NOTE ON EXISTENCE OF THE UNSTABLE ADAMS MAP†

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0. Introduction

We denote by $M^n$ a Moore space of type $(Z_n, n-1)$. The purpose of the present note is to give a short proof of the existence of the unstable Adams map $A : M^{13} \rightarrow M^5$ [O].

The key point of our method is simply to use Toda brackets in the relative homotopy group $\pi_{12}(M^5, S^4)$.

We denote by $i : S^{n-1} \rightarrow M^n$ an inclusion and by $p : M^n \rightarrow S^n$ a collapsing map. Let $\sigma''$ be the generator of the 2-component of $\pi_{12}(S^5)$ [T2]. We shall prove the following theorem.

THEOREM. ([O].) There exists a mapping $A : M^{13} \rightarrow M^5$ with a homotopy commutative square

$$
\begin{array}{ccc}
M^{13} & \xrightarrow{A} & M^5 \\
\uparrow{i} & & \downarrow{p} \\
S^{12} & \xrightarrow{\sigma''} & S^5.
\end{array}
$$

1. A summary of fundamental results

We denote by $\iota_X \in [X, X]$ the identity class of $X$. We set $\iota_X = \iota_n$ for $X = S^n$ and $\iota_X = \iota'_n$ for $X = M^n$. Let $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$ be a Hopf map and $\eta_n^2 = \eta_n \eta_{n+1} \in \pi_{n+2}(S^n)$.

First we recall the following ([M2], [T3]).

LEMMA 1.1. (i) $\pi_3(M^3) = \mathbb{Z}_4[i\eta_2]$ and $\pi_n(M^n) = \mathbb{Z}_2[i\eta_{n-1}]$ for $n \geq 4$.

(ii) $[M^n, M^n] = \mathbb{Z}_4[\iota'_n]$ for $n \geq 3$, where $2\iota'_n = i\eta_{n-1}p$.

† Dedicated to Professor Seiya Sasa on his 60th birthday.
Let $CX$ be a cone of a space $X$. For $\alpha \in \pi_{n-1}(X)$, we denote by $\hat{\alpha} \in \pi_n(CX, X)$ an element satisfying $\partial \hat{\alpha} = \alpha$, where $\partial : \pi_n(CX, X) \to \pi_{n-1}(X)$ is the connecting isomorphism.

Let $\beta \in \pi_p(X)$, $\alpha \in \pi_{k-1}(S^{p-1})$ and $j : (X, *) \to (X, A)$ be an inclusion. Then, from the definition, we have

**Lemma 1.2.** $j_* (\beta \circ \Sigma \alpha) = j_* \beta \circ \hat{\alpha}.$

Let $A$ be an $m$-connected CW-complex for $m \geq 1$ and let $X = A \cup e^n$ be the complex formed by attaching an $n$-cell. Let $p : X \to S^n$ be a map induced by shrinking $A$ to the base point. Let $\omega_n \in \pi_n(X, A)$ be the characteristic map of the $n$-cell of $X$. Then, by Theorem 2.1 of [J], we have the following lemma.

**Lemma 1.3.** (i) Suppose that $k \leq \min(3n - 4, 2m + n) - 1$. Let $\tilde{\alpha} \in \pi_k(X)$ be an element satisfying $p_* \tilde{\alpha} = \Sigma \alpha$. Then there exists an element $\beta \in \pi_{k-n+1}(A)$ satisfying $j_* \tilde{\alpha} \equiv \omega_n \beta \mod [\omega_n, \beta]$.

Let $A_1$ be a finite CW-complex. Let $X_1 = A_1 \cup \bigcup e^n$ be a mapping cone of $\theta \in \pi_{n-2}(A_1)$ and $i_1 : A_1 \to X_1$ be an inclusion. We set $A = \Sigma A_1$, $\tau = \Sigma \theta$, $X = \Sigma X_1 = A \cup \bigcup e^n$ and $i = \Sigma i_1$. Let $j' : (CX_1, X_1) \to (X, A)$ be a map corresponding to $j : (X, X) \to (X, A)$. Then we have the following lemma.

**Lemma 1.4.** Suppose that $\tau \alpha = 0$ for $\alpha \in \pi_{k-1}(S^{n-1})$. Let $\tilde{\alpha} \in \{i, \tau, \alpha\}$ be a co-extension of $\alpha$. Then we have $j_* \tilde{\alpha} = \omega_n \tilde{\tau}$ and $j_* [i, \tau, \alpha] = -[j', \tilde{i}_1, \tilde{\theta}] \circ \hat{\alpha}$.

**Proof.** The assertion is a direct consequence of Proposition 1.4 of [T2]. To make sure of it, we provide a proof.

Let $M_\tau$ be a mapping cylinder of $\tau : S^{n-1} \to A$. We consider a homotopy commutative diagram

\[
\begin{array}{ccc}
(CS^{k-1}, S^{k-1}) & \xrightarrow{\varphi} & (M_\tau, S^{n-1}) \\
\downarrow \cong & & \downarrow \varphi' \\
(CS^{k-1}, S^{k-1}) & \xrightarrow{\hat{\alpha}} & (CS^{n-1}, S^{n-1}) \xrightarrow{\omega_n} (X, A),
\end{array}
\]

where $\varphi$ is induced from the relation $\alpha \simeq 0$ in $M_\tau$, $\varphi'$ and $\omega_n$ are the canonical maps.

We have $p'_* \varphi = \hat{\alpha}$ and $j_* \tilde{\alpha} = j_* p'_* \varphi = \omega_n \hat{\alpha}$. $-\omega_n$ is a representative of $\{j', \tilde{i}_1, \hat{\theta}\}$, because $p''_* \{j', \tilde{i}_1, \hat{\theta}\} \subset \{p''_* j', i, \tau\} \ni -\iota_n$ for a collapsing map $p'' : (X, A) \to (S^n, *)$. This completes the proof. \qed
A smash product $M^2 \land M^2$ is regarded as a mapping cone $M^3 \cup_{2i_3} CM^3$. We have $\pi_2(M^2 \land M^2) \cong \pi_2(M^3) = Z_2[i]$. Let $i' : M^3 \hookrightarrow M^2 \land M^2$ be an inclusion and let $i' \in \{i', 2i_3, i\} \subset \pi_3(M^2 \land M^2)$ be a coextension of $i$. Then we have the following.

**Lemma 1.5.** $\pi_3(M^2 \land M^2) = Z_8[i]$ and $\pi_{r+3}(\Sigma^n(M^2 \land M^2)) = Z_4(\Sigma^n i')$ for $n \geq 1$, where $2i = \pm i'' \eta_2$ with $i'' = i' i$.

**Proof.** Since $\{p, i, 2i_3\} \ni -i_3$, we have $2i \in -i'\{2i_3, i, 2i_3\} \ni -i'i_2 \{p, i, 2i_3\} \ni i'' \eta_2 \mod 2i_4 \pi_5(M^3)$. So, by Lemma 1.1, we have $2i = \pm i'' \eta_2$. We have $M^2 \land M^2 = (M^3 \lor S^3) \cup_\alpha e^4$, where $\alpha = i \eta_2 \lor 2i_3$. Hence, by Lemma 1.1, we have $\pi_3(M^2 \land M^2) \cong Z_8$. This leads us to the first half.

For $n \geq 1$, we set $\beta = \Sigma^n \alpha$. Then we have $\pi_{n+3}(\Sigma^n(M^2 \land M^2)) = \pi_{n+3}((\Sigma^{n+3} \lor S^{n+3}) \cup_\beta e^{n+4}) = Z_4(\Sigma^n i')$, where $2\Sigma^n i' = i'' \eta_{n+2}$. This completes the proof. \[\square\]

We will now prepare a useful formula about relative Whitehead products.

**Lemma 1.6.** Let $\alpha \in \pi_p(X, A)$, $\beta \in \pi_q(A)$, $\gamma \in \pi_{i-1}(S^{p-1})$ and $\delta \in \pi_{j-1}(S^{q-1})$. Then we have

$$[\alpha \gamma, \beta \circ \Sigma \delta] = [\alpha, \beta] \gamma \text{ for } \tau = \gamma \land \delta.$$

**Proof.** We use the same notation as [BM]. Let $f : (E^p, E_+^{p-1}) \rightarrow (X, A, *)$, $g : (E^q, E_q) \rightarrow (A, *)$, $c : S^{i-1} \rightarrow S^{p-1}$ and $d : S^{j-1} \rightarrow S^{q-1}$ be representatives of $\alpha$, $\beta$, $\gamma$ and $\delta$, respectively. We set $B' = E^p \times S^{q-1} \cup E_+^{p-1} \times E^q$ and $C' = E' \times S^{j-1} \cup E_+^{i-1} \times E'$, where $r = p + q - 1$ and $h = i + j - 1$. From the definition, $[\alpha, \beta]$ is represented by a mapping $\Phi : (B', B') \rightarrow (X, A)$ defined as

$$\Phi(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in E^p \times S^{q-1}, \\ g(y) & \text{if } (x, y) \in E_+^{p-1} \times E^q. \end{cases}$$

On the other hand, $[\alpha \circ \gamma, \beta \circ \Sigma \delta]$ is represented by a composition $\Phi \circ \theta$, where $\theta : (C', \hat{C}') \rightarrow (B', B')$ is defined as a restriction of a join $\hat{c} \star \hat{d}$: $E' \times S^{j-1} \cup S^{j-1} \times E' \cong E' \star E' \rightarrow E^p \star E^q \cong E^p \times S^{q-1} \cup S^{p-1} \times E^q$. This completes the proof. \[\square\]

**Remark.** By [A] and by Proposition 3.2 of [B], we have the following: let $\alpha \in [\Sigma K, X]$, $\beta \in [\Sigma L, X]$, $\gamma \in [P, K]$ and $\delta \in [Q, L]$, where $K$, $L$, $P$ and $Q$ are polyhedra. Then we have

$$[\alpha \circ \Sigma \gamma, \beta \circ \Sigma \delta] = [\alpha, \beta] \circ \Sigma (\gamma \land \delta).$$
2. Some relative homotopy groups

First we recall the following fact ([M3]) which partially corrects the statement of Theorem 2 of [M3].

**Proposition 2.1.** Let \( \rho_n \in \pi_{4n-2}(M^{2n}) \) for \( n \geq 2 \) be an attaching map in the Stiefel manifold \( V_{2n+1,2} = M^{2n} \cup_{\rho_n} e^{4n-1} \) of 2-frames. Then we have

(i) \( \pi_{4n-2}(M^{2n}) \cong \mathbb{Z}_{4n(a)} \{ \rho_n \} \oplus \pi_{4n-2}(V_{2n+1,2}) \), where \( a(n) = 1 \) for even \( n \) and \( a(n) = 2 \) for odd \( n \);

(ii) \( \Sigma \rho_n = i(t_{2n}, \rho_n) \), \( \Delta((\Sigma^{n-3})^2) = \pm 2\rho_n \), \( j_{*}\rho_n = \pm [\omega_{2n}, t_{2n-1}] \) and \( H(\rho_n) = \pm \Sigma^{4n-5} \).

Let \( \tilde{\eta}_3 \subseteq \{ i, 2t_3, \eta_3 \} \subseteq \pi_3(M^4) \) be a coextension of \( \eta_3 \). We set \( \tilde{\eta}_n = \Sigma^{n-3} \tilde{\eta}_3 \) for \( n \geq 3 \). We have \( \pi_5(M^4) = \mathbb{Z}_{4}(\tilde{\eta}_3) \), where \( 2\tilde{\eta}_3 = i\tilde{\eta}_3^2 \). We recall [T2] that \( \pi_6(S^3) = \mathbb{Z}[v] \oplus \mathbb{Z}_2[\alpha] \), \( \pi_7(S^4) = \mathbb{Z}[v] \oplus \mathbb{Z}_4[v'] \oplus \mathbb{Z}_3[\alpha] \) with \( [t_4, t_4] = \pm 2v_4 - v' \) and \( \pi_{n+3}(M^3) = \mathbb{Z}[\nu_n] \oplus \mathbb{Z}_3[\alpha] \) for \( n \geq 5 \).

We denote by \( \Sigma' : \pi_n(X, A) \rightarrow \pi_{n+1}(X, A) \) a relative suspension homomorphism [T1]. Hereafter we shall simply use the notation \( \hat{\alpha} \in \pi_n(X, A) \) for \( X = A \cup e^6 \) to represent \( \omega_{6} \hat{\alpha} \), unless otherwise stated. We show the following lemma which partially corrects the results of [M2].

**Lemma 2.2.** (i) \( \pi_7(M^4, S^3) = \mathbb{Z}[v'] \oplus \mathbb{Z}_2[\omega_3, \eta_3] \oplus \mathbb{Z}_3[\tilde{\alpha}] \).

(ii) \( \pi_8(M^5, S^4) = \mathbb{Z}[\omega_5, t_4] \oplus \mathbb{Z}_8[v_4] \oplus \mathbb{Z}_3[\tilde{\alpha}] \), where \( \partial v_4 = 2(\omega_5, t_4) \).

(iii) \( \pi_9(M^5, S^4) = \mathbb{Z}_2[\omega_5, \eta_4] \oplus \mathbb{Z}_2[v_4 \eta_4] \), where \( [t_4, \eta_4] = [\omega_5, t_4] \).

(iv) \( \pi_8(M^5) = \mathbb{Z}_2[\tilde{\eta}_4^2] \oplus \mathbb{Z}_2[v_4 \eta_4] \) and \( \Sigma : \pi_8(M^5) \rightarrow \pi_9(M^6) \) is an isomorphism, where \( J(\tilde{\eta}_4 \eta_4^2) = 4(\tilde{\eta}_4 \pm \omega_5, t_4) \).

(v) \( \pi_6(M^4) = \mathbb{Z}_2[\tilde{\eta}_3] \oplus \mathbb{Z}_2[\tilde{\eta}_3] \) and \( \pi_7(M^4) = \mathbb{Z}_2[\tilde{\eta}_3] \oplus \mathbb{Z}_2[\tilde{\eta}_3] \), where \( \delta = \rho_2 \).

(vi) \( \pi_9(M^5) = \mathbb{Z}_2[\omega_5, \eta_4] \oplus \mathbb{Z}_2[\omega_5, \eta_4] \), where \( J(\omega_5, \eta_4) = [\omega_5, \eta_4] \).

**Proof:** We always use Theorem 2.1 of [J] and homotopy exact sequences of a pair \( (M^n, S^{n-1}) \).

(i) is immediately obtained. In \( \pi_6(M^4, S^4) \) there exist elements \( \hat{v}_4 \) and \( [\omega_5, t_4] \). We have \( \partial \hat{v}_4 = 2(\omega_5, t_4) = 4v_4 - v' \) since \( H(\omega_5, t_4) = 4(\omega_5, t_4) \) and \( 2v_4 = \Sigma v' \). We also have \( \partial \omega_5, t_4 = -2(\omega_5, t_4) = \pm 2(\Sigma v' - 2v_4) \). So we have \( \partial(\hat{v}_4 \pm [\omega_5, t_4]) = v' \). Since \( p(2\hat{v}_4 - v') = 0 \) for a collapsing map \( p' : (M^5, S^4) \rightarrow (S^5, *) \), we set \( 2\hat{v}_4 - v' = a[\omega_5, t_4] \) for an integer \( a \). By applying the boundary homomorphism to this relation, we have \( 8v_4 = \pm 2a(\Sigma v' - 2v_4) \). So we have \( a = \pm 2 \). This leads us to (ii).
The groups of (iii) and (iv) are obviously obtained. By Lemma 1.5, we have the relation of (iii). By using the relation $2\nu' = \eta_3^5$ and by (i) or by Lemma 1.3, we have $j_*(\eta_3^2 \eta_3^2) \equiv 2\nu' \ mod \ [\omega_4, \eta_3]$. Since $\Sigma'[\omega_4, \eta_3] = 0$ [T1], we have the relation of (iv) by the use of the last relation of (ii).

The first half of (v) is obtained by Proposition 2.1. By Lemmas 1.2 and 1.6, we have $j_*(\delta \eta_6) = j_* \delta \circ \hat{\gamma}_5 = [\omega_4, \iota_5] \circ \hat{\gamma}_5 = [\omega_4, \eta_3]$. So we have the second half of (v).

We have $\partial(\hat{\nu}_4 \hat{\eta}_7) = 2\iota_4 \circ \nu_4 \circ \eta_7 = \Sigma \nu \eta_7$. By (3.3) and (3.10) of [BM] and by Lemmas 1.4 and 1.6, we have $j_*[\hat{\eta}_4, i] = [\hat{\eta}_4, \iota_4] = [\omega_5, \iota_4] \circ \hat{\gamma}_7 = [\omega_5, \eta_4]$. So we have (vi). This completes the proof. \qed

3. Proof of the theorem

Roughly speaking, Oka found an unstable Adams map $A : M^{13} \rightarrow M^5$ to be a coextension of an extension of $\sigma'''$. We assert that $A$ is also obtained from an extension of a coextension of $\sigma'''$. Our idea is only to use a Toda bracket in $\pi_{12}(M^5, S^4)$.

By Lemma 2.2, we can define a Toda bracket $\{\varphi, 8 \iota_7, \hat{\nu}_7\} \subset \pi_{12}(M^5, S^4)$, where $\varphi = \hat{\nu}_4 \pm [\omega_5, \iota_4]$. Let $p' : (M^5, S^4) \rightarrow (S^5, \ast)$ be a collapsing map. We can now show the following lemma.

**Lemma 3.1.** (i) $p'_*\{\varphi, 8 \iota_7, \hat{\nu}_7\} \ni \sigma''' \mod 0$.
(ii) $2\{\varphi, 8 \iota_7, \hat{\nu}_7\} = 0$.

**Proof.** We have

\[
p'_*\{\varphi, 8 \iota_7, \hat{\nu}_7\} \subset \{\nu_5, 8 \iota_8 \circ p'', \hat{\nu}_7\}
\supseteq \{\nu_5, 8 \iota_8, \nu_3\}
\ni \sigma''' \mod \nu_5 \pi_{12}(S^8) + \pi_9(S^5, \ast) \circ \Sigma \hat{\nu}_7
\equiv \sigma''' \mod 0.
\]

We note that the indeterminacy consists of $\{\nu_5 \eta_8 \nu_9\} = 0$ by Lemma 2.2 of [T2]. So we have (i).

By (ii) of Lemma 2.2, we have

\[
2\{\varphi, 8 \iota_7, \hat{\nu}_7\} = \{\varphi, 8 \iota_7, \hat{\nu}_7\} \circ 2\iota_{11}
\subset \{\varphi, 16 \iota_7, \hat{\nu}_7\}
\]
Since $\pi_{12}(CS^4, S^4) \cong \pi_{11}(S^4) \cong \mathbb{Z}_{15}$, $\nu^{\prime} v_6 = 0$ and $\hat{\eta}_8 \nu_7 = 0$, we have $2\{\varphi, 8\hat{\iota}_7, \nu_7\} \equiv 0$ mod $\varphi \pi_{12}(CS^7, S^7) + \pi_9(M^5, S^4) \circ \nu_8 = 0$ by (iii) of Lemma 2.2. This completes the proof.

Next we show

**Lemma 3.2.** There exists an element $\theta \in \pi_{12}(M^5)$ such that $j_* \theta \in \{\varphi, 8\hat{\iota}_7, \nu_7\}$ and $\theta$ is of order 2.

**Proof.** In a homotopy exact sequence

$$
\pi_{12}(S^4) \xrightarrow{i_*} \pi_{12}(M^5) \xrightarrow{j_*} \pi_{12}(M^5, S^4) \xrightarrow{\partial} \pi_{11}(S^4),
$$

we have $\partial\{\varphi, 8\hat{\iota}_7, \nu_7\} = 0$ by (ii) of Lemma 3.1. So we have the first assertion.

Suppose that $\theta$ is of order 4. Since $\pi_{12}(S^4) = \mathbb{Z}_2\{e_4\}$ [T2], we have $2\theta = i_* e_4$.

By stabilizing this relation, we have $2\Sigma^\infty \theta = \Sigma^\infty (i_* e_4)$. This contradicts the following fact: $\pi_8^S(M^2) = \mathbb{Z}_2\{\Sigma^\infty \theta\} \oplus \mathbb{Z}_2\{i \eta \sigma\} \oplus \mathbb{Z}_2\{i \varepsilon\}$ [M1]. This completes the proof.

**Proof of the theorem.** In a commutative diagram:

$$
\begin{array}{ccc}
\pi_{12}(M^5) & \xrightarrow{j_*} & \pi_{12}(M^5, S^4) \\
\rho_* \downarrow & & \downarrow \rho_* \\
\pi_{12}(S^5) & & \\
\end{array}
$$

we have $\rho_* \theta = \sigma^m$ by Lemma 3.1. By Lemma 3.2, $\theta$ has an extension over $M^{13}$. This completes the proof of the theorem.

**4. Appendix**

In this appendix, we shall determine $\pi_4(M^3)$ and give a comment about the results of [O].

First we show the following fact which overlaps with Proposition 2.5 of [CN].

**Lemma 4.1.** $\pi_4(M^3) = \mathbb{Z}_4\{\alpha\}$, where $\Sigma \alpha = \tilde{\eta}_3$, $j_* \alpha = \tilde{\eta}_2 + [\omega_3, \nu_2]$, $2\alpha = i \eta_2$, $p_* \alpha = \eta_3$ and $H(\alpha) = \pm \Sigma^7$


Proof. We consider an EHP-sequence
\[ \pi_4(M^3) \xrightarrow{\Sigma} \pi_5(M^4) \xrightarrow{H} \pi_5(\Sigma(M^3 \land M^3)) \xrightarrow{\Delta} \pi_5(M^3). \]
\[ \pi_5(\Sigma(M^3 \land M^3)) \cong \pi_5(M^6) = \mathbb{Z}_2[i] \] and \( \Delta(i) = i \Delta(i_5) = 2i \eta_2 \neq 0 \) by Lemma 1.1. So \( H \) is trivial and \( \Sigma \) is an epimorphism. Therefore, there exists an element \( \alpha \in \pi_4(M^3) \) satisfying \( \Sigma \alpha = \hat{\eta}_3 \) and \( p_5 \alpha = \eta_3 \). By Lemma 1.3, we have \( j_4 \alpha = \hat{\eta}_2 \mod \{ \omega_3, \iota_2 \} \). In a homotopy exact sequence
\[ \pi_4(S^2) \xrightarrow{j} \pi_4(M^3) \xrightarrow{h} \pi_4(M^3, S^2) \xrightarrow{\partial} \pi_5(S^2), \]
we have \( \pi_4(M^3, S^2) = \mathbb{Z}_2[\omega_3, \iota_2] \oplus \mathbb{Z}_2[\eta_2 + [\omega_3, \iota_2]] \), because \( \partial[\omega_3, \iota_2] = -2[\iota_2, \iota_2] = -4 \eta_2 \) ([BM]) and \( \partial \eta_2 = 2 \iota_2 \circ \eta_2 = 4 \eta_2 \). So we have \( j_4 \alpha = \hat{\eta}_2 + [\omega_3, \iota_2] \), \( 2 \alpha = i \eta_2^2 \) and \( \pi_4(M^3) \cong \mathbb{Z}_4 \).

Since \( 2H(\alpha) = H(2 \alpha) = H(i \eta_2^2) = i \circ H(\eta_2) \circ \eta_3 = i \eta_3 = 2 \Sigma \hat{\eta} \), we have \( H(\alpha) = \pm \Sigma \hat{\eta} \). This completes the proof. \( \Box \)

Let \( F \) be a homotopy fiber of \( p : M^5 \rightarrow S^5 \). We consider a fiber sequence
\[ \cdots \rightarrow \Omega S^5 \xrightarrow{d} F \xrightarrow{i} M^5 \xrightarrow{p} S^5. \]
Let \( Y = S^4 \cup_\alpha e^0 \) be the 11-skeleton of \( F \). Here \( \alpha = 2[\iota_4, \iota_4] + a \eta_4^2 \) for \( a = 0 \) or 1 [O]. We show

**Lemma 4.2.** (i) \( d_4 \nu_4 = \pm i' \Sigma \nu' \), where \( i' : S^4 \hookrightarrow Y \) is an inclusion.
(ii) \( \pi_{10}(M^5, S^4) = \mathbb{Z}_2[\hat{\nu}_4 \eta_4^2] \oplus \mathbb{Z}_2[[\omega_5, \eta_4^2]] \), where \( \partial(\hat{\nu}_4 \eta_4^2) = \Sigma \nu' \eta_4^2 \).
(iii) \( \alpha = 0, \alpha = 2[\iota_4, \iota_4] \) and \( \pi_9(F) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \).

**Proof.** We consider an exact sequence
\[ \cdots \rightarrow \pi_7(\Omega S^5) \xrightarrow{d_4} \pi_7(Y) \xrightarrow{j_4} \pi_7(M^5) \rightarrow \cdots. \]
We know \( \pi_7(M^5) = \mathbb{Z}_4[i \nu_4] \oplus \mathbb{Z}_2[\hat{\nu}_4 \eta_6] \) and \( ii' \Sigma \nu' = 0 \) [M2]. So we have (i).

By [J], we obtain \( \pi_{10}(M^5, S^4) \). We have \( \partial(\hat{\nu}_4 \eta_4^2) = 2 \iota_4 \circ \nu_4 \circ \eta_4^2 = \Sigma \nu' \eta_4^2 \) by Lemma 2.2. This leads us to (ii).

We consider a homotopy exact sequence of a pair \((Y, S^4)\):
\[ \pi_{10}(Y, S^4) \xrightarrow{\partial} \pi_9(S^4) \xrightarrow{i_5} \pi_9(Y) \xrightarrow{j_4} \pi_9(Y, S^4) \rightarrow \cdots. \]
We have \( \partial \hat{\eta}_4 = \alpha \eta_4 = 0 \) and \( \partial \hat{\eta}_4^2 = \alpha \eta_4^2 = 0 \). Let \( \hat{\eta}_4 \in \{ i', \alpha, \eta_4 \} \subset \pi_9(Y) \) be a coextension of \( \eta_4 \). We have \( \{ 2[\iota_4, \iota_4], \eta_4, 2 \iota_4 \} \supset \{ \iota_4, \iota_4 \} \{ 2 \iota_4, \eta_4, 2 \iota_4 \} \supset \cdots \).
$\Sigma \nu' \eta_2^2 \mod 2\pi_9(S^4) = 0$ and $\{\eta_4^3, \eta_7, 2\iota_8\} \equiv 2\Sigma \nu', \eta_7, 2\iota_8 \equiv 0$. So we have $2\tilde{\eta}_7 = -i'(\alpha, \eta_7, 2\iota_8) \subset -i'(2\iota_4, \eta_7, 2\iota_8) + \{an_4^3, \eta_7, 2\iota_8\} \equiv (1 + a)i'\Sigma \nu' \eta_2^2 \mod 0$.

In an exact sequence

$$\pi_9(\Omega M^5) \xrightarrow{\Omega p_*} \pi_9(\Omega S^5) \xrightarrow{d_*} \pi_3(Y) \xrightarrow{i_*} \pi_9(M^5) \xrightarrow{p_*} \pi_9(S^5),$$

we have $\Im \Omega p_* = 0$ by (ii). So, by (i), we have $d_*(\nu_5 \eta_2^2) = i'\Sigma \nu' \eta_2^2 \neq 0$. Therefore, by Lemma 2.2, we have $\pi_9(Y) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and $2\tilde{\eta}_7 = i'\Sigma \nu' \eta_2^2$. This completes the proof. $\square$

Remark. By making use of a parallel argument to [O] and by use of an argument of §3, we have the following: $\pi_{12}(F) = \mathbb{Z}_2\{i \varepsilon_4\}$ and $\pi_{12}(M^5) = \mathbb{Z}_2[\theta] \oplus \mathbb{Z}_2\{i \varepsilon_4\}$, where $\theta$ is an element satisfying $p_*\theta = \sigma''$.

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Note on existence of the unstable Adams map


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