THE TERWILLIGER ALGEBRAS OF GROUP ASSOCIATION SCHEMES

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The Terwilliger algebra of an association scheme was introduced by Paul Terwilliger [7] in order to study P- and Q-polynomial association schemes. The purpose of this paper is to discuss in detail properties of the Terwilliger algebra of the group association scheme of a finite group. We shall give bounds on the dimension of the Terwilliger algebra, and define triple regularity.

Let G be a finite group, $C_0 = \{e\}$, $C_1, \dots, C_d$ the conjugacy classes of G . For $i=0, 1, \dots, d$ define

$$R_i = \{(x, y) | yx^{-1} \in C_i\}.$$

Then $\chi(G) = (G, \{R_i\}_{0 \leq i \leq d})$ becomes a commutative association scheme, and it is called the group association scheme of the finite group G . Define the adjacency matrix A_i of the relation R_i :

$$(A_i)_{x,y} = \begin{cases} 1 & (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

Then there exist nonnegative integers p_{ij}^k such that $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$. This is equivalent to the relation $\underline{C}_i \underline{C}_j = \sum_{k=0}^d p_{ij}^k \underline{C}_k$ where $\underline{C}_i = \sum_{x \in C_i} x \in \mathbf{C}G$ and the multiplication is performed as elements of the group algebra $\mathbf{C}G$. If we put $\mathcal{A} = \langle A_0, \dots, A_d \rangle_{\mathbf{C}}$, then $\mathcal{A}$ becomes a $(d+1)$ -dimensional subalgebra of the matrix algebra $M_{|G|}(\mathbf{C})$, and is called the Bose-Mesner algebra. It is isomorphic to the center of the group algebra. Let $E_0, \dots, E_d$ be the primitive idempotents of $\mathcal{A}$. Since $\mathcal{A}$ is closed under the Hadamard multiplication, we have $E_i \circ E_j \in \mathcal{A}$, so that there exist complex numbers q_{ij}^k such that

$$E_i \circ E_j = \frac{1}{|G|} \sum_{k=0}^d q_{ij}^k E_k$$

(it is known that q_{ij}^k are indeed real and nonnegative, see [2]).

We define the diagonal matrices E_i^*, A_i^* by

$$(E_i^*)_{xx} = \begin{cases} 1 & x \in C_i \\ 0 & \text{otherwise} \end{cases}$$

$$(A_i^*)_{xx} = |G|(E_i)_{e,x}$$

Then we have

$$A_i^* A_j^* = \sum_{k=0}^d q_{ij}^k A_k^*$$

and the algebra

$$\mathcal{A}^* = \langle A_0^*, \dots, A_d^* \rangle_{\mathbb{C}} = \langle E_0^*, \dots, E_d^* \rangle_{\mathbb{C}}$$

is called the dual Bose-Mesner algebra.

Now the Terwilliger algebra T is the subalgebra of $M_{|G|}(\mathbb{C})$ generated by $\mathcal{A}$ and $\mathcal{A}^*$. Since T is closed under the conjugate-transpose, T is semisimple, and one can easily see that T is non-commutative if $G \neq 1$.

1 Bounds on $\dim T$

In this section we give upper and lower bounds on the dimension of the Terwilliger algebra of a group association scheme. The following identities will be used in the proof of the next lemma. See [2] for a proof.

$$\begin{aligned} \bar{E}_i &= E_i^T = E_{\bar{i}} \text{ for some } \bar{i} \in \{0, 1, \dots, d\}, \\ \text{rank } E_j &= |G|(E_j)_{e,e}, \\ q_{ik}^j \text{rank } E_j &= q_{ij}^k \text{rank } E_k. \end{aligned}$$

Lemma 1 (i) $\text{tr}(E_i^* A_j E_k^* \overline{(E_l^* A_m E_n^*)^T}) = \delta_{il} \delta_{jm} \delta_{kn} p_{ij}^k |C_k|$.
(ii) $\text{tr}(E_i A_j^* E_k \overline{(E_l A_m^* E_n)^T}) = \delta_{il} \delta_{jm} \delta_{kn} q_{ij}^k \text{rank } E_k$

Proof. (i) This follows directly from the definition.

(ii) By the identities mentioned above,

$$\begin{aligned} \text{tr}(E_i A_j^* E_k \overline{(E_l A_m^* E_n)^T}) &= \text{tr}(E_l E_i A_j^* E_k E_n \overline{A_m^*}) \\ &= \delta_{il} \delta_{kn} \text{tr}(E_i A_j^* E_k \overline{A_m^*}) \end{aligned}$$

$$\begin{aligned}
&= \delta_{il} \delta_{kn} |G|^2 \sum_{x,y \in G} (E_i)_{x,y} (E_j)_{e,y} (E_k)_{y,x} (E_m)_{x,e} \\
&= \delta_{il} \delta_{kn} |G|^2 (E_j (E_i \circ E_k^T) E_m)_{e,e} \\
&= \delta_{il} \delta_{kn} \delta_{jm} |G| q_{ik}^j (E_j)_{e,e} \\
&= \delta_{il} \delta_{kn} \delta_{jm} q_{ij}^k \text{rank } E_k.
\end{aligned}$$

This completes the proof. $\square$

Consider the subspaces T_0 and T_0^* defined by

$$\begin{aligned}
T_0 &= \text{span}_{\mathbb{C}} \{E_i^* A_j E_k^* \mid 0 \leq i, j, k \leq d\}, \\
T_0^* &= \text{span}_{\mathbb{C}} \{E_i A_j^* E_k \mid 0 \leq i, j, k \leq d\}.
\end{aligned}$$

By Lemma 1 we have

$$\begin{aligned}
\dim T_0 &= |\{(i, j, k) \mid p_{ij}^k \neq 0\}|, \\
\dim T_0^* &= |\{(i, j, k) \mid q_{ij}^k \neq 0\}|.
\end{aligned}$$

In other words, $\dim T_0$ is the number of triples (i, j, k) such that $(C_i C_j) \cap C_k \neq \emptyset$. There is a one-to-one correspondence between the set of primitive idempotents $\{E_i\}_{0 \leq i \leq d}$ and the set of complex irreducible characters of G , say $E_i \leftrightarrow \chi_i$. As shown in [2], $q_{ij}^k = \frac{\chi_i(1)\chi_j(1)}{\chi_k(1)}(\chi_i \chi_j, \chi_k)$ holds. Thus, $\dim T_0^*$ is the number of triples (i, j, k) such that $(\chi_i \chi_j, \chi_k) \neq 0$.

Let $\tilde{T} = \text{End}_G(\mathbf{C}G)$ be the centralizer algebra of the permutation representation of G acting on G itself by conjugation. The dimension of $\tilde{T}$ is the number of orbits of G acting on $G \times G$ by simultaneous conjugation. As is well-known, this is equal to the average of fixed points, i.e.,

$$\dim \tilde{T} = \frac{1}{|G|} \sum_{a \in G} |C_G(a)|^2 = \sum_{i=0}^d \frac{|G|}{|C_i|}.$$

Theorem 2 *We have the following bounds on the dimension of the Terwilliger algebra T .*

- (i) $|\{(i, j, k) \mid p_{ij}^k \neq 0\}| \leq \dim T$.
- (ii) $|\{(i, j, k) \mid q_{ij}^k \neq 0\}| \leq \dim T$.
- (iii) $\dim T \leq \sum_{i=0}^d |G|/|C_i|$.

Proof. These are direct consequences of $T_0 \subset T$, $T_0^* \subset T$ and $T \subset \tilde{T}$. $\square$

If G is abelian, then Theorem 2 implies $\dim T_0 = |G|^2$, i.e., T coincides with the full matrix algebra $M_{|G|}(\mathbb{C})$.

2 Triple regularity

If the finite group G acts transitively on the set

$$S_{ijk} = \{(g, h) \in C_i \times C_j \mid gh \in C_k\}$$

for any $i, j, k \in \{0, 1, \dots, d\}$ with $S_{ijk} \neq \emptyset$, we say that G is triply transitive. Note that, since $G \times G = \bigcup_{i,j,k} S_{ijk}$ and $\dim T_0 = |\{(i, j, k) \mid S_{ijk} \neq \emptyset\}|$, G is triply transitive if and only if $\dim T_0 = \dim \tilde{T}$. In this case $T_0 = T = \tilde{T}$ holds. We call the finite group G triply regular if $T_0 = T$, and dually triply regular if $T_0^* = T$. Since T_0 or T_0^* generates T as an algebra, G is triply regular (resp. dually triply regular) precisely when the subspace T_0 (resp. T_0^*) is a subalgebra.

A combinatorial meaning of the triple regularity is as follows. Given i, j, k, l, m, n , the size of the set

$$\{z \in C_n \mid (y, z) \in R_l, (x, z) \in R_m\}$$

depends only on i, j, k, l, m, n , and is independent of the choice of $(x, y) \in S_{ijk}$. This property, when reformulated for association schemes, plays a central role in the theory of spin models (see [4], [5]).

If G is abelian, then $\dim T_0 = |G|^2$, so that G is triply transitive.

Examples. (i) Let $G = A_4$ be the alternating group on four letters. G is triply regular but not triply transitive.

(ii) All finite groups of order 16 are triply transitive.

As for the dual triple regularity, we give a sufficient condition in terms of character products.

Theorem 3 *Let $\chi_0, \dots, \chi_d$ be the complex irreducible characters of the finite group G . If $\chi_i \chi_j$ is multiplicity-free for any i, j , then $T_0^* = \tilde{T}$ holds, in particular, G is dually triply regular. Conversely, $T_0^* = \tilde{T}$ implies that $\chi_i \chi_j$ is multiplicity-free for any i, j .*

Proof. Write $\chi_i \chi_j = \sum_{k=0}^d N_{ij}^k \chi_k$. Then we have

$$\sum_{i,j,k} (N_{ij}^k)^2 = \sum_{i,j=0}^d (\chi_i \chi_j, \chi_i \chi_j)$$

$$\begin{aligned}
&= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{i=0}^d \chi_i(g) \overline{\chi_i(g)} \right) \left(\sum_{j=0}^d \chi_j(g) \overline{\chi_j(g)} \right) \\
&= \frac{1}{|G|} \sum_{g \in G} |C_G(g)|^2 \\
&= \sum_{i=0}^d \frac{|G|}{|C_i|}
\end{aligned}$$

Thus, if all $N_{ij}^k \in \{0, 1\}$, then

$$\begin{aligned}
\dim \tilde{T} &= \sum_{i=0}^d |G| / |C_i| \\
&= \sum_{i,j,k} N_{ij}^k \\
&= |\{(i, j, k) \mid N_{ij}^k \neq 0\}| \\
&= |\{(i, j, k) \mid q_{ij}^k \neq 0\}| \\
&= \dim T_0^*
\end{aligned}$$

as desired. The converse can also be seen from the above equalities. $\square$

Proposition 4 *Let G_1 and G_2 be finite groups.*

- (i) *If G_1 and G_2 are triply transitive, so is $G_1 \times G_2$.*
- (ii) *If G_1 and G_2 are triply regular, so is $G_1 \times G_2$.*
- (iii) *If G_1 and G_2 are dually triply regular, so is $G_1 \times G_2$.*

Proof. (i) This follows immediately from the definition.

(ii) If G_1 and G_2 are triply regular, then $T_0(G_1)$ and $T_0(G_2)$ are subalgebras. Since $T_0(G_1 \times G_2) = T_0(G_1) \otimes T_0(G_2)$, it follows that $G_1 \times G_2$ is triply regular.

(iii) Similar as (ii). $\square$

Theorem 5 *Let $D_{2n} = \langle \sigma, \tau \mid \sigma^n = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$ be the dihedral group of order $2n$. Then D_{2n} is triply transitive and dually triply regular.*

Proof. First suppose that n is odd, say $n = 2m + 1$. Then the conjugacy classes of D_{2n} are $C_0 = \{e\}$, $C_i = \{\sigma^i, \sigma^{-i}\}$ ($1 \leq i \leq m$) and $C_{m+1} = \tau\langle\sigma\rangle$. Thus $\dim \tilde{T} = 2m^2 + 5m + 4$. In order to compute $\dim T_0$, let us consider the product $C_i C_j$. If $1 \leq i \leq m$, then

$$C_i C_j = \begin{cases} C_i & \text{if } j=0 \\ C_0 \cup \{\sigma^{2i}, \sigma^{-2i}\} & \text{if } j=i \\ \{\sigma^{i+j}, \sigma^{-i-j}\} \cup \{\sigma^{i-j}, \sigma^{j-i}\} & \text{if } 1 \leq j \leq m \text{ and } j \neq i \\ C_{m+1} & \text{if } j=m+1. \end{cases}$$

Also

$$C_{m+1} C_j = \begin{cases} C_{m+1} & \text{if } 0 \leq j \leq m \\ C_0 \cup \dots \cup C_m & \text{if } j=m+1. \end{cases}$$

We can easily see that the number of triples (i, j, k) with $(C_i C_j) \cap C_k \neq \emptyset$ is $(m+2) + m(2m+2) + (2m+2) = 2m^2 + 5m + 4 = \dim \tilde{T}$, so that D_{2n} is triply transitive.

To show that D_{2n} is dually triply regular, it suffices to prove that $\chi_i \chi_j$ is multiplicity-free, where $\{\chi_0, \dots, \chi_{m+1}\}$ is the set of complex irreducible characters of D_{2n} . We may assume $\chi_i(1) = \chi_j(1) = 2$, otherwise one of χ_i or χ_j has degree 1, so that $\chi_i \chi_j$ is irreducible. But all irreducible characters of degree 2 are obtained by inducing a linear character of the subgroup $\langle \sigma \rangle$ to D_{2n} . It is now straightforward to check that $\chi_i \chi_j$ is multiplicity-free. Hence D_{2n} is dually triply regular by Theorem 3.

Next suppose that n is even, say $n = 2m$. Then the conjugacy classes of D_{2n} are $C_0 = \{e\}$, $C_i = \{\sigma^i, \sigma^{-i}\}$ ($1 \leq i \leq m-1$), $C_m = \{\sigma^m\}$, $C_{m+1} = \tau \langle \sigma^2 \rangle$, and $C_{m+2} = \tau \sigma \langle \sigma^2 \rangle$. Thus $\dim \tilde{T} = 2m^2 + 6m + 8$. A tedious calculation similar to the case $n = 2m + 1$ establishes $\dim T_0 = \dim T_0^* = 2m^2 + 6m + 8$, hence D_{2n} is triply transitive and dually triply regular. $\square$

We have computed $\dim T_0$, $\dim T_0^*$, and $\dim \tilde{T}$ for all nonabelian indecomposable finite groups of order at most 100 using **GAP** [6]. The results are tabulated in the Appendix. We have not found any group for which $\dim T_0 = \dim \tilde{T} > \dim T_0^*$ holds.

To conclude this section, we give a list of indecomposable finite groups of order at most 24 which are not triply transitive. Balmaceda and Oura [1] have determined the Terwilliger algebra for $G = S_5$ and A_5 .

	$\dim T_0$	$\dim T_0^*$	$\dim T$	$\dim \tilde{T}$
A_4	19	19	19	22
5.4	29	29	29	37
7.3	35	35	37	41
$SL(2, 3)$	75	73	75	76
S_4	42	43	43	43

3 Relationship with the quantum double

Let $\tilde{\mathcal{A}}$ be the complex vector space with basis $G \times G \times G$, and define the multiplication in $\tilde{\mathcal{A}}$ by

$$(x, g, a)(y, h, b) = \delta_{x^{-1}ga, h}(xy, g, ab)$$

and extend it linearly to $\tilde{\mathcal{A}}$. Then $\tilde{\mathcal{A}}$ becomes an associative algebra. The subalgebra of $\tilde{\mathcal{A}}$ defined by

$$\mathcal{D} = \langle (h, g, h) \mid g, h \in G \rangle_c \subset \tilde{\mathcal{A}}$$

is known as the quantum double of the finite group G (precisely speaking, the quantum double is defined for a Hopf algebra, and $\mathcal{D}$ is the quantum double of the group Hopf algebra of G , see [3]). On the other hand, let

$$\mathcal{T} = \langle (1, g, h) \mid g, h \in G \rangle_c \subset \tilde{\mathcal{A}},$$

and denote by $\mathcal{T}^G$ the G -fixed subspace of $\mathcal{T}$. Then $\mathcal{T}^G$ is isomorphic to the centralizer algebra $\tilde{T}$ defined in Section 2. Therefore, $\tilde{\mathcal{A}}$ is an algebra containing both the quantum double $\mathcal{D}$ and the Terwilliger algebra T .

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Appendix

Group	ord.	T_0	T_0^*	$\tilde{T}$	Group	ord.	T_0	T_0^*	$\tilde{T}$
S_3	6	11	11	11	(2)	48	368	368	368
D_8, Q_8	8	28	28	28	(4)	48	233	227	236
D_{10}	10	22	22	22	$GL(2, 3)$	48	130	118	136
6.2	12	44	44	44	(1)	48	130	118	136
A_4	12	19	19	22	(2)	48	119	119	134
D_{14}	14	37	37	37	D_{50}	50	352	352	352
(4)	16	112	112	112	$5^2:2$	50	352	352	352
(3)	16	64	64	64	26.2	52	424	424	424
$D_{18}, 3^2:2$	18	56	56	56	13:4	52	76	76	103
10.2	20	88	88	88	$(3^2:3):2$	54	208	192	216
5:4	20	29	29	37	$3^2:6, 9:6$	54	127	127	162
7:3	21	35	35	41	(3)	54	407	407	407
D_{22}	22	79	79	79	11:5	55	63	63	97
12.2	24	176	176	176	28.2	56	592	592	592
(3)	24	116	116	116	(3)	56	484	484	484
$SL(2, 3)$	24	75	73	76	$2^3:7$	56	71	71	106
S_4	24	42	43	43	19:3	57	159	159	177
D_{26}	26	106	106	106	D_{58}	58	466	466	466
$3^2:3, 9:3$	27	137	137	153	30.2	60	548	548	548
14.2	28	148	148	148	(1)	60	132	135	161
D_{30}	30	137	137	137	A_5	60	71	65	77
(7)	32	448	448	448	D_{62}	62	529	529	529
(16)	32	256	256	256	21.3	63	315	315	369
(2)	32	304	304	304	(10)	64	1792	1792	1792
(5)	32	142	142	160	(38)	64	1024	1024	1024
(3)	32	184	184	184	(5)	64	1216	1216	1216
D_{34}	34	172	172	172	(33)	64	568	568	640
$(2^2 \times 3).3$	36	171	171	198	(30)	64	736	736	736
(2)	36	224	224	224	(12)	64	784	784	784
$3^2:4$	36	48	48	66	(5)	64	406	406	496
D_{38}	38	211	211	211	(31)	64	538	526	544
13:3	39	85	85	97	(9)	64	340	340	448
20.2	40	352	352	352	(3)	64	583	592	592
10.4	40	116	116	148	(9)	64	361	364	400
(3)	40	268	268	268	(6)	64	223	226	280
7:6	42	55	55	79	(6)	64	367	376	376
D_{42}	42	254	254	254	(3)	64	616	616	616
22.2	44	316	316	316	D_{66}	66	596	596	596
D_{46}	46	301	301	301	34.2	68	688	688	688
24.2	48	704	704	704	17:4	68	112	112	148
(7)	48	464	464	464	D_{70}	70	667	667	667
(1)	48	300	292	304	(2)	72	896	896	896
(1)	48	168	172	172	(1)	72	675	657	684
(2)	48	308	308	308	(1)	72	484	484	484
D_{48}	48	368	368	368	(1)	72	192	192	264

Appendix

Group	ord.	T_0	T_0^*	$\tilde{T}$
(6)	72	764	764	764
(3)	72	394	382	400
(1)	72	89	89	137
(1)	72	138	156	156
(1)	72	44	44	101
(2)	72	160	149	179
D_{74}	74	742	742	742
$5^2:3$	75	257	257	281
38.2	76	844	844	844
$13:6$	78	80	80	134
D_{78}	78	821	821	821
40.2	80	1408	1408	1408
(2)	80	464	464	592
(7)	80	1072	1072	1072
(2)	80	616	616	616
(4)	80	278	284	352
(4)	80	514	502	520
(3)	80	928	928	928
$2^4:5$	80	94	94	148
(4)	81	1233	1233	1377
(4)	81	443	443	513
D_{82}	82	904	904	904
$(14.2):3$	84	220	220	316
42.2	84	1016	1016	1016
(1)	84	315	315	342
D_{86}	86	991	991	991
44.2	88	1264	1264	1264
(3)	88	1108	1108	1108
(2)	90	1082	1082	1082
46.2	92	1204	1204	1204
$31:3$	93	379	379	409
D_{94}	94	1177	1177	1177
(1)	96	2816	2816	2816
(2)	96	304	269	344
(7)	96	1232	1232	1232
(12)	96	1856	1856	1856
(1)	96	1200	1168	1216
(3)	96	704	704	704
(16)	96	1472	1472	1472
(34)	96	932	908	944
(3)	96	141	144	216
(4)	96	974	992	992
(2)	96	486	520	568
(1)	96	259	281	296
(12)	96	692	698	752

Group	ord.	T_0	T_0^*	$\tilde{T}$
(4)	96	497	506	560
(3)	96	1304	1304	1304
(4)	96	599	581	608
(3)	96	520	472	544
(3)	96	340	352	352
(2)	96	234	231	243
(1)	96	672	688	688
(3)	96	366	346	400
$D_{98}, 7^2:2$	98	1276	1276	1276
(2)	100	1408	1408	1408
(1)	100	214	214	262
(1)	100	210	210	262
(1)	100	323	329	373
$25:4$	100	209	209	262

A number in parentheses (n) in the column "Group" indicates that there are n groups up to isomorphism which have the order, $\dim T_0$, $\dim T_0^*$, $\dim \tilde{T}$ specified in that row.