ON BEHAVIORS OF CELLULAR AUTOMATA
WITH RULE 27

Tatsuro SATO

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1. Introduction

Let $m$ be a positive integer. A finite cellular automaton $CA-27(m)$ has states 0 or 1 and works on a linear array of $m$ cells with the triplet local transition rule $f$. The rule $f$ is defined as follows:

\[
\begin{array}{cccccccc}
111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}
\]

and the rule number 27 is decided according to

\[2^4 + 2^3 + 2^1 + 2^0 = 27.\]

The global transition function $\delta$ of $CA-27(m)$ is given by

\[\delta(x) = f(x_0,x_1,x_2)f(x_1,x_2,x_3) \cdots f(x_{m-1},x_m,x_{m+1})\]

for a configuration $x = x_1x_2 \cdots x_m$ of $CA-27(m)$, where $x_i$ is a state of the $i$th cell $(1 \leq i \leq m)$. When $x_0$ and $x_{m+1}$ are fixed for $a$ and $b$ ($a = 0$ or $1$ and $b = 0$ or $1$), $CA-27(m)$ has the boundary condition $a-b$. Since $CA-27(m)$ has $2^m$ possible configurations, each configuration reaches a stable state (that is, a fixed point or a limit cycle) after some transition steps. Let $h(x)$ be the least number of transition steps needed by a configuration $x$ to fall into a stable state. Then the transient length $H$ of $CA-27(m)$ is defined by $\text{Max}_x h(x)$.

One-dimensional finite cellular automata with a triplet local transition rule have been studied under fixed boundary conditions [1–3]. A lot of results have been reported under the boundary condition 0–0, especially in [1] and [2]. In [3] a study was made of limit cycles, regarding the changed states of each cell as a column vector; however, there was no reference to transition steps. Since the transient length is an important concept, the author has studied one-dimensional finite cellular automata with a triplet local transition rule having limit cycles of period length 3, and
transient lengths under the fixed boundary conditions. Above all, he has investigated \( CA-27(m) \) under the boundary condition 0–0 and 0–1 taking note of configurations after every three steps; this is a new approach to these problems. The main results are as follows:

- \( CA-27(m) \) with boundary condition 1–1 has a unique limit cycle of period length 2 and its transient length is \( 2m - 3 \).
- \( CA-27(m) \) with boundary condition 1–0 has only two limit cycles of period length 2 and its transient length is \( 2m - 4 \).
- \( CA-27(m) \) with boundary condition 0–1 has a unique limit cycle of period length 3 and its transient length is \( 3m - 7 \).
- \( CA-27(m) \) with boundary condition 0–0 has a unique limit cycle of period length 6 and its transient length is \( 3m - 7 \).

(Cf. Theorem 2.4, 2.5, 3.13, 4.8.)

2. Boundary condition 1–0 and 1–1

The notations used are as follows:

1. \( a^k = \underbrace{a \cdots a}_{k \text{-times}} \) (\( a = 0 \) or 1).

   Let \( A \) be a subsequence.

2. \( A^k = \underbrace{AA \cdots A}_{k \text{-times}} \).

3. \( (A_l^m) \) : sequence composed of \( l \) bits taken from the right edge when some \( A \)'s are arranged.

**LEMMA 2.1.** For any configuration \( x \) of \( CA-27(m) \), there exists an integer \( k \) with \( 0 \leq k \leq 4 \) such that

\[
\delta^k(x) = 111\ p,
\]

where \( p \) is a sequence with a length of \( m - 3 \).

**Proof.** We set \( \delta(x) = y \) and \( \delta(y) = z \).

1. When \( x_1 x_2 x_3 = 000 \).

   Since \( y_1 y_2 y_3 = 111 \), we have \( k = 1 \).

2. In the case \( x_1 x_2 x_3 = 001 \).

   If \( x_4 = 1 \), then \( y_1 y_2 y_3 y_4 = 1110 \). So we have \( k = 1 \).

   If \( x_4 = 0 \) and \( x_5 = 0 \), then \( y_1 y_2 y_3 y_4 = 11011 \) and \( z_1 z_2 z_3 z_4 z_5 = 00010 \). From 1, we have \( k = 3 \).
If $x_4 = 0$ and $x_5 = 1$, then $y_1y_2y_3y_4 = 1100$ and $z_1z_2z_3z_4 = 0011$. From the case of $x_4 = 1$, we have $k = 3$. Therefore $k = 1, 3$.

3. In the case $x_1x_2x_3 = 010$.
   If $x_4 = 0$, then $y_1y_2y_3y_4 = 0011$. From 2, we have $k = 2$.
   If $x_4 = 1$, then $y_1y_2y_3 = 000$. From 1, we have $k = 2$.
   Therefore we have $k = 2$.

4. In the case $x_1x_2x_3 = 011$.
   $y_1y_2y_3 = 010$ holds. From 3, we have $k = 3$.

5. In the case $x_1x_2x_3 = 100$.
   $y_1y_2y_3y_4 = 011$ holds. From 4, we have $k = 4$.

6. In the case $x_1x_2x_3 = 101$.
   If $x_4 = 0$, then $y_1y_2y_3 = 000$. From 1, we have $k = 2$.
   If $x_4 = 1$, then $y_1y_2y_3y_4 = 0010$. From 2, we have $k = 4$.
   Therefore we have $k = 2, 4$.

7. In the case $x_1x_2x_3 = 110$.
   If $x_4 = 0$, then $y_1y_2y_3y_4 = 0011$. From 2, we have $k = 2$.
   If $x_4 = 1$, then $y_1y_2y_3 = 000$. From 1, we have $k = 2$.
   Therefore we have $k = 2$.

8. In the case $x_1x_2x_3 = 111$.
   Obviously $k = 0$.

Thus, it is sufficient to study the behaviors of configurations $x = 111p$ for a sequence $p$ with a length of $m - 3$.

**Lemma 2.2.** If $x = 111p$, then

$$
\delta^{2i}(x) = 1^{i+3}q \text{ for } 1 \leq i \leq m - 4,
$$

where $q$ is a sequence with a length of $m - i - 3$.

**Proof.** (By induction on $i$.)

1. When we set $\delta(x) = y$ and $\delta(y) = z$, we have $y_1y_2y_3 = 000$ and $z_1z_2z_3 = 111$.
   If $y_4 = 0$, we have $z_4 = f(00y_5) = 1$. If $y_4 = 1$, then $x_4x_5 = 00$ from $y_4 = f(1x_4x_5) = 1$. So we have $y_5 = f(00x_6) = 1$. Hence $z_4 = f(011) = 1$.
   Since $z_1z_2z_3z_4 = 1^4$, the statement holds when $i = 1$.

2. Assume that the statement holds when $1 \leq i \leq m - 5$. When we set $\delta^{2i+1}(x) = u$ and $\delta(u) = v$, we have $u_1u_2\cdots u_{i+3} = 0^{i+3}$, $v_1v_2\cdots v_{i+3} = 1^{i+3}$ and $v_{i+4} = 1$ in the same way as 1. Since $\delta^{2(i+1)}(x) = v$ and $v_1v_2\cdots v_{i+4} = 1^{i+4}$, the statement holds for $i + 1$. □
LEMMA 2.3. For any configuration \( x \), there exists an integer \( k \) with \( 0 \leq k \leq 2m - 4 \) such that
\[
\delta^k(x) = i^{m-1} a.
\]

Proof. From Lemma 2.1, there exists an integer \( k_1 \) (\( 0 \leq k_1 \leq 4 \)) for \( x \) such that
\[
\delta^{k_1}(x) = 111p
\]
where \( q \) is a sequence with a length of \( m - 3 \). Let \( l = m - 4 \) in Lemma 2.2; it follows that
\[
\delta^{2(m-4)}(111p p) = i^{m-1} a.
\]
Since \( k_1 + 2(m-4) \leq 4 + 2m - 8 = 2m - 4 \), there exists an integer \( k \) (\( 0 \leq k \leq 2m-4 \)) such that
\[
\delta^k(x) = i^{m-1} a.
\]

\(\square\)

THEOREM 2.4. CA–27(m) with boundary condition 1–1 has a unique limit cycle of period length 2 and its transient length is \( 2m - 3 \).

Proof. First, for any configuration \( x \), from Lemma 2.3, we have
\[
\delta^k(x) = i^{m-1} a \quad \text{for} \quad 0 \leq k \leq 2m - 4.
\]
If \( a = 0 \), then \( \delta^k(x) = 1^{m-1} 0 \), so we have \( \delta(1^{m-1} 0) = 0^m \), \( \delta(0^m) = 1^m \) and \( \delta(1^m) = 0^m \). If \( a = 1 \), \( \delta^k(x) = 1^m \) holds. Therefore \( 0^m \) and \( 1^m \) make a limit cycle of period length 2. Thus, there exists a unique limit cycle of period length 2.

Next, as we have \( h(x) = k + 1 \leq 2m - 3 \) from the above discussion, the condition \( H \leq 2m - 3 \) holds. Let \( x = 10^{m-1} \), then \( \delta(x) = 01^{m-1} \), \( \delta(01^{m-1}) = 010^{m-2} \), \( \delta(010^{m-2}) = 001^{m-2} \) and \( \delta(001^{m-2}) = 1110^{m-3} \). Generally both \( \delta(1^l 0^{m-l}) = 0^l 1^{m-l} \) and \( \delta(0^l 1^{m-l}) = 1^l + 10^{m-l-1} \) hold for \( 2 \leq l \leq m - 2 \) by \( f(110) = f(111) = 0 \) and \( f(000) = f(100) = f(001) = f(011) = 1 \). Then we have \( \delta^2(1110^{m-3}) = \delta^{2m-8}(1110^{m-3}) = 1^{m-10} \), \( \delta(1^{m-10}) = 0^m \) and \( h(1^{3} 0^{m-3}) = 2m - 7 \) by using the above repeatedly for \( 1^3 0^{m-3} \).

Finally, we have \( h(x) = 4 + 2m - 7 = 2m - 3 = H \).

\(\square\)

THEOREM 2.5. CA–27(m) with boundary condition 1–0 has only two limit cycles of period length 2 and its transient length is \( 2m - 3 \).
Proof. First, for any configuration $x$, from Lemma 2.3, we have

$$\delta^k(x) = 1^{m-1}a \quad \text{for} \quad 0 \leq k \leq 2m - 4.$$ 

If $a = 0$, that is $\delta^k(x) = 1^{m-1}0$, we have $\delta(1^{m-1}0) = 0^{m-1}1$ and $\delta(0^{m-1}1) = 1^{m-1}0$. If $a = 1$, then $\delta^k(x) = 1^m$, we have $\delta(1^m) = 0^m$ and $\delta(0^m) = 1^m$. Therefore both $1^{m-1}0$, $0^{m-1}1$ and $0^m$, $1^m$ make limit cycles of period length 2. Hence, there exist only two limit cycles of period length 2.

Next, as we have $h(x) = k \leq 2m - 4$ from the above discussion, it holds that $H \leq 2m - 4$. Let $x = 10^{m-1}$, then we have $\delta^{2m-k}(x) = 1^{m-1}0$ and that $h(x) = 2m - 4 = H$ in a similar way to Theorem 2.4. □

3. Boundary condition 0–1

Lemma 3.1. (Common to both boundary conditions 0–1 and 0–0.)

For any configuration $x$ of CA–27(m), there exists an integer $k$ with $0 \leq k \leq 5$ such that

$$\delta^k(x) = 110p,$$

where $p$ is a sequence with a length of $m - 3$.

Proof. We set $\delta(x) = y$, $\delta(y) = z$, $\delta(z) = u$ and $\delta(u) = v$.

1. In the case $x_1x_2x_3 = 000$.
   Since $y_1y_2y_3 = 111$, $z_1z_2z_3 = 100$, $u_1u_2u_3 = 011$ and $v_1v_2v_3 = 110$, we have $k = 4$.

2. In the case $x_1x_2x_3 = 001$.
   If $x_4 = 0$, then $y_1y_2y_3 = 110$ and we have $k = 1$.
   If $x_4 = 1$, then $y_1y_2y_3 = 111$. From 1, we have $k = 4$.
   Thus, we have $k = 1, 4$.

3. In the case $x_1x_2x_3 = 010$.
   If $x_4 = 0$, then $y_1y_2y_3y_4 = 1011$ and $z_1z_2z_3z_4 = 0010$. From 2, we have $k = 3$.
   If $x_4 = 1$, then $y_1y_2y_3 = 100$. From 1, we have $k = 3$.
   Therefore we have $k = 3$.

4. In the case $x_1x_2x_3 = 011$.
   From 1, we have $k = 1$.

5. In the case $x_1x_2x_3 = 100$.
   From 1, we have $k = 2$. 


6. In the case $x_1x_2x_3 = 101$.
   If $x_4 = 0$, then $y_1y_2y_3 = 000$. From 1, we have $k = 5$.
   If $x_4 = 1$, then $y_1y_2y_3y_4 = 0010$. From 2, we have $k = 2$.
   Therefore we have $k = 2, 5$.
7. In the case $x_1x_2x_3 = 110$.
   Obviously $k = 0$.
8. In the case $x_1x_2x_3 = 111$.
   From 1, we have $k = 3$.

Thus, it is sufficient to first study the behaviors of configurations $x = 110p$ for a sequence $p$ with a length of $m - 3$.

When $m = 3k + r$ ($r = 0, 1$ or $2$), for a configuration $x$ of $CA_{−27}(m)$, we set

$$x_{3i−2}x_{3i−1}x_{3i} = X_i \quad \text{for } 1 \leq i \leq k.$$ 

Then we have $x = X_1X_2 \cdots X_kx'$ for a sequence $x'$

$$x' = \begin{cases} 
\varepsilon & (m = 3k) \\
x_m & (m = 3k + 1) \\
x_{m−1}x_m & (m = 3k + 2), 
\end{cases}$$

where $\varepsilon$ is an empty sequence. We use this rotation for a configuration of $CA_{−27}(m)$ from now on. When we set $A = 110$, $B = 010$ and $C = 000, 001, 011, 100, 101$ or 111, configurations $110p$ are divided into three cases as follows:

(a) $X_1 = A, X_i = A$ or $B$ ($2 \leq i \leq k$) and $x' = \begin{cases} 
\varepsilon & (m = 3k) \\
0 \text{ or } 1 & (m = 3k + 1) \\
01 \text{ or } 11 & (m = 3k + 2). 
\end{cases}$

(b) $X_1 = A, X_i = A$ or $B$ ($2 \leq i \leq k$) and $x' = 00$ or 10 ($m = 3k + 2$).

(c) $X_1 = A, X_j = A$ or $B$ ($2 \leq j \leq i − 1$) and $X_i = C$ ($2 \leq i \leq k$).

We set $\Delta = \delta^3$ and define mappings $F_1, F_2, F_3$ and $F$ as follows:

$$F_1(x_1x_2 \cdots x_9) = f(x_1x_2x_3)f(x_2x_3x_4) \cdots f(x_7x_8x_9)$$
$$F_2(x_1x_2 \cdots x_7) = f(x_1x_2x_3)f(x_2x_3x_4) \cdots f(x_5x_6x_7)$$
$$F_3(x_1x_2 \cdots x_5) = f(x_1x_2x_3)f(x_2x_3x_4)f(x_3x_4x_5)$$

and

$$F = F_3 \circ F_2 \circ F_1.$$
When \( y = \Delta(x) \), we have \( y = Y_1 Y_2 \cdots Y_k y' \),

\[
F(X_{i-1}X_iX_{i+1}) = Y_i \quad \text{for } 2 \leq i \leq k - 1
\]

and

\[
\Delta(x) = \begin{cases} 
Y_1 F(X_1 X_2 X_3) F(X_2 X_3 X_4) \cdots F(X_{k-2} X_{k-1} X_k) Y_k & (m = 3k) \\
Y_1 F(X_1 X_2 X_3) F(X_2 X_3 X_4) \cdots F(X_{k-2} X_{k-1} X_k) Y_k y_m & (m = 3k + 1) \\
Y_1 F(X_1 X_2 X_3) F(X_2 X_3 X_4) \cdots F(X_{k-2} X_{k-1} X_k) Y_k y_{m-1} y_m & (m = 3k + 2)
\end{cases}
\]

First, we investigate case (a).

**Lemma 3.2.** When \( \Delta(x) = y \), \( X_1 = A \) and \( X_i = A \) or \( B \) (\( 2 \leq i \leq k \)) for a configuration \( x \) of CA-27(m), and we define \( \bar{A} = B \) and \( \bar{B} = A \), then the following properties hold:
1. \( F(X_{i-1}X_iX_{i+1}) = \bar{X}_{i+1} \) (\( 2 \leq i \leq k - 1 \)).
2. \( Y_1 = A \).
3. If \( m = 3k \), then \( Y_k = B \).
4. If \( m = 3k + 1 \) and \( x_m = 0 \), then \( Y_k y_m = A0 \).
5. If \( m = 3k + 1 \) and \( x_m = 1 \), then \( Y_k y_m = B0 \).
6. If \( m = 3k + 2 \) and \( x_{m-1} x_m = 01 \), then \( Y_k y_{m-1} y_m = A11 \).
7. If \( m = 3k + 2 \) and \( x_{m-1} x_m = 11 \), then \( Y_k y_{m-1} y_m = B11 \).

**Proof.** By direct calculations.

**Proposition 3.3.** For any configuration \( x \) of case (a), the following holds.
1. When \( m = 3k \) and \( k = 2l + r \) (\( r = 0 \) or \( 1 \)), we have

\[
\Delta^{k-1}(x) = A^r(AB)^l.
\]

2. When \( m = 3k + 1 \) and \( k = 2l + r \) (\( r = 0 \) or \( 1 \)), we have

\[
\Delta^k(x) = A^{1-r}(AB)^{l+r-1}A0.
\]

3. When \( m = 3k + 2 \) and \( k = 2l + r \) (\( r = 0 \) or \( 1 \)), we have

\[
\Delta^k(x) = A^r(AB)^l11.
\]
Proof.
1. Let $x = AX_2X_3 \cdots X_k$, then the following holds:

$$\Delta^i(x) = \begin{cases} 
    A\overline{X}_{i+2} \cdots \overline{X}_kB(AB)^{\frac{i-1}{2}} & (i : \text{odd}) \\
    AX_{i+2} \cdots X_k(AB)^\frac{i}{2} & (i : \text{even})
\end{cases} \quad (1 \leq i \leq k - 1).$$

By induction on $i$ we have the following.

(a) When $i = 1$ or 2, we have $\Delta(x) = A\overline{X}_3 \cdots \overline{X}_kB$ and $\Delta^2(x) = AX_4 \cdots X_kAB$. So the statement holds when $i = 1$ or 2.

(b) Assume that the statement holds when $1 \leq i \leq k - 2$.

(1) When $i$ is odd, then it follows that

$$\Delta^{i+1}(x) = AX_{i+3} \cdots X_kA(BA)^{\frac{i-1}{2}}B$$
$$= AX_{i+3} \cdots X_k(AB)^{\frac{i+1}{2}}.$$

(2) When $i$ is even, then it follows that

$$\Delta^{i+1}(x) = A\overline{X}_{i+3} \cdots \overline{X}_k(BA)^\frac{i}{2}B$$
$$= A\overline{X}_{i+3} \cdots \overline{X}_kB(AB)^\frac{i}{2}.$$

Thus, the statement holds for $i + 1$.

Let $i = k - 1$, then we have

$$\Delta^{k-1}(x) = \begin{cases} 
    ABA(AB)^{\frac{k-2}{2}} & (k - 1 : \text{odd}) \\
    A(AB)^{\frac{k-1}{2}} & (k - 1 : \text{even})
\end{cases}$$

$$= \begin{cases} 
    (AB)^\frac{k}{2} & (k : \text{even}) \\
    A(AB)^{\frac{k-1}{2}} & (k : \text{odd}).
\end{cases}$$

Thus, we have $\Delta^{k-1}(x) = A^r(AB)^{k}$ when $k = 2l + r$ ($r = 0$ or 1).

2. Parts 2 and 3 are proved in a similar way. \hfill \square

Remarks.
1. When $m = 3k$, $k = 2l + r (r = 0$ or 1) and $X_k = A$, the configuration $x$ reaches $A^r(AB)^l$ with just $m - 3$ steps.

2. When $m = 3k + 1$, $k = 2l + r (r = 0$ or 1) and $X = AX_2 \cdots X_kA$, the configuration $x$ reaches $A^{1-r}(AB)^{l+r-1}A0$ with just $m - 1$ steps.

3. When $m = 3k + 2$, $k = 2l + r (r = 0$ or 1) and $X = AX_2 \cdots X_k01$, the configuration $x$ reaches $A^r(AB)^l11$ with just $m - 2$ steps.
For each configuration $x$ of case (b), we have that $\Delta(x)$ is a configuration of case (a) by direct calculations.

Finally, we investigate the configurations of case (c).

**Lemma 3.4.** We set $y = \delta^2(x)$. If $x_i x_{i+1} x_{i+2} x_{i+3} = 0111$, then $y_i y_{i+1} y_{i+2} y_{i+3} y_{i+4} = 0111$ for $0 \leq i \leq m - 3$.

**Proof.** Let $z = \delta(x)$ and $y = \delta(z)$, then we have $z_{i+1} z_{i+2} z_{i+3} = 011$ and $y_{i+1} y_{i+2} y_{i+3} = 011$. If $z_{i+4} = 0$, we have $y_{i+4} = f(00z_{i+5}) = 1$. If $z_{i+4} = 1$, we have $x_{i+4} x_{i+5} = 00$ for $z_{i+4} = f(1x_{i+4} x_{i+5}) = 1$. Since $z_{i+5} = f(00x_{i+6}) = 1$, we have $y_{i+4} = f(011) = 1$. Therefore, we have $y_{i+1} y_{i+2} y_{i+3} y_{i+4} = 0111$. □

Concerning $F$ and $\Delta$, the following lemma holds.

**Lemma 3.5. (Common to both boundary conditions 0–1 and 0–0.)**

1. When $X = A$ or $B$ and $Y = A$ or $B$, it follows that

$$F(XYC) = \begin{cases} A & (C = 000, 001 \text{ or } 011) \\ B & (C = 100, 101 \text{ or } 111) \end{cases}$$

2. For $x = AX_2 X_3 \cdots X_{n+1} X_{n+3} \cdots X_k x'$ where $X_i = A$ or $B$ ($2 \leq i \leq n + 1$),

   (a) if $C = 000, 001$ or $011$, then

$$\Delta(x) = A X_3 X_4 \cdots X_{n+1} A 111 X'_{n+3} \cdots X_k x'';$$

   (b) if $C = 100$ or $101$, then

$$\Delta^2(x) = A X_4 X_5 \cdots X_{n+1} A 111 X''_{n+3} \cdots X_k x''' .$$

**Remark.** When $n = 0$, $x = ACX_3 X_4 \cdots X_k x'$.

**Proof.**

1. The result is calculated directly.

2. By direct calculations using item 1 above, Lemma 3.2 item 1 and Lemma 3.4, we can obtain the results. □

**Lemma 3.6. (Common to both boundary conditions 0–1 and 0–0.)**

*For any configuration $x = AX_2 X_3 \cdots X_{n+1} 111 X_{n+3} \cdots X_k x'$ where $X_i = A$ or $B$ ($2 \leq i \leq n + 1$), we have

$$\Delta^{2i}(x) = AX_{2i+2} \cdots X_{n+i} A 111 X'_{n+i+3} \cdots X_k x''',$$

for $1 \leq i \leq k - n - 2$.***
Proof. (By induction on \(i\).)
1. When \(i = 1\), \(\Delta^2(x) = AX_4 \cdots X_{n+1} AAA111 X'_{n+4} \cdots X'_{k}x''\) holds from Lemma 3.4 and Lemma 3.5 item 1. So the statement holds when \(i = 1\).
2. Assume that \(\Delta^{2i}(x) = AX_{2i+2} \cdots X_{n+1} A^{3i}111X'_{n+i+3} \cdots X'_{k}x''\). In the same way as for \(i = 1\), it follows that \(\Delta^{2i+2}(x) = AX_{2i+4} \cdots X_{n+1} A^{3i} AAA111X''_{n+i+4} \cdots X''_{k}x''\). Thus, the statement holds for \(i + 1\).

Lemma 3.7. For any configuration \(x = A'111X_{s+2} \cdots X_kx'\) with \(s \geq 0\), we have
\[
\delta^{2i}(x) = (A)_{3s+i}^* 111q \quad \text{for } 1 \leq i \leq m - 3s - 3,
\]
where \(q\) is a sequence with a length of \(m - 3s - i - 3\).

Proof. (By induction on \(i\).)
1. When \(i = 1\), from \(\delta(x) = (100)^*100X'_{s+2} \cdots X'_{k}x''\) and Lemma 3.4, we have
\[
\delta^2(x) = (110)^*0111q
= 0(110)^*111q
= (A)_{3s+1}^* 111q,
\]
for a sequence \(q\) with a length of \(m - 3s - 4\). So, the statement holds when \(i = 1\).
2. Assume that it holds when \(1 \leq i \leq m - 3s - 4\).
   (a) When \(i = 3j\), as \(\delta^{2i}(x) = A^{i+j}111X'_{s+j+2} \cdots X'_{k}x''\), it follows that \(\delta^{2i+2}(x) = (A)_{3(i+j)+1}^* 111q'\) for a sequence \(q'\) with a length of \(m - 3(s + j) - 4\) in the same way as for the case \(i = 1\). So we have \(\delta^{2(i+1)}(x) = (A)_{3s+i+1}^* 111q'\).
   (b) When \(i = 3j + 1\) or \(3j + 2\), we can prove in a similar way to (a) above. Thus, the statement holds for \(i + 1\). \(\square\)

Lemma 3.8. For the configuration \(x = (A)_{m-3}^* 111\), we have
\[
\delta^4(x) = (A)_{m-1}^* 0.
\]

Proof.
1. When \(m = 3k\), since \(x = (A)_{3k-3}^* 111 = A^{k-1}111\), it follows that \(\delta(x) = (100)^{k-1}100\), \(\delta^2(x) = (011)^{k-1}011\), \(\delta^3(x) = 110(010)^{k-2}010\) and \(\delta^4(x) = 101(101)^{k-2}100 = 10(110)^{k-1}0 = (A)_{3(k-1)+2}^* 0 = (A)_{m-1}^* 0.\)
2. When \( m = 3k + 1 \) or \( 3k + 2 \), we can prove in a similar way to 1 above.

**Proposition 3.9.** For any configuration \( x = AX_2X_3 \cdots X_{n+1}111X_{n+3} \cdots X_kx' \) where \( X_i = A \) or \( B \) \((2 \leq i \leq n + 1)\), we have

\[
\delta^{2m-6n-8}(x) = (A)^{m-1}_m 0.
\]

**Proof.**

1. If \( n = 2j \) and \( i = j \) in Lemma 3.6, then we have \( y = \Delta^4(x) = AA^3Y_{3j+3} \cdots Y_ky' = A^{3j+1}Y_{3j+3} \cdots Y_ky' \). Moreover, let \( i = m - 3(3j + 1) - 3 = m - 9j - 6 \) in Lemma 3.7, then we have

\[
\delta^{2(m-9j-6)}(y) = \delta^{2m-9n-12}(y) = (A)^{m-3}_m 111.
\]

By Lemma 3.8, it follows that

\[
\delta^{3n+(2m-9n-12)+4}(x) = \delta^{2m-6n-8}(x) = (A)^{m-1}_m 0.
\]

2. If \( n = 2j + 1 \) and \( i = j \) in Lemma 3.6, then we have \( y = \Delta^2(x) = AX_nA^3Y_{3j+4} \cdots Y_ky' \) and \( z = \Delta^2(y) = \Delta^{n+1}(x) = AA^3Z_{3j+5} \cdots Z_kz' = A^{3j+3}Z_{3j+5} \cdots Z_kz' \). Let \( i = m - 3(3j + 3) - 3 = m - 9j - 12 \) in Lemma 3.7, and we have \( \delta^{2m-9n-15}(z) = \delta^{2m-6n-8}(x) = (A)^{m-1}_m 111 \). By Lemma 3.8, it follows that

\[
\delta^{3(n+1)+(2m-9n-15)+4}(x) = \delta^{2m-6n-8}(x) = (A)^{m-1}_m 0.
\]

**Proposition 3.10.** For the configuration \( x = (A)^{m-1}_m 0 \), the following holds.

1. When \( m = 3k \) and \( k = 2l + r \) \((r = 0 \text{ or } 1)\), we have

\[
\delta^{m-7}(x) = A' (AB)^l.
\]

2. When \( m = 3k + 1 \) and \( k = 2l + r \) \((r = 0 \text{ or } 1)\), we have

\[
\delta^{m-7}(x) = A^{1-r} (AB)^l + r-1 A0.
\]

3. When \( m = 3k + 2 \) and \( k = 2l + r \) \((r = 0 \text{ or } 1)\), we have

\[
\delta^{m-7}(x) = A' (AB)^l 111.
\]

**Proof.**

1. As \( x = (A)^{3k-1}_m 0 = 10(110)^{k-1} 0 = 10(110)^{k-2} 1100 \), so \( \delta(x) = 00(100)^{k-2} 1011 \) and \( \delta^2(x) = 11(011)^{k-2} 0010 = (110)^{k-1} 010 = A^{k-1}B \). Since this shows \( \Delta^2(X) = AA^{k-3}AB \) in Proposition 3.3 item 1, it follows that \( \Delta^{(k-1)-2}(A^{k-1}B) = \Delta^{k-3}(A^{k-1}B) = A' (AB)^l \). Thus, we have \( \delta^{2+3(k-3)}(x) = \delta^{3k-7}(x) = \delta^{m-7}(x) = A' (AB)^l \).

2. Items 2 and 3 are proved in a similar way.
When $m = 3k$, each configuration $x$ reaches $y = AY_2Y_3 \cdots Y_k$ within 5 steps by Lemma 3.1, and $y$ is a configuration of case (a), case (b) or case (c) stated above. After some steps, $y$ reaches $A'(AB)^l$ by Proposition 3.3, Lemma 3.5 item 2, Proposition 3.9 and Proposition 3.10. Since

$$ \Delta(A'(AB)^l) = A'(AB)^l, $$

configuration $A'(AB)^l$ is a fixed point or a configuration on a limit cycle of period length 3 about $\delta$. However, $A'(AB)^l$ is not a fixed point by $\delta(110p) = 10q$, so $A'(AB)^l$ is a configuration on a limit cycle of period length 3. Thus, there exists a unique limit cycle of period length 3.

Let $x = A'(AB)^l$, then the other configurations on the limit cycle are as follows.

- When $r = 0$, we have

$$ \delta(x) = (101100)^l, \quad \delta^2(x) = (001011)^l. $$

- When $r = 1$, we have

$$ \delta(x) = 100(101100)^l, \quad \delta^2(x) = 011(001011)^l. $$

To see whether there are entrances to the limit cycle other than $A'(AB)^l$, recall the discussion in Proposition 3.3 item 1. We have

$$ \Delta^{k-2}(x) = \begin{cases} A\overline{X}_kB(AB)^{l-1} & (k - 2 : \text{odd}) \\ AX_k(AB)^{l-1} & (k - 2 : \text{even}) \end{cases} $$

is not a configuration on the limit cycle and the remaining configurations on the limit cycle must appear while $\Delta^{k-2}(x)$ transfers to $\Delta^{k-1}(x)$. $\Delta^{k-2}(x)$ is not a configuration on the limit cycle when $X_k = A$. So we have the following.

- When $r = 0$ and $x = A^2(AB)^{l-1}$,

$$ \delta(x) = 100100(101100)^{l-1}, \quad \delta^2(x) = 011011(001011)^{l-1}. $$

- When $r = 1$ and $x = AB^2(AB)^{l-1}$,

$$ \delta(x) = 101101100(101100)^{l-1}, \quad \delta^2(x) = 001001011(001011)^{l-1}. $$

The above two cases contain no configurations on the limit cycle. Thus, the transient length is the maximum of the transition steps which each configuration needs to reach $A'(AB)^l$.

When $m = 3k + 1$ or $m = 3k + 2$, we can consider in a similar manner.
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PROPOSITION 3.11. For the configuration \( x = ACX_3X_4 \cdots X_kx' \), we have

\[
h(x) = \begin{cases} 
3m - 12 & (C = 000, \, 001 \text{ or } 011) \\
3m - 9 & (C = 100 \text{ or } 101) \\
3m - 15 & (C = 111).
\end{cases}
\]

Proof. By Lemma 3.5 item 2, Proposition 3.9 and Lemma 3.10, it follows that

\[
h(x) = \begin{cases} 
3 + (2m - 6n - 8) + (m - 7) = 3m - 6n - 12 & (C = 000, \, 001 \text{ or } 011) \\
6 + (2m - 6n - 8) + (m - 7) = 3m - 6n - 9 & (C = 100 \text{ or } 101) \\
(2m - 6n - 8) + (m - 7) = 3m - 6n - 15 & (C = 111),
\end{cases}
\]

for \( x = AX_2Y_3 \cdots X_{n+1}CX_{n+3} \cdots X_kx' \) where \( X_i = A \) or \( B \) (\( 2 \leq i \leq n + 1 \)). Let \( n = 0 \), then we have \( x = ACX_3X_4 \cdots X_kx' \) and

\[
h(x) = \begin{cases} 
3m - 12 & (C = 000, \, 001 \text{ or } 011) \\
3m - 9 & (C = 100 \text{ or } 101) \\
3m - 15 & (C = 111).
\end{cases}
\]

\( \square \)

LEMMA 3.12. (Common to both boundary conditions 0–1 and 0–0.) \( \delta(x) \) does not contain the subsequence 1010 for any configuration \( x \).

Proof. Let \( y = \delta(x) \) and \( y_{i+1}y_{i}y_{i+1}y_{i+2}y_{i+3} = 1010 \) with \( 1 \leq i \leq m - 3 \). For \( y_{i+1} = 0 \), it follows that \( x_i x_{i+1} x_{i+2} = 111, 110, 101 \) or 010. As \( f(110) = f(111) = 0, x_i x_{i+1} x_{i+2} = 111 \) contradicts \( y_{i+1} = 1 \). When \( x_i x_{i+1} x_{i+2} = 110 \) or 010, we have that \( x_{i+3} = 0 \) for \( y_{i+2} = 1 \). However, this contradicts \( y_{i+3} = 0 \) by \( f(000) = f(001) = 1 \). As \( f(010) = f(110) = 0, x_i x_{i+1} x_{i+2} = 101 \) contradicts \( y_{i} = 1 \). \( \square \)

THEOREM 3.13. CA–27(m) with boundary condition 0–1 has a unique limit cycle of period length 3 and its transient length is \( 5m - 7 \).

Proof. When \( C = 100, \, 101 \) in Proposition 3.11, then \( x = 110100X_3X_4 \cdots X_kx' \) or \( 110101X_3X_4 \cdots X_kx' \). However, no predecessor of the configuration \( x \) exists by Lemma 3.12. About the other cases we have \( H \leq 5 + (3m - 12) = 3m - 7 \) from Lemma 3.1 and Proposition 3.11. In particular, if we take \( x = 1010q \), it follows that \( \delta(x) = 000q' \) and that \( \delta^2(x) = 111q'' \). Let \( s = 0 \) and \( i = 3 \) in Lemma 3.7, then we have \( \delta^6(111q'') = 110111q''' \). Thus, it follows that \( h(x) = 2 + 6 + 3m - 15 = 3m - 7 = H \) from the case \( C = 111 \) in Proposition 3.11. \( \square \)
Remarks.
1. When \( m = 3k \) and \( k = 2l + r \) (\( r = 0 \) or \( 1 \)),
   \[
   A^r(AB)^{l_1}, \quad (100)^r(101100)^l, \quad (011)^r(001011)^l
   \]
   make a limit cycle.
2. When \( m = 3k + 1 \) and \( k = 2l + r \) (\( r = 0 \) or \( 1 \)),
   \[
   A^{1-r}(AB)^{l_1+r-1}A0, \quad (100)^{1-r}(101100)^{l_1+r-1}1011, \quad (011)^{1-r}(001011)^{l_1+r-1}0010
   \]
   make a limit cycle.
3. When \( m = 3k + 2 \) and \( k = 2l + r \) (\( r = 0 \) or \( 1 \)),
   \[
   A^r(AB)^{l_111}, \quad (100)^r(101100)^{l_110}, \quad (011)^r(001011)^{l_100}
   \]
   make a limit cycle.

4. Boundary condition 0–0

We can consider the boundary condition 0–0 in a similar way to the boundary condition 1–0. We adopt the same notations \( A, B, C, \Delta, F \) and \( \overline{X}_i \) as in the preceding section.

**Lemma 4.1.** When \( \Delta(x) = y \), \( X_1 = A \) and \( X_i = A \) or \( B \) (\( 2 \leq i \leq k \)) for a configuration \( x \) of \( CA–27(m) \), the following properties hold.
1. \( F(X_{i-1}X_iX_{i+1}) = \overline{X}_{i+1} \) (\( 2 \leq i \leq k - 1 \)).
2. \( Y_1 = A \).
3. If \( m = 3k \), then \( Y_k = 111 \).
4. If \( m = 3k \), then \( F(X_{i-1}X_i111) = B \) (\( 2 \leq i \leq k - 1 \)).
5. If \( m = 3k + 1 \) and \( x_m = 0 \), then \( Y_ky_m = A1 \).
6. If \( m = 3k + 1 \) and \( x_m = 1 \), then \( Y_ky_m = B0 \).
7. If \( m = 3k + 2 \) and \( x_{m-1}x_m = 01 \), then \( Y_ky_{m-1}y_m = A00 \).
8. If \( m = 3k + 2 \) and \( x_{m-1}x_m = 11 \), then \( Y_ky_{m-1}y_m = B00 \).
9. If \( m = 3k + 2 \) and \( x_{m-1}x_m = 00 \), then \( Y_ky_{m-1}y_m = A11 \).

**Proof.** The results can be obtained by direct calculations. \( \square \)

**Lemma 4.2.** When \( \delta(x) = y \), we have
\[
y_m = 1 - x_m
\]
for any configuration \( x \).
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Proof. Since \( f(000) = f(100) = 1 \), we have \( y_m = 1 = 1 - x_m \) when \( x_m = 0 \). Since
\( f(010) = f(110) = 0 \), we have \( y_m = 0 = 1 - x_m \) when \( x_m = 1 \).

\[ \square \]

**Lemma 4.3.** For any configuration \( x = A^i111X_{s+2} \cdots X_kx' \) with \( s \geq 0 \), we have
\[
\delta^{2i}(x) = (A|_{s+1}^{s+i}111q \quad \text{for}\ 1 \leq i \leq m - 3s - 4,
\]
where \( q \) is a sequence with a length of \( m - 3s - i - 3 \).

**Proof.** Cf. Lemma 3.7.

\[ \square \]

**Proposition 4.4.** For any configuration \( x \) of case (a), the following holds.
1. When \( m = 3k \), we have
\[
\Delta^{k-1}(x) = A^{k-1}111 \text{ or } AB^{k-1}.
\]
2. When \( m = 3k + 1 \), we have
\[
\Delta^k(x) = A^k1 \text{ or } AB^{k-1}0.
\]
3. When \( m = 3k + 2 \), we have
\[
\Delta^k(x) = A^k11 \text{ or } AB^{k-1}00.
\]

**Proof.**
1. Let \( x = AX_2X_3 \cdots X_k \), then the following holds:
\[
\Delta^i(x) = \begin{cases} 
A\overline{X}_{i+2} \cdots \overline{X_k}A^{i-1}111 & (i : \text{odd}) \\
AX_{i+2} \cdots X_kB^i & (i : \text{even}) 
\end{cases} \quad (1 \leq i \leq k - 1).
\]
By induction on \( i \) we have the following.

(a) When \( i = 1 \) or 2, we have \( \Delta(x) = A\overline{X}_3 \cdots \overline{X_k}111 \) and that \( \Delta^2(x) = AX_4 \cdots X_kBB \). So the statement holds when \( i = 1 \) or 2.

(b) Assume that the subject holds when \( 1 \leq i \leq k - 2 \).

1. When \( i \) is odd, it follows that
\[
\Delta^{i+1}(x) = AX_{i+3} \cdots X_kB^{i-1}BB = AX_{i+3} \cdots X_kB^{i+1}.
\]

2. When \( i \) is even, it follows that
\[
\Delta^{i+1}(x) = A\overline{X}_{i+3} \cdots \overline{X_k}A'^{111}.
\]
Thus, the given equation holds for \( i + 1 \).
Let \( i = k - 1 \). Then we have

\[
\Delta^{k-1}(x) = \begin{cases} 
AA^{k-2}111 & (k - 1 \text{: odd}) \\
AB^{k-1} & (k - 1 \text{: even})
\end{cases}
\]

Thus, we have \( \Delta^{k-1}(x) = A^{k-1}111 \) or \( AB^{k-1} \) when \( m = k \).

2. Items 2 and 3 are shown in a similar way.

Remarks.
1. When \( m = 3k \) and \( X_k = A \), the configuration \( x \) reaches \( A^{k-1}111 \) or \( AB^{k-1} \) with just \( m - 3 \) steps.
2. When \( m = 3k + 1 \) and \( x = AX_2 \cdots X_{k-1}A0 \) or \( AX_2 \cdots X_{k-1}B1 \), the configuration \( X \) reaches \( A^k \) or \( AB^{k-1}0 \) with just \( m - 4 \) steps.
3. When \( m = 3k + 2 \) and \( x = AX_2 \cdots X_k01 \), the configuration \( X \) reaches \( A^k11 \) or \( AB^{k-1}00 \) with just \( m - 2 \) steps.

**Proposition 4.5.** For any configuration \( x = AX_2X_3 \cdots X_{n+1}111X_{n+3} \cdots X_kx' \) where \( X_i = A \) or \( B \) (\( 2 \leq i \leq n + 1 \)), we have

\[
\delta^{2m-6n-14}(x) = [A]_{m-4}^{*}111a.
\]

**Proof.**
1. If \( n = 2j \) and \( i = j \) in Lemma 3.6, then we have \( y = \Delta^n(x) = AA^{3j+1}111Y_{3j+3} \cdots Y_ky' = A^{3j+1}111Y_{3j+3} \cdots Y_ky' \). Furthermore, let \( i = m - 3(3j + 1) - 4 = m - 9j - 7 \) in Lemma 4.3, then it follows that \( \delta^{2(m-9j-7)}(y) = \delta^{2m-9n-14}(y) = (A)_{m-4}^{*}111a \). Thus, we have \( \delta^{3n+2m-9n-14}(x) = \delta^{2m-6n-14}(x) = (A)_{m-3}^{*}111a \).
2. If \( n = 2j + 1 \) and \( i = j \) in Lemma 3.6, then we have \( y = \Delta^{2j}(x) = AX_{n+1}A^{3j+1}111Y_{3j+4} \cdots Y_ky' \) and \( z = \Delta^2(y) = \Delta^{n+1}(x) = AA^{3j+1}A^{3j+1}111Z_{3j+5} \cdots Z_kz' = A^{3j+1}111Z_{3j+5} \cdots Z_kz' \). Let \( i = m-3(3j+3)-4 = m-9j-13 \) in Lemma 4.3, it follows that \( \delta^{2(m-9j-13)}(z) = \delta^{2m-9n-17}(z) = (A)_{m-3}^{*}111a \). Thus, we have \( \delta^{3(n+1)+(2m-9n-17)}(x) = \delta^{2m-6n-14}(x) = (A)_{m-3}^{*}111a \).

**Proposition 4.6.** For configurations \( x = (A)_{m-4}^{*}111x_m \), the following holds.
1. When \( m = 3k \), we have

\[
\delta^{m-1}(x) = AB^{k-1} \text{ or } A^{k-1}111.
\]
2. When \( m = 3k + 1 \), we have \( \delta^{m-1}(x) = A B^{k-1}0 \text{ or } A^k1 \).

3. When \( m = 3k + 2 \), we have \( \delta^{m-1}(x) = A B^{k-1}100 \text{ or } A^k11 \).

**Proof.**

1. Since \( x = (110)^{3k}1111x_m = 10(110)^{k-2}1111x_m \), if \( x_m = 1 \), then \( x = 10(110)^{k-2}1111 \) and it follows that \( \delta(x) = 00(100)^{k-2}1000 = (001)^{k-1}000 \) and that \( \delta^2(x) = A^{k-1}111 \). Thus, we have \( \delta^{m-1}(x) = \delta^{3k-3}(A^{k-1}111) = A^{k-1}111 \) or \( AB^{k-1} \). If \( x_m = 0 \), then \( x = 10(110)^{k-2}1110 \), \( \delta(x) = 00(100)^{k-2}1001 \) and \( \delta^2(x) = 11(011)^{k-2}0110 = A^{k} \). Thus, we have \( \delta^{2+3(k-1)}(x) = \delta^{m-1}(x) = A^{k-1}111 \) or \( AB^{k-1} \) from Proposition 4.4 item 1.

2. Items 2 and 3 are shown in a similar way.

**Remarks.**

1. When \( m = 3k \) and \( x_m = 0 \), the configuration \( x \) reaches \( A^{k-1}111 \) or \( AB^{k-1} \) with just \( m - 1 \) steps.

2. When \( m = 3k + 1 \) and \( x_m = 0 \), the configuration \( x \) reaches \( A^k1 \) or \( AB^{k-1}0 \) with just \( m - 1 \) steps.

3. When \( m = 3k + 2 \) and \( x_m = 0 \), the configuration \( x \) reaches \( A^k11 \) or \( AB^{k-1}100 \) with just \( m - 1 \) steps.

When \( m = 3k \), each configuration \( x \) reaches \( y = AY_2Y_3\cdots Y_k \) within 5 steps by Lemma 3.1, and \( y \) is a configuration of case (a), case (b) or case (c) stated above. After some steps, \( y \) reaches \( A^{k-1}111 \) or \( AB^{k-1} \) by Proposition 4.4, Lemma 3.4 item 2, and Propositions 4.5 and 4.6. Since

\[
\Delta(A^{k-1}111) = AB^{k-1} \quad \text{and} \quad \Delta(AB^{k-1}) = A^{k-1}111,
\]

configurations \( A^{k-1}111 \) and \( AB^{k-1} \) are on a limit cycle of period length 6. Thus, there exists a unique limit cycle of period length 6.

Let \( x = A^{k-1}111 \), then the remaining configurations on the limit cycle are as follows:

\[
\delta(x) = (100)^k, \quad \delta^2(x) = (011)^k, \quad \delta^3(x) = AB^{k-1}, \quad \delta^4(x) = (101)^k, \quad \delta^5(x) = (001)^{k-1}000.
\]
We can consider the transient length when $x_m = 0$ in Proposition 4.6. To see whether there are entrances to the limit cycle other than $A^{k-1}111$ or $AB^{k-1}$, recall the discussion in Proposition 4.4 item 1. We have that

$$\Delta^{k-2}(X) = \begin{cases} 
ABX_kA^{k-3}111 & (k - 2 : \text{odd}) \\
AX_kB^{k-2} & (k - 2 : \text{even}) 
\end{cases}$$

is not a configuration on the limit cycle and the other configurations on the limit cycle must appear while $\Delta^{k-2}(x)$ transfers to $\Delta^{k-1}(x)$. As $\Delta^{k-2}(x)$ is not a configuration on the limit cycle when $x_k = A$, so for $x = ABA^{k-3}111$, we have

$$\delta(x) = 101(100)^{k-1}, \quad \delta^2(x) = 001(011)^{k-1}$$

and for $x = A^2B^{k-2}$, we have

$$\delta(x) = 100(101)^{k-1}, \quad \delta^2(x) = 011(001)^{k-2}000.$$ 

Neither of them contains a configuration on the limit cycle. Thus, the transient length is the maximum of the transition steps which each configuration needs to reach $A^{k-1}111$ or $AB^{k-1}$.

When $m = 3k + 1$ or $3k + 2$, we can consider in a similar manner.

**PROPOSITION 4.7.** For the configuration $x = ACX_3X_4\cdots X_kx'$, we have

$$h(x) \leq \begin{cases} 
3m - 12 & (C = 000, \; 001 \text{ or } 011) \\
3m - 9 & (C = 100 \text{ or } 101) \\
3m - 15 & (C = 111). 
\end{cases}$$

**Proof.** By Lemma 3.5 item 2, Proposition 3.9 and Lemma 3.10, it follows that

$$h(x) \leq \begin{cases} 
3 + (2m - 6n - 14) + (m - 1) = 3m - 6n - 12 & (C = 000, \; 001 \text{ or } 011) \\
6 + (2m - 6n - 14) + (m - 1) = 3m - 6n - 9 & (C = 100 \text{ or } 101) \\
(2m - 6n - 14) + (m - 1) = 3m - 6n - 15 & (C = 111). 
\end{cases}$$

for $x = AX_2X_3\cdots X_{n+1}CX_{n+3}\cdots X_kx'$ where $X_i = A$ or $B$ $(2 \leq i \leq n + 1)$. Let $n = 0$, then we have $x = ACX_3X_4\cdots X_kx'$ and

$$h(x) \leq \begin{cases} 
3m - 12 & (C = 000, \; 001 \text{ or } 011) \\
3m - 9 & (C = 100 \text{ or } 101) \\
3m - 15 & (C = 111). 
\end{cases}$$

$\square$
Remark. When \( C = 000, 001 \) or \( 011 \) and \( z_m = 1 \), the above equality holds. For the other cases, the above equality holds when \( x_m = 0 \).

**Theorem 4.8.** CA-27\((m)\) with boundary condition 0-0 has a unique limit cycle of period length 6 and its transient length is \( 5m - 7 \).

**Proof.** When \( C = 100, 101 \) in Proposition 4.7, then \( x = 110100X_3 \cdots X_kx' \) or \( 110101X_3 \cdots X_kx' \). However, from Lemma 3.12, no predecessor of the configuration \( x \) exists. Regarding the other cases, we have \( H \leq 5 + (3m - 12) = 3m - 7 \) from Lemma 3.1 and Proposition 4.7. If we take \( x = 1010q0 \) considering Lemma 4.1, it follows that \( \delta(x) = 000q'1 \) and that \( \delta^2(x) = 111q''0 \). Let \( s = 0 \) and \( i = 3 \) in Lemma 4.3, then we have \( \delta^6(111q''0) = 110111q''0 \). Thus, it follows that \( h(x) = 2 + 6 + 3m - 15 = 3m - 7 = H \) from \( C = 111 \) in Proposition 4.7. □

**Remarks.**
1. When \( m = 3k \),

\[
A^{k-1}111, \ (100)^k, \ (011)^k, \ AB^{k-1}, \ (101)^k, \ (001)^{k-1}000
\]

make a limit cycle.
2. When \( m = 3k + 1 \),

\[
A^k1, \ (100)^k0, \ (011)^k1, \ AB^{k-1}0, \ (101)^k1, \ (001)^k0
\]

make a limit cycle.
3. When \( m = 3k + 2 \),

\[
A^k11, \ (100)^k10, \ (011)^k01, \ AB^{k-1}00, \ (101)^k11, \ (001)^k00
\]

make a limit cycle.

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Tatsuro Sato
Department of Mathematics
Oita National College of Technology
1666 Maki Oita 870–01, Japan