EQUIVARIANT SK GROUP OF MANIFOLDS WITH BOUNDARY

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1. Introduction

Kosniowski [5] has studied an equivariant cutting-and-pasting theory SK_*^G based on unoriented closed manifolds for any finite abelian group G . We can naturally extend cutting-and-pasting theory for unoriented singular G manifolds with boundary in a pair (X, A) of G spaces. When X is the one-point set and $A = \emptyset$, this coincides with the original state already mentioned.

In §2, we define a group $SK_*^G(X, A)$ obtained by these objects and obtain a short split exact sequence including this theory and the original theory (Theorem 2.3). Using this sequence, we see that there is an isomorphism $SK_*(X, A) \cong SK_*(pt, pt) \oplus 2$ -torsion for a pair (X, A) of arcwise connected spaces. Here $SK_*(-, -) = SK_*^{[e]}(-, -)$, $\{e\}$ is the trivial group and pt the one-point set.

In §3, we treat the case $G = \mathbb{Z}_2$ and show that an element $[M, \partial M]$ in $SK_*^{\mathbb{Z}_2}(pt, pt)$ is determined by the class $\{\chi(M, \partial M), \chi(F_i, \partial F_i)\}$ modulo the elements of order two, where χ is the Euler characteristic of a pair of spaces and $\{F_i\}$ are the fixed point sets of the involution (Theorem 3.2).

Finally we mention a ring structure of $SK_*^{\mathbb{Z}_2}(pt, pt) \otimes \mathbb{Z}[\frac{1}{2}]$ induced by the product of manifolds (Proposition 3.4). Here $\mathbb{Z}[\frac{1}{2}]$ is a subring of rationals $\mathbb{Q}$ generated by the integers and $\frac{1}{2}$.

We assume that all manifolds are smooth, compact and unoriented.

2. An equivariant cutting-and-pasting theory

Let G be, in general, a compact Lie group. Let M^n be an n -dimensional smooth G manifold with boundary ∂M . Let $(L, \partial L) \subset (M, \partial M)$ satisfy the following properties.

- (i) $(L, \partial L)$ is a G invariant codimension-1 submanifold of $(M, \partial M)$. Here we admit the case $\partial L = \emptyset$ and $\partial M \neq \emptyset$.

- (ii) The normal bundle of $(L, \partial L)$ in $(M, \partial M)$ is G equivalent to $(L, \partial L) \times \mathbb{R}$ with trivial action G on the real numbers $\mathbb{R}$.

We assume that $M = N_1 \cup N_2$ (pasting along the common parts L) for some G invariant submanifolds N_i of codimension zero as usual, and denote this decomposition simply by $(M, \partial M) = (N_1, \partial N_1) \cup (N_2, \partial N_2)$.

Definition 2.1. Let (X, A) be a pair of topological G spaces. We say that the singular n -dimensional G manifolds $(M_1, \partial M_1; f_1)$ and $(M_2, \partial M_2; f_2)$ are obtained from each other by a G equivariant cutting-and-pasting in (X, A) if

- (i) $(M_1, \partial M_1)$ has been obtained from $(M_2, \partial M_2)$ by a G equivariant cutting-and-pasting along the codimension-1 G submanifold $(L, \partial L)$ as already mentioned, that is, $(M_1, \partial M_1) = (N_1, \partial N_1) \cup_\varphi (N_2, \partial N_2)$, $(M_2, \partial M_2) = (N_1, \partial N_1) \cup_\psi (N_2, \partial N_2)$ for some G diffeomorphisms $\varphi, \psi : L \rightarrow L$ where $(L, \partial L) \subset (M_i, \partial M_i)$ ($i = 1, 2$), and
- (ii) there are G homotopies

$$f_1|_{(N_1, \partial N_1 \setminus \text{Int} L)} \simeq f_2|_{(N_1, \partial N_1 \setminus \text{Int} L)} \text{ and } f_1|_{(N_2, \partial N_2 \setminus \text{Int} L)} \simeq f_2|_{(N_2, \partial N_2 \setminus \text{Int} L)}$$

where $\text{Int} L = L \setminus \partial L$.

Let $\mathcal{M}_n^G(X, A)$ be a set of singular G -manifolds with dimension n . We say that $(M_1, \partial M_1; f_1)$ and $(M_2, \partial M_2; f_2) \in \mathcal{M}_n^G(X, A)$ are called SK^G equivalent in (X, A) if $(M_2, \partial M_2; f_2)$ can be obtained from $(M_1, \partial M_1; f_1)$ by a finite sequence of G -equivariant cutting-and-pasting in (X, A) . This relation is an equivalence relation ([5, Ch. 2.2.1]).

Example A. If singular closed G manifolds $(M_1; f_1)$ and $(M_2; f_2)$ in a G space X are SK^G equivalent in the original sense, then $(M_1 \times I, 2M_1; F_1)$ and $(M_2 \times I, 2M_2; F_2)$ are naturally SK^G equivalent in (X, X) where $I = [0, 1]$ is the unit interval with trivial action and $F_i : M_i \times I \rightarrow X$ are defined by $F_i(x, t) = f_i(x)$ for $i = 1, 2$.

Definition 2.2. We define $SK_n^G(X, A)$ to be the Grothendieck group of $\Upsilon_n^G(X, A)$, the quotient semigroup $\mathcal{M}_n^G(X, A)/SK^G$ -equivalence. An element of $SK_n^G(X, A)$ is denoted by $[M^n, \partial M; f]$.

When $A = \emptyset$, $SK_n^G(X, \emptyset) = SK_n^G(X)$ as mentioned in §1.

THEOREM 2.3. For any pair (X, A) of G spaces and integer n with $n \geq 1$, the sequence

$$0 \longrightarrow SK_n^G(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{i \otimes (id)} SK_n^G(X, A) \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\partial \otimes (id)} SK_{n-1}^G(A) \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow 0$$

is split exactly where $i([M; f]) = [M, \emptyset; f]$, $\partial([M, \partial M; f]) = [\partial M; f|_{\partial M}]$ and id is the identity map.

Proof. First we see that i is injective since our SK process on a manifold without boundary is the same as the original one. Hence, $i \otimes (id)$ is also injective. It is trivial that $(\partial \otimes (id)) \circ (i \otimes (id)) = 0$. We may write each element in $SK_*^G(X, A)$ as $[M, \partial M; f] - [N, \partial N; g]$ by definition. Now let $\sum_{k \geq 0} ([M_k, \partial M_k; f_k] - [N_k, \partial N_k; g_k]) \otimes \frac{1}{2^k} \in \text{Ker}(\partial \otimes (id))$ so $\sum_{k \geq 0} [\partial M_k; f_k|_{\partial M_k}] \otimes \frac{1}{2^k} = \sum_{k \geq 0} [\partial N_k; g_k|_{\partial N_k}] \otimes \frac{1}{2^k}$ in $SK_{n-1}^G(A) \otimes \mathbb{Z}[\frac{1}{2}]$.

In general we have

$$2[M, \partial M; f] = [DM, \emptyset; f \cup f] + [\partial M \times I, 2\partial M; (f|_{\partial M}) \circ pr] \quad (1)$$

by considering a G collar $\partial M \times I$ in M , where DM is the double of M . Hence

$$\begin{aligned} & \sum_{k \geq 0} ([M_k, \partial M_k; f_k] - [N_k, \partial N_k; g_k]) \otimes \frac{1}{2^k} \\ &= \sum_{k \geq 0} ([DM_k, \emptyset; f_k \cup f_k] - [DN_k, \emptyset; g_k \cup g_k]) \otimes \frac{1}{2^{k+1}} \in \text{Im}(i \otimes (id)). \end{aligned}$$

Therefore, $\text{Ker}(\partial \otimes (id)) \subset \text{Im}(i \otimes (id))$. Finally we define $r : SK_{n-1}^G(A) \rightarrow SK_n^G(X, A)$ by $r([M^{n-1}; f]) = [M^{n-1} \times I, 2M; F]$ where F is defined in Example A, and then $\frac{1}{2}r \otimes (id)$ is a right inverse of $\partial \otimes (id)$. Hence we conclude the proof of the theorem. $\square$

PROPOSITION 2.4. *For any pair (X, A) of arcwise connected spaces, there are isomorphisms:*

$$SK_*(X) \cong SK_*(pt) \oplus 2\text{-torsion}, \quad SK_*(X, A) \cong SK_*(pt, pt) \oplus 2\text{-torsion}.$$

Proof. Throughout the proof, G is the trivial group. Let $\overline{SK}_n(A)$ be $SK_n(A)$ factored by the bordism relation and $I_n(A)$ be the kernel of the natural projection map $SK_n(A) \rightarrow \overline{SK}_n(A)$, then we have an exact sequence:

$$0 \longrightarrow SK_n(X) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{i \otimes (id)} SK_n(X, A) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\partial \otimes (id)} I_{n-1}(A) \otimes \mathbb{Z}[\frac{1}{2}] \longrightarrow 0$$

([4, Theorem 1.3]). Combine this and the exact sequence in Theorem 2.3, then $I_*(A) \otimes \mathbb{Z}[\frac{1}{2}] \cong SK_*(A) \otimes \mathbb{Z}[\frac{1}{2}]$. Since $I_*(A) \otimes \mathbb{Z}[\frac{1}{2}] \cong I_*(pt) \otimes \mathbb{Z}[\frac{1}{2}] \cong SK_*(pt) \otimes \mathbb{Z}[\frac{1}{2}]$ in general, we have $SK_*(A) \otimes \mathbb{Z}[\frac{1}{2}] \cong SK_*(pt) \otimes \mathbb{Z}[\frac{1}{2}]$. This implies that $SK_*(A) \cong SK_*(pt) \oplus 2\text{-torsion}$ ([2, Theorems (1.1) and (1.3a)]). Consider again an exact sequence in Theorem 2.3 when a pair of spaces is (X, A)

or (pt, pt) respectively, then $SK_*(X, A) \otimes \mathbb{Z}[\frac{1}{2}] \cong SK_*(pt) \otimes \mathbb{Z}[\frac{1}{2}]$ by the five lemma. Therefore, $SK_*(X, A) \cong SK_*(pt, pt) \oplus 2\text{-torsion}$ and its torsion-free part is generated by the n -disk $[D^n, S^{n-1}; *]$ for $n \geq 0$ where $*$ is a constant map ([4, Theorem 1.2]). $\square$

Remark. $SK_*(X) \cong SK_*(pt) \oplus 2\text{-torsion}$, in Proposition 2.4, was obtained in a different way in [2, Theorem (2.10)].

3. On involutions

For $G = \mathbb{Z}_2$, an SK_* module $SK_*^{\mathbb{Z}_2}(pt)$ of closed manifolds with involutions has been determined as follows.

PROPOSITION 3.1. ([5, Theorem 5.3.1.]) $SK_*^{\mathbb{Z}_2}(pt)$ is a free SK_* module with basis $\{\mathbb{Z}_2, \mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^j); j \geq 0\}$ where $\tilde{\mathbb{R}}$ denotes the real numbers with $\mathbb{Z}_2$ acting by multiplication by -1 , and $\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^j)$ is the real projective space with induced involution.

In this section we consider a similar group $SK^{\mathbb{Z}_2}(pt, pt)$ for involutions on manifolds with a boundary. We simply denote an element of this group by $[M, \partial M]$ and by $[M]$ when the manifold M is closed.

Now we consider the Euler characteristic $\chi(M, \partial M)$ for $(M, \partial M)$. Note that, in general, $\chi(M, \partial M) = \chi(M) - \chi(\partial M)$ ([1, p. 105]). Given an involution $(M, \partial M)$, we see that $\chi(M, \partial M)$ and $\chi(F_i, \partial F_i)$ are $SK^{\mathbb{Z}_2}$ invariant where F_i is the fixed-point set with codimension i . Conversely, we have the following.

THEOREM 3.2. For any involutions $(M, \partial M)$ and $(M', \partial M')$, if $\chi(M, \partial M) = \chi(M', \partial M')$ and $\chi(F_i, \partial F_i) = \chi(F'_i, \partial F'_i)$ for all i then $(M, \partial M)$ is $SK^{\mathbb{Z}_2}$ equivalent to $(M', \partial M')$ modulo the elements of order two in $SK_*^{\mathbb{Z}_2}(pt, pt)$.

Proof. (i) When $\dim M = 2m$, following from (1), Proposition 3.1 and Example A, we may write $[M^{2m}, \partial M]$ as

$$\begin{aligned} 2[M, \partial M] &= \sum_{j=0}^m a_{2j} [\mathbb{R}P^2]^{m-j} [\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^{2j})] + a [\mathbb{R}P^2]^m [\mathbb{Z}_2] \\ &\quad + \sum_{j=1}^m b_{2j} [\mathbb{R}P^2]^{m-j} [\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^{2j-1}) \times I, 2\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^{2j-1})] \end{aligned}$$

where $SK_* \cong \mathbb{Z}[\alpha]$, $\alpha = [\mathbb{R}P^2]$ [5] and $a, a_{2j}, b_{2j} \in \mathbb{Z}$.

Considering the fixed-point sets, we have

$$2[F_i, \partial F_i] = \begin{cases} a_{2j}[\mathbb{R}P^2]^{m-j} & (i = 2j) \\ b_2[\mathbb{R}P^2]^{m-1}[pt \times I, 2pt] \\ \quad + \sum_{j=1}^m b_{2j}[\mathbb{R}P^2]^{m-j}[\mathbb{R}P^{2j-2} \times I, 2\mathbb{R}P^{2j-2}] & (i = 1) \\ b_{2j}[\mathbb{R}P^2]^{m-j}[pt \times I, 2pt] & (i = 2j - 1 \neq 1). \end{cases}$$

Now we apply the Euler characteristic χ , then

$$2\chi(M, \partial M) = \sum_{j=0}^m a_{2j} + 2a \quad (2)$$

and

$$2\chi(F_i, \partial F_i) = \begin{cases} a_{2j} & (i = 2j) \\ -b_2 - \sum_{j=1}^m b_{2j} & (i = 1) \\ -b_{2j} & (i = 2j - 1 \neq 1). \end{cases} \quad (3)$$

(ii) When $\dim M = 2m + 1$, we may write $[M^{2m+1}, \partial M]$ as

$$\begin{aligned} 2[M, \partial M] &= \sum_{j=1}^m a_{2j+1}[\mathbb{R}P^2]^{m-j}[\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^{2j+1})] + b[\mathbb{R}P^2]^m[\mathbb{Z}_2 \times I, 2\mathbb{Z}_2] \\ &\quad + \sum_{j=0}^m b_{2j+1}[\mathbb{R}P^2]^{m-j}[\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^{2j}) \times I, 2\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^{2j})] \end{aligned}$$

where $b, a_{2j+1}, b_{2j+1} \in \mathbb{Z}$, and

$$2[F_i, \partial F_i] = \begin{cases} a_{2j+1}[\mathbb{R}P^2]^{m-j} & (i = 2j + 1 \neq 1) \\ \sum_{j=1}^m a_{2j+1}[\mathbb{R}P^2]^{m-j}[\mathbb{R}P^{2j}] \\ \quad + \sum_{j=1}^m b_{2j+1}[\mathbb{R}P^2]^{m-j}[\mathbb{R}P^{2j-1} \times I, 2\mathbb{R}P^{2j-1}] & (i = 1) \\ b_{2j+1}[\mathbb{R}P^2]^{m-j}[I, \partial I] & (i = 2j). \end{cases}$$

Therefore,

$$2\chi(M, \partial M) = - \sum_{j=0}^m b_{2j+1} - 2b \quad (4)$$

and

$$2\chi(F_i, \partial F_i) = \begin{cases} a_{2j+1} & (i = 2j + 1 \neq 1) \\ \sum_{j=1}^m a_{2j+1} & (i = 1) \\ -b_{2j+1} & (i = 2j). \end{cases} \quad (5)$$

Thus, the result follows by (2) ~ (5). $\square$

We may have to consider some other method to show that the class $\{\chi(M, \partial M), \chi(F_i, \partial F_i)\}$ determines an element $[M, \partial M]$ in $SK_*^{\mathbb{Z}_2}(pt, pt)$.

Let $y = [\mathbb{Z}_2] \in SK_0^{\mathbb{Z}_2}(pt)$, $y_i = [\mathbb{R}P(\mathbb{R} \times \tilde{\mathbb{R}}^i)] \in SK_i^{\mathbb{Z}_2}(pt)$ for $i \geq 0$ and $\alpha = [\mathbb{R}P^2] \in SK_2$, then we have the following relations.

PROPOSITION 3.3. ([3, Theorem 3.]) *For any integers $m, n \geq 0$,*

$$\begin{aligned} (1) \quad & y^2 = 2y \quad (2) \quad yy_{2m+1} = 0 \quad (3) \quad yy_{2m} = \alpha^m y \quad (4) \quad y_{2m} = y_2^m \\ (5) \quad & y_{2m+1}y_{2n} = y_{2m+2n+1} + \alpha^m y_{2n+1} - \alpha^{m+n} y_1 \\ (6) \quad & y_{2m+1}y_{2n+1} = \alpha^{m+n} y_2 + \alpha^m y_2^{n+1} + \alpha^n y_2^{m+1} + y_2^{m+n+1} - 2\alpha^{m+n+1} y \end{aligned}$$

as a ring structure of $SK_^{\mathbb{Z}_2}(pt)$.*

We consider the multiplication in $SK_*^{\mathbb{Z}_2}(pt, pt)$ induced by the product of manifolds.

Let $\bar{y} = y \times I$, $\bar{y}_i = y_i \times I$. By Theorem 2.3 and Proposition 3.3, we have the following proposition.

PROPOSITION 3.4. $SK_*^{\mathbb{Z}_2}(pt, pt) \otimes \mathbb{Z}[\frac{1}{2}]$ is a free $SK_*(pt) \otimes \mathbb{Z}[\frac{1}{2}]$ module generated by $\{y, \bar{y}, y_i, \bar{y}_i; i \geq 0\}$. The multiplicative relations are determined by those in Proposition 3.3 and the fact that $[D^2, S^1] = \alpha$.

Note that $2[D^2, S^1] = [S^2, \emptyset] + [S^1 \times I, 2S^1]$ in $SK_*(pt, pt)$ by (1). This implies that $[D^2, S^1] = \alpha + t$ for some element t of order two since $[S^1, \emptyset] = 0$ and $[S^2, \emptyset] = 2\alpha$ in $SK_*(pt)$ ([2, Ch.1]). Hence, the previous relation is obtained.

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