COMMUTANTS AND DOUBLE COMMUTANTS OF REFLEXIVE ALGEBRAS

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1. Introduction and preliminaries

In this paper we discuss questions concerning the commutant and the double commutant of a reflexive algebra. We mainly consider the lattice-theoretic conditions on a subspace lattice $\mathcal{L}$ which imply that $(\text{alg}\mathcal{L})''$ is reflexive or $(\text{alg}\mathcal{L})'$ is reflexive or $\text{alg}\mathcal{L}$ is abelian. In [3] Lambrou proved that if $\mathcal{L}$ is completely distributive then $(\text{alg}\mathcal{L})''$ is reflexive and obtained an equivalent condition such that $\text{alg}\mathcal{L}$ is abelian. A proof of the equivalence of this condition (that $\text{alg}\mathcal{L}$ is abelian) was first given in [4]. We observe that some of Lambrou's results can be achieved under considerably weaker hypotheses on $\mathcal{L}$. As an application of the results which we obtain, we prove that if $\mathcal{L}$ is a pentagon subspace lattice, then $(\text{alg}\mathcal{L})'' = L(X)$. The final example of the paper represents the first known reflexive operator algebra whose commutant and double commutant are not reflexive.

Throughout the paper, X denotes a fixed non-zero Banach space over a field $\mathbf{F}$ (where $\mathbf{F}$ is real or complex). Let $L(X)$ denote the set of all operators on X . If $T \in L(X)$, then $R(T)$ denotes the norm closure of $T(X)$. The topological dual of X is denoted by X^* . For any family $\{M_r\}$ of subspaces of X , $\bigvee M_r$ denotes the closed linear span of $\{M_r\}$ and $\bigwedge M_r$ denotes their set-theoretic intersection. For any vectors $e \in X$ and $f^* \in X^*$, $f^* \otimes e$ denotes the operator on X defined by $(f^* \otimes e)x = f^*(x)e$. A complete lattice of subspaces of X containing (0) and X is called a subspace lattice. The lattice of all subspaces of X is denoted by $C(X)$. If $E \subseteq X$, the annihilator $E^\perp$ of E is the set $E^\perp = \{f^* \in X^*, f^*(x) = 0, \text{ for all } x \in E\}$. If $F \subseteq X^*$, the pre-annihilator $F^\perp$ of F is the set $F^\perp = \{x \in X : f^*(x) = 0, \text{ for all } f^* \in F\}$. In this paper $\mathcal{L}$ denotes a subspace lattice.

For any subset $\mathcal{F}$ of $C(X)$ the set of operators on X that leaves every member of $\mathcal{F}$ invariant is denoted by $\text{alg}\mathcal{F}$. Thus

$$\text{alg}\mathcal{F} = \{T \in L(X) : TM \subseteq M, \text{ for every } M \in \mathcal{F}\}.$$

It is easy to show that $\text{alg}\mathcal{F}$ is a unital subalgebra of $L(X)$ which is closed in the

strong operator topology. Dually, for any subset $\mathcal{A}$ of $L(X)$, the set of invariant subspaces of $\mathcal{A}$ is denoted by $\text{lat}\mathcal{A}$. Thus

$$\text{lat}\mathcal{A} = \{M \in C(X) : TM \subseteq M, \text{ for every } T \in \mathcal{A}\}.$$

An operator algebra contained in $L(X)$ is called reflexive if $\mathcal{A} = \text{alglat}\mathcal{A}$. It is easy to verify that an algebra $\mathcal{A}$ is reflexive if and only if $\mathcal{A} = \text{alg}\mathcal{F}$ for some subset $\mathcal{F}$ of $C(X)$. Dually we say that $\mathcal{L}$ is reflexive if $\mathcal{L} = \text{latalg}\mathcal{L}$. The commutant of $\mathcal{A} \subseteq L(X)$ is denoted by $\mathcal{A}'$ and the double commutant by $\mathcal{A}''$. For a subspace lattice $\mathcal{L}$ and for $E \in \mathcal{L}$ define

$$\begin{aligned} E_- &= \bigvee \{F \in \mathcal{L} : E \not\subseteq F\}, \\ E_* &= \bigwedge \{F_- : F \in \mathcal{L} \text{ and } F \not\subseteq E\}, \\ E_{\sharp} &= \bigvee \{F \in \mathcal{L} : E \not\subseteq F_-\}. \end{aligned}$$

We use the convention $\bigvee \emptyset = (0)$, $\bigwedge \emptyset = X$. $\mathcal{L}$ is called $\bigvee$ -distributive if and only if $E \bigwedge (\bigvee_{i \in I} F_i) = \bigvee_{i \in I} (E \bigwedge F_i)$, ($E, F_i \in \mathcal{L}$). An element L of $\mathcal{L}$ is called an atom if the condition $(0) \subseteq K \subseteq L$ ($K \in \mathcal{L}$) implies either $K = (0)$ or $K = L$.

Let $\mathcal{J}$ denote the subset of $\mathcal{L}$ defined by

$$\mathcal{J} = \{L \in \mathcal{L} : L \neq (0) \text{ and } L_- \neq X\}.$$

In what follows we will often refer to this subset $\mathcal{J}$. The importance of $\mathcal{J}$ lies in its intimate relation with the set of rank-one operators in $\text{alg}\mathcal{L}$.

THEOREM 1.1. (see [3, 5]). *If $\mathcal{L}$ is a subspace lattice, then $f^* \otimes e$ belongs to $\text{alg}\mathcal{L}$ if and only if there is a subspace $M \in \mathcal{L}$ such that $e \in M$ and $f^* \in (M_-)^\perp$.*

As for completely distributive lattices, we need not be concerned with the actual definition here. For our purpose it is enough to mention the following characterization for them.

THEOREM 1.2. (see [5]). *Let $\mathcal{L}$ be a subspace lattice on X . The following statements are equivalent.*

- (1) $\mathcal{L}$ is completely distributive.
- (2) For every $L \in \mathcal{L}$, $L_{\sharp} = L$.
- (3) For every $L \in \mathcal{L}$, $L_* = L$.

If in Theorem 1.2(2) we take X to be L , then we conclude that, in a completely distributive subspace lattice, $X = \bigvee \{K : K \in \mathcal{J}\}$. Also, Theorem 1.2(3) implies $(0) = \bigwedge \{K_- : K \in \mathcal{J}\}$.

2. Some results

In this section we discuss the properties of the commutant and the double commutant of $\text{alg}\mathcal{L}$.

PROPOSITION 2.1. *Suppose A and T are operators on X , and K is a subspace of X which is invariant under A . Then for each scalar λ in $\mathbb{F}$ we have*

- (1) $K \subseteq \text{Ker}(T - \lambda I)$ implies $K \subseteq \text{Ker}(AT - TA)$,
- (2) $R(T - \lambda I) \subseteq K$ implies $R(AT - TA) \subseteq K$.

Proof. (1) is obvious.

(2) Let x be an arbitrary element in X . Since $R(T - \lambda I) \subseteq K$, it follows that $(AT - TA)x = A(T - \lambda I)x - (T - \lambda I)Ax \in K$; hence $R(AT - TA) \subseteq K$. This completes the proof. $\square$

PROPOSITION 2.2. *Suppose that $T \in (\text{alg}\mathcal{L})'$ and $M \in \mathcal{J}$. Then there is a scalar λ satisfying*

- (1) $M \subseteq \text{Ker}(T - \lambda I)$ and
- (2) $R(T - \lambda I) \subseteq M_-$.

Proof. (1) Let $f, g \in M$. Choose $e^* \in (M_-)^\perp$ arbitrarily with $e^* \neq 0$. By Theorem 1.1 we have that $e^* \otimes f$ and $e^* \otimes g$ belong to $\text{alg}\mathcal{L}$. By assumption, $T(e^* \otimes f) = e^* \otimes (Tf) = (T^*e^*) \otimes f = (e^* \otimes f)T$ and similarly for $e^* \otimes g$. Hence there exist scalars λ and ϵ such that

$$\begin{aligned} T^*e^* &= \lambda e^*, & Tf &= \lambda f, \\ T^*e^* &= \epsilon e^*, & Tg &= \epsilon g. \end{aligned}$$

Therefore $\lambda = \epsilon$. Since f and g are arbitrary, it follows that $M \subseteq \text{Ker}(T - \lambda I)$.

(2) It suffices to prove that $\text{Ker}(T^* - \lambda I) \supseteq (M_-)^\perp$. Suppose that $f \in M$ with $f \neq 0$. For $e^*, h^* \in (M_-)^\perp$, by Theorem 1.1, $e^* \otimes f$ and $h^* \otimes f$ belong to $\text{alg}\mathcal{L}$. Thus T commutes with $e^* \otimes f$ and $h^* \otimes f$ respectively. Similar to (1), we can show that there exists δ such that $Tf = \delta f$, $T^*e^* = \delta e^*$ and $T^*h^* = \delta h^*$. By (1), since $M \subseteq \text{Ker}(T - \lambda I)$, we have that $\lambda = \delta$. Hence $\text{Ker}(T^* - \lambda I) \supseteq (M_-)^\perp$. This completes the proof. $\square$

COROLLARY 2.3. (1) *Suppose that $X_\# = X$. Then $T \in (\text{alg}\mathcal{L})'$ if and only if for each $M \in \mathcal{J}$ there is a scalar λ satisfying $M \subseteq \text{Ker}(T - \lambda I)$.*

- (2) Suppose that $(0)_* = (0)$. Then $T \in (\text{alg}\mathcal{L})'$ if and only if for each $M \in \mathcal{J}$ there is a scalar λ satisfying $R(T - \lambda) \subseteq M_-$.
- (3) Suppose that $X_{\sharp} = X$ or $(0)_* = (0)$. Then $T \in (\text{alg}\mathcal{L})'$ if and only if T commutes with all rank-one operators of $\text{alg}\mathcal{L}$.

Proof. By Propositions 2.1 and 2.2, (1) is immediate.

(2) The forward implication is clear by Proposition 2.2.

Conversely, if $A \in \text{alg}\mathcal{L}$, $M \in \mathcal{J}$, then $A^*(M_-)^\perp \subseteq (M_-)^\perp$. It follows that if $x^* \in (M_-)^\perp$, then $A^*T^*x^* = \lambda A^*x^* = T^*A^*x^*$. Since $(AT)^*$ and $(TA)^*$ are weak*-continuous, $(0)_* = (0)$ implies that the linear span of $\{(M_-)^\perp : M \in \mathcal{J}\}$ is weak*-dense in X^* ; hence $A^*T^* = T^*A^*$. We have thus proved that $AT = TA$.

(3) If T commutes with all rank-one operators of $\text{alg}\mathcal{L}$, by Proposition 2.2 and its proof, we have that for every $M \in \mathcal{J}$ there exists a scalar λ satisfying $M \subseteq \text{Ker}(T - \lambda I)$ and $R(T - \lambda I) \subseteq M_-$. By (1) and (2), we have that $T \in (\text{alg}\mathcal{L})'$. This completes the proof. $\square$

COROLLARY 2.4. *If $\mathcal{L}$ is $\bigvee$ -distributive such that $X_{\sharp} = X$, then $(\text{alg}\mathcal{L})' \subseteq \text{alg}\mathcal{L}$.*

Proof. Suppose that $T \in (\text{alg}\mathcal{L})'$. Since $\mathcal{L}$ is $\bigvee$ -distributive such that $X_{\sharp} = X$, we have that for every $N \in \mathcal{L}$, $N = \bigvee\{N \cap M : M \in \mathcal{J}\}$. By Proposition 2.2(1), it follows that $T(N \cap M) \subseteq N \cap M$, hence $TN \subseteq N$, $T \in \text{alg}\mathcal{L}$. This completes the proof. $\square$

Let $\mathcal{F}$ denote the subspace lattice generated by the set $\{\text{Ker}(T), R(T) : T \in (\text{alg}\mathcal{L})'\}$. We have the following results.

PROPOSITION 2.5. (1) $(\text{alg}\mathcal{L})'' \subseteq \text{alg}\mathcal{F}$.

(2) Suppose $T \in (\text{alg}\mathcal{L})'$ and $A \in \text{alg}\mathcal{F}$. Then

- (a) $\text{Ker}(AT - TA) \supseteq X_{\sharp}$ and
 (b) $R(AT - TA) \subseteq (0)_*$.

Proof. (1) is clear.

(2)(a) By Proposition 2.2, if $M \in \mathcal{J}$, then there exists λ satisfying $M \subseteq \text{Ker}(T - \lambda I)$. It follows that for each $x \in M$, $ATx = \lambda Ax = TAx$. Hence $\text{Ker}(AT - TA) \supseteq \bigvee\{M : M \in \mathcal{J}\} = X_{\sharp}$.

In a similar way we can show (b). This completes the proof. $\square$

COROLLARY 2.6. *Suppose that $X_{\sharp} = X$ or $(0)_* = (0)$. Then $(\text{alg}\mathcal{L})'' = \text{alg}\mathcal{F}$. In particular, $(\text{alg}\mathcal{L})''$ is reflexive.*

Proof. Suppose that $X_{\sharp} = X$. By Proposition 2.5(1), we have $(\text{alg}\mathcal{L})'' \subseteq \text{alg}\mathcal{F}$. For the reverse inclusion, let $T \in (\text{alg}\mathcal{L})'$ and $A \in \text{alg}\mathcal{F}$. By Proposition 2.5(2) and $X_{\sharp} = X$, we have $TA = AT$; hence $(\text{alg}\mathcal{L})'' \supseteq \text{alg}\mathcal{F}$. Thus $(\text{alg}\mathcal{L})'' = \text{alg}\mathcal{F}$.

Suppose that $(0)_{*} = (0)$. Then in a similar way we may show that $(\text{alg}\mathcal{L})'' \subseteq \text{alg}\mathcal{F}$. This completes the proof. $\square$

Remark. In Corollary 2.6, if $X_{\sharp} = X$, we can take other subspace lattices to play the role of $\mathcal{F}$. For example, suppose $M \in \mathcal{J}$. By Corollary 2.3 it follows that for each $T \in (\text{alg}\mathcal{L})'$ there is a unique eigenspace M_T containing M . Define

$$K_M = \bigcap \{M_T : T \in (\text{alg}\mathcal{L})'\}.$$

Let $\tilde{\mathcal{F}}$ be the subspace lattice generated by (0) , X and all K_M . We have $\tilde{\mathcal{F}} \subseteq \mathcal{F}$. Hence by Proposition 2.5, $(\text{alg}\mathcal{L})'' \subseteq \text{alg}\tilde{\mathcal{F}}$. For the reverse inclusion let $T \in (\text{alg}\mathcal{L})'$ and let $A \in \text{alg}\tilde{\mathcal{F}}$. Then for every $M \in \mathcal{J}$ and every $x \in K_M$, we have $Ax \in K_M \subseteq M_T$ and $ATx = A\lambda x = TAx$. So, since $M \subseteq K_M$ and $X_{\sharp} = X$, then $(\text{alg}\mathcal{L})'' \supseteq \text{alg}\tilde{\mathcal{F}}$. Hence $(\text{alg}\mathcal{L})'' = \text{alg}\tilde{\mathcal{F}}$. We next show that if N and M belong to $\mathcal{J}$ then either $K_M \cap K_N = (0)$ or $K_M = K_N$. If N and M belong to $\mathcal{J}$, there are two possibilities: either for each $T \in (\text{alg}\mathcal{L})'$, M_T and N_T are such that $M_T \cap N_T \neq (0)$ in which case $M_T = N_T$, or there is a $T \in (\text{alg}\mathcal{L})'$ such that $M_T \cap N_T = (0)$. Thus there are two possibilities for K_M and K_N : either $K_M = K_N$ or $K_M \cap K_N = (0)$. If $\mathcal{L}$ is reflexive, it is evident that $\mathcal{L}$ contains $\tilde{\mathcal{F}}$.

COROLLARY 2.7. (Lambrou [3]). *If $\mathcal{L}$ is a completely distributive lattice on X , then there exists a complete atomic Boolean sublattice $\tilde{\mathcal{F}}$ of $\mathcal{L}$ such that $(\text{alg}\mathcal{L})'' = \text{alg}\tilde{\mathcal{F}}$.*

Proof. From the above Remark, since $\tilde{\mathcal{F}} \subseteq \mathcal{L}$, and $\mathcal{L}$ is completely distributive, then $\tilde{\mathcal{F}}$ is also completely distributive. Since $\tilde{\mathcal{F}}$ is completely distributive by our Remark, it follows that the set $\{K_M : M \in \mathcal{J}\}$ is a set of atoms of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ is atomic. It is easy to show that a completely distributive atomic lattice is Boolean, hence $\tilde{\mathcal{F}}$ is a complete atomic Boolean lattice. This completes the proof. $\square$

The following example shows that as far as the reflexivity of $(\text{alg}\mathcal{L})''$ is concerned, the condition of Corollary 2.6 is strictly weaker than that of Corollary 2.7.

Example 1. Let $\mathcal{L}$ be a pentagon with Hasse diagram

$\mathcal{L}$ is not distributive, hence $\mathcal{L}$ is not completely distributive. Note that $\mathcal{J} = \{N, L\}$ and that $M = N_-$, $N = L_-$, hence $(0)_* = (0)$ and $X = X_\sharp$. By Corollary 2.6 we have that $(\text{alg}\mathcal{L})''$ is reflexive.

In fact, we may further prove:

PROPOSITION 2.8. *Suppose that $\mathcal{L}$ is as in Example 1. Then $(\text{alg}\mathcal{L})'' = L(X)$.*

Proof. By our Remark we have that there exists $\tilde{\mathcal{F}}$ such that $(\text{alg}\mathcal{L})'' = \text{alg}\tilde{\mathcal{F}}$ where $\tilde{\mathcal{F}}$ is generated by (0) , X , K_L and K_N .

(i) If $K_N = K_L$, we have that $\tilde{\mathcal{F}}$ is a nest, then $(\text{alg}\mathcal{L})'' = (\text{alg}\tilde{\mathcal{F}})'' = L(X)$.

(ii) If $K_N \cap K_L = (0)$, by our Remark we have that there exists an operator $T \in (\text{alg}\mathcal{L})'$ and distinct scalars λ and δ such that $L \subseteq \text{Ker}(T - \lambda I)$ and $N \subseteq \text{Ker}(T - \delta I)$. Letting $P = (T - \lambda I)/(\delta - \lambda)$, we have $P|_{K_N} = I$, $P|_{K_L} = 0$, so the sum $K_N + K_L$ is closed. Since $N \subseteq K_N$, $L \subseteq K_L$, we have that $N + L$ is closed. This is impossible. This proves that (ii) is impossible. Combining the two cases, we have $(\text{alg}\mathcal{L})'' = L(X)$. This completes the proof. $\square$

PROPOSITION 2.9. *Suppose that $\mathcal{L}$ is a subspace lattice with a non-trivial comparable element M . If there is a subspace N of X such that $X = M \oplus N$, then $(\text{alg}\mathcal{L})' = \mathbf{FI}$. In particular $(\text{alg}\mathcal{L})'$ is reflexive.*

Proof. Let $\mathcal{L}_1 = \{L : L \in \mathcal{L}, 0 \subseteq L \subseteq M\}$, $\mathcal{L}_2 = \{L \cap N : L \in \mathcal{L}\}$, then we have $\mathcal{L} = \mathcal{L}_1 \oplus_{\text{ord}} \mathcal{L}_2$ and

$$\text{alg}\mathcal{L} = \text{alg}(\mathcal{L}_1 \oplus_{\text{ord}} \mathcal{L}_2) = \left\{ \begin{pmatrix} A & P \\ 0 & B \end{pmatrix} \mid A \in \text{alg}\mathcal{L}_1, B \in \text{alg}\mathcal{L}_2, P \in L(N, M) \right\}.$$

It is not difficult to show $(\text{alg}\mathcal{L})' = (\text{alg}\mathcal{L}_1 \oplus_{\text{ord}} \mathcal{L}_2)' = \mathbf{FI}$. Hence $(\text{alg}\mathcal{L})'$ is reflexive. This completes the proof. $\square$

PROPOSITION 2.10. *If $X_{\sharp} = X$, then $\text{alg}\mathcal{L}$ is abelian if and only if $\dim M = 1$, for every $M \in \mathcal{J}$.*

Proof. Suppose that $\text{alg}\mathcal{L}$ is abelian. Let M be an element of $\mathcal{J}$. We first show that $M \not\subseteq M_-$. Suppose on the contrary that $M \subseteq M_-$. Let $f \in M$, $e^* \in (M_-)^\perp$ such that $f \neq 0$, $e^* \neq 0$. For any $L \in \mathcal{J}$, if $q \in L$, $p^* \in (L_-)^\perp$, $p^* \neq 0$, $q \neq 0$, then $p^* \otimes q \in \text{alg}\mathcal{L}$. If $L \subseteq M$, then $L \subseteq M_-$, so $e^*(L) = 0$. If $L \not\subseteq M$, we have $M \subseteq L_-$, and

$$(p^* \otimes q)(e^* \otimes f) = p^*(f)(e^* \otimes q) = 0.$$

Since $p^* \otimes q$ and $e^* \otimes f$ commute, it follows that

$$(e^* \otimes f)(p^* \otimes q) = e^*(q)(p^* \otimes f) = 0, \quad \text{so } e^*(q) = 0.$$

Hence, once again, $e^*(L) = 0$. Since $X_{\sharp} = X$, combining the previous two cases gives $e^* = 0$, and this is a contradiction. Thus for any $M \in \mathcal{J}$, $M \not\subseteq M_-$. We now show that $\dim M = 1$. Since $M \not\subseteq M_-$, we can choose $e^* \in (M_-)^\perp$, $g \in M$ such that $e^*(g) \neq 0$. Since $e^* \otimes g$ and $e^* \otimes h$ belong to $\text{alg}\mathcal{L}$ for every $h \in M$ and $\text{alg}\mathcal{L}$ is abelian, we have

$$e^*(h)g = e^*(g)h.$$

Hence $\dim M = 1$.

Conversely, let $A, B \in \text{alg}\mathcal{L}$. If $\dim M = 1$, we have that for every $x \in M$, $ABx = BAx$. Since $\bigvee\{M : M \in \mathcal{J}\} = X$, it follows that $AB = BA$. Thus $\text{alg}\mathcal{L}$ is abelian. This completes the proof. $\square$

PROPOSITION 2.11. *If $(0)_* = (0)$ and $\text{alg}\mathcal{L}$ is abelian, then $\dim M = 1$ for every $M \in \mathcal{J}$.*

The proof is similar to that of Proposition 2.10 and we omit its proof here.

If X is a complex Hilbert space and $\mathcal{A}$ is a reflexive algebra on X , we may have that $\mathcal{A}'$ and $\mathcal{A}''$ are not reflexive.

Example 2. Let $\mathcal{A} = \text{alg}\{(0), N, X\}$ be a nest algebra on X where $N \neq (0), X$. Define

$$\tilde{\mathcal{A}} = \left\{ \begin{pmatrix} \lambda I & a \\ 0 & \lambda I \end{pmatrix} \mid \lambda \in \mathbb{C}, a \in \mathcal{A} \right\}.$$

By [1, Proposition 3.10], we have that $\tilde{\mathcal{A}}$ is reflexive. It is easily proved that

$$\tilde{\mathcal{A}}' = \tilde{\mathcal{A}}'' = \left\{ \begin{pmatrix} \lambda I & a \\ 0 & \lambda I \end{pmatrix} \mid \lambda \in \mathbb{C}, a \in L(H) \right\}.$$

By [1, Proposition 3.10], we have that $\tilde{\mathcal{A}}'$ and $\tilde{\mathcal{A}}''$ are not reflexive.

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