AN ESTIMATE OF A DIRICHLET SERIES OF KLOOSTERMANN TYPE

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0. Introduction

Let Γ be a Fuchsian group of the first kind which has a cusp at ∞ , and let q be the smallest positive number such that $\begin{pmatrix} 1 & q \\ & 1 \end{pmatrix} \in \Gamma$. For $s \in \mathbb{C}$ and $m \in \mathbb{Z} - \{0\}$, we define the Dirichlet series $\Phi_m(s, \Gamma)$ as

$$\Phi_m(s, \Gamma) = \sum_{c>0} \frac{S(m, c, \Gamma)}{c^{2s}}, \quad (\text{A})$$

where

$$S(m, c, \Gamma) = \sum_{0 \leq a < qc} e\left(\frac{ma}{qc}\right), \quad \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma \ (c > 0)$$

and $e(x) = \exp(2\pi ix)$. This series appears in the m th Fourier coefficient of the Eisenstein series $E(z, s, \Gamma)$ (cf (1.10)), as well as in the constant term of the non-holomorphic Poincaré series $P_m(z, s, \Gamma)$ (cf (1.6)).

From the trivial estimate $|S(m, c, \Gamma)| \leq qc$, it follows that $\Phi_m(s, \Gamma)$ is absolutely convergent and therefore holomorphic for $\operatorname{Re} s > 1$. It is the purpose of this paper to continue $\Phi_m(s)$ meromorphically to the larger domain $\operatorname{Re} s > 1/2$, and to estimate it as a function of $|m|$ and $|\operatorname{Im} s|$. To this end we consider the inner product between the Poincaré series $P_m(z, s, \Gamma)$ and the function $E_b(z, \bar{w}, \Gamma)$ defined in (1.7). Based on the fact that Φ_m occurs in the constant term of the series $P_m(z, s, \Gamma)$, the inner product can be expressed in terms of $\Phi_m(s, \Gamma)$ (Theorem 2.3). Using this formula, we obtain our main result (Theorem 3.6):

$$\Phi_m(s, \Gamma) = O\left(\frac{|m|^\sigma |\tau|^{1/2}}{(\sigma - 1/2)^{5/2}}\right) \quad (\text{B})$$

for $s = \sigma + i\tau$, $1/2 < \sigma < M$ and $|\tau| \geq 1$, where $M > 1$ is arbitrary and the implied constant depends on M and Γ .

On the other hand, according to Hejhal [2, p. 164, Theorem 12.9, Note for chapter six p. 131]), the following is known to be true:

$$\Phi_m(s, \Gamma) = O\left(\omega(\tau)^{1/2} \exp(\varepsilon|\tau| + 5\pi|m|\eta)\right) \quad (\text{C})$$

for $s = \sigma + i\tau$, $1/2 \leq \sigma \leq 3/2$ and $|\tau| \geq 1$; the implied constant depends on Γ and $\mathcal{D}$ (the fundamental domain of Γ); the symbols ε and η mean positive constants with η depending only on Γ , but not on $|m|$; $\omega(\tau)$ is a function of τ satisfying the properties $\omega(-\tau) = \omega(\tau)$, $\omega(\tau) \geq 1$ and $\int_0^R \omega(\tau) d\tau = O(R^2)$. A more explicit estimate of $\omega(\tau)$ is not known. In comparison to (C), our estimate (B) is explicit in $|\tau|$ and does not include the factor $\exp(\varepsilon|\tau|)$ appearing in (C). Moreover, with respect to $|m|$, the estimate (B) is better than (C) for $\sigma > 1/2$.

1. Preliminaries

Let $H = \{z = x + iy \in \mathbb{C} \mid \text{Im } z = y > 0\}$ be the complex upper half plane given the Riemannian structure

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \quad (1.1)$$

The group $G = PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$ acts on H by means of a linear fractional transformation:

$$\gamma z = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

The metric (1.1) gives the G -invariant measure

$$d\mu(z) = \frac{dx dy}{y^2}. \quad (1.2)$$

Throughout this paper, we always assume that Γ is a Fuchsian group of the first kind with a cusp ∞ . Therefore the fundamental domain $\mathcal{D} = \Gamma \backslash H$ is noncompact. Let q be the smallest positive number such that $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in \Gamma$, then the stabilizer Γ_∞ of a cusp ∞ is expressed as

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & nq \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

The Laplace operator D associated with the metric (1.1) is

$$D = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \quad (1.3)$$

For $m \in \mathbb{Z} - \{0\}$, we will define the exponential sum $S(m, c, \Gamma)$ and the Dirichlet series $\Phi_m(s, \Gamma)$ as follows:

$$S(m, c, \Gamma) := \sum_{0 \leq a < qc} e\left(\frac{ma}{qc}\right), \quad \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma \quad (c > 0), \quad (1.4)$$

$$\Phi_m(s, \Gamma) := \sum_{c > 0} \frac{S(m, c, \Gamma)}{c^{2s}}, \quad (1.5)$$

where $e(x) = \exp(2\pi i x)$. Since $|S(m, c, \Gamma)| \leq qc$, the series (1.5) converges absolutely for $\operatorname{Re} s > 1$; it is holomorphic in the region.

Suppose that $m \in \mathbb{Z} - \{0\}$. For $z \in H$ and $s \in \mathbb{C}$, the non-holomorphic Poincaré series $P_m(z, s, \Gamma)$ is defined by

$$P_m(z, s, \Gamma) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \exp\left(\frac{2\pi}{q}(imx(\gamma z) - |m|y(\gamma z))\right) y(\gamma z)^s, \quad (1.6)$$

where $x(\gamma z) = \operatorname{Re}(\gamma z)$ and $y(\gamma z) = \operatorname{Im}(\gamma z)$.

Suppose that $b \in \mathbb{R} - \{0\}$, $z \in H$ and $s \in \mathbb{C}$. We define the series $E_b(z, s, \Gamma)$ as

$$E_b(z, s, \Gamma) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \exp\left(-2\pi \frac{|b|}{q} y(\gamma z)\right) y(\gamma z)^s. \quad (1.7)$$

Let $L^2(\mathcal{D})$ be the Hilbert space consisting of all functions which are Γ -automorphic and square-integrable on $\mathcal{D}$:

$$L^2(\mathcal{D}) = \left\{ f \left| \begin{array}{l} f(\gamma z) = f(z), \gamma \in \Gamma, \\ \int_{\mathcal{D}} |f(z)|^2 d\mu(z) < \infty \end{array} \right. \right\}.$$

The series (1.6) and (1.7) converge absolutely for $\operatorname{Re} s > 1$ and belong to the space $L^2(\mathcal{D})$. Moreover the series (1.6) satisfies the recursion relation for $\operatorname{Re} s > 1$:

$$\{s(1-s) + D\} P_m(z, s, \Gamma) = -4\pi \frac{|m|}{q} s P_m(z, s+1, \Gamma),$$

or equivalently

$$P_m(z, s, \Gamma) = -4\pi \frac{|m|}{q} s \mathcal{R}_\lambda P_m(z, s+1, \Gamma), \quad (1.8)$$

where $\mathcal{R}_\lambda := (\lambda + D)^{-1}$, with $\lambda = s(1 - s)$, denotes the resolvent of the Laplace operator D (cf (1.3)). The resolvent $\mathcal{R}_\lambda$ is holomorphic in $\operatorname{Re} s > 1/2$ except at most a finite number of simple poles at the points $s = s_j$ for $1/2 < s_j < 1$.

We set $a_m(y, s, n, \Gamma)$ to be the n th Fourier coefficient of $P_m(z, s, \Gamma)$, i.e.,

$$\begin{cases} P_m(z, s, \Gamma) = \sum_{n=-\infty}^{\infty} a_m(y, s, n, \Gamma) e\left(\frac{n}{q}x\right), \\ a_m(y, s, n, \Gamma) = \frac{1}{q} \int_0^q P_m(x + iy, s, \Gamma) e\left(-\frac{n}{q}x\right) dx. \end{cases} \quad (1.9)$$

For $z \in H$ and $s \in \mathbb{C}$, the Eisenstein series $E(z, s, \Gamma)$ is defined by

$$E(z, s, \Gamma) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^s.$$

This converges absolutely for $\operatorname{Re} s > 1$ ([3, Theorem 2.1.1]) and has the following Fourier expansion ([3, §2.2]):

$$\begin{cases} E(z, s, \Gamma) = \sum_{n=-\infty}^{\infty} c(y, s, n, \Gamma) e\left(\frac{n}{q}x\right), \\ c(y, s, n, \Gamma) = \delta_{0n} y^s + \frac{1}{q} y^{1-s} \sum_{c>0} S(n, c, \Gamma) c^{-2s} \int_{-\infty}^{\infty} \frac{e(-nyt/q)}{(1+t^2)^s} dt, \end{cases} \quad (1.10)$$

where $\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma$ and δ_{0n} is the Kronecker symbol. The Fourier coefficient $c(y, s, n, \Gamma)$ can be expressed in terms of the modified Bessel function, however we will use formula (1.10) in this paper. One notes that the Dirichlet series $\Phi_n(s, \Gamma)$ appears in the second term of $c(y, s, n, \Gamma)$.

2. Calculation of the inner product $\langle P_m(z, s, \Gamma), E_b(z, \bar{w}, \Gamma) \rangle$

The aim of this section is to state an expression of the inner product $\langle P_m(z, s, \Gamma), E_b(z, \bar{w}, \Gamma) \rangle$ in terms of the Dirichlet series $\Phi_m(z, s)$ (Theorem 2.3). The Rankin–Selberg method gives

$$\langle P_m(z, s, \Gamma), E_b(z, \bar{w}, \Gamma) \rangle = q \int_0^\infty a_m(y, s, 0, \Gamma) \exp\left(-2\pi \frac{|b|}{q} y\right) y^{w-2} dy \quad (2.1)$$

for $\operatorname{Re} s > 1$ and $\operatorname{Re} w > 1$, where $m \in \mathbb{Z} - \{0\}$, $b \in \mathbb{R} - \{0\}$, $\bar{w}$ is the complex conjugate of w , and $a_m(y, s, 0, \Gamma)$ is the constant term of the series $P_m(z, s, \Gamma)$. For the coefficient $a_m(y, s, 0, \Gamma)$, we state the following lemma.

LEMMA 2.1. *Under the condition $1 < \operatorname{Re} s < 2$, the constant term $a_m(y, s, 0, \Gamma)$ of the non-holomorphic Poincaré series $P_m(z, s, \Gamma)$ is expressed as*

$$\begin{aligned} a_m(y, s, 0, \Gamma) &= 2^{2(1-s)} \frac{\pi s}{q \Gamma(s) \Gamma(2-s)} y^{1-s} \sum_{c>0} S(m, c, \Gamma) c^{-2s} \\ &\quad \times \int_0^1 \exp\left(-\frac{\pi |m|u}{q c^2 y}\right) u^{2(s-1)} (1-u)^{1-s} du \\ &\quad - 2^{2(1-s)} \frac{\pi^2}{q^2 \Gamma(s) \Gamma(2-s)} |m| y^{-s} \sum_{c>0} S(m, c, \Gamma) c^{-2(1+s)} \\ &\quad \times \int_0^1 \exp\left(-\frac{\pi |m|u}{q c^2 y}\right) u^{2s-1} (1-u)^{1-s} du. \end{aligned}$$

Proof. From [5, Theorem B], we have

$$\begin{aligned} a_m(y, s, 0, \Gamma) &= 2^{3-2s} \frac{\pi}{q \Gamma(s)} \left(\pi \frac{|m|}{q}\right)^{(1-s)/2} y^s \sum_{c>0} S(m, c, \Gamma) c^{-(1+s)} \\ &\quad \times \int_0^\infty e^{-y t^2} t^{3s-2} J_{s-1} \left(\left(4\pi \frac{|m|}{q}\right)^{1/2} \frac{t}{c} \right) dt \end{aligned} \quad (2.2)$$

for $\operatorname{Re} s > 1$, where J_{s-1} denotes the Bessel function. The function J_ν satisfies the recursion formula ([1, vol 2, p. 12(56)]):

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu z^{-1} J_\nu(z). \quad (2.3)$$

One knows from [1, vol 2, p. 50(22)] that

$$\int_0^\infty e^{-a^2 t^2} t^{\mu-1} J_\nu(bt) dt = \frac{\Gamma((\mu+\nu)/2) b^\nu}{2^{\nu+1} a^{\mu+\nu} \Gamma(\nu+1)} {}_1F_1\left(\frac{\mu+\nu}{2}, \nu+1; -\frac{b^2}{4a^2}\right) \quad (2.4)$$

for $\operatorname{Re} a^2 > 0$ and $\operatorname{Re}(\mu+\nu) > 0$, where ${}_1F_1$ indicates the confluent hypergeometric function. Moreover, ${}_1F_1$ has an integral representation for $\operatorname{Re} \gamma > \operatorname{Re} \alpha > 0$ ([1, vol 1, p. 255(1)]):

$${}_1F_1(\alpha, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{xu} u^{\alpha-1} (1-u)^{\gamma-\alpha-1} du. \quad (2.5)$$

Using the above three formulas, we calculate the integral part of (2.2). If we put $\nu = s$ and $z = (4\pi|m|/q)^{1/2} t c^{-1}$ in (2.3), the integral part of (2.2) is transformed

into

$$2sc \left(4\pi \frac{|m|}{q}\right)^{-1/2} \int_0^\infty e^{-yt^2} t^{3(s-1)} J_s \left(\left(4\pi \frac{|m|}{q}\right)^{1/2} \frac{t}{c} \right) dt \\ - \int_0^\infty e^{-yt^2} t^{3s-2} J_{s+1} \left(\left(4\pi \frac{|m|}{q}\right)^{1/2} \frac{t}{c} \right) dt. \quad (2.6)$$

We apply formula (2.4) to the first term of (2.6) by putting $a^2 = y$, $b = (4\pi|m|q^{-1})^{1/2}c^{-1}$, $\mu = 3s - 2$ and $\nu = s$, and next apply (2.5) under the identification $\alpha = 2s - 1$, $\gamma = s + 1$ and $x = -\pi|m|(qc^2y)^{-1}$. We analogously apply the formulas (2.4) and (2.5) to the second term of (2.6), under $a^2 = y$, $b = (4\pi|m|q^{-1})^{1/2}c^{-1}$, $\mu = 3s - 1$, $\nu = s + 1$, and under $\alpha = 2s$, $\gamma = s + 2$, $x = -\pi|m|(qc^2y)^{-1}$ respectively. Then formula (2.6) is reduced to

$$\frac{s}{2\Gamma(2-s)} \left(\pi \frac{|m|}{q}\right)^{(s-1)/2} y^{1-2s} c^{1-s} \int_0^1 \exp\left(-\frac{\pi|m|u}{qc^2y}\right) u^{2(s-1)} (1-u)^{1-s} du \\ - \frac{1}{2\Gamma(2-s)} \left(\pi \frac{|m|}{q}\right)^{(s+1)/2} y^{-2s} c^{-(s+1)} \int_0^1 \exp\left(-\frac{\pi|m|u}{qc^2y}\right) u^{2s-1} (1-u)^{1-s} du$$

for $1/2 < \operatorname{Re} s < 2$. After substituting this formula into the integral part of (2.2), we obtain the expression of the lemma. $\square$

PROPOSITION 2.2. *Suppose that $m \in \mathbb{Z} - \{0\}$, $b \in \mathbb{R} - \{0\}$, and let $P_m(z, s, \Gamma)$ and $E_b(z, \bar{w}, \Gamma)$ be the series defined by (1.6) and (1.7). Then for $1 < \operatorname{Re} s < 2$ and $\operatorname{Re} w > 1$, we have*

$$\langle P_m(z, s, \Gamma), E_b(z, \bar{w}, \Gamma) \rangle \\ = 2^{(6-3s-w)/2} \frac{\pi s}{\Gamma(s)\Gamma(2-s)} |b|^{(s-w)/2} |m|^{(w-s)/2} \sum_{c>0} S(m, c, \Gamma) c^{-(s+w)} \\ \times \int_0^1 u^{(3s+w-4)/2} (1-u)^{1-s} K_{s-w} \left(\frac{2\pi(2|bm|u)^{1/2}}{qc} \right) du \\ - 2^{(7-3s-w)/2} \frac{\pi^2}{q\Gamma(s)\Gamma(2-s)} |b|^{(1+s-w)/2} |m|^{(1-s+w)/2} \sum_{c>0} S(m, c, \Gamma) c^{-(1+s+w)} \\ \times \int_0^1 u^{(3s+w-3)/2} (1-u)^{1-s} K_{s-w+1} \left(\frac{2\pi(2|bm|u)^{1/2}}{qc} \right) du, \quad (2.7)$$

where K_ν denotes the modified Bessel function.

Proof. We first verify absolute convergence of the integral on the right-hand side of (2.1) after replacing the integrand $a_m(y, s, 0, \Gamma)$ by the formula of Lemma 2.1. Since $|S(m, c, \Gamma)| \leq qc$, it follows that

$$a_m(y, s, 0, \Gamma) = O \left\{ \frac{2^{2(1-\sigma)}}{|\Gamma(s)\Gamma(2-s)|} \left(|s|y^{1-\sigma} \sum_{c>0} c^{1-2\sigma} \int_0^1 u^{2(\sigma-1)}(1-u)^{1-\sigma} du \right. \right. \\ \left. \left. + y^{-\sigma} \sum_{c>0} c^{-1-2\sigma} \int_0^1 u^{2\sigma-1}(1-u)^{1-\sigma} du \right) \right\}$$

for $1 < \sigma < 2$, where $\sigma = \operatorname{Re} s$. The integrand on the right-hand side of (2.1) can be estimated, after substituting the above into (2.1), by

$$O \left\{ \frac{2^{2(1-\sigma)}}{|\Gamma(s)\Gamma(2-s)|} \left(|s|e^{-\beta y} y^{\sigma'-\sigma-1} \sum_{c>0} c^{1-2\sigma} \int_0^1 u^{2(\sigma-1)}(1-u)^{1-\sigma} du \right. \right. \\ \left. \left. + e^{-\beta y} y^{\sigma'-\sigma-2} \sum_{c>0} c^{-1-2\sigma} \int_0^1 u^{2\sigma-1}(1-u)^{1-\sigma} du \right) \right\}$$

at least for $1 < \sigma < 2$, where $\sigma = \operatorname{Re} s$, $\sigma' = \operatorname{Re} w$ and $\beta = 2\pi|b|/q$. If we temporarily suppose that $\sigma' > \sigma + 1$ and $1 < \sigma < 2$, the absolute convergence of the integral in (2.1) follows. Thus, under these conditions, we can change the order of summation over c and the order of integrations over y and u . Then we have

$$\begin{aligned} & \langle P_m(z, s, \Gamma), E_b(z, \bar{w}, \Gamma) \rangle \\ &= 2^{2(1-s)} \frac{\pi s}{\Gamma(s)\Gamma(2-s)} \sum_{c>0} S(m, c, \Gamma) c^{-2s} \\ & \quad \times \int_0^1 u^{2(s-1)}(1-u)^{1-s} du \int_0^\infty \exp \left(-2\pi \frac{|b|}{q} y - \frac{\pi |m|u}{qc^2 y} \right) y^{w-s-1} dy \\ & \quad - 2^{2(1-s)} \frac{\pi^2}{q\Gamma(s)\Gamma(2-s)} |m| \sum_{c>0} S(m, c, \Gamma) c^{-2(1+s)} \\ & \quad \times \int_0^1 u^{2s-1}(1-u)^{1-s} du \int_0^\infty \exp \left(-2\pi \frac{|b|}{q} y - \frac{\pi |m|u}{qc^2 y} \right) y^{w-s-2} dy \quad (2.8) \end{aligned}$$

for $1 < \operatorname{Re} s < 2$ and $\operatorname{Re} w > \operatorname{Re} s + 1$. The modified Bessel function $K_\nu(z)$ has the integral representation:

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2} \right)^\nu \int_0^\infty \exp \left(-t - \frac{z^2}{4t} \right) t^{-\nu-1} dt$$

for $|\arg(z)| < \pi/4$ ([4, p. 183(15)]). The double integral on the first term of (2.8) is equal to

$$\left(2\pi \frac{|b|}{q}\right)^{s-w} \int_0^1 u^{2(s-1)}(1-u)^{1-s} du \int_0^\infty \exp\left(-y - \frac{2\pi^2|bm|u}{q^2c^2y}\right) y^{w-s-1} dy$$

for $1/2 < \operatorname{Re} s < 2$ and $\operatorname{Re} w > \operatorname{Re} s$. In general, for $a > 0$, we have the estimate

$$\int_0^\infty \exp\left(-t - \frac{a}{t}\right) t^{\nu-1} dt = \begin{cases} O(1) & \text{for } \operatorname{Re} \nu > 0 \\ O(1 + |\log a|) & \text{for } \operatorname{Re} \nu = 0 \\ O(a^{\operatorname{Re} \nu}) & \text{for } \operatorname{Re} \nu < 0. \end{cases} \quad (2.9)$$

By using this, the above integral is estimated by

$$O\left(c^{-2(\sigma'-\sigma)} \int_0^1 u^{\sigma+\sigma'-2}(1-u)^{1-\sigma} du\right)$$

for $\sigma' < \sigma$, and

$$O\left(|\log c| \int_0^1 u^{2(\sigma-1)}(1-u)^{1-\sigma} du + \int_0^1 u^{2(\sigma-1)} |\log u| (1-u)^{1-\sigma} du\right)$$

for $\sigma' = \sigma$, where $\sigma = \operatorname{Re} s$ and $\sigma' = \operatorname{Re} w$. This implies that the first term on the right-hand side of (2.8) is absolutely convergent under the conditions $1 < \operatorname{Re} s < 2$ and $\operatorname{Re} w > 1$. Therefore the condition $\operatorname{Re} w > \operatorname{Re} s$ is weakened to $\operatorname{Re} w > 1$. Moreover the inner integral over y coincides with

$$2(\pi(2|bm|u)^{1/2}(qc)^{-1})^{w-s} K_{s-w}(2\pi(2|b||m|u)^{1/2}(qc)^{-1}).$$

Hence, the double integral over y and u on the first term of (2.8) is reduced to

$$2^{(s-w+2)/2} |b|^{(s-w)/2} |m|^{(w-s)/2} c^{s-w} \times \int_0^1 u^{(3s+w-4)/2} (1-u)^{1-s} K_{s-w} \left(\frac{2\pi(2|bm|u)^{1/2}}{qc} \right) du, \quad (2.10)$$

for $1/2 < \operatorname{Re} s < 2$ and $\operatorname{Re} w > 1$. In the same way, the double integral on the second term of (2.8) is transformed into

$$2^{(s-w+3)/2} |b|^{(s-w+1)/2} |m|^{(w-s-1)/2} c^{s-w+1} \times \int_0^1 u^{(3s+w-3)/2} (1-u)^{1-s} K_{s-w+1} \left(\frac{2\pi(2|bm|u)^{1/2}}{qc} \right) du, \quad (2.11)$$

for $0 < \operatorname{Re} s < 2$ and $\operatorname{Re} w > 1$. Here the condition $\operatorname{Re} w > \operatorname{Re} s + 1$ is weakened to $\operatorname{Re} w > 1$. Substituting (2.10) and (2.11) into each term of (2.8), we obtain the assertion of the proposition. $\square$

The following theorem is the main result of this section.

THEOREM 2.3. *Suppose that Γ is a Fuchsian group of the first kind with a cusp ∞ . Let $P_m(z, s, \Gamma)$ ($m \in \mathbb{Z} - \{0\}$) and $E_b(z, \bar{w}, \Gamma)$ ($b \in \mathbb{R} - \{0\}$) be the series defined by (1.6) and (1.7), and let $\Phi_m(s, \Gamma)$ be the Dirichlet series defined by (1.5). Then under the conditions $\operatorname{Re} s > 1$, $\operatorname{Re} w > 1$ and $\operatorname{Re} s - \operatorname{Re} w \notin \mathbb{Z}$, we have the equality*

$$\begin{aligned} & 2^{2-s-w} \pi^{1+s-w} q^{w-s} \frac{\Gamma(w-s)\Gamma(2s-1)}{\Gamma(s)^2} |b|^{s-w} \Phi_m(s, \Gamma) \\ &= \langle P_m(z, s, \Gamma), E_b(z, \bar{w}, \Gamma) \rangle - R_m(s, w, b, c, \Gamma), \end{aligned}$$

where

$$\begin{aligned} & R_m(s, w, b, c, \Gamma) \\ &= 2^{2(1-s)} \pi^{1-s+w} q^{s-w} \frac{\Gamma(s-w+1)\Gamma(s+w-1)}{\Gamma(s)\Gamma(w+1)} |m|^{w-s} \sum_{c>0} S(m, c, \Gamma) c^{-2w} \\ & \quad \times \left\{ \frac{s}{s-w} \sum_{n=0}^{\infty} \frac{(s+w-1)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (w-s+1)_n (w+1)_n} \right. \\ & \quad \left. + \sum_{n=0}^{\infty} \frac{(s+w-1)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (w-s)_n (w+1)_n} \right\} \\ & \quad + 2^{2-s-w} \pi^{s-w+1} q^{w-s} \frac{\Gamma(w-s)\Gamma(2s-1)}{\Gamma(s)^2} |b|^{s-w} \\ & \quad \times \sum_{c>0} S(m, c, \Gamma) c^{-2s} \sum_{n=1}^{\infty} \frac{(2s-1)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (s-w+1)_n (s+1)_n} \\ & \quad + 2^{3-s-w} \pi^{s-w+3} q^{w-s-2} \frac{\Gamma(2s)\Gamma(w-s-1)}{\Gamma(s)\Gamma(s+2)} |b|^{s-w+1} |m| \\ & \quad \times \sum_{c>0} S(m, c, \Gamma) c^{-2(1+s)} \sum_{n=0}^{\infty} \frac{(2s)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (s-w+2)_n (s+2)_n} \end{aligned} \quad (2.12)$$

and $(\alpha)_n = (\alpha) \cdot (\alpha+1) \cdots (\alpha+n-1)$, $(\alpha)_0 = 1$.

Proof. The modified Bessel function $K_\nu(y)$ is defined by

$$\begin{cases} K_\nu(y) = \frac{\pi}{2} \frac{I_{-\nu}(y) - I_\nu(y)}{\sin(\nu\pi)} \\ I_\nu(y) = \left(\frac{y}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(y/2)^{2n}}{n! \Gamma(\nu+n+1)}, \end{cases}$$

where I_ν is the modified Bessel function of the first kind. Consider the first term on the right-hand side of (2.7). Putting $\nu = s - w$ and $y = 2\pi(2|bm|u)^{1/2}(qc)^{-1}$, we replace the modified Bessel function by the series stated above. Then we can change the order of integration over u and summation over n if $1 < \operatorname{Re} s < 2$ and $\operatorname{Re} w > 1$. After that, we apply the integral representation of the Beta function:

$$B(p, q) = \int_0^1 u^{p-1}(1-u)^{q-1} du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (\operatorname{Re} p > 0, \operatorname{Re} q > 0),$$

and apply further the properties $\sin(\nu\pi) = \pi(\Gamma(\nu)\Gamma(1-\nu))^{-1}$ and $\Gamma(\nu+1) = \nu\Gamma(\nu)$. Then the first term of (2.7) is transformed into

$$\begin{aligned} & 2^{2(1-s)} \pi^{1-s+w} q^{s-w} \frac{\Gamma(s-w)\Gamma(s+w-1)}{\Gamma(s)\Gamma(w+1)} s|m|^{w-s} \\ & \times \sum_{c>0} S(m, c, \Gamma) c^{-2w} \sum_{n=0}^{\infty} \frac{(s+w-1)_n (2\pi^2|bm|(qc)^{-2})^n}{n!(w-s+1)_n(w+1)_n} \\ & + 2^{2-s-w} \pi^{s-w+1} q^{w-s} \frac{\Gamma(w-s)\Gamma(2s-1)}{\Gamma(s)^2} |b|^{s-w} \\ & \times \sum_{c>0} S(m, c, \Gamma) c^{-2s} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(2s-1)_n (2\pi^2|bm|(qc)^{-2})^n}{n!(s-w+1)_n(s+1)_n} \right\}, \quad (2.13) \end{aligned}$$

for $1 < \operatorname{Re} s < 2$, $\operatorname{Re} w > 1$ and $\operatorname{Re} s - \operatorname{Re} w \notin \mathbb{Z}$. Put

$$\begin{aligned} \alpha_1 &= \max \left\{ \left\{ |s+w-1+n|(|w-s+1+n| \cdot |w+1+n|)^{-1}, n \geq 0 \right\}, 1 \right\}, \\ \alpha_2 &= \max \left\{ \left\{ |2s-1+n|(|s-w+1+n| \cdot |s+1+n|)^{-1}, n \geq 0 \right\}, 1 \right\} \end{aligned}$$

and

$$c_0 = \min \left\{ c > 0 \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right. \right\}.$$

The values α_1 and α_2 are bounded if $\operatorname{Re} s > 1$, $\operatorname{Re} w > 1$ and $\operatorname{Re} s \neq \operatorname{Re} w$. In fact, it follows that

$$\begin{aligned} & \frac{|s+w-1+n|}{|w-s+1+n| \cdot |w+1+n|} \\ & < \frac{1}{|w-s|} \cdot \left| \frac{s+w-1+n}{w+1+n} \right| \\ & < \frac{1}{|w-s|} \left(1 + \left| \frac{s-2}{w} \right| \right) = \frac{1}{|w-s|} + \left| \frac{s-2}{s-w} \right| \cdot \frac{1}{|w|} \end{aligned}$$

$$\begin{aligned}
&< \frac{1}{|\sigma' - \sigma|} + \left(1 + \left|\frac{w-2}{s-w}\right| \frac{1}{|w|}\right) \\
&< \frac{1}{|\sigma' - \sigma|} + 1 + \left|1 - \frac{2}{w}\right| \cdot \frac{1}{|\sigma' - \sigma|} \\
&< 1 + \frac{4}{|\sigma' - \sigma|},
\end{aligned}$$

where $\sigma = \operatorname{Re} s$ and $\sigma' = \operatorname{Re} w$. The same also holds for α_2 . Then, since $|S(m, c, \Gamma)| \leq qc$, formula (2.13) is estimated by

$$\begin{aligned}
O \left(2^{2(1-\sigma)} \pi^{1-\sigma+\sigma'} q^{\sigma-\sigma'+1} \left| \frac{\Gamma(s-w)\Gamma(s+w-1)}{\Gamma(s)\Gamma(w+1)} s \right| \cdot |m|^{\sigma-\sigma'} \sum_{c>0} c^{1-2\sigma'} \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha_1 a)^n \right. \\
\left. + 2^{2-\sigma-\sigma'} \pi^{\sigma-\sigma'+1} q^{\sigma'-\sigma+1} \left| \frac{\Gamma(w-s)\Gamma(2s-1)}{\Gamma(s)^2} \right| \cdot |b|^{\sigma-\sigma'} \sum_{c>0} c^{1-2\sigma} \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha_2 a)^n \right)
\end{aligned}$$

for $\sigma > 1$, $\sigma' > 1$ and $\sigma - \sigma' \notin \mathbb{Z}$, where $\sigma = \operatorname{Re} s$, $\sigma' = \operatorname{Re} w$ and $a = 2\pi^2 |bm| (qc_0)^{-2}$. Therefore, (2.13) is absolutely convergent for $\operatorname{Re} s > 1$, $\operatorname{Re} w > 1$ and $\operatorname{Re} s - \operatorname{Re} w \notin \mathbb{Z}$. In the same way, we can transform the second term on the right-hand side of (2.7) into

$$\begin{aligned}
&2^{2(1-s)} \pi^{1-s+w} q^{s-w} \frac{\Gamma(s-w+1)\Gamma(s+w-1)}{\Gamma(s)\Gamma(w+1)} |m|^{w-s} \\
&\times \sum_{c>0} S(m, c, \Gamma) c^{-2w} \sum_{n=0}^{\infty} \frac{(s+w-1)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (w-s)_n (w+1)_n} \\
&+ 2^{3-s-w} \pi^{s-w+3} q^{w-s-2} \frac{\Gamma(2s)\Gamma(w-s-1)}{\Gamma(s)\Gamma(s+2)} |b|^{s-w+1} |m| \\
&\times \sum_{c>0} S(m, c, \Gamma) c^{-2(1+s)} \sum_{n=0}^{\infty} \frac{(2s)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (s-w+2)_n (s+2)_n} \quad (2.14)
\end{aligned}$$

for $\operatorname{Re} s > 1$, $\operatorname{Re} w > 1$ and $\operatorname{Re} s - \operatorname{Re} w \notin \mathbb{Z}$. Combining formulas (2.13) and (2.14) with (2.7), we arrive at the assertion of the theorem. $\square$

3. Estimation of $\Phi_m(s, \Gamma)$

The infinite series (1.5) defining $\Phi_m(s, \Gamma)$ is absolutely convergent in the domain $\operatorname{Re} s > 1$. Our purpose in this section is to continue the function $\Phi_m(s, \Gamma)$ to the larger domain $\operatorname{Re} s > 1/2$, and to estimate it for $|m|$ and $|\operatorname{Im} s|$ (Theorem 3.6). The

approach is based on Theorem 2.3; we will put $w = s + 1/2$, $b = 1/m$ throughout this section.

3.1. Estimation of $R_m(s, s + 1/2, 1/m, c, \Gamma)$

From Theorem 2.3, the series R_m defined by (2.12) makes sense at least for $\operatorname{Re} s > 1$ subject to putting $w = s + 1/2$ and $b = 1/m$. From now on we give an estimate for $R_m(s, s + 1/2, 1/m, c, \Gamma)$ in the wider region $\operatorname{Re} s > 1/2$; this also gives the holomorphy of R_m in $\operatorname{Re} s > 1/2$. We start with the first term. Put $s = \sigma + i\tau$ and set

$$a_1 = \max \{ |(2s - 1/2 + n)/(s + 3/2 + n)|, n \geq 0 \}, 1 \},$$

$$c_0 = \min \left\{ c > 0 \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right. \right\}.$$

Then a_1 is a constant independent of $|\tau|$. In fact, it is easily seen that

$$\begin{aligned} \left| \frac{2s - 1/2 + n}{s + 3/2 + n} \right| &= \left| 1 + \frac{s - 2}{s + 3/2 + n} \right| \\ &< 1 + \left| \frac{s - 2}{s} \right| < 2 + \frac{2}{|s|} < 4, \end{aligned}$$

for $\sigma > 1/2$. Since $|S(m, c, \Gamma)| \leq qc$, we have

$$\begin{aligned} &\left| \sum_{c>0} S(m, c, \Gamma) c^{-2w} \sum_{n=0}^{\infty} \frac{(s + w - 1)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (w - s + 1)_n (w + 1)_n} \right| \\ &\leq q \sum_{c>0} c^{-2\sigma} \sum_{n=0}^{\infty} \frac{|2s - 1/2|_n}{n! |s + 3/2|_n} \left(\frac{2\pi^2}{q^2 c^2} \right)^n \\ &\leq q \sum_{c>0} c^{-2\sigma} \exp \left(\frac{2\pi^2}{q^2 c_0^2} a_1 \right) \\ &= O \left(\frac{1}{\sigma - 1/2} \right), \end{aligned}$$

for $\sigma > 1/2$, where $w = s + 1/2$. Thus

$$\left| \sum_{c>0} S(m, c, \Gamma) c^{-2w} \sum_{n=0}^{\infty} \frac{(s + w - 1)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (w - s)_n (w + 1)_n} \right|$$

$$\begin{aligned}
&\leq 2q \sum_{c>0} c^{-2\sigma} \exp\left(\frac{2\pi^2}{q^2 c_0^2} a_1\right) \\
&= O\left(\frac{1}{\sigma - 1/2}\right),
\end{aligned}$$

for $\sigma > 1/2$, where $w = s + 1/2$. From the results above, we can estimate the first term of R_m in $\sigma > 1/2$ as

$$O\left(\left|\frac{\Gamma(2s - 1/2)}{\Gamma(s)\Gamma(s + 3/2)}\right|^s \frac{|m|^{1/2}}{\sigma - 1/2}\right). \quad (3.1)$$

We next estimate the second term of R_m . For $\sigma > 0$, we have

$$\begin{aligned}
&\left| \sum_{c>0} S(m, c, \Gamma) c^{-2s} \sum_{n=1}^{\infty} \frac{(2s - 1)_n (2\pi^2 |bm| (qc)^{-2})^n}{n! (s - w + 1)_n (s + 1)_n} \right| \\
&\leq 2q \sum_{c>0} c^{1-2\sigma} \sum_{n=1}^{\infty} \frac{|2s - 1|_n}{n! |s + 1|_n} \left(\frac{2\pi^2}{q^2 c^2}\right)^n \\
&\leq 2q \sum_{c>0} c^{1-2\sigma} \left(\frac{2\pi^2}{q^2 c^2} a_2\right) \exp\left(\frac{2\pi^2}{q^2 c^2} a_2\right) \quad (\because |e^z - 1| \leq |z| e^{|z|}) \\
&= O\left(\sum_{c>0} c^{-1-2\sigma} \exp\left(\frac{2\pi^2}{q^2 c_0^2} a_2\right)\right) \\
&= O\left(\frac{1}{\sigma}\right),
\end{aligned}$$

where $a_2 = \max\{|(2s - 1 + n)/(s + 1 + n)|, n \geq 0\}$ and it is independent of $|\tau|$. Therefore we derive the estimate of the second term of R_m as

$$O\left(\left|\frac{\Gamma(2s - 1)}{\Gamma(s)^2}\right| \frac{|m|^{1/2}}{\sigma}\right), \quad (3.2)$$

for $\sigma > 0$. In the same way, we can estimate the third term of R_m as

$$O\left(\left|\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s + 2)}\right| \frac{|m|^{1/2}}{\sigma}\right), \quad (3.3)$$

for $\sigma > 0$. In view of the estimates (3.1), (3.2) and (3.3) together with Stirling's formula, we obtain the following proposition.

PROPOSITION 3.1. *The series $R_m(s, s + 1/2, 1/m, \Gamma)$ defined by (2.12) determines a holomorphic function of s in $1/2 < \operatorname{Re} s < M$, and satisfies the estimate*

$$R_m(s, s + 1/2, 1/m, c, \Gamma) = O\left(\frac{|m|^{1/2}|\tau|^{-1/2}}{\sigma - 1/2}\right),$$

where $\sigma = \operatorname{Re} s$ and the O -symbol depends on M and Γ .

3.2. Estimation of the inner product $\langle P_m(z, s, \Gamma), E_{1/m}(z, \bar{s} + 1/2, \Gamma) \rangle$

The Cauchy–Schwartz inequality yields

$$\langle P_m(z, s, \Gamma), E_{1/m}(z, \bar{s} + 1/2, \Gamma) \rangle \leq \|P_m(z, s, \Gamma)\| \cdot \|E_{1/m}(z, \bar{s} + 1/2, \Gamma)\|, \quad (3.4)$$

where $\|\cdot\|$ means the norm of $L^2(\mathcal{D})$. From now on, we evaluate the norms $\|P_m(z, s, \Gamma)\|$ and $\|E_{1/m}(z, \bar{s} + 1/2, \Gamma)\|$ in the domain $\operatorname{Re} s > 1/2$.

PROPOSITION 3.2. *Suppose that $M > 1$ is an arbitrary constant, and put $s = \sigma + i\tau \in \mathbb{C}$. Then we have*

$$\|P_m(z, s, \Gamma)\| = O\left(\frac{|m|^{1/2}}{(\sigma - 1/2)^2}\right)$$

for all s such that $1/2 < \sigma < M$ and $|\tau| \geq 1$, where the O -symbol depends on M and Γ .

Proof. By (1.8),

$$\|P_m(z, s, \Gamma)\| = 4\pi \frac{|m|}{q} |s| \cdot \|\mathcal{R}_\lambda\| \cdot \|P_m(z, s + 1, \Gamma)\|.$$

Due to [2, p. 672], we know the inequality $\|\mathcal{R}_\lambda\| \leq 1/\{|\tau|(2\sigma - 1)\}$ for $\sigma > 1/2$ and $|\tau| \geq 1$. Hence from using [6, Lemma], we have

$$\begin{aligned} \|P_m(z, s, \Gamma)\| &\leq 4\pi \frac{|m|}{q} |s| \frac{1}{|\tau|(2\sigma - 1)} O\left(\frac{|m|^{-1/2}}{\sigma - 1/2}\right) \\ &= O\left(\frac{|m|^{1/2}}{(\sigma - 1/2)^2}\right) \end{aligned}$$

under the same conditions of the proposition. This completes the proof. $\square$

LEMMA 3.3. *Let $b \in \mathbb{R} - \{0\}$, $z \in H$, $s \in \mathbb{C}$ and let $d_b(y, s, n, \Gamma)$ be the n th Fourier coefficient of the series $E_b(z, s, \Gamma)$. Then for $\operatorname{Re} s > 1$, we have*

$$\begin{aligned} d_b(y, s, n, \Gamma) &= \delta_{0n} \exp\left(-2\pi \frac{|b|}{q} y\right) y^s + \frac{1}{q} y^{1-s} \sum_{c>0} S(n, c, \Gamma) c^{-2s} \\ &\quad \times \int_{-\infty}^{\infty} \exp\left(-2\pi i \frac{n}{q} yt - \frac{2\pi |b|}{q c^2 y (1+t^2)}\right) \frac{1}{(1+t^2)^s} dt. \end{aligned}$$

Proof. Under the condition $\operatorname{Re} s > 1$, we have

$$E_b(z, s, \Gamma) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-2\pi \frac{|b|}{q}\right)^k E(z, s+k, \Gamma).$$

From expressing the Fourier coefficient of $E_b(z, s, \Gamma)$ by using that of $E(z, s+k, \Gamma)$ (see (1.10)), it follows that

$$\begin{aligned} d_b(y, s, n, \Gamma) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-2\pi \frac{|b|}{q}\right)^k c(y, s+k, n, \Gamma) \\ &= \delta_{0n} y^s \sum_{k=0}^{\infty} \frac{1}{k!} \left(-2\pi \frac{|b|}{q} y\right)^k + \frac{1}{q} y^{1-s} \sum_{c>0} S(n, c, \Gamma) c^{-2s} \\ &\quad \times \int_{-\infty}^{\infty} \frac{e(-nyt/q)}{(1+t^2)^s} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{2\pi |b|}{q c^2 y (1+t^2)}\right)^k dt \\ &= \delta_{0n} \exp\left(-2\pi \frac{|b|}{q} y\right) y^s + \frac{1}{q} y^{1-s} \sum_{c>0} S(n, c, \Gamma) c^{-2s} \\ &\quad \times \int_{-\infty}^{\infty} \exp\left(-2\pi i \frac{n}{q} yt - \frac{2\pi |b|}{q c^2 y (1+t^2)}\right) \frac{1}{(1+t^2)^s} dt \end{aligned}$$

for $\operatorname{Re} s > 1$. This completes the proof. $\square$

PROPOSITION 3.4. *Suppose that $M > 1$ is an arbitrary constant. For all $s \in \mathbb{C}$ such that $1/2 < \operatorname{Re} s < M$, we have*

$$\|E_{1/m}(z, \bar{s} + 1/2, \Gamma)\| = O\left(q^{\sigma+1/2} \frac{|m|^\sigma}{(\sigma - 1/2)^{1/2}}\right),$$

where $\sigma = \operatorname{Re} s$ and the O -symbol depends on M and Γ .

Proof. For $\operatorname{Re} s > 1/2$ and $\operatorname{Re} s' > 1$, we have

$$\begin{aligned} & \langle E_{1/m}(z, \bar{s} + 1/2, \Gamma), E_{1/m}(z, \bar{s}', \Gamma) \rangle \\ &= q \int_0^\infty d_{1/m}(y, \bar{s} + 1/2, 0, \Gamma) \exp\left(-\frac{2\pi}{q|m|}y\right) y^{s'-2} dy. \end{aligned}$$

The integrand on the right-hand side can be estimated, after substituting the formula of Lemma 3.3 into $d_{1/m}(y, \bar{s} + 1/2, 0, \Gamma)$, by

$$O\left(e^{-2\beta y} y^{\sigma+\sigma'-3/2} + e^{-\beta y} y^{\sigma'-\sigma-3/2} \sum_{c>0} c^{-2\sigma} \int_{-\infty}^\infty \frac{dt}{(1+t^2)^{\sigma+1/2}}\right),$$

where $\sigma = \operatorname{Re} s$, $\sigma' = \operatorname{Re} s'$ and $\beta = 2\pi(q|m|)^{-1}$. The absolute convergence holds if we temporarily suppose that $\sigma' > \sigma + 1/2$ and $\sigma > 1/2$. Under these conditions, we have

$$\begin{aligned} & \langle E_{1/m}(z, \bar{s} + 1/2, \Gamma), E_{1/m}(z, \bar{s}', \Gamma) \rangle \\ &= q \int_0^\infty \exp\left(-\frac{4\pi}{q|m|}y\right) y^{\bar{s}+s'-3/2} dy + \sum_{c>0} S(0, c, \Gamma) c^{-2\bar{s}-1} \\ & \quad \times \int_{-\infty}^\infty \frac{dt}{(1+t^2)^{\bar{s}+1/2}} \int_0^\infty \exp\left(-\frac{2\pi}{q|m|}y - \frac{2\pi}{qc^2|m|(1+t^2)}y\right) y^{s'-\bar{s}-3/2} dy. \end{aligned} \quad (3.5)$$

The double integral in the second term of (3.5) is equal to

$$\left(\frac{2\pi}{q|m|}\right)^{\bar{s}-s'+1/2} \int_{-\infty}^\infty \frac{dt}{(1+t^2)^{\bar{s}+1/2}} \int_0^\infty \exp\left(-y - \frac{a}{y}\right) y^{s'-\bar{s}-3/2} dy,$$

where $a = 4\pi^2(q^2c^2|m|^2(1+t^2))^{-1}$. Recalling (2.9), we see that the above formula converges absolutely in both cases $\operatorname{Re} s' = \operatorname{Re} s + 1/2$ and $\operatorname{Re} s' < \operatorname{Re} s + 1/2$ if $\operatorname{Re} s > 1/2$ and $\operatorname{Re} s' > 1$. Therefore we can put $s' = s + 1/2$ in (3.5) under the condition $\operatorname{Re} s > 1/2$. After that, the first term of (3.5) is estimated by

$$O\left(\frac{|m|^{2\sigma}}{\sigma}\right), \quad (3.6)$$

for $0 < \sigma < M$, where the implied constant depends on M . The second term of (3.5) is majorized, in terms of (2.9), by

$$O\left(\sum_{c>0} S(0, c, \Gamma) c^{-2\sigma-1} \int_{-\infty}^\infty \frac{O(1+|\log a|)}{(1+t^2)^{\sigma+1/2}} dt\right)$$

$$\begin{aligned}
&= O\left(\sum_{c>0} c^{-2\sigma} \int_{-\infty}^{\infty} \frac{\{qc|m|(1+t^2)\}^\varepsilon}{(1+t^2)^{\sigma+1/2}} dt\right) + O\left(\frac{1}{\sigma-1/2}\right) \\
&= O\left(\frac{|m|^\varepsilon}{\sigma-(1+\varepsilon)/2}\right), \tag{3.7}
\end{aligned}$$

for an arbitrary $\varepsilon > 0$, where the O -symbol depends on M and Γ . In view of the estimates (3.6) and (3.7), we arrive at the assertion of the proposition. $\square$

Combining (3.4), Proposition 3.2 and Proposition 3.4, we obtain the following proposition.

PROPOSITION 3.5. *Suppose that $M > 1$ is an arbitrary constant. Then we have*

$$\langle P_m(z, s, \Gamma), E_{1/m}(z, \bar{s} + 1/2, \Gamma) \rangle = O\left(\frac{|m|^{\sigma+1/2}}{(\sigma-1/2)^{5/2}}\right)$$

for all $s = \sigma + i\tau \in \mathbb{C}$ such that $1/2 < \sigma < M$ and $|\tau| \geq 1$, where the implied constant depends on M and Γ .

3.3. Estimation of $\Phi_m(s, \Gamma)$

The next theorem is the main result of this paper.

THEOREM 3.6. *Suppose that Γ is a Fuchsian group of the first kind with a cusp ∞ , and let $\Phi_m(s, \Gamma)$ be the Dirichlet series defined by (1.5). Then $\Phi_m(s, \Gamma)$ is continued meromorphically to the domain $\operatorname{Re} s > 1/2$ with at most a finite number of simple poles at $s = s_j$ lying in $(1/2, 1)$, and satisfies the estimate*

$$\Phi_m(s, \Gamma) = O\left(\frac{|m|^\sigma |\tau|^{1/2}}{(\sigma-1/2)^{5/2}}\right)$$

for all $s = \sigma + i\tau \in \mathbb{C}$ such that $1/2 < \sigma < M$ and $|\tau| \geq 1$, where $M > 1$ is an arbitrary constant and the O -symbol depends only on M .

Proof. In view of Theorem 2.3, Proposition 3.1 and Proposition 3.5, we have

$$q^{1/2} \left| \frac{\Gamma(2s-1)}{\Gamma(s)^2} \right| |m|^{1/2} \Phi_m(s, \Gamma) = O\left(\frac{|m|^{\sigma+1/2}}{(\sigma-1/2)^{5/2}}\right) + O\left(\frac{|m|^{1/2} |\tau|^{-1/2}}{\sigma-1/2}\right).$$

Applying Stirling's formula to the Γ -function, we achieve the assertion of the theorem. $\square$

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