ON THE UNIVERSAL ORDINARY DISTRIBUTION

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1. Introduction

Let $m \geq 3$ be an integer and let A_m be an abelian group generated by the set of indeterminates

$$\left\{ g\left(\frac{a}{m}\right) \mid \frac{a}{m} \in \frac{1}{m}Z/Z \right\},$$

namely,

$$A_m = Z \left\langle g\left(\frac{a}{m}\right) \mid \frac{a}{m} \in \frac{1}{m}Z/Z \right\rangle.$$

Let

$$E(r, x) = -g\left(\frac{x}{r}\right) + \sum_{j=0}^{\frac{m}{r}-1} g\left(\frac{x+rj}{m}\right)$$

for $r \mid m, r \neq m, 1 \leq x \leq r$.

We then define the subgroup D_m of A_m generated by the elements $E(r, x)$;

$$D_m = Z \langle E(r, x) \mid r \mid m, r \neq m, 1 \leq x \leq r \rangle.$$

We call the factor group A_m/D_m the universal ordinary distribution ([2], [7]).

The cyclotomic numbers $1 - e^{2\pi i/m}$ satisfy the following relations

$$\prod_{j=0}^{m/r-1} (1 - e^{2\pi i(x+rj)/m}) = 1 - e^{2\pi ix/r},$$

where $m, r (\neq 1), x (\neq r)$ are as above. These relations are called ‘distribution relations’ ([3], [4]), and, for another example, Gauss sums satisfy a similar relation which is also called a distribution relation ([6]).

Ennola [1] characterized the universal punctured ordinary distribution and gave certain necessary and sufficient homogeneous linear conditions for the coefficients

of $R = \sum_{a=1}^{m-1} C_a g\left(\frac{a}{m}\right) \in A_m^0$ to satisfy $R \in A_1$, where

$$A_m^0 = Z \left\langle g\left(\frac{a}{m}\right) \mid \frac{a}{m} \in \frac{1}{m}Z/Z, a \not\equiv 0 \pmod{m} \right\rangle, \\ A_1 = Z \langle E(r, x) \mid r \mid m, 1 < r < m, 1 \leq x \leq r-1 \rangle.$$

In [5], we characterized the universal non-punctured ordinary distribution following Ennola [1]. The purpose of this article is to obtain the characterization of the universal distribution of weight k according to [5], and this result is a generalization of [5].

2. Universal distribution of weight k

Let m be an integer ≥ 3 , k an integer ≥ 0 and let A_m be an abelian group generated by the set of indeterminates

$$\left\{ g\left(\frac{a}{m}\right) \mid \frac{a}{m} \in \frac{1}{m}Z/Z \right\}.$$

Let

$$E^{(k)}(r, x) = -g\left(\frac{x}{r}\right) + \left(\frac{m}{r}\right)^k \sum_{j=0}^{r-1} g\left(\frac{x+rj}{m}\right) \quad (1)$$

for $r \mid m$, $r \neq m$, $1 \leq x \leq r$.

Let $D_m^{(k)}$ be the subgroup of A_m generated by the elements $E^{(k)}(r, x)$ and let $\tilde{D}_m^{(k)} = \mathbb{C} \otimes_Z D_m^{(k)}$, where $\mathbb{C}$ denotes the complex number field.

We call the factor group $A_m/D_m^{(k)}$ the universal distribution of weight k .

We want to obtain the necessary and sufficient homogeneous linear conditions for the coefficients of $R = \sum_{a=1}^m C_a g\left(\frac{a}{m}\right) \in A_m$ to satisfy $R \in D_m^{(k)}$.

As in [5], we always assume that χ denotes a primitive Dirichlet character the conductor f_χ of which divides m .

For any character χ , d such that $d \mid m$, $f_\chi \mid d$ and $R = \sum_{a=1}^m C_a g\left(\frac{a}{m}\right) \in A_m$, we define

$$T(\chi, d, R) = \sum_{\substack{x=1 \\ (x,d)=1}}^d \bar{\chi}(x) C_{\frac{m}{d}x}, \quad (2)$$

where $\bar{\chi}$ denotes the complex conjugate of χ and

$$Y^{(k)}(\chi, R) = \sum_{\substack{d \\ f_\chi | d, d | m}} \frac{1}{\varphi(d)d^k} \sum_{\delta | d} \mu(\delta) \chi(\delta) \delta^k T(\chi, d, R), \quad (3)$$

where φ denotes Euler's totient function and μ denotes the Möbius function.

Then we prove the following

THEOREM 2.1. *For any element $R = \sum_{a=1}^m C_a g\left(\frac{a}{m}\right) \in A_m$, we have $R \equiv 0 \pmod{D_m^{(k)}}$ if and only if $Y^{(k)}(\chi, R) = 0$ for any character χ .*

Proof. For any character χ and d such that $d | m$, $f_\chi | d$, we define

$$G^{(k)}(\chi, d) = \sum_{\substack{x=1 \\ (x,d)=1}}^d \chi(x) d^k g\left(\frac{x}{d}\right). \quad (4)$$

LEMMA 2.2. *For any character χ and d such that $d | m$, $f_\chi | d$, we have*

$$G^{(k)}(\chi, d) \equiv \sum_{\delta | d} \mu(\delta) \chi(\delta) \delta^k G^{(k)}(\chi, f_\chi) \pmod{\tilde{D}_m^{(k)}}. \quad (5)$$

This lemma can be proved by the same method as that in the lemma presented in [5].

For any integer a such that $1 \leq a \leq m$, there exists uniquely d, x such that $d | m$, $(x, d) = 1$ and $a = \frac{m}{d}x$.

Therefore

$$R = \sum_{a=1}^m C_a g\left(\frac{a}{m}\right) = \sum_{d|m} \sum_{\substack{x=1 \\ (x,d)=1}}^d C_{\frac{m}{d}x} g\left(\frac{x}{d}\right).$$

For any d, x such that $d | m$, $1 \leq x \leq d$, $(x, d) = 1$, it is easily seen that

$$g\left(\frac{x}{d}\right) = \frac{1}{\varphi(d)d^k} \sum_{\substack{\chi \\ f_\chi | d}} \bar{\chi}(x) G^{(k)}(\chi, d).$$

Hence

$$R = \sum_{d|m} \sum_{\substack{x=1 \\ (x,d)=1}}^d C_{\frac{m}{d}x} \frac{1}{\varphi(d)d^k} \sum_{\substack{\chi \\ f_\chi | d}} \bar{\chi}(x) G^{(k)}(\chi, d)$$

$$\begin{aligned}
&= \sum_{\chi} \sum_{\substack{d \\ f_{\chi} | d, d | m}} \frac{1}{\varphi(d)d^k} \sum_{\substack{x=1 \\ (x,d)=1}}^d \bar{\chi}(x) C_{\frac{m}{d}x} G^{(k)}(\chi, d) \\
&= \sum_{\chi} \sum_{\substack{d \\ f_{\chi} | d, d | m}} \frac{1}{\varphi(d)d^k} T(\chi, d, R) G^{(k)}(\chi, d).
\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}
R &\equiv \sum_{\chi} \sum_{\substack{d \\ f_{\chi} | d, d | m}} \frac{1}{\varphi(d)d^k} T(\chi, d, R) \sum_{\delta | d} \mu(\delta) \chi(\delta) \delta^k G^{(k)}(\chi, f_{\chi}) \pmod{\tilde{D}_m^{(k)}} \\
&\equiv \sum_{\chi} Y^{(k)}(\chi, R) G^{(k)}(\chi, f_{\chi}) \pmod{\tilde{D}_m^{(k)}}.
\end{aligned}$$

Hence

$$R \equiv \sum_{\chi} Y^{(k)}(\chi, R) G^{(k)}(\chi, f_{\chi}) \pmod{\tilde{D}_m^{(k)}}. \quad (6)$$

It is clear from (6) that $R \equiv 0 \pmod{\tilde{D}_m^{(k)}}$ if any character χ satisfies the condition $Y^{(k)}(\chi, R) = 0$.

We have proved the sufficiency of the conditions.

We now consider the necessity of the conditions. By the homogeneity of the conditions, we only have to check that for each generator $E^{(k)}(r, x)$ satisfies the conditions.

For $r \mid m$, $1 \leq x \leq r$, we put $s = (r, x)$, $x = x's$, $r = r's$.

$$\begin{aligned}
E^{(k)}(r, x) &= -g\left(\frac{x'}{r'}\right) + \left(\frac{m}{r}\right)^k \sum_{\substack{u'=1 \\ u' \equiv x' \pmod{r'}}}^{\frac{m}{s}} g\left(\frac{su'}{m}\right) \\
&= -g\left(\frac{x'}{r'}\right) + E^{(k)*}.
\end{aligned}$$

It is easy to see that

$$T(\chi, d, E^{(k)} - E^{(k)*}) = \begin{cases} -\bar{\chi}(x') & \text{if } d = r', \\ 0 & \text{otherwise.} \end{cases}$$

for $\chi, f_{\chi} \mid d, d \mid m$.

We put $su' = \frac{m}{d}v$, $1 \leq v \leq d$, $(v, d) = 1$. Then we have $d \mid \frac{m}{s}$, $(\frac{m}{sd}, r') = 1$. For such d , there exists uniquely $z \pmod{r'}$ such that

$$\frac{m}{sd}z \equiv x' \pmod{r'}, \quad z \equiv v \pmod{r'}.$$

Therefore

$$E^{(k)*} = \left(\frac{m}{r}\right)^k \sum_{\substack{d \mid \frac{m}{s} \\ (\frac{m}{sd}, r')=1}} \sum_{\substack{v=1 \\ (v,d)=1 \\ v \equiv z(r')}}^d g\left(\frac{v}{d}\right).$$

We have

$$T(\chi, d, E^{(k)*}) = \left(\frac{m}{r}\right)^k \sum_{\substack{v=1 \\ (v,d)=1 \\ v \equiv z(r')}}^d \bar{\chi}(v).$$

If $f_\chi \nmid r'$, then we can take v' such that $v' \equiv 1 \pmod{r'}$, $(v', d) = 1$ and $\chi(v') \neq 1$,

$$\sum_{\substack{v=1 \\ (v,d)=1 \\ v \equiv z(r')}}^d \bar{\chi}(v) = \sum_{\substack{v=1 \\ (v,d)=1 \\ v \equiv z(r')}}^d \bar{\chi}(vv') = \bar{\chi}(v') \sum_{\substack{v=1 \\ (v,d)=1 \\ v \equiv z(r')}}^d \bar{\chi}(v).$$

Hence

$$\sum_{\substack{v=1 \\ (v,d)=1 \\ v \equiv z(r')}}^d \bar{\chi}(v) = 0.$$

We obtain

$$T(\chi, d, E^{(k)*}) = 0 \quad \text{if } f_\chi \nmid r'.$$

On the other hand, if $f_\chi \mid r'$ then $d \neq r'$. Therefore $T(\chi, d, E^{(k)} - E^{(k)*}) = 0$, namely, $Y^{(k)}(\chi, E^{(k)}) = T(\chi, d, E^{(k)*}) + T(\chi, d, E^{(k)} - E^{(k)*}) = 0$ if $f_\chi \nmid r'$.

By the latter calculation, we only consider the case $f_\chi \mid r'$. In this case

$$\begin{aligned} T(\chi, d, E^{(k)*}) &= \left(\frac{m}{r}\right)^k \sum_{\substack{v=1 \\ (v,d)=1 \\ v \equiv z(r')}}^d \bar{\chi}(v) \\ &= \left(\frac{m}{r}\right)^k \frac{\varphi(d)}{\varphi(r')} \bar{\chi}(z) \\ &= \left(\frac{m}{r}\right)^k \frac{\varphi(d)}{\varphi(r')} \bar{\chi}(x') \chi\left(\frac{m}{sd}\right). \end{aligned}$$

Then we have

$$T(\chi, d, E^{(k)*}) = \begin{cases} \left(\frac{m}{r}\right)^k \overline{\chi}(x') \chi\left(\frac{m}{sd}\right) - \overline{\chi}(x') & \text{if } r' = d, \\ \left(\frac{m}{r}\right)^k \frac{\varphi(d)}{\varphi(r')} \overline{\chi}(x') \chi\left(\frac{m}{sd}\right) & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} Y^{(k)}(\chi, E^{(k)}) &= \sum_{\substack{d \mid \frac{m}{s} \\ (\frac{m}{sd}, r')=1}} \frac{1}{\varphi(d)d^k} \sum_{\delta \mid d} \mu(\delta) \chi(\delta) \delta^k T(\chi, d, E^{(k)}) \\ &= \left(\frac{m}{r}\right)^k \sum_d' \frac{1}{\varphi(r')d^k} \sum_{\delta \mid d} \mu(\delta) \chi(\delta) \delta^k \overline{\chi}(x') \chi\left(\frac{m}{sd}\right) \\ &\quad + \frac{1}{\varphi(r')(r')^k} \sum_{\delta \mid d} \mu(\delta) \chi(\delta) \delta^k \overline{\chi}(x') \left\{ \left(\frac{m}{r}\right)^k \chi\left(\frac{m}{sr'}\right) - 1 \right\} \\ &= \frac{\overline{\chi}(x')}{\varphi(r')} \left(\frac{m}{r}\right)^k \left\{ \sum_d' d^{-k} \sum_{\delta \mid d} \mu(\delta) \chi(\delta) \delta^k \chi\left(\frac{m}{sd}\right) \right. \\ &\quad \left. - r'^{-k} \sum_{\delta \mid r'} \mu(\delta) \chi(\delta) \delta^k \left(\frac{r}{m}\right)^k \right\}, \end{aligned}$$

where d runs over $d \mid \frac{m}{s}$, $d \neq r'$, $(\frac{m}{sd}, r') = 1$ in the previous summation.

Then we put $ns = m$, $tsd = m$, so we obtain the following

$$\begin{aligned} Y^{(k)}(\chi, E^{(k)}) &= \frac{\overline{\chi}(x')}{\varphi(r')} \left(\frac{m}{r}\right)^k \left\{ \sum_{t \mid n, (t, r')=1} \left(\frac{t}{n}\right)^k \sum_{\delta \mid \frac{n}{t}} \mu(\delta) \chi(\delta) \delta^k \chi(t) \right. \\ &\quad \left. - r'^{-k} \sum_{\delta \mid r'} \mu(\delta) \chi(\delta) \delta^k \left(\frac{r}{m}\right)^k \right\}. \end{aligned}$$

Here we see that

$$\sum_{t \mid n, (t, r')=1} \left(\frac{t}{n}\right)^k \sum_{\delta \mid \frac{n}{t}} \mu(\delta) \chi(\delta) \delta^k \chi(t) = n^{-k} \sum_{h \mid n} \chi(h) h^k \sum_{\substack{\delta \mid h \\ (\frac{h}{\delta}, r')=1}} \mu(\delta), \quad (7)$$

where we set $h = \delta t$, and

$$r'^{-k} \sum_{\delta \mid r'} \mu(\delta) \chi(\delta) \delta^k \left(\frac{r}{m}\right)^k = n^{-k} \sum_{\delta \mid r'} \mu(\delta) \chi(\delta) \delta^k. \quad (8)$$

By (7) and (8), we only have to show the following:

$$\sum_{\substack{\delta|h \\ (\frac{h}{\delta}, r')=1}} \mu(\delta) = \begin{cases} \mu(h) & \text{if } h \mid r', \\ 0 & \text{otherwise.} \end{cases}$$

Actually, if $h \mid r'$ then we have $\sum_{\substack{\delta|h \\ (\frac{h}{\delta}, r')=1}} \mu(\delta) = \sum_{\delta=h} \mu(\delta) = \mu(h)$.

If $h \nmid r'$, we set $h = p_1^{e_1} \cdots p_\alpha^{e_\alpha} p_{\alpha+1}^{e_{\alpha+1}} \cdots p_\beta^{e_\beta}$, $r' = p_1^{f_1} \cdots p_\alpha^{f_\alpha} q_{\alpha+1}^{f_{\alpha+1}} \cdots q_\gamma^{f_\gamma}$ as the prime factorization. Then we have $h \nmid r'$ if and only if $\beta > \alpha$ or there exists ν such that $e_\nu > f_\nu$ ($1 \leq \nu \leq \alpha$). We now put $g = p_1^{e_1} \cdots p_\alpha^{e_\alpha}$, so we have $(g, \frac{h}{g}) = 1$.

Hence $\sum_{\substack{\delta|h \\ (\frac{h}{\delta}, r')=1}} \mu(\delta) = \sum_{\substack{\delta|h \\ g|\delta}} \mu(\delta) = \mu(g) \sum_{\tau|\frac{h}{g}} \mu(\tau)$.

Then, if $\beta > \alpha$ we have $\frac{h}{g} > 1$ so $\sum_{\tau|\frac{h}{g}} \mu(\tau) = 0$, and if $e_\nu > f_\nu$ (for some ν) so $e_\nu \geq 2$, then we have $\mu(g) = 0$.

Therefore we obtain $\sum_{\substack{\delta|h \\ (\frac{h}{\delta}, r')=1}} \mu(\delta) = 0$ if $h \nmid r'$.

This completes the proof of our theorem. $\square$

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