-3$ variables associated with the Grassmannian $G_{2,n}$. His approach was carried out by considering a quantum analogue of the space $G_{2,n}$ and the representation of $U_q(gl(n))$. He use the *Casimir* element or the central element of the subalgebra of $U_q(gl(n))$, isomorphic to $U_q(gl(2))$, instead of Laplacian.

In the classical case, the hypergeometric function $F_{3,6}$, associated with the $G_{3,6}$, is now investigated from several points of view and various new properties have been known (see [MSY]). So, in this paper, we will discuss a quantum analogue of $G_{3,6}$ and the associated hypergeometric functions, relying Noumi's work.

In the first section, we recall the definition of the equations of type (k, n) and the hypergeometric function $F_{3,6}$. Furthermore we give a short preview of this paper.

In the second section, we define a noncommutative algebra generated by quantum analogues of the Plücker coordinates of the $G_{3,6}$ and compute the representation of $U_q(gl(6))$ on it. Furthermore we define a localization of this

noncommutative algebra and two types of decomposition of it. Here remark that, among several possibilities, we have treated the two types of decomposition, because the explicit form of the action of $U_q(gl(6))$ is seen to be simpler.

In the third section, we translate the action of $U_q(gl(6))$ on the above localization to that on the commutative algebra. By defining two isomorphisms of vector spaces, associated with the decompositions, between the localization and commutative algebras, we exhibit the action of $U_q(gl(6))$ in terms of usual q -difference operators.

In the fourth section, we define the Casimir operators and, in the fifth section, consider the q -hypergeometric functions of type (3,6), denoted by φ_λ^I and φ_λ^{II} . One of the functions is defined by the series

$$\begin{aligned} \varphi_\lambda^I &= \varphi_\lambda^I(x_1, x_2, x_3, x_4) \\ &= \sum_{a,b,c,d=0}^{\infty} \frac{(q^{-2\lambda_5}; q^2)_{a+c} (q^{-2\lambda_6}; q^2)_{b+d} (q^{2\lambda_2+2}; q^2)_{a+b} (q^{2\lambda_3+2}; q^2)_{c+d}}{(q^{2\lambda_2+2\lambda_3+2\lambda_4+6}; q^2)_{a+b+c+d} (q^2; q^2)_a (q^2; q^2)_b (q^2; q^2)_c (q^2; q^2)_d} \\ &\quad \times q^{2bc} x_1^a x_2^b x_3^c x_4^d. \end{aligned} \quad (5.3)$$

In the last section, we compute the contiguity relations and, in view of these relations, propose to define the q -hypergeometric system of type (3,6;I) and (3,6;II). Furthermore we define another system of q -difference equations, which has q -hypergeometric functions as solutions. Refer to Proposition 6.6.

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1. Preliminaries

In this section, we give a short review of the hypergeometric system of differential equations associated with the Grassmannian $G_{k,n}$ ($k \leq n$) and a preview of this paper.

Let us denote by $M(k, n)$ the space of all $k \times n$ matrices of the form

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ & & \cdots & \cdots \\ t_{k1} & t_{k2} & \cdots & t_{kn} \end{pmatrix}$$

The *hypergeometric system of differential equations of type (k, n)* , denoted by $E_{k,n}$, is the following system of differential equations defined on the space $M(k, n)$:

$$\Phi(gT) = \det(g)^{-1} \Phi(T) \quad (g \in GL(k)) \quad (1.1)$$

$$\Phi(T \operatorname{diag}(c_1, \dots, c_n)) = \Phi(T) c_1^{\lambda_1} \dots c_n^{\lambda_n} \quad (1.2)$$

$$\frac{\partial^2}{\partial t_{ri} \partial t_{sj}} \Phi(T) = \frac{\partial^2}{\partial t_{si} \partial t_{rj}} \Phi(T) \quad (1 \leq r < s \leq k, 1 \leq i < j \leq n), \quad (1.3)$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a set of complex numbers. For the conditions (1.1) and (1.2) to be compatible, it is necessary to assume $\lambda_1 + \dots + \lambda_n = -k$. Roughly speaking, multivalued holomorphic solutions $\Phi(T)$ of the system (1.1)-(1.3), defined on an appropriate Zariski open subset of the space $M(k, n)$, are called the hypergeometric functions of type (k, n) .

Now let T be a general $k \times n$ matrix. We can decompose T as follows:

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1k} \\ t_{21} & t_{22} & \dots & t_{2k} \\ \dots & \dots & \dots & \dots \\ t_{k1} & t_{k2} & \dots & t_{kk} \end{pmatrix} \begin{pmatrix} 1 & & 1 & 1 & \dots & 1 \\ & 1 & x_{2k+1} & x_{2k+2} & \dots & x_{2n} \\ & \ddots & & & \dots & \\ & & 1 & x_{kk+1} & x_{kk+2} & \dots & x_{kn} \end{pmatrix} \quad (1.4)$$

$$\times \operatorname{diag}(1, 1, \dots, 1, -\xi_{1\dots k}^{-1} \xi_{1k+1}, \dots, -\xi_{1\dots k}^{-1} \xi_{1n}),$$

where

$$x_{rj} = (-1)^{r+1} \xi_{1j}^{-1} \xi_{\hat{r}j} \quad \text{for } r=2, \dots, k \quad j=k+1, \dots, n,$$

$$\xi_{i_1 i_2 \dots i_k} = \sum_{\omega \in S_k} (-1)^{l(\omega)} t_{\omega(1)i_1} t_{\omega(2)i_2} \dots t_{\omega(k)i_k} \quad \text{for } 1 \leq i_1, \dots, i_k \leq n,$$

$$\xi_{\hat{r}j} = \xi_{1\dots \hat{r} \dots kj}.$$

Here S_k is the permutation group of k letters and, for each $\omega \in S_k$, $l(\omega)$ denotes the number of inversions in ω . Note that $\{\xi_{i_1 i_2 \dots i_k}\}$ are the Plücker coordinates of the $G_{k,n}$, and satisfy the Plücker relations as follows:

$$\sum_{i=1}^{k+1} (-1)^{i-1} \xi_{\alpha_1 \dots \alpha_{k-1} \beta_i} \xi_{\hat{\beta}_i} = 0 \quad (1.5)$$

for $1 \leq \alpha_1, \dots, \alpha_{k-1} \leq n$, $1 \leq \beta_1 < \beta_2 < \dots < \beta_{k+1} \leq n$ and where $\xi_{\hat{\beta}_i} = \xi_{\beta_1 \dots \beta_{i-1} \beta_{i+1} \dots \beta_{k+1}}$. The relative invariances (1.1), (1.2) imply that $\Phi(T)$ can be written in the form

$$\Phi(T) = G(x_{2k+1}, x_{2k+2}, \dots, x_{kn}) \xi_{1\dots k}^{\lambda_1 + \lambda_2 + \dots + \lambda_k + k - 1} \xi_{1k+1}^{\lambda_{k+1}} \dots \xi_{1n}^{\lambda_n},$$

for some function $G(x_{2k+1}, \dots, x_{kn})$.

Furthermore, because of invariances (1.1), (1.2) again, $G(x_{2k+1}, \dots, x_{kn})$ has the homogeneity:

$$G(x_{2k+1}, \dots, x_{r-1n}, cx_{rk+1}, \dots, cx_{rn}, x_{r+1k+1}, \dots, x_{kn}) = c^{-\lambda_{r-1}} G(x_{2k+1}, \dots, x_{kn}), \quad (1.6)$$

for $2 \leq r \leq k$. Then $G(x_{2k+1}, \dots, x_{kn})$ takes the form

$$G(x_{2k+1}, \dots, x_{kn}) = x_{2k+1}^{-\lambda_2-1} \dots x_{kk+1}^{-\lambda_k-1} F(z_{2k+2}, z_{2k+3}, \dots, z_{kn}), \quad (1.7)$$

where

$$z_{rj} = x_{rk+1}^{-1} x_{rj}$$

for $2 \leq r \leq k$, $k+2 \leq j \leq n$. In the coordinates $z = (z_{2k+2}, z_{2k+3}, \dots, z_{kn})$, it is known that a solution of the equation (1.3), holomorphic around the origin $z=0$, is given as follows:

$$F_{k,n}(z, \lambda) = \sum_{a_{rj} \geq 0} \frac{(-\lambda_{k+2})_{a_{2 \dots k, k+2}} \dots (-\lambda_n)_{a_{2 \dots k, n}} (\lambda_2+1)_{a_{2, k+2 \dots n}} \dots (\lambda_k+1)_{a_{k, k+2 \dots n}}}{(\lambda_2 + \lambda_3 + \dots + \lambda_{k+1} + k)_{a_{2 \dots k, k+2 \dots n}} (1)_{a_{2k+2}} (1)_{a_{2k+2}} (1)_{a_{2k+3}} \dots (1)_{a_{kn}}} \\ \times z_{2k+2}^{a_{2k+2}} z_{2k+3}^{a_{2k+3}} \dots z_{kn}^{a_{kn}} \quad (1.8)$$

where $a_{2 \dots k, j} = a_{2j} + \dots + a_{kj}$, $a_{r, k+2 \dots n} = a_{rk+2} + \dots + a_{rn}$, $a_{2 \dots k, k+2 \dots n} = \sum a_{rj}$ over $2 \leq r \leq k$, $k+2 \leq n$, and $(\alpha)_n = \alpha(\alpha+1) \dots (\alpha+n-1)$.

The hypergeometric function F_D associated with $G_{2,n}$ has been generalized to the hypergeometric function $F_{k,n}$ associated with the $G_{k,n}$, and it is known that the contiguity relation for $F_{k,n}$ is described in terms of the Lie algebra $gl(n)$. Refer to [G, S]. Therefore the contiguity relation for the function $F_{3,6}$, defined by the series

$$F_{3,6}(z, \lambda) = \sum_{a,b,c,d \geq 0} \frac{(-\lambda_5)_{a+c} (-\lambda_6)_{b+d} (\lambda_2+1)_{a+b} (\lambda_3+1)_{c+d}}{(\lambda_2 + \lambda_3 + \lambda_4 + 3)_{a+b+c+d} (1)_a (1)_b (1)_c (1)_d} z_{25}^a z_{26}^b z_{35}^c z_{36}^d, \quad (1.9)$$

is described in terms of the Lie algebra $gl(6)$.

Now let us set

$$E_{ij} = \sum_{r=1}^k t_{ri} \frac{\partial}{\partial t_{rj}}.$$

Then $\{E_{ij}\}_{1 \leq i, j \leq n}$ generate a Lie algebra isomorphic to $gl(n)$ and define, as differential operators, a $U(gl(n))$ -module structure on the polynomial algebra $\mathbb{C}[t_{11}, t_{12}, \dots, t_{kn}]$ of variables $(t_{rj})_{1 \leq r \leq k, 1 \leq j \leq n}$. It holds a remarkable identity

$$(E_{ii}+1)E_{jj}-E_{ji}E_{ij}=\sum_{1 \leq r < s \leq k}(t_{ri}t_{sj}-t_{si}t_{rj})\left(\frac{\partial^2}{\partial t_{ri}\partial t_{sj}}-\frac{\partial^2}{\partial t_{si}\partial t_{rj}}\right), \quad (1.10)$$

for each $i \neq j$. Then the function

$$G(x_{2k+1}, \dots, x_{kn}) = x_{2k+1}^{-\lambda_2-1} \dots x_{kk+1}^{-\lambda_k-1} F_{k,n}(z, \lambda)$$

is a solution of the system of equation (1.6) and

$$\{(E_{ii}+1)E_{jj}-E_{ji}E_{ij}\}(G(x_{2k+1}, \dots, x_{kn}))=0. \quad (1.11)$$

In order to define a quantum analogue of the hypergeometric system $E_{3,6}$, we define in the next section a noncommutative algebra as a quantum analogue of the space $G_{3,6}$, which has a structure of left $U_q(gl(6))$ -module. Furthermore we give two kinds of left $U_q(gl(6))$ -module structure on the commutative algebra $\mathbb{C}[x_{24}, x_{25}, x_{26}, x_{34}, x_{35}, x_{36}]$ of polynomials in 6 variables.

In view of the identity (1.11), we look for the analogous elements in $U_q(gl(n))$; they are the central elements of the subalgebra of $U_q(gl(6))$, isomorphic to $U_q(gl(2))$, defined by

$$(q-q^{-1})^2 C_{ij} = (q^{1+\epsilon_i}-q^{-1-\epsilon_i})(q^{\epsilon_j}-q^{-\epsilon_j}) - (q-q^{-1})^2 \widehat{E}_{ji} \widehat{E}_{ij} \quad (1.12)$$

for $1 \leq i \neq j \leq 6$. In this paper, we call such elements the *Casimir* elements. When we need to regard them as the operators, we also call them the *Casimir* operators. In the last section, we define the *q-hypergeometric system of type* $(3,6; *)$:

$$T_{q,x_{24}} T_{q,x_{25}} T_{q,x_{26}} u = q^{-\lambda_2-1} u, \quad T_{q,x_{34}} T_{q,x_{35}} T_{q,x_{36}} u = q^{-\lambda_3-1} u, \quad (6.5)$$

$$\rho_\lambda^*(C_{ij})u = 0 \quad (1 \leq i \neq j \leq 6), \quad (6.6*)$$

where $*$ = I or II and $\{T_{q,x_{rj}}\}$ are the *q-shift* operators:

$$(T_{q,x_{rj}} f)(x) = f(x_{24}, \dots, qx_{rj}, \dots, x_{36}).$$

(6.5) and (6.6) correspond to (1.6) and (1.11) respectively.

2. The algebra $U_q(gl(6))$, Quantum Grassmannians, and localization

We first in this section recall the definition of $U_q(gl(6))$ briefly. In what follows, we fix a complex number q with $0 < |q| < 1$. The quantum analogue of $U(gl(6))$, denoted by $U_q(gl(6))$, is the algebra generated by e_k, f_k ($1 \leq k \leq 5$) and $q^{\pm \epsilon_j}$ ($1 \leq j \leq 6$). We use the notation $q^h = q^{a_1 \epsilon_1} \cdots q^{a_6 \epsilon_6}$ for any linear combination $h = a_1 \epsilon_1 + \cdots + a_6 \epsilon_6$ with integral coefficients. Then the relations for the generators are given by

$$\begin{aligned}
 q^0 &= 1, \quad q^h q^{h'} = q^{h+h'}, \\
 q^h e_j q^{-h} &= q^{\langle h, \epsilon_j - \epsilon_{j+1} \rangle} e_j, \quad q^h f_j q^{-h} = q^{\langle h, -\epsilon_j + \epsilon_{j+1} \rangle} f_j \quad (1 \leq j \leq 5), \\
 e_i f_j - f_j e_i &= \delta_{i,j} \frac{q^{\epsilon_i - \epsilon_{i+1}} - q^{-\epsilon_i + \epsilon_{i+1}}}{q - q^{-1}} \quad (1 \leq i, j \leq 5), \\
 e_i e_j &= e_j e_i, \quad f_i f_j = f_j f_i \quad (|i-j| \geq 2), \\
 e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 &= 0 \quad (|i-j| = 1), \\
 f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 &= 0 \quad (|i-j| = 1),
 \end{aligned} \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the canonical symmetric bilinear form such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j}$. Refer to [N, NUW]. This algebra has the structure of Hopf algebra with the following convention of the *coproduct* Δ , the *counit* ε , and the *antipode* S :

$$\begin{aligned}
 \Delta(q^h) &= q^h \otimes q^h, \\
 \Delta(e_j) &= e_j \otimes 1 + q^{\epsilon_j - \epsilon_{j+1}} \otimes e_j, \\
 \Delta(f_j) &= f_j \otimes q^{-\epsilon_j + \epsilon_{j+1}} + 1 \otimes f_j, \\
 \varepsilon(q^h) &= 1, \quad \varepsilon(e_j) = \varepsilon(f_j) = 0, \\
 S(q^h) &= q^{-h}, \quad S(e_j) = -q^{-\epsilon_j + \epsilon_{j+1}} e_j, \quad S(f_j) = -f_j q^{\epsilon_j - \epsilon_{j+1}},
 \end{aligned} \tag{2.2}$$

for $1 \leq j \leq 5$.

Now we define the noncommutative algebra, which is the quantum analogue of the coordinate ring $A(M(3, 6))$.

Let $A_q(M(3, 6))$ denote the algebra defined as follows:

$$\begin{aligned}
 \text{generators:} \quad & t_{rj} \quad (1 \leq r \leq 3, 1 \leq j \leq 6) \\
 \text{relations:} \quad & ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \\
 & bc = cb, \quad ad - da = (q - q^{-1})bc,
 \end{aligned} \tag{2.3}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t_{ri} & t_{rj} \\ t_{si} & t_{sj} \end{pmatrix} \begin{pmatrix} 1 \leq r < s \leq 3 \\ 1 \leq i < j \leq 6 \end{pmatrix}$$

The algebra $A_q(M(3, 6))$ has a structure of a left $U_q(gl(6))$ -module; it is determined by assuming the $U_q(gl(6))$ -symmetry in the sense that

- (a) $a \cdot 1 = \varepsilon(a) 1$ for all $a \in U_q(gl(6))$;
 (b) if $a \in U_q(gl(6))$ and $\Delta(a) = \sum_i a'_i \otimes a''_i$, then

$$a \cdot (\varphi\psi) = \sum_i a'_i \cdot \varphi a''_i \cdot \psi \text{ for all } \varphi, \psi \in A_q(M(3, 6))$$

and by the action of the generators of $U_q(gl(6))$ on t_{rj} of $A_q(M(3, 6))$ given by the formula

$$q^h \cdot t_{rj} = q^{\langle h, \varepsilon_j \rangle} t_{rj}, \quad e_k \cdot t_{rj} = \delta_{k+1,j} t_{rk}, \quad f_k \cdot t_{rj} = \delta_{k,j} t_{rk+1}. \quad (2.4)$$

Since we can regard the ordinary Grassmann manifold as the space whose coordinate ring is the algebra generated by the Plücker coordinates, we define the quantum Grassmannian by giving the quantum analogue of the Plücker coordinates.

We denote the subalgebra in $A_q(M(3, 6))$ generated by the *quantum minor determinants*

$$\xi_{ijk} = \sum_{\omega \in S_3} (-q)^{l(\omega)} t_{\omega(1)i} t_{\omega(2)j} t_{\omega(3)k} \quad (1 \leq i, j, k \leq 6)$$

by $\mathcal{A}$. In the classical case, the Plücker coordinates satisfy the Plücker relations (1.5). Similarly, these minors satisfy the Plücker relations as follows:

$$\sum_{i=1}^4 (-q)^{i-1} \xi_{\alpha_1 \alpha_2 \beta_i} \xi_{\hat{\beta}_i} = 0 \quad (2.5)$$

for $1 \leq \alpha_1, \alpha_2 \leq 6$ and $1 \leq \beta_1 < \beta_2 < \beta_3 < \beta_4 \leq 6$, where $\xi_{\hat{\beta}_i} = \xi_{\beta_1 \dots \beta_{i-1} \beta_{i+1} \dots \beta_4}$.

The subalgebra $\mathcal{A}$ is then a left $U_q(gl(6))$ -submodule of $A_q(M(3, 6))$; the action of $U_q(gl(6))$ on the quantum minors is determined by

$$\begin{aligned} q^h \cdot \xi_{ijk} &= q^{\langle h, \varepsilon_i + \varepsilon_j + \varepsilon_k \rangle} \xi_{ijk}, \\ e_l \cdot \xi_{ijk} &= \delta_{l+1,i} \xi_{ljk} + \delta_{l+1,j} q^{\langle \varepsilon_l - \varepsilon_{l+1}, \varepsilon_i \rangle} \xi_{ilk} + \delta_{l+1,k} q^{\langle \varepsilon_l - \varepsilon_{l+1}, \varepsilon_i + \varepsilon_j \rangle} \xi_{ijl}, \\ f_l \cdot \xi_{ijk} &= \delta_{l,i} q^{-\langle \varepsilon_l - \varepsilon_{l+1}, \varepsilon_j + \varepsilon_k \rangle} \xi_{l+1jk} + \delta_{l,j} q^{-\langle \varepsilon_l - \varepsilon_{l+1}, \varepsilon_i \rangle} \xi_{il+1k} + \delta_{l,k} \xi_{ijl+1}. \end{aligned} \quad (2.6)$$

In order to give the decomposition of the matrix $T = (t_{rj})_{1 \leq r \leq 3, 1 \leq j \leq 6}$ of generators for $A_q(M(3, 6))$, in a similar way to (1.4), we introduce some localization of $\mathcal{A}$. In the decomposition (1.4), the ξ_{123}^{-1} in the diagonal matrix has played a special role. Also in the present case, a first step is to invert the principal minor ξ_{123} . By the Plücker relations (2.5), the localization $\mathcal{A}[\xi_{123}^{-1}]$ has the monomial basis

$$\xi_{i6}^{a_{i6}} \xi_{i5}^{a_{i5}} \xi_{i4}^{a_{i4}} \xi_{i26}^{a_{i26}} \xi_{i25}^{a_{i25}} \xi_{i24}^{a_{i24}} \xi_{i36}^{a_{i36}} \xi_{i35}^{a_{i35}} \xi_{i34}^{a_{i34}} \xi_{i23}^{\mu} \quad (2.7)$$

where $a_{ij} \in \mathbf{N}$, $\mu \in \mathbf{Z}$ and $\xi_{ij} = \xi_{1 \dots i \dots 3j}$. Here note that we have fixed an ordering of the variables ξ_{ij} , called a normal ordering. By the Plücker relations (2.5) again, the commutation relations among ξ_{123} and ξ_{ij} are generated by the following relations:

$$\xi_{123} \xi_{ij} = q \xi_{ij} \xi_{123} \quad \text{for } 1 \leq i \leq 3, 4 \leq j \leq 6, \quad (2.8)$$

and

$$\begin{aligned} \xi_{i1j} \xi_{i2j} &= q^{-1} \xi_{i2j} \xi_{i1j}, \quad \xi_{i1j_1} \xi_{i2j_2} = q \xi_{i2j_2} \xi_{i1j_1}, \\ \xi_{i1j_1} \xi_{i2j_2} &= \xi_{i2j_2} \xi_{i1j_1}, \quad \xi_{i2j_1} \xi_{i1j_2} - \xi_{i1j_2} \xi_{i2j_1} = (q - q^{-1}) \xi_{i1j_1} \xi_{i2j_2}, \\ &\text{for } 1 \leq i_1 < i_2 \leq 3, 4 \leq j_1 < j_2 \leq 6. \end{aligned} \quad (2.9)$$

In order to get variables analogous to x_{2j} and x_{3j} in (1.4), we next adjoin the inverses of the quantum minors ξ_{i4} , ξ_{i5} , ξ_{i6} . The localized algebra is $\mathcal{A}[\xi_{123}^{-1}, \xi_{i4}^{-1}, \xi_{i5}^{-1}, \xi_{i6}^{-1}]$ and, by the commutation relations, it has the monomial base of type (2.9) with a_{2j} , $a_{3j} \in \mathbf{N}$ and $a_{1j}, \mu \in \mathbf{Z}$. Refer to [N, § 3].

With this localization and the commutation relations the matrix $T = (t_{rj})_{1 \leq r \leq 3, 1 \leq j \leq 6}$ of generators for $A_q(M(3, 6))$ can be decomposed as follows:

$$\begin{aligned} T &= \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & u_{24} & u_{25} & u_{26} \\ 0 & 0 & 1 & u_{34} & u_{35} & u_{36} \end{pmatrix} \\ &\times \text{diag}(1, 1, 1, -q^{-2} \xi_{123}^{-1} \xi_{i4}, -q^{-2} \xi_{123}^{-1} \xi_{i5}, -q^{-2} \xi_{123}^{-1} \xi_{i6}), \end{aligned} \quad (2.10)$$

where

$$u_{2j} = -\xi_{i1j}^{-1} \xi_{i2j} \quad \text{and} \quad u_{3j} = q \xi_{i1j}^{-1} \xi_{i3j}. \quad (2.11)$$

Since $A_q(M(3, 6))$ is the algebra with $U_q(gl(6))$ -symmetry, we can extend the action of $U_q(gl(6))$ on $\mathcal{A}$ to the action on $\mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]$. For an element a of $U_q(gl(6))$, we denote its action on $\mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]$ by

$$\bar{\rho}(a) : \mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}] \rightarrow \mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]. \quad (2.12)$$

The algebra $\mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]$, as a vector space, has several kinds of decomposition. But the commutation relations among u_{rj} are not so simple. So, in order to check the action of $U_q(gl(6))$ on $\mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]$ explicitly, we here treat the following two decompositions.

$$(I) \quad \mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}] = \bigoplus_{\mu, \lambda_j \in \mathbb{Z}} R_2 \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} R_3 \quad (2.13)$$

R_r is the *noncommutative* algebra generated by u_{rj} ($j=4, 5, 6$)

$$(II) \quad \mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}] = \bigoplus_{\mu, \lambda_j \in \mathbb{Z}} \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} R \quad (2.14)$$

R is the *noncommutative* algebra generated by u_{rj} ($r=2, 3$ $j=4, 5, 6$)

Remark 2.1. For the Grassmannian $G_{2,n}$, Noumi [N] treated the second decomposition above. However, since choice of bases of a noncommutative algebra depends on normal ordering of generators, we should be concerned with how to fix a normal ordering. Considering the action on $\mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]$ again, we choose in the next section two types of monomial bases. In the forthcoming paper we shall propose another choice that works for general (k, n) .

3. Translation of the action of $U_q(gl(6))$

The action of $U_q(gl(6))$ on the *noncommutative* algebra $\mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]$ is translated to that on the *commutative* algebra $\mathbb{C}[x_{24}, x_{25}, x_{26}, x_{34}, x_{35}, x_{36}]$ of polynomials in terms of usual q -difference operators in this section.

To do this we define a vector space

$$\mathcal{G} = \bigoplus_{\mu, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{Z}} \mathcal{G}_{\mu, \lambda_4, \lambda_5, \lambda_6}, \quad (3.1)$$

where $\mathcal{G}_{\mu, \lambda_4, \lambda_5, \lambda_6} = \mathbb{C}[x_{24}, \dots, x_{36}]$ for all $\mu, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{Z}$ and the two isomorphisms of vector spaces

$$\omega^I : \mathcal{G} \rightarrow \mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}], \quad (3.2)$$

$$\omega^{\text{II}}: \mathcal{G} \rightarrow \mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}], \quad (3.3)$$

as follows. Let us fix the monomial bases of $\mathcal{A}[\xi_{123}^{-1}, \xi_{14}^{-1}, \xi_{15}^{-1}, \xi_{16}^{-1}]$ in two ways, referring to the decompositions (2.13) and (2.14):

$$\begin{aligned} \text{I} \quad & u_{24}^{\nu_{24}} u_{25}^{\nu_{25}} u_{26}^{\nu_{26}} \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} u_{36}^{\nu_{36}} u_{35}^{\nu_{35}} u_{34}^{\nu_{34}} \quad (\nu_{rj} \in \mathbf{N}), \\ \text{II} \quad & \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} u_{26}^{\nu_{26}} u_{36}^{\nu_{36}} u_{25}^{\nu_{25}} u_{35}^{\nu_{35}} u_{24}^{\nu_{24}} u_{34}^{\nu_{34}} \quad (\nu_{rj} \in \mathbf{N}), \end{aligned} \quad (3.4)$$

where $u_{2j} = -\xi_{1j}^{-1} \xi_{2j}$ and $u_{3j} = q \xi_{1j}^{-1} \xi_{3j}$. Then the isomorphism ω^{I} is determined as the direct sum of linear mappings

$$\omega_{\mu, \lambda_4, \lambda_5, \lambda_6}^{\text{I}}: \mathbf{C}[x_{24}, \dots, x_{36}] \rightarrow R_2 \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} R_3$$

such that

$$\begin{aligned} \omega_{\mu, \lambda_4, \lambda_5, \lambda_6}^{\text{I}}(x_{24}^{\nu_{24}} x_{25}^{\nu_{25}} x_{26}^{\nu_{26}} x_{34}^{\nu_{34}} x_{35}^{\nu_{35}} x_{36}^{\nu_{36}}) \\ = u_{24}^{\nu_{24}} u_{25}^{\nu_{25}} u_{26}^{\nu_{26}} \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} u_{36}^{\nu_{36}} u_{35}^{\nu_{35}} u_{34}^{\nu_{34}}, \end{aligned}$$

and the isomorphism ω^{II} as the direct sum of linear mappings

$$\omega_{\mu, \lambda_4, \lambda_5, \lambda_6}^{\text{II}}: \mathbf{C}[x_{24}, \dots, x_{36}] \rightarrow \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} R$$

such that

$$\begin{aligned} \omega_{\mu, \lambda_4, \lambda_5, \lambda_6}^{\text{II}}(x_{24}^{\nu_{24}} x_{25}^{\nu_{25}} x_{26}^{\nu_{26}} x_{34}^{\nu_{34}} x_{35}^{\nu_{35}} x_{36}^{\nu_{36}}) \\ = \xi_{16}^{\lambda_6} \xi_{15}^{\lambda_5} \xi_{14}^{\lambda_4} \xi_{123}^{\mu} u_{26}^{\nu_{26}} u_{36}^{\nu_{36}} u_{25}^{\nu_{25}} u_{35}^{\nu_{35}} u_{24}^{\nu_{24}} u_{34}^{\nu_{34}}. \end{aligned}$$

Through the isomorphisms ω^{I} and ω^{II} thus defined, we obtain two kinds of left $U_q(\mathfrak{gl}(6))$ -module structure on the vector space $\mathcal{G}$. More precisely, for an element a of $U_q(\mathfrak{gl}(6))$ of weight κ , i.e., $q^h a q^{-h} = q^{\langle h, \kappa \rangle} a$, its action on $\mathcal{G}$ induces a family of operators

$$\tilde{\rho}_{\mu, \lambda_4, \lambda_5, \lambda_6}^*(a) = \omega_{\mu', \lambda'_4, \lambda'_5, \lambda'_6}^*{}^{-1} \circ \tilde{\rho}(a) \circ \omega_{\mu, \lambda_4, \lambda_5, \lambda_6}^*: \mathcal{G}_{\mu, \lambda_4, \lambda_5, \lambda_6} \rightarrow \mathcal{G}_{\mu', \lambda'_4, \lambda'_5, \lambda'_6}, \quad (3.5)$$

where $\mu' = \mu + \kappa_1 + \kappa_2 + \kappa_3$ and $\lambda'_j = \lambda_j + \kappa_j$ ($j=4, 5, 6$) for $\kappa = \kappa_1 \epsilon_1 + \dots + \kappa_6 \epsilon_6$ ($\kappa_j \in \mathbf{Z}$), and $*$ = I or II. The explicit formulas of the operators will now be given. They are rationally written by q -difference operators with coefficients in $\mathbf{C}[x_{24}, \dots, x_{36}]$.

For each 6-tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_6)$ of complex numbers with $\sum \lambda_j = -3$, let

$\mathcal{F}_\lambda$ be the vector space of polynomials $G=G(x_{24}, \dots, x_{36})$ such that

$$T_{q, x_{24}} T_{q, x_{25}} T_{q, x_{26}} G = q^{-\lambda_2-1} G, \quad T_{q, x_{34}} T_{q, x_{35}} T_{q, x_{36}} G = q^{-\lambda_3-1} G. \quad (3.6)$$

Then, for each element $a \in U_q(gl(6))$ of weight κ , the operators

$$\rho_\lambda^I(a) = \bar{\rho}_{\lambda_1+\lambda_2+\lambda_3+2, \lambda_4, \lambda_5, \lambda_6}^I(a), \quad \rho_\lambda^{II}(a) = \bar{\rho}_{\lambda_1+\lambda_2+\lambda_3+2, \lambda_4, \lambda_5, \lambda_6}^{II}(a) \quad (3.7)$$

define mappings

$$\rho_\lambda^I(a), \quad \rho_\lambda^{II}(a): \mathcal{F}_\lambda \rightarrow \mathcal{F}_{\lambda+\kappa}. \quad (3.8)$$

For instance, for the operator $\rho_\lambda^I(q^{\epsilon_1})$, we set

$$\begin{aligned} \xi_{\lambda, \mu} &= \xi_{i6}^{\lambda_6} \xi_{i5}^{\lambda_5} \xi_{i4}^{\lambda_4} \xi_{i3}^{\lambda_3} \xi_{i2}^{\lambda_2} \xi_{i1}^{\lambda_1}, \\ R_2(\nu_{24}, \nu_{25}, \nu_{26}) &= u_{24}^{\nu_{24}} u_{25}^{\nu_{25}} u_{26}^{\nu_{26}}, \\ R_3(\nu_{36}, \nu_{35}, \nu_{34}) &= u_{36}^{\nu_{36}} u_{35}^{\nu_{35}} u_{34}^{\nu_{34}}. \end{aligned}$$

Then we have

$$\begin{aligned} \bar{\rho}(q^{\epsilon_1}) \circ \omega_{\mu, \lambda_4, \lambda_5, \lambda_6}^I(x_{24}^{\nu_{24}} x_{25}^{\nu_{25}} x_{26}^{\nu_{26}} x_{34}^{\nu_{34}} x_{35}^{\nu_{35}} x_{36}^{\nu_{36}}) \\ = q^{\mu+\nu_{24}+\nu_{25}+\dots+\nu_{36}} R_2(\nu_{24}, \nu_{25}, \nu_{26}) \xi_{\lambda, \mu} R_3(\nu_{36}, \nu_{35}, \nu_{34}), \end{aligned}$$

for $x_{24}^{\nu_{24}} x_{25}^{\nu_{25}} \dots x_{36}^{\nu_{36}} \in \mathcal{F}_\lambda$. Here we remark that (3.6) means $\nu_{24} + \nu_{25} + \nu_{26} = -\lambda_2 - 1$ and $\nu_{34} + \nu_{35} + \nu_{36} = -\lambda_3 - 1$. So,

$$\begin{aligned} \rho_\lambda^I(q^{\epsilon_1}) &= \bar{\rho}_{\lambda_1+\lambda_2+\lambda_3+2, \lambda_4, \lambda_5, \lambda_6}^I(q^{\epsilon_1}) \\ &= q^{\lambda_1+\lambda_2+\lambda_3+2+\nu_{24}+\nu_{25}+\dots+\nu_{36}} \\ &= q^{\lambda_1}. \end{aligned}$$

This way of computation gives the explicit formulas of the operators

$$\begin{aligned} \rho_\lambda^*(q^{\epsilon_j}): \mathcal{F}_\lambda &\rightarrow \mathcal{F}_\lambda \quad (1 \leq j \leq 6), \\ \rho_\lambda^*(e_j): \mathcal{F}_\lambda &\rightarrow \mathcal{F}_{\lambda+\alpha_j} \quad (1 \leq j \leq 5), \\ \rho_\lambda^*(f_j): \mathcal{F}_\lambda &\rightarrow \mathcal{F}_{\lambda-\alpha_j} \quad (1 \leq j \leq 5), \end{aligned} \quad (3.9)$$

where $\alpha_j = \epsilon_j - \epsilon_{j+1}$ and $*$ = I or II, as follows.

We first consider the case (I). Put

$$T_{rj} = T_{q^2, x_{rj}};$$

then we see

$$\begin{aligned}
\rho_\lambda^1(q^{\epsilon_j}) &= q^{\lambda_j} \quad (1 \leq j \leq 6), \\
(q - q^{-1})\rho_\lambda^1(e_1) &= x_{24}(q^{-2\lambda_4} T_{24} T_{34} - 1) + x_{25} q^{-2\lambda_4} T_{24} T_{34} (q^{-2\lambda_5} T_{25} T_{35} - 1) \\
&\quad + x_{26} q^{-2\lambda_4 - 2\lambda_5} T_{24} T_{34} T_{25} T_{35} (q^{-2\lambda_6} T_{26} T_{36} - 1), \\
(q - q^{-1})\rho_\lambda^1(e_2) &= q^{-\lambda_1 - \lambda_3 - 3} \left[\frac{1}{x_{24}} \{ (x_{36} - x_{35}) (1 - q^{-2\lambda_6} T_{26} T_{36}) \right. \\
&\quad + (x_{35} - x_{34}) (1 - q^{-2\lambda_6 - 2\lambda_5} T_{26} T_{36} T_{25} T_{35}) + x_{34} \} (1 - T_{24}) \\
&\quad + \frac{1}{x_{25}} \{ (x_{36} - x_{35}) (1 - q^{-2\lambda_6} T_{26} T_{36}) + x_{35} \} (1 - T_{25}) T_{24} \\
&\quad \left. + \frac{1}{x_{26}} x_{36} T_{24} T_{25} (1 - T_{26}) \right], \\
(q - q^{-1})\rho_\lambda^1(e_3) &= q^{-\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + 1} \frac{1}{x_{34}} (T_{34} - 1), \\
(q - q^{-1})\rho_\lambda^1(e_j)_{(j=4,5)} &= q^{\lambda_{j+1}} T_{2j+1}^{-1} - q^{-\lambda_{j+1}} T_{3j+1} \\
&\quad + \frac{x_{2j}}{x_{2j+1}} q^{\lambda_{j+1}} (1 - T_{2j+1}^{-1}) + \frac{x_{3j}}{x_{3j+1}} q^{-\lambda_{j+1}} (T_{3j+1} - 1), \\
(q - q^{-1})\rho_\lambda^1(f_1) &= -q^{-1 - \lambda_1 - \lambda_2} \left\{ \frac{1}{x_{24}} T_{25}^{-1} T_{26}^{-1} (1 - T_{24}^{-1}) \right. \\
&\quad \left. + \frac{1}{x_{25}} T_{26}^{-1} (1 - T_{25}^{-1}) + \frac{1}{x_{26}} (1 - T_{26}^{-1}) \right\}, \\
(q - q^{-1})\rho_\lambda^1(f_2) &= q^{2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6} \times \left[\frac{x_{26}}{x_{36}} (1 - T_{36}) \right. \\
&\quad + \frac{1}{x_{35}} \{ x_{25} q^{2\lambda_6} T_{26}^{-1} T_{36}^{-1} + x_{26} (1 - q^{2\lambda_6} T_{26}^{-1} T_{36}^{-1}) \} T_{36} (1 - T_{35}) \\
&\quad + \frac{1}{x_{34}} \{ x_{24} q^{2\lambda_6 + 2\lambda_5} T_{26}^{-1} T_{36}^{-1} T_{25}^{-1} T_{35}^{-1} \} \\
&\quad + x_{25} q^{2\lambda_6} T_{26}^{-1} T_{36}^{-1} (1 - q^{2\lambda_5} T_{25}^{-1} T_{35}^{-1}) \\
&\quad \left. + x_{26} (1 - q^{2\lambda_6} T_{26}^{-1} T_{36}^{-1}) \} T_{36} T_{35} (1 - T_{34}) \right], \\
(q - q^{-1})\rho_\lambda^1(f_3) &= q^{-2 - \lambda_3 - \mu} \times \\
&\quad [x_{36} (q^{-2\lambda_6} T_{36} - T_{26}^{-1}) T_{25}^{-1} \\
&\quad + x_{35} (q^{-2\lambda_5} T_{35} - T_{25}^{-1}) q^{-2\lambda_6} T_{36} + x_{34} (q^{2\mu} - q^{-2\lambda_6 - 2\lambda_5} T_{36} T_{35})]
\end{aligned}$$

$$\begin{aligned}
& + \frac{x_{24}}{x_{25}} \{ x_{36} (q^{-2\lambda_6} T_{36} - T_{26}^{-1}) - x_{35} q^{-2\lambda_6} T_{36} \} (1 - T_{25}^{-1}) \\
& - \frac{x_{24}}{x_{26} x_{36}} (1 - T_{26}^{-1}) \Big], \\
& (q - q^{-1}) \rho_\lambda^1(f_j)_{(j=4,5)} = (q^{\lambda_j} T_{3j}^{-1} - q^{-\lambda_j} T_{2j}) \\
& + \frac{x_{3j+1}}{x_{3j}} q^{\lambda_j} (1 - T_{3j}^{-1}) + \frac{x_{2j+1}}{x_{2j}} q^{-\lambda_j} (T_{2j} - 1).
\end{aligned}$$

We should remark here that the coefficients belong to the polynomial ring, because x_{rj} in the denominators cancels with the operator $(1 - T_{rj})$ or $(1 - T_{rj}^{-1})$ as difference operators, as often as it appears. We have already seen the formula for q^{ϵ_1} . The actions of $e_1 \in U_q(\mathfrak{gl}(6))$, for instance, on u_{2j} and u_{3j} is determined by

$$\bar{\rho}(e_1) u_{2j}^{\nu_{2j}} = (q^{2\nu_{2j}} - 1) u_{2j}^{\nu_{2j}}, \quad \bar{\rho}(e_1) u_{3j}^{\nu_{3j}} = (q^{2\nu_{3j}} - 1) u_{2j} u_{3j}^{\nu_{3j}-1}.$$

So, the second formula is seen by the computation

$$\begin{aligned}
& (q - q^{-1}) \bar{\rho}(e_1) R_2(\nu_{24}, \nu_{25}, \nu_{26}) \xi_{\lambda, \mu} R_3(\nu_{36}, \nu_{35}, \nu_{34}) \\
& = \sum_{j=4}^6 q^{2\nu_{24} + \dots + 2\nu_{2j-1}} (q^{2\nu_{2j}} - 1) R_2(\dots, \nu_{2j} + 1, \dots) \xi_{\lambda, \mu} R_2(\nu_{36}, \nu_{35}, \nu_{34}) \\
& - q^{2\nu_{24} + 2\nu_{25} + 2\nu_{26}} R_2(\nu_{24}, \nu_{25}, \nu_{26}) \\
& \times \sum_{j=4}^6 q^{-2\lambda_4 - \dots - 2\lambda_{j-1}} (1 - q^{-2\lambda_j}) u_{2j} \xi_{\lambda, \mu} R_3(\nu_{36}, \nu_{35}, \nu_{34}) \\
& + q^{2\nu_{24} + 2\nu_{25} + 2\nu_{26} - \lambda_4 - \lambda_5 - \lambda_6} R_2(\nu_{24}, \nu_{25}, \nu_{26}) \xi_{\lambda, \mu} \\
& \times \sum_{j=4}^6 q^{2\nu_{34} + \dots + 2\nu_{3j-1}} (q^{2\nu_{3j}} - 1) u_{2j} R_3(\nu_{36}, \nu_{35}, \nu_{34}) \\
& + q^{2\nu_{24} + 2\nu_{25} + 2\nu_{26} - \lambda_4 - \lambda_5 - \lambda_6} R_2(\nu_{24}, \nu_{25}, \nu_{26}) \xi_{\lambda, \mu} \\
& \times \sum_{j=4}^6 q^{2\nu_{34} + \dots + 2\nu_{3j-1}} (q^{2\nu_{3j}} - 1) u_{2j} R_3(\nu_{36}, \nu_{35}, \nu_{34}) \\
& = (q^{2\nu_{24} + 2\nu_{34} - 2\lambda_4} - 1) R_2(\nu_{24} + 1, \nu_{25}, \nu_{26}) \xi_{\lambda, \mu} R_3(\nu_{36}, \nu_{35}, \nu_{34}) \\
& + q^{2\nu_{24} + 2\nu_{34} - 2\lambda_4} (q^{2\nu_{25} + 2\nu_{35} - 2\lambda_5} - 1) R_2(\nu_{24}, \nu_{25} + 1, \nu_{26}) \xi_{\lambda, \mu} R_3(\nu_{36}, \nu_{35}, \nu_{34}) \\
& + q^{2\nu_{24} + 2\nu_{34} + 2\nu_{25} + 2\nu_{35} - 2\lambda_4 - 2\lambda_5} (q^{2\nu_{26} + 2\nu_{36} - 2\lambda_6} - 1) \\
& R_2(\nu_{24}, \nu_{25}, \nu_{26} + 1) \xi_{\lambda, \mu} R_3(\nu_{36}, \nu_{35}, \nu_{34}).
\end{aligned}$$

Namely, we have

$$\begin{aligned}
& (q - q^{-1}) \rho_\lambda^I(e_1) (x_{24}^{\nu_{24}} \cdots x_{36}^{\nu_{36}}) \\
&= (q - q^{-1}) \omega_{\lambda_1 + \lambda_2 + \lambda_3 + 2, \lambda_4, \lambda_5, \lambda_6}^{-1} \circ \bar{\rho}(e_1) \circ \omega_{\lambda_1 + \lambda_2 + \lambda_3 + 2, \lambda_4, \lambda_5, \lambda_6} (x_{24}^{\nu_{24}} \cdots x_{36}^{\nu_{36}}) \\
&= (q - q^{-1}) \omega_{\lambda_1 + \lambda_2 + \lambda_3 + 2, \lambda_4, \lambda_5, \lambda_6}^{-1} \circ \bar{\rho}(e_1) R_2(\nu_{24}, \nu_{25}, \nu_{26}) \xi_{\lambda, \lambda_1 + \lambda_2 + \lambda_3 + 2} R_3(\nu_{36}, \nu_{35}, \nu_{34}) \\
&= (q^{2\nu_{24} + 2\nu_{34} - 2\lambda_4} - 1) x_{24}^{\nu_{24} + 1} \cdots x_{36}^{\nu_{36}} \\
&\quad + q^{2\nu_{24} + 2\nu_{34} - 2\lambda_4} (q^{2\nu_{25} + 2\nu_{35} - 2\lambda_5} - 1) x_{24}^{\nu_{24}} x_{25}^{\nu_{25} + 1} \cdots x_{36}^{\nu_{36}} \\
&\quad + q^{2\nu_{24} + 2\nu_{34} + 2\nu_{25} + 2\nu_{35} - 2\lambda_4 - 2\lambda_5} (q^{2\nu_{26} + 2\nu_{36} - 2\lambda_6} - 1) x_{24}^{\nu_{24}} \cdots x_{26}^{\nu_{26} + 1} \cdots x_{36}^{\nu_{36}},
\end{aligned}$$

which shows the second formula. The remaining formulas can be shown similarly.

We second consider the case (II). Set

$$T_{rj} = T_{q, x_{rj}};$$

then we see

$$\begin{aligned}
& \rho_\lambda^{\text{II}}(q^{\epsilon_j}) = q^{\lambda_j} \quad (1 \leq j \leq 6), \\
& (q - q^{-1}) \rho_\lambda^{\text{II}}(e_1) = q^{\lambda_4 + \lambda_5} x_{26} (q^{-\lambda_6} T_{26}^{-2} T_{36}^{-2} - q^{\lambda_6}) \\
& \quad + q^{\lambda_4 - \lambda_6} x_{25} T_{26}^{-2} T_{36} (q^{-\lambda_5} T_{25}^{-2} T_{35}^{-2} - q^{\lambda_5}) \\
& \quad + q^{-\lambda_5 - \lambda_6} x_{24} T_{26}^{-2} T_{36} T_{25}^{-2} T_{35} (q^{-\lambda_4} T_{24}^{-2} T_{34}^{-2} - q^{\lambda_4}), \\
& (q - q^{-1}) \rho_\lambda^{\text{II}}(e_2) = q^{-1} \left\{ \frac{x_{36}}{x_{26}} T_{26}^{-1} (1 - T_{26}^{-2}) \frac{x_{35}}{x_{25}} T_{26}^{-1} T_{25}^{-1} T_{36} (1 - T_{25}^{-2}) \right. \\
& \quad \left. + \frac{x_{34}}{x_{24}} T_{26}^{-1} T_{25}^{-1} T_{24}^{-1} T_{36} T_{35} (1 - T_{24}^{-2}) \right\}, \\
& (q - q^{-1}) \rho_\lambda^{\text{II}}(e_3) = q^{-\lambda_1 + \lambda_3 - \lambda_4 + 2} \frac{1}{x_{34}} (T_{34}^{-2} - 1), \\
& (q - q^{-1}) \rho_\lambda^{\text{II}}(e_j)_{(j=4,5)} = q^{\lambda_{j+1}} - q^{-\lambda_{j+1}} T_{3j+1}^{-2} T_{2j+1}^{-2} \\
& \quad - q^{-\lambda_{j+1}} \left\{ \frac{x_{2j}}{x_{2j+1}} T_{3j+1} (1 - T_{2j+1}^{-2}) + \frac{x_{3j}}{x_{3j+1}} T_{2j} (1 - T_{3j+1}^{-2}) \right\}, \\
& (q - q^{-1}) \rho_\lambda^{\text{II}}(f_1) = -q \left\{ \frac{1}{x_{26}} T_{25}^{-2} T_{24}^{-2} T_{34}^{-1} T_{35}^{-1} T_{36}^{-1} (1 - T_{26}^{-2}) \right. \\
& \quad + \frac{1}{x_{25}} T_{24}^{-2} T_{35}^{-1} T_{34}^{-1} (1 - T_{25}^{-2}) \\
& \quad \left. + \frac{1}{x_{24}} T_{34}^{-1} (1 - T_{24}^{-2}) \right\},
\end{aligned}$$

$$\begin{aligned}
(q - q^{-1}) \rho_\lambda^{\text{II}}(f_2) &= q \left\{ \frac{x_{26}}{x_{36}} T_{24} T_{25} T_{34}^{-1} T_{35}^{-1} T_{36}^{-1} (1 - T_{36}^2) \right. \\
&\quad \left. + \frac{x_{25}}{x_{35}} T_{24} T_{34}^{-1} T_{35}^{-1} (1 - T_{35}^2) + \frac{x_{24}}{x_{34}} T_{34}^{-1} (1 - T_{34}^2) \right\}, \\
(q - q^{-1}) \rho_\lambda^{\text{II}}(f_3) &= q^{-2-\mu-\lambda_3} \times \\
&\quad \{ x_{36} T_{26} (1 - q^{-2\lambda_6} T_{26}^2 T_{36}^2) \\
&\quad + q^{-2\lambda_6} x_{35} T_{26} T_{25} T_{36}^2 (1 - q^{-2\lambda_5} T_{25}^2 T_{35}^2) \\
&\quad - q^{2\mu} x_{34} T_{26} T_{25} T_{24} (1 - q^{-2\mu-2\lambda_6-2\lambda_5} T_{36}^2 T_{35}^2) \\
&\quad - q^{-1-2\lambda_6-2\lambda_5} \frac{x_{24} x_{35}}{x_{25}} T_{26} T_{25} T_{36}^2 T_{35} (1 - T_{25}^2) \\
&\quad - q^{-1-2\lambda_6-2\lambda_5} \frac{x_{24} x_{36}}{x_{26}} T_{26} T_{25}^2 T_{36} T_{35} (1 - T_{26}^2) \\
&\quad - q^{-1-2\lambda_6} \frac{x_{25} x_{36}}{x_{26}} T_{26} T_{36} (1 - T_{26}^2) (1 - q^{-2\lambda_5} T_{25}^2 T_{35}^2) \}, \\
(q - q^{-1}) \rho_\lambda^{\text{II}}(f_j)_{(j=4,5)} &= q^{\lambda_j} T_{3j}^{-2} T_{2j}^{-2} - q^{\lambda_j} \\
&\quad + q^{\lambda_j} \left\{ \frac{x_{2j+1}}{x_{2j}} T_{3j+1} T_{3j}^{-2} (1 - T_{2j}^{-2}) + \frac{x_{3j+1}}{x_{3j}} T_{2j}^{-1} (1 - T_{3j}^{-2}) \right\}.
\end{aligned}$$

Thus we have translated the action of $U_q(gl(6))$ on the commutative algebra $\bigoplus_\lambda \mathcal{F}_\lambda$.

4. Casimir operators

In this section we introduce the Casimir operators, instead of Laplacian (1.3) in the classical case, and compute their action on the algebra $\bigoplus_\lambda \mathcal{F}_\lambda$. The Casimir elements are defined by

$$(q - q^{-1})^2 C_j = (q^{1+\epsilon_j} - q^{-1-\epsilon_j}) (q^{\epsilon_{j+1}} - q^{-\epsilon_{j+1}}) - (q - q^{-1})^2 f_j e_j, \quad (4.1)$$

for $1 \leq j \leq 5$. They are central elements of the subalgebra $\mathbf{C}[q^{\pm\epsilon_j}, q^{\pm\epsilon_{j+1}}, e_j, f_j]$ of $U_q(gl(6))$, isomorphic to $U_q(gl(2))$.

Here recall the homogeneity condition (3.6); to take care of this, we introduce new coordinates by

$$z_1 = \frac{x_{25}}{x_{24}}, \quad z_2 = \frac{x_{26}}{x_{24}}, \quad z_3 = \frac{x_{35}}{x_{34}}, \quad z_4 = \frac{x_{36}}{x_{34}},$$

and the new unknown φ by

$$G = x_{24}^{-\lambda_2-1} x_{34}^{-\lambda_3-1} \varphi(z_1, z_2, z_3, z_4). \quad (4.2)$$

Then, relative to these coordinates, we can compute the actions of the Casimir operators by using the concrete forms of the actions of $q^{\pm \epsilon_j}$, e_j , and f_j , which have been given in the previous section.

Proposition 4.1. For the case (I), define the q -difference operators by

$$\begin{aligned} T_{25} &= T_{q^2, z_1} = T_1, \quad T_{26} = T_{q^2, z_2} = T_2, \quad T_{35} = T_{q^2, z_3} = T_3, \\ T_{36} &= T_{q^2, z_4} = T_4, \quad T_{24} = q^{-2\lambda_2-2} T_1^{-1} T_2^{-1}, \quad T_{34} = q^{-2\lambda_3-2} T_3^{-1} T_4^{-1}, \\ \mathbf{T} &= T_1 T_2 T_3 T_4. \end{aligned} \quad (4.3)$$

Then we have

$$\begin{aligned} (q - q^{-1})^2 \rho_\lambda^I(C_4) &= q^{-2\lambda_2-\lambda_4+\lambda_5-1} (1 - z_1^{-1}) T_1^{-2} T_2^{-1} \times \\ &\quad \{ (1 - q^{2\lambda_2+2\lambda_3+2\lambda_4+4} \mathbf{T}) (1 - T_1) - q^2 z_1 (1 - q^{2\lambda_2+2} T_1 T_2) (1 - q^{-2\lambda_5} T_1 T_3) \} \\ &\quad + q^{-2\lambda_2-\lambda_4-\lambda_5-3} (1 - z_3^{-1}) T_1^{-1} \times \\ &\quad \{ T_2^{-1} (1 - q^{2\lambda_2+2\lambda_3+2\lambda_4+4} \mathbf{T}) (1 - T_3) \\ &\quad - q^{2\lambda_2+2\lambda_4+2\lambda_5+4} z_3 (1 - q^{2\lambda_3+2} T_3 T_4) (1 - q^{-2\lambda_5} T_1 T_3) \} \\ &\quad + q^{-2\lambda_2-\lambda_4-\lambda_5-1} \left(\frac{1}{z_3} - \frac{1}{z_1} \right) T_1^{-1} \times \\ &\quad \{ z_1 T_2^{-1} (1 - q^{2\lambda_2+2} T_1 T_2) (1 - T_3) \\ &\quad - q^{2\lambda_2+2\lambda_4+2\lambda_5+2} z_3 (1 - q^{2\lambda_3+2} T_3 T_4) (1 - T_1) \}, \\ (q - q^{-1})^2 \rho_\lambda^I(C_5) &= q^{\lambda_5+\lambda_6+1} \left(\frac{1}{z_2} - \frac{1}{z_1} \right) T_2^{-1} \times \\ &\quad \{ z_1 T_3^{-1} (1 - q^{-2\lambda_5} T_1 T_3) (1 - T_2) - q^{-2\lambda_5} z_2 (1 - q^{-2\lambda_6} T_2 T_4) (1 - T_1) \} \\ &\quad + q^{\lambda_5-\lambda_6+1} \left(\frac{1}{z_4} - \frac{1}{z_3} \right) T_2^{-1} \times \\ &\quad \{ z_3 (1 - q^{-2\lambda_5} T_1 T_3) (1 - T_4) - q^{2\lambda_6} z_4 T_2^{-1} (1 - q^{-2\lambda_6} T_2 T_4) (1 - T_3) \} \\ &\quad + q^{\lambda_5+\lambda_6+1} \left(\frac{1}{z_2 z_3} - \frac{1}{z_1 z_4} \right) \times \\ &\quad \{ q^{-2\lambda_5-2\lambda_6-2} z_2 z_3 (1 - T_1) (1 - T_4) - z_1 z_4 T_2^{-1} T_3^{-1} (1 - T_2) (1 - T_3) \}. \end{aligned}$$

For the case (II), define the difference operators by

$$\begin{aligned} T_{25} &= T_{q,z_1} = T_1, & T_{26} &= T_{q,z_2} = T_2, & T_{35} &= T_{q,z_3} = T_3, \\ T_{36} &= T_{q,z_4} = T_4, & T_{24} &= q^{-\lambda_2-1} T_1^{-1} T_2^{-1}, & T_{34} &= q^{-\lambda_3-1} T_3^{-1} T_4^{-1}, \end{aligned} \quad (4.4)$$

Then we have

$$\begin{aligned} (q-q^{-1})^2 \rho_\lambda^{\text{II}}(C_4) &= q^{-\lambda_4-\lambda_5-1} \left(T_3 - \frac{1}{z_1} \right) T_3 \times \\ &\quad \{ (1 - q^{2\lambda_2+2\lambda_3+2\lambda_4+4} \mathbf{T}^2) (1 - T_1^2) \\ &\quad - q^{2\lambda_3+2\lambda_4+2\lambda_5+4} T_3 T_4^2 z_1 (1 - q^{2\lambda_2+2} T_1^2 T_2^2) (1 - q^{-2\lambda_5} T_1^2 T_3^2) \} \\ &+ q^{-\lambda_2-\lambda_4-\lambda_5-2} \left(T_1 T_2 - \frac{1}{z_3} \right) \times \\ &\quad \{ T_1^{-1} T_2^{-1} (1 - q^{2\lambda_2+2\lambda_3+2\lambda_4+4} \mathbf{T}^2) (1 - T_3^2) \\ &\quad - q^{\lambda_2+2\lambda_4+2\lambda_5+3} z_3 (1 - q^{2\lambda_3+2} T_3^2 T_4^2) (1 - q^{-2\lambda_5} T_1^2 T_3^2) \} \\ &+ q^{-\lambda_2+2\lambda_3+\lambda_4-\lambda_5-1} \left(\frac{1}{z_3} T_3 - \frac{1}{z_1} T_1 T_2 \right) T_3 \times \\ &\quad \{ z_1 T_1^{-1} T_2^{-1} T_3 T_4^2 (1 - q^{2\lambda_2+2} T_1^2 T_2^2) (1 - T_3^2) \\ &\quad - q^{\lambda_2-2\lambda_3} z_3 (1 - q^{2\lambda_3+2} T_3^2 T_4^2) (1 - T_1^2) \}, \\ (q-q^{-1})^2 \rho_\lambda^{\text{II}}(C_5) &= q^{\lambda_5-\lambda_6+1} \left(\frac{1}{z_2} - \frac{1}{z_1} T_4^{-1} \right) T_1^{-2} T_3^{-2} \times \\ &\quad \{ z_1 T_4 (1 - q^{-2\lambda_5} T_1^2 T_3^2) (1 - T_2^2) - q^{2\lambda_6} z_2 (1 - q^{-2\lambda_6} T_2^2 T_4^2) (1 - T_1^2) \} \\ &+ q^{\lambda_5-\lambda_6+1} \left(\frac{1}{z_4} - \frac{1}{z_3} T_1^{-1} \right) T_3^{-2} \times \\ &\quad \{ z_3 T_1^{-1} (1 - q^{-2\lambda_5} T_1^2 T_3^2) (1 - T_4^2) - q^{2\lambda_6} z_4 (1 - q^{-2\lambda_6} T_2^2 T_4^2) (1 - T_3^2) \} \\ &+ q^{\lambda_5-\lambda_6+2} \left(\frac{1}{z_2 z_3} - \frac{1}{z_1 z_4} T_1^{-1} T_4^{-1} \right) T_1^{-2} T_3^{-2} \times \\ &\quad \{ q^{-1} z_2 z_3 (1 - T_1^2) (1 - T_4^2) - z_1 z_4 T_1 T_4 (1 - T_1^2) (1 - T_3^2) \}. \end{aligned}$$

Proof. We exhibit only one case $\rho_\lambda^{\text{I}}(C_5)$ as follows. By definition, we have

$$\begin{aligned} (q-q^{-1})^2 \rho_\lambda^{\text{I}}(f_5 e_5) &= -q^{\lambda_5+\lambda_6-1} T_{26}^{-2} T_{35}^{-2} \frac{x_{25}}{x_{26}} (1 - q^{-2\lambda_5} T_{25}^2 T_{35}^2) (1 - T_{26}^2) \\ &\quad - q^{-\lambda_5+\lambda_6+1} T_{26}^{-2} \frac{x_{26}}{x_{25}} (1 - q^{-2\lambda_6} T_{26}^2 T_{36}^2) (1 - T_{25}^2) \\ &\quad - q^{\lambda_5-\lambda_6+1} T_{35}^{-2} \frac{x_{35}}{x_{36}} (1 - q^{-2\lambda_5} T_{25}^2 T_{35}^2) (1 - T_{36}^2) \end{aligned}$$

$$\begin{aligned}
& -q^{\lambda_5+\lambda_6-1} T_{26}^{-2} T_{35}^{-2} \frac{x_{36}}{x_{35}} (1-q^{-2\lambda_6} T_{26}^2 T_{36}^2) (1-T_{35}^2) \\
& + q^{-\lambda_5-\lambda_6-1} \frac{x_{26} x_{35}}{x_{25} x_{36}} (1-T_{25}^2) (1-T_{36}^2) \\
& + q^{\lambda_5+\lambda_6-3} \frac{x_{25} x_{36}}{x_{26} x_{35}} (1-T_{26}^2) (1-T_{35}^2) \\
& + q^{\lambda_5+\lambda_6+1} T_{26}^{-2} T_{35}^{-2} (1-q^{-2\lambda_5-2} T_{25}^2 T_{35}^2) (1-q^{-2\lambda_6-2} T_{26}^2 T_{36}^2) \\
& + q^{-\lambda_5+\lambda_6-1} T_{26}^{-2} (1-q^2 T_{25}^2) (1-T_{26}^2) \\
& + q^{\lambda_5-\lambda_6-1} T_{35}^{-2} (1-q^2 T_{35}^2) (1-T_{36}^2) \\
& = (q^{\lambda_5+1} - q^{-\lambda_5-1}) (q^{\lambda_6} - q^{-\lambda_6}) \\
& \quad - q^{\lambda_5+\lambda_6+1} \left(\frac{1}{z_2} - \frac{1}{z_1} \right) T_2^{-1} \times \\
& \quad \{ z_1 T_3^{-1} (1-q^{-2\lambda_5} T_1 T_3) (1-T_2) - q^{-2\lambda_5} z_2 (1-q^{-2\lambda_6} T_2 T_4) (1-T_1) \} \\
& \quad - q^{\lambda_5-\lambda_6+1} \left(\frac{1}{z_4} - \frac{1}{z_3} \right) T_2^{-1} \times \\
& \quad \{ z_3 (1-q^{-2\lambda_5} T_1 T_3) (1-T_4) - q^{2\lambda_6} z_4 T_2^{-1} (1-q^{-2\lambda_6} T_2 T_4) (1-T_3) \} \\
& \quad - q^{\lambda_5+\lambda_6+1} \left(\frac{1}{z_2 z_3} - \frac{1}{z_1 z_4} \right) \times \\
& \quad \{ q^{-2\lambda_5-2\lambda_6-2} z_2 z_3 (1-T_1) (1-T_4) - z_1 z_4 T_2^{-1} T_3^{-1} (1-T_2) (1-T_3) \}.
\end{aligned}$$

and, then, this combined with (4.1) shows the formula for the case $\rho_\lambda^1(C_5)$.

Note that we translated all actions of the elements of $U_q(gl(6))$ in terms of q -difference operators. Therefore, we can extend the action of $U_q(gl(6))$ to the space of formal power series.

5. q -hypergeometric functions

In this section we define two kinds of functions, which are both q -analogue of $F_{3,6}$. Let us consider the following system of difference equations:

$$T_{q,x_{24}} T_{q,x_{25}} T_{q,x_{26}} G = q^{-\lambda_2-1} G, \quad T_{q,x_{34}} T_{q,x_{35}} T_{q,x_{36}} G = q^{-\lambda_3-1} G, \quad (5.1)$$

$$(q-q^{-1})^2 \rho_\lambda^*(C_j) G = 0 \quad (1 \leq j \leq 5), \quad (5.2*)$$

for $*$ = I or II. We define functions that are solutions of this system, assuming that $\lambda_2 + \lambda_3 + \lambda_4 + 3 \neq 0, -1, -2, \dots$. Put

$$\begin{aligned}
\varphi_\lambda^{\text{I}} &= \varphi_\lambda^{\text{I}}(x_1, x_2, x_3, x_4) \\
&= \sum_{a,b,c,d=0}^{\infty} \frac{(q^{-2\lambda_5}; q^2)_{a+c} (q^{-2\lambda_6}; q^2)_{b+d} (q^{2\lambda_2+2}; q^2)_{a+b} (q^{2\lambda_3+2}; q^2)_{c+d}}{(q^{2\lambda_2+2\lambda_3+2\lambda_4+6}; q^2)_{a+b+c+d} (q^2; q^2)_a (q^2; q^2)_b (q^2; q^2)_c (q^2; q^2)_d} \\
&\quad \times q^{2bc} x_1^a x_2^b x_3^c x_4^d, \tag{5.3}
\end{aligned}$$

where

$$\begin{aligned}
x_1 &= q^2 z_1, \quad x_2 = q^{2-2\lambda_5} z_2, \\
x_3 &= q^{2\lambda_2+2\lambda_4+2\lambda_5+4} z_3, \quad x_4 = q^{2\lambda_2+2\lambda_4+2\lambda_5+2\lambda_6+4} z_4, \tag{5.4}
\end{aligned}$$

and

$$\begin{aligned}
\varphi_\lambda^{\text{II}} &= \varphi_\lambda^{\text{II}}(x_1, x_2, x_3, x_4) \\
&= \sum_{a,b,c,d=0}^{\infty} \frac{(q^{-2\lambda_5}; q^2)_{a+c} (q^{-2\lambda_6}; q^2)_{b+d} (q^{2\lambda_2+2}; q^2)_{a+b} (q^{2\lambda_3+2}; q^2)_{c+d}}{(q^{2\lambda_2+2\lambda_3+2\lambda_4+6}; q^2)_{a+b+c+d} (q^2; q^2)_a (q^2; q^2)_b (q^2; q^2)_c (q^2; q^2)_d} \\
&\quad \times q^{c(a+b)+d(2a+b)} x_1^a x_2^b x_3^c x_4^d, \tag{5.5}
\end{aligned}$$

where

$$\begin{aligned}
x_1 &= q^{2\lambda_3+2\lambda_4+2\lambda_5+4} z_1, \quad x_2 = q^{2\lambda_3+2\lambda_4+2\lambda_5+2\lambda_6+4} z_2, \\
x_3 &= q^{\lambda_2+2\lambda_4+2\lambda_5+3} z_3, \quad x_4 = q^{\lambda_2+2\lambda_4+2\lambda_5+2\lambda_6+3} z_4. \tag{5.6}
\end{aligned}$$

Here $(\alpha, q)_n = (1-\alpha)(1-\alpha q) \cdots (1-\alpha q^{n-1})$.

Then we can see the next proposition by a simple but a little longer direct computation.

Proposition 5.1. The functions

$$G_\lambda^{\text{I}} = x_{24}^{-\lambda_2-1} x_{34}^{-\lambda_3-1} \varphi_\lambda^{\text{I}}, \quad G_\lambda^{\text{II}} = x_{24}^{-\lambda_2-1} x_{34}^{-\lambda_3-1} \varphi_\lambda^{\text{II}}$$

satisfy the q -difference equations (5.1) and (5.2*) for $\ast = \text{I}$ and II , respectively.

Remark 5.2. For $1 \leq j \leq 3$, we can check $(q - q^{-1})^2 \rho_j^*(C_j) G_\lambda^* = 0$ by use of the contiguity relations, that will be introduced in the last section.

Definition 5.3. We call the functions $\varphi_\lambda^{\text{I}}$ and $\varphi_\lambda^{\text{II}}$ defined above the q -hypergeometric series of type $(3,6;\text{I})$ and type $(3,6;\text{II})$, respectively.

Remark 5.4. When q tends to 1, $(1-q^\alpha)/(1-q)$ converges to α . Hence, both of φ_λ^I and φ_λ^{II} converge to the function $F_{3,6}$ in (1.9), when q tends to 1.

We can, furthermore, show that, if a is an element of $U_q(gl(6))$ of weight κ , then the q -difference operators $\rho_\lambda^*(a)$ transform the solutions G_λ^* to a constant multiple of $G_{\lambda+\kappa}^*$. In this sense, the family of q -difference operators $\rho_\lambda^*(a)$ describe the contiguity relations for the family of the q -hypergeometric series φ_λ^I or φ_λ^{II} . In the next section, we examine these relations in terms of the functions φ_λ^I and φ_λ^{II} .

6. Contiguity relations and q -hypergeometric systems

Define the operators $\pi_\lambda^I(a)$ and $\pi_\lambda^{II}(a)$ as follows:

$$\begin{aligned}\rho_\lambda^I(a) G_\lambda^I &= x_{24}^{-\lambda_2^I-1} x_{34}^{-\lambda_3^I-1} \pi_\lambda^I(a) \varphi_\lambda^I, \\ \rho_\lambda^{II}(a) G_\lambda^{II} &= x_{24}^{-\lambda_2^{II}-1} x_{34}^{-\lambda_3^{II}-1} \pi_\lambda^{II}(a) \varphi_\lambda^{II},\end{aligned}\tag{6.1}$$

where a is an element of $U_q(gl(6))$ of weight κ and $\lambda' = (\lambda'_j) = \lambda + \kappa$. Then we have

Proposition 6.1. The following is the list of contiguity relations for the q -hypergeometric series φ_λ^I :

$$\begin{aligned}\pi_\lambda^I(q^{\epsilon_j}) \varphi_\lambda^I &= q^{\lambda_j} \varphi_\lambda^I \quad (1 \leq j \leq 6), \\ \pi_\lambda^I(e_1) \varphi_\lambda^I &= -q^{-\mu} [\mu] \varphi_{\lambda+\alpha_1}^I, \\ \pi_\lambda^I(e_2) \varphi_\lambda^I &= q^{\lambda_4-\lambda_5-\lambda_6-1} [\lambda_2+1] \varphi_{\lambda+\alpha_2}^I, \\ \pi_\lambda^I(e_3) \varphi_\lambda^I &= -q^{\lambda_3+\lambda_5+\lambda_6+3} [\lambda_3+1] \varphi_{\lambda+\alpha_3}^I, \\ \pi_\lambda^I(e_4) \varphi_\lambda^I &= q^{2\lambda_2+\lambda_4-\mu} \frac{[\lambda_4+1][\lambda_5]}{[\mu+1]} \varphi_{\lambda+\alpha_4}^I, \\ \pi_\lambda^I(e_5) \varphi_\lambda^I &= [\lambda_6] \varphi_{\lambda+\alpha_5}^I, \\ \pi_\lambda^I(f_1) \varphi_\lambda^I &= -q^{\mu+1} \frac{[\lambda_1][\lambda_2+1]}{[\mu+1]} \varphi_{\lambda-\alpha_1}^I, \\ \pi_\lambda^I(f_2) \varphi_\lambda^I &= q^{-\lambda_4+\lambda_5+\lambda_6+1} [\lambda_3+1] \varphi_{\lambda-\alpha_2}^I, \\ \pi_\lambda^I(f_3) \varphi_\lambda^I &= -q^{-\lambda_3-\lambda_5-\lambda_6-2} [\lambda_4+1] \varphi_{\lambda-\alpha_3}^I, \\ \pi_\lambda^I(f_4) \varphi_\lambda^I &= q^{-2\lambda_2-\lambda_4+\mu-2} [\mu] \varphi_{\lambda-\alpha_4}^I, \\ \pi_\lambda^I(f_5) \varphi_\lambda^I &= [\lambda_5] \varphi_{\lambda-\alpha_5}^I.\end{aligned}$$

The contiguity relations for the q -hypergeometric series φ_λ^Π are given as follows :

$$\begin{aligned}
\pi_\lambda^\Pi(q^{\epsilon_j})\varphi_\lambda^\Pi &= q^{\lambda_j}\varphi_\lambda^\Pi \quad (1 \leq j \leq 6), \\
\pi_\lambda^\Pi(e_1)\varphi_\lambda^\Pi &= -q^{\lambda_1+\lambda_4+1}[\mu]\varphi_{\lambda+\alpha_1}^\Pi, \\
\pi_\lambda^\Pi(e_2)\varphi_\lambda^\Pi &= q^{-1}[\lambda_2+1]\varphi_{\lambda+\alpha_2}^\Pi, \\
\pi_\lambda^\Pi(e_3)\varphi_\lambda^\Pi &= q^{-\lambda_1-\lambda_4+1}[\lambda_3+1]\varphi_{\lambda+\alpha_3}^\Pi, \\
\pi_\lambda^\Pi(e_4)\varphi_\lambda^\Pi &= q^{-\lambda_2-\lambda_3-2}\frac{[\lambda_4+1][\lambda_5]}{[\mu+1]}\varphi_{\lambda+\alpha_4}^\Pi, \\
\pi_\lambda^\Pi(e_5)\varphi_\lambda^\Pi &= [\lambda_6]\varphi_{\lambda+\alpha_5}^\Pi, \\
\pi_\lambda^\Pi(f_1)\varphi_\lambda^\Pi &= -q^{-\lambda_1-\lambda_4}\frac{[\lambda_1][\lambda_2+1]}{[\mu+1]}\varphi_{\lambda-\alpha_1}^\Pi, \\
\pi_\lambda^\Pi(f_2)\varphi_\lambda^\Pi &= q[\lambda_3+1]\varphi_{\lambda-\alpha_2}^\Pi, \\
\pi_\lambda^\Pi(f_3)\varphi_\lambda^\Pi &= q^{\lambda_1+\lambda_4}[\lambda_4+1]\varphi_{\lambda-\alpha_3}^\Pi, \\
\pi_\lambda^\Pi(f_4)\varphi_\lambda^\Pi &= q^{\lambda_2+\lambda_3+2}[\mu]\varphi_{\lambda-\alpha_4}^\Pi, \\
\pi_\lambda^\Pi(f_5)\varphi_\lambda^\Pi &= [\lambda_5]\varphi_{\lambda-\alpha_5}^\Pi.
\end{aligned}$$

Here $\mu = \lambda_2 + \lambda_3 + \lambda_4 + 2$, and $[a] = (q^a - q^{-a})/(q - q^{-1})$.

The proof can also be given by computation, which we omit to reproduce here.

So far we have considered only the generators of the algebra $U_q(gl(6))$; now we introduce the elements $\widehat{E}_{ij} \in U_q(gl(6))$ ($i \neq j$) determined recursively by the following formulas :

$$\begin{aligned}
\widehat{E}_{jj+1} &= e_j, \quad \widehat{E}_{j+1j} = f_j \\
\widehat{E}_{ij} &= \widehat{E}_{ik}\widehat{E}_{kj} - q^{\pm 1}\widehat{E}_{kj}\widehat{E}_{ik} \quad (i \leqslant^{\vee} k \leqslant j).
\end{aligned} \tag{6.2}$$

The element $\widehat{E}_{ij}$ is of weight $\alpha_{ij} = \epsilon_i - \epsilon_j$, and can be seen as an analogue of the root vector E_{ij} of $gl(6)$ defined in § 1. When being considered as q -difference operators, they define endomorphisms

$$\rho_\lambda^*(\widehat{E}_{ij}): \mathcal{F}_\lambda \rightarrow \mathcal{F}_{\lambda+\alpha_{ij}} \tag{6.3}$$

for $\ast = \text{I}$ and II . Their actions on q -hypergeometric series are symmarized in the following list.

Proposition 6.2. The following is the list of contiguity relations appending those in Proposition 6.1 for the q -hypergeometric series $\varphi_\lambda^{\text{I}}$:

$$\begin{aligned}
\pi_\lambda^{\text{I}}(\widehat{E}_{13})\varphi_\lambda^{\text{I}} &= -q^{-\lambda_2+\lambda_4-\lambda_5-\lambda_6-\mu-1}[\mu]\varphi_{\lambda+\alpha_{13}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{31})\varphi_\lambda^{\text{I}} &= -q^{\lambda_2-\lambda_4+\lambda_5+\lambda_6+\mu+2}\frac{[\lambda_1][\lambda_3+1]}{[\mu+1]}\varphi_{\lambda+\alpha_{31}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{14})\varphi_\lambda^{\text{I}} &= q^{-2\lambda_2-\lambda_3-1}[\mu]\varphi_{\lambda+\alpha_{14}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{41})\varphi_\lambda^{\text{I}} &= q^{2\lambda_2+\lambda_3+1}\frac{[\lambda_1][\lambda_4+1]}{[\mu+1]}\varphi_{\lambda+\alpha_{41}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{15})\varphi_\lambda^{\text{I}} &= q^{-\mu-\lambda_3+1}[\lambda_5]\varphi_{\lambda+\alpha_{15}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{51})\varphi_\lambda^{\text{I}} &= q^{\mu+\lambda_3-1}[\lambda_1]\varphi_{\lambda+\alpha_{51}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{16})\varphi_\lambda^{\text{I}} &= q^{-\mu-\lambda_3-\lambda_5+1}[\lambda_6]\varphi_{\lambda+\alpha_{16}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{61})\varphi_\lambda^{\text{I}} &= q^{\mu+\lambda_3+\lambda_5-1}[\lambda_1]\varphi_{\lambda+\alpha_{61}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{24})\varphi_\lambda^{\text{I}} &= -q^{\lambda_4+1}[\lambda_2+1]\varphi_{\lambda+\alpha_{24}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{42})\varphi_\lambda^{\text{I}} &= -q^{-\lambda_4-2}[\lambda_4+1]\varphi_{\lambda+\alpha_{42}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{25})\varphi_\lambda^{\text{I}} &= -q^{\lambda_2-\lambda_3+2}\frac{[\lambda_2+1][\lambda_5]}{[\mu+1]}\varphi_{\lambda+\alpha_{25}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{52})\varphi_\lambda^{\text{I}} &= -q^{-\lambda_2+\lambda_4-1}[\mu]\varphi_{\lambda+\alpha_{52}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{26})\varphi_\lambda^{\text{I}} &= -q^{\lambda_2-\lambda_3-\lambda_5+2}\frac{[\lambda_2+1][\lambda_6]}{[\mu+1]}\varphi_{\lambda+\alpha_{26}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{62})\varphi_\lambda^{\text{I}} &= -q^{-\lambda_2+\lambda_3+\lambda_5-1}[\mu]\varphi_{\lambda+\alpha_{62}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{35})\varphi_\lambda^{\text{I}} &= -q^{\lambda_2-\lambda_4+\lambda_5+\lambda_6+2}\frac{[\lambda_3+1][\lambda_5]}{[\mu+1]}\varphi_{\lambda+\alpha_{35}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{53})\varphi_\lambda^{\text{I}} &= -q^{-\lambda_2+\lambda_4-\lambda_5-\lambda_6-3}[\mu]\varphi_{\lambda+\alpha_{53}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{36})\varphi_\lambda^{\text{I}} &= -q^{\lambda_2-\lambda_4+\lambda_6+2}\frac{[\lambda_3+1][\lambda_6]}{[\mu+1]}\varphi_{\lambda+\alpha_{36}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{63})\varphi_\lambda^{\text{I}} &= -q^{-\lambda_2-\lambda_4-\lambda_6-3}[\mu]\varphi_{\lambda+\alpha_{63}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{46})\varphi_\lambda^{\text{I}} &= q^{\lambda_2-\lambda_3-\lambda_5}\frac{[\lambda_4+1][\lambda_6]}{[\mu+1]}\varphi_{\lambda+\alpha_{46}}^{\text{I}}, \\
\pi_\lambda^{\text{I}}(\widehat{E}_{64})\varphi_\lambda^{\text{I}} &= q^{-\lambda_2+\lambda_3+\lambda_5}[\mu]\varphi_{\lambda+\alpha_{64}}^{\text{I}}.
\end{aligned}$$

For the q -hypergeometric series $\varphi_\lambda^{\text{II}}$, we have

$$\begin{aligned}
\pi_\lambda^{\text{II}}(\widehat{E}_{13})\varphi_\lambda^{\text{II}} &= -q^{\lambda_1-\lambda_2+\lambda_4}[\mu]\varphi_{\lambda+\alpha_{13}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{31})\varphi_\lambda^{\text{II}} &= -q^{-\lambda_1+\lambda_2-\lambda_4+1}\frac{[\lambda_1][\lambda_3+1]}{[\mu+1]}\varphi_{\lambda+\alpha_{31}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{14})\varphi_\lambda^{\text{II}} &= -q^{-\lambda_2-\lambda_3}[\mu]\varphi_{\lambda+\alpha_{14}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{41})\varphi_\lambda^{\text{II}} &= -q^{\lambda_2+\lambda_3}\frac{[\lambda_1][\lambda_4+1]}{[\mu+1]}\varphi_{\lambda+\alpha_{41}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{15})\varphi_\lambda^{\text{II}} &= -q^{-2\mu+\lambda_4+2}[\lambda_5]\varphi_{\lambda+\alpha_{15}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{51})\varphi_\lambda^{\text{II}} &= -q^{2\mu-\lambda_4-2}[\lambda_1]\varphi_{\lambda+\alpha_{51}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{16})\varphi_\lambda^{\text{II}} &= -q^{-2\mu+\lambda_4-\lambda_5+2}[\lambda_6]\varphi_{\lambda+\alpha_{16}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{61})\varphi_\lambda^{\text{II}} &= -q^{2\mu-\lambda_4+\lambda_5-2}[\lambda_1]\varphi_{\lambda+\alpha_{61}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{24})\varphi_\lambda^{\text{II}} &= q^{-\lambda_1-\lambda_3-\lambda_4}[\lambda_2+1]\varphi_{\lambda+\alpha_{24}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{42})\varphi_\lambda^{\text{II}} &= q^{\lambda_1+\lambda_3+\lambda_4+1}[\lambda_4+1]\varphi_{\lambda+\alpha_{42}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{25})\varphi_\lambda^{\text{II}} &= q^{-\lambda_1-\lambda_3-\lambda_4-\mu-1}\frac{[\lambda_2+1][\lambda_5]}{[\mu+1]}\varphi_{\lambda+\alpha_{25}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{52})\varphi_\lambda^{\text{II}} &= q^{\lambda_1+\lambda_3+\lambda_4+\mu}[\mu]\varphi_{\lambda+\alpha_{52}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{26})\varphi_\lambda^{\text{II}} &= q^{-\lambda_1-\lambda_3-\lambda_4-\lambda_5-\mu-1}\frac{[\lambda_2+1][\lambda_6]}{[\mu+1]}\varphi_{\lambda+\alpha_{26}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{62})\varphi_\lambda^{\text{II}} &= q^{\lambda_1+\lambda_3+\lambda_4+\lambda_5+\mu}[\mu]\varphi_{\lambda+\alpha_{62}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{35})\varphi_\lambda^{\text{II}} &= q^{-\lambda_1-\lambda_4-\mu}\frac{[\lambda_3+1][\lambda_5]}{[\mu+1]}\varphi_{\lambda+\alpha_{35}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{53})\varphi_\lambda^{\text{II}} &= q^{\lambda_1+\lambda_4+\mu-1}[\mu]\varphi_{\lambda+\alpha_{53}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{36})\varphi_\lambda^{\text{II}} &= q^{-\lambda_1-\lambda_4-\lambda_5-\mu}\frac{[\lambda_3+1][\lambda_6]}{[\mu+1]}\varphi_{\lambda+\alpha_{36}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{63})\varphi_\lambda^{\text{II}} &= q^{\lambda_1+\lambda_4+\lambda_5+\mu-1}[\mu]\varphi_{\lambda+\alpha_{63}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{46})\varphi_\lambda^{\text{II}} &= q^{\lambda_4-\lambda_5-\mu}\frac{[\lambda_4+1][\lambda_6]}{[\mu+1]}\varphi_{\lambda+\alpha_{46}}^{\text{II}}, \\
\pi_\lambda^{\text{II}}(\widehat{E}_{64})\varphi_\lambda^{\text{II}} &= q^{-\lambda_4+\lambda_5+\mu}[\mu]\varphi_{\lambda+\alpha_{64}}^{\text{II}}.
\end{aligned}$$

We can now rewrite these relations in a much simpler form; namely, referring to these relations, define the elements of $U_q(gl(6))$ by

$$(q-q^{-1})^2 C_{ij} = (q^{1+\epsilon_i} - q^{-1-\epsilon_i})(q^{\epsilon_j} - q^{-\epsilon_j}) - (q-q^{-1})^2 \widehat{E}_{ji} \widehat{E}_{ij}, \quad (1.12)$$

for $1 \leq i \neq j \leq 6$. Then we can define a quantum analogue of the classical hypergeometric system $E_{3,6}$ as follows.

Definition 6.3. We call the following system the q -hypergeometric system of type $(3,6;*)$:

$$T_{q,x_{24}} T_{q,x_{25}} T_{q,x_{26}} u = q^{-\lambda_2-1} u, \quad T_{q,x_{34}} T_{q,x_{35}} T_{q,x_{36}} u = q^{-\lambda_3-1} u, \quad (6.4)$$

$$\rho_\lambda^*(C_{ij})u = 0 \quad (1 \leq i \neq j \leq 6), \quad (6.6*)$$

where $*$ = I or II.

A direct computation by use of the contiguity relations above shows

Theorem 6.4. The functions G_λ^* is a solution of the q -hypergeometric system of type $(3,6;*)$ for $*$ = I or II.

Proof. We have only to check $\pi_\lambda^*(C_{ij})\varphi_\lambda^* = 0$ for $*$ = I or II. We compute only one case $\pi_\lambda^1(C_{15})\varphi_\lambda^1 = 0$ by the contiguity relations, as follows.

$$\begin{aligned} \pi_\lambda^1(\widehat{E}_{51} \widehat{E}_{15}) \varphi_\lambda^1 &= q^{-\lambda_2-2\lambda_3-\lambda_4-1} [\lambda_5] \pi_{\lambda+\alpha_{15}}^1(\widehat{E}_{51}) \varphi_{\lambda+\alpha_{15}}^1 \\ &= [\lambda_1+1] [\lambda_5] \varphi_\lambda^1. \end{aligned}$$

Remark 6.5. It is still an open problem to calculate the rank of the systems above.

Finally we define the q -difference equation as follows.

Proposition 6.6. For $J = (j_1, j_2, j_3, j_4)$ ($j_2 < j_4$, $j_k \neq j_l$ if $k \neq l$), we have

$$\pi_\lambda^*(\widehat{E}_{j_1 j_2} \widehat{E}_{j_3 j_4} - q^{a(J, \lambda)} \widehat{E}_{j_1 j_4} \widehat{E}_{j_3 j_2}) \varphi_\lambda^* = 0$$

for each $*$ = I or II, where

$$a(J, \lambda) = \begin{cases} -1 & (j_1, j_3 < j_2 < j_4 \text{ or } j_2 < j_4 < j_1, j_3) \\ \lambda_{j_2} + \lambda_{j_3} & (j_1 < j_2 < j_3 < j_4) \\ -\lambda_{j_1} - \lambda_{j_2} & (j_3 < j_2 < j_1 < j_4) \\ -\lambda_{j_1} - \lambda_{j_4} & (j_2 < j_1 < j_4 < j_3) \\ \lambda_{j_3} + \lambda_{j_4} & (j_2 < j_3 < j_4 < j_1) \\ -\lambda_{j_1} + \lambda_{j_3} + 1 & (j_2 < j_1, j_3 < j_4) \\ \lambda_{j_2} - \lambda_{j_4} - 1 & (j_1 < j_2 < j_4 < j_3 \text{ or } j_3 < j_2 < j_4 < j_1). \end{cases}$$

We can compute these equations by the contiguity relations. Let us notice that $a(J, \lambda)$ does not depend on the type of the monomial basis I and II.

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