CENTRAL EXTENSIONS OF H-SPACES

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1. Introduction

We work in the category of spaces of the homotopy type of connected CW-complexes with a base point $*$. An H-space consists of a space X and a multiplication $\mu = \mu_X : X \times X \rightarrow X$ such that the restrictions $\mu|_{*\times X}$ and $\mu|_{X\times*}$ are both homotopic to the identity map $1_X : X \rightarrow X$.

Given a space X , if it admits a multiplication μ , it may admit others which are not homotopic to μ . For the study of the enumeration of multiplications and essentially different multiplications see, for example, Arkowitz and Curjel [1], Curjel [3] and Copeland [2].

In this paper, we are concerned with the set of homotopy classes of multiplications for H-spaces.

We denote by $[A, X]$ the homotopy set of all maps from A to X . Let (X, μ_X) and (Y, μ_Y) be H-spaces and let $HD : [X, Y] \rightarrow [X \wedge X, Y]$ be the H-deviation morphism. Then we have the main result of this paper.

THEOREM 4.3. *Let (X_1, μ_1) be a homotopy associative and homotopy commutative H-space and let (X_2, μ_2) be an H-space. Then the set of equivalence classes of central H-extensions of (X_1, μ_1) by (X_2, μ_2) is in one-to-one correspondence with the left quotient set $[X_2 \wedge X_2, X_1]/\text{Im } HD$.*

This is a generalization of Lemma 3.4.2. of Zabrodsky [11]. As applications of the theorem, we have the following corollaries.

COROLLARY 4.4. ([9]) *Let $CP(3)$ denote the complex projective 3-space. Then the loop space $\Omega CP(3)$ is homotopy commutative.*

COROLLARY 4.5. ([11]) *$K(Z, 4n) \times K(Z, 2n)$ admits only two essentially different multiplications.*

COROLLARY 4.6. *$K(Z, 2n) \times S^n$ ($n = 1, 3, 7$) admits infinitely many essentially different multiplications.*

The paper is organized as follows. In Section 2, we prepare for fundamental facts on H-spaces. In Section 3, we define the set of equivalence classes of central H-extensions, and in Section 4, we prove the main theorem and related results.

2. Preliminaries

In this paper all spaces are assumed to be connected and of the homotopy type of CW-complexes with a nonsingular base point $*$. We denote by $[X, Y]$ the homotopy set of all homotopy classes $[f]$ of based maps $f : X \rightarrow Y$, and f and $[f]$ are frequently denoted by the same letter.

We use the notations

$$A \vee B = \{(x, y) \in A \times B; x = * \text{ or } y = *\}, \quad A \wedge B = A \times B / A \vee B.$$

We denote by i the inclusion $i : A \vee B \rightarrow A \times B$ and by q the identification map $q : A \times B \rightarrow A \wedge B$. We use Δ as the notation for the diagonal map $\Delta : X \rightarrow X \times X$ defined by $\Delta(x) = (x, x)$. A space $X = (X, \mu_X)$ is an *H-space* with multiplication $\mu = \mu_X : X \times X \rightarrow X$ if $\mu|_{X \times *} \simeq 1_X \simeq \mu|_{* \times X}$ and is *homotopy associative* (respectively *commutative*) if

$$\mu(\mu \times 1_X) \simeq \mu(1_X \times \mu) : X \times X \times X \rightarrow X \quad (\text{respectively } \mu \simeq \mu T : X \times X \rightarrow X).$$

Here $\simeq$ means homotopic, 1_X is the identity map and $T : X \times X \rightarrow X \times X$ is a commuting map defined by $T(x, y) = (y, x)$. A map $f : X = (X, \mu_X) \rightarrow Y = (Y, \mu_Y)$ between H-spaces is called an *H-map* if $f\mu_X \simeq \mu_Y(f \times f)$.

Now we recall the necessary details on H-spaces from, for example, Zabrodsky [11] and Stasheff [9].

Let (X, μ) be an H-space. Then for any space A , $[A, X]$ has a multiplication with unit

$$\mu_* : [A, X] \times [A, X] = [A, X \times X] \rightarrow [A, X]$$

defined by $\mu_*(f, g) = \mu(f \times g) \Delta$.

Although $[A, X]$ is not abelian in general, in order to distinguish between the multiplication in $[A, X]$ and the composition of maps we use $+$ for the operation μ_* , $\mu_*(f, g) = f + g$.

THEOREM 2.1. ([6], [9]) *If (X, μ) is an H-space, then for any space A , $[A, X]$ is an algebraic loop, i.e., for any pair $f, g \in [A, X]$ there exists a unique element $D_{f,g} \in [A, X]$ such that $D_{f,g} + g = \mu_*(D_{f,g}, g) = f$.*

The element $D_{f,g}$ is referred to as the difference of f and g . If (X, μ) is a homotopy associative and homotopy commutative H-space, then the algebraic loop $[A, X]$ is an abelian group. Hence we have $D_{g,f} = -D_{f,g}$.

LEMMA 2.2. ([11]) *Let (X, μ) be an H-space. Then for any spaces A, B*

$$e \longrightarrow [A \wedge B, X] \xrightarrow{q^*} [A \times B, X] \xrightarrow{i^*} [A \vee B, X] \longrightarrow e$$

is a short exact sequence as the algebraic loop in the following sense: i^ is surjective, q^* is injective and $i^*f = i^*g$ if and only if $D_{f,g} \in \text{Im } q^*$, where $i : A \vee B \rightarrow A \times B$ is the inclusion and $q : A \times B \rightarrow A \wedge B$ is the identification map.*

Let $(X, \mu), (Y, \mu')$ be H-spaces. Let $f : X \rightarrow Y$ be a map of spaces. Consider the exact sequence of Lemma 2.2,

$$e \longrightarrow [X \wedge X, Y] \xrightarrow{q^*} [X \times X, Y] \xrightarrow{i^*} [X \vee X, Y] \longrightarrow e.$$

Then $i^*(D_{f\mu, \mu'(f \times f)}) = D_{f\mu i, \mu'(f \times f)i} = 0$, because $f\mu i = \mu'(f \times f)i$. The deviation of f is the element $HD(f) \in [X \wedge X, Y]$ defined by $HD(f)q = D_{f\mu, \mu'(f \times f)}$.

The deviation $HD(f)$ is the obstruction for f to be an H-map, i.e., f is an H-map if and only if $HD(f) = 0$.

LEMMA 2.3. *Let $f : X \rightarrow Y$ be a map. If f and $\mu'(f \times f)$ are contained in the center of $[X, Y]$ and $[X \times X, Y]$ respectively, then, for any map $g : X \rightarrow Y$,*

$$HD(f + g) = HD(f) + HD(g).$$

Proof. Let p_1, p_2 be the projections of $X \times X$ on the first and second factors respectively. Then

$$\mu'(f \times f) = fp_1 + fp_2$$

and since f is contained in the center of $[X, Y]$, it follows that

$$fp_2 + gp_1 = gp_1 + fp_2.$$

Hence

$$\mu'((f + g) \times (f + g)) = \mu'(f \times f) + \mu'(g \times g).$$

Thus we have

$$D_{(f+g)\mu, \mu'((f+g) \times (f+g))} = D_{f\mu+g\mu, \mu'(f \times f) + \mu'(g \times g)}.$$

Since $\mu'(f \times f)$ is contained in the center of $[X \times X, Y]$, it follows that

$$D_{f\mu, \mu'(f \times f)} + D_{g\mu, \mu'(g \times g)} + \mu'(f \times f) + \mu'(g \times g) = f\mu + g\mu.$$

From the uniqueness of the difference, we obtain

$$D_{(f+g)\mu, \mu'((f+g) \times (f+g))} = D_{f\mu, \mu'(f \times f)} + D_{g\mu, \mu'(g \times g)}.$$

Therefore, we have

$$HD(f + g) = HD(f) + HD(g).$$

This completes the proof. $\square$

COROLLARY 2.4. *If (Y, μ') is a homotopy associative and homotopy commutative H-space, then the deviation morphism*

$$HD : [X, Y] \longrightarrow [X \wedge X, Y]$$

is a homomorphism.

Proof. Any map $f : X \rightarrow Y$ satisfies the condition of Lemma 2.3. This completes the proof. $\square$

Let $\nabla : X \vee X \rightarrow X$ be the folding map defined by $\nabla(x, *) = x = \nabla(*, x)$. The set of multiplications for an H-space X is the set $i^{*-1}(\nabla) \subset [X \times X, X]$ where $i : X \vee X \rightarrow X \times X$ is the inclusion.

THEOREM 2.5. ([2]) *If (X, μ) is an H-space, then the set of multiplications for X is in one-to-one correspondence with the algebraic loop $[X \wedge X, X]$.*

Two multiplications μ_1 and μ_2 for X are said to be *H-equivalent* if there exists an H-equivalence $h : (X, \mu_1) \rightarrow (X, \mu_2)$, that is, $h : (X, \mu_1) \rightarrow (X, \mu_2)$ is an H-map and a homotopy equivalence. This relation is an equivalence relation and the set $i^{*-1}(\nabla)$ modulo this relation is called the set of essentially different multiplications for X .

In the next section we consider essentially different multiplications for products of H-spaces.

3. Central H-extensions

Let $(X_1, \mu_1), (X_2, \mu_2)$ be H-spaces. The H-space (X, μ) is called an *H-extension* of (X_1, μ_1) by (X_2, μ_2) if there exists a sequence of H-maps

$$(X_1, \mu_1) \xrightarrow{f_1} (X, \mu) \xrightarrow{f_2} (X_2, \mu_2)$$

such that the short sequence

$$e \longrightarrow [A, X_1] \xrightarrow{f_{1*}} [A, X] \xrightarrow{f_{2*}} [A, X_2] \longrightarrow e$$

is exact as an algebraic loop for any space A , where f_{i*} is the loop homomorphism induced by the H-map f_i ($i = 1, 2$).

(X, μ) is called a *central H-extension* if the image of f_{1*} is contained in the center of $[A, X]$. If (X, μ) is the central H-extension, then (X_1, μ_1) is a homotopy associative and homotopy commutative H-space.

On the other hand, if (X_1, μ_1) is a homotopy associative and homotopy commutative H-space, then the sequence

$$(X_1, \mu_1) \xrightarrow{i_1} (X_1 \times X_2, \mu_1 \times \mu_2) \xrightarrow{p_2} (X_2, \mu_2)$$

is a central H-extension (called a trivial central H-extension), where i_1 is the inclusion defined by $i_1(x) = (x, *)$ and p_2 is the projection of $X_1 \times X_2$ on the second factor.

PROPOSITION 3.1. *Let $(X_1, \mu_1) \xrightarrow{f_1} (X, \mu) \xrightarrow{f_2} (X_2, \mu_2)$ be an H-extension. Then X is homotopy equivalent to $X_1 \times X_2$.*

Proof. From the definition of an H-extension, we have $f_2 f_1 \simeq 0$. Since $f_{2*} : [X_2, X] \rightarrow [X_2, X_2]$ is an epimorphism, there exists a mapping $g_2 : X_2 \rightarrow X$ such that $f_2 g_2 \simeq 1_{X_2}$.

We consider the exact sequence

$$e \longrightarrow [X, X_1] \xrightarrow{f_{1*}} [X, X] \xrightarrow{f_{2*}} [X, X_2] \longrightarrow e.$$

Since $f_2(g_2 f_2) \simeq f_2(1_{X_2})$, there exists a unique mapping $g_1 : X \rightarrow X_1$ such that $f_1 g_1 + g_2 f_2 \simeq 1_X$. Then $f_1 \simeq (f_1 g_1 + g_2 f_2) f_1 \simeq f_1 g_1 f_1$. Since f_{1*} is a monomorphism, we have $g_1 f_1 \simeq 1_{X_1}$. Moreover, $g_2 \simeq (f_1 g_1 + g_2 f_2) g_2 \simeq f_1 g_1 g_2 + g_2$. The uniqueness of the difference implies that $f_1 g_1 g_2 \simeq 0$. Thus we obtain that $g_1 g_2 \simeq 0$.

Then the map $(g_1, f_2) = (g_1 \times f_2)\Delta : X \rightarrow X_1 \times X_2$ is a homotopy equivalence and the mapping $\mu(f_1 \times g_2) : X_1 \times X_2 \rightarrow X$ is a homotopy inverse of (g_1, f_2) . Moreover, we obtain that the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X & \xrightarrow{f_2} & X_2 \\ \downarrow = & & \downarrow (g_1, g_2) & & \downarrow = \\ X_1 & \xrightarrow{i_1} & X_1 \times X_2 & \xrightarrow{p_2} & X_2 \end{array}$$

is homotopy commutative. □

Definition. A (central) H-extension

$$(X_1, \mu_1) \xrightarrow{f_1} (X, \mu) \xrightarrow{f_2} (X_2, \mu_2)$$

is called *equivalent* to a (central) H-extension

$$(X_1, \mu_1) \xrightarrow{f'_1} (X', \mu) \xrightarrow{f'_2} (X_2, \mu_2)$$

if there exists an H-map $h : (X, \mu) \rightarrow (X', \mu')$ (called an equivalence) such that the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X & \xrightarrow{f_2} & X_2 \\ \downarrow = & & \downarrow h & & \downarrow = \\ X_1 & \xrightarrow{f'_1} & X' & \xrightarrow{f'_2} & X_2 \end{array}$$

is homotopy commutative.

We note that the above H-map h is a homotopy equivalence and its homotopy inverse is an H-map. Therefore this relation is an equivalence relation.

We denote by $CH(X_1, \mu_1; X_2, \mu_2)$ the set of equivalence classes of central H-extensions of (X_1, μ_1) by (X_2, μ_2) , and denote by $[(X, \mu), f_1, f_2]$ the equivalence class of a central H-extension

$$(X_1, \mu_1) \xrightarrow{f_1} (X, \mu) \xrightarrow{f_2} (X_2, \mu_2).$$

The underlying spaces of two equivalence central H-extensions are homotopy equivalent and they are homotopy equivalent to $X_1 \times X_2$. Thus one is led to consider the essentially different multiplications on $X_1 \times X_2$. We can identify the set $CH(X_1, \mu_1; X_2, \mu_2)$ with a subset of the set of essentially different multiplications of $X_1 \times X_2$.

4. Classification of central H-extensions

Let $(X_1, \mu_1) \xrightarrow{f_1} (X, \mu) \xrightarrow{f_2} (X_2, \mu_2)$ be a central H-extension. As is seen in the proof of Proposition 3.1, there exists a mapping $g : X_2 \rightarrow X$ such that $f_2g \simeq 1_{X_2}$.

Consider the exact sequence

$$e \longrightarrow [X_2 \wedge X_2, X_1] \xrightarrow{f_{1*}} [X_2 \wedge X_2, X] \xrightarrow{f_{2*}} [X_2 \wedge X_2, X_2] \longrightarrow e.$$

Then

$$f_{2*}(HD(g)) = HD(f_2g) = HD(1_{X_2}) = 0$$

by Proposition 1.4.2 of [11]. Hence there exists a unique element $\theta \in [X_2 \wedge X_2, X_1]$ such that

$$f_{1*}(\theta) = HD(g).$$

Let $g' : X_2 \rightarrow X$ be another map such that $f_2g' \simeq 1_{X_2}$. Similarly, we have that there exists a unique element $\theta' \in [X_2 \wedge X_2, X_1]$ such that $f_{1*}(\theta') = HD(g')$.

Consider the exact sequence

$$e \rightarrow [X_2, X_1] \xrightarrow{f_{1*}} [X_2, X] \xrightarrow{f_{2*}} [X_2, X_2] \rightarrow e.$$

Then

$$f_{2*}(D_{g,g'}) = D_{f_2g, f_2g'} = D_{1_{X_2}, 1_{X_2}} = 0$$

by Lemma 1.3.4 of [11]. Hence there exists an element $s \in [X_2, X_1]$ such that

$$f_{1*}(s) = D_{g,g'}.$$

Now f_1s satisfies the conditions of Lemma 2.3. Thus we have

$$f_{1*}(\theta) = HD(g) = HD(f_1s + g') = HD(f_1s) + HD(g') = f_{1*}(HD(s)) + f_{1*}(\theta').$$

Since f_{1*} is a monomorphism, we obtain

$$\theta = HD(s) + \theta'.$$

Let $(X_1, \mu_1) \xrightarrow{f'_1} (X', \mu') \xrightarrow{f'_2} (X_2, \mu_2)$ be a central H-extension which is equivalent to the central H-extension (X, μ) and let the H-map $h : (X, \mu) \rightarrow (X', \mu')$ be its equivalence. We denote that $g' = hg$. Then $f'_2g' \simeq 1_{X_2}$ and there exists a unique element $\theta' \in [X_2 \wedge X_2, X_1]$ such that $f'_{1*}(\theta') = HD(g')$.

Then, by Proposition 1.4.2 of [11],

$$f'_{1*}(\theta) = h_* f_{1*}(\theta) = h_*(HD(g)) = HD(hg) = HD(g') = f'_{1*}(\theta').$$

Since f'_{1*} is a monomorphism, it follows that

$$\theta = \theta'.$$

Now we may define a map

$$\Phi : CH(X_1, \mu_1; X_2, \mu_2) \longrightarrow [X_2 \wedge X_2, X_1]/ImHD$$

by

$$\Phi([(X, \mu), f_1, f_2]) = [-\theta],$$

where $f_2g \simeq 1_{X_2}$, $f_{1*}(\theta) = HD(g)$, $\theta \in [X_2 \wedge X_2, X_1]$.

Let (X_1, μ_1) be a homotopy associative and homotopy commutative H-space and let (X_2, μ_2) be an H-space.

Consider $w \in [X_2 \wedge X_2, X_1]$. We define a map

$$\mu_w : (X_1 \times X_2) \times (X_1 \times X_2) \longrightarrow X_1 \times X_2$$

by

$$\mu_w((x_1, x_2), (y_1, y_2)) = (\mu_1(wq((x_2, y_2)), \mu_1((x_1, y_1))), \mu_2((x_2, y_2)))$$

for $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, where $q : X_2 \times X_2 \rightarrow X_2 \wedge X_2$ is the identification map.

Then we have the following lemma.

LEMMA 4.1. $(X_1 \times X_2, \mu_w)$ is an H-space and

$$(X_1, \mu_1) \xrightarrow{i_1} (X_1 \times X_2, \mu_w) \xrightarrow{p_2} (X_2, \mu_2)$$

is a central H-extension of (X_1, μ_1) by (X_2, μ_2) , where i_1 is the inclusion defined by $i_1(x) = (x, *)$ and p_2 is the projection of $X_1 \times X_2$ on the second factor.

Proof. If (x_1, x_2) or (y_1, y_2) is the base point of $X_1 \times X_2$, then $wq(x_2, y_2) = *$. Thus we obtain that μ_w is a multiplication of $X_1 \times X_2$. Clearly i_1 and p_2 are H-maps. $\square$

LEMMA 4.2. Consider $w' = HD(s) + w$ for $s \in [X_2, X_1]$ and $w \in [X_2 \wedge X_2, X_1]$. We define a map

$$h_s : X_1 \times X_2 \longrightarrow X_1 \times X_2$$

by

$$h_s((x_1, x_2)) = (\mu_1(x_1, s(x_2)), x_2)$$

for $(x_1, x_2) \in X_1 \times X_2$.

Then the map

$$h_s : (X_1 \times X_2, \mu_w) \longrightarrow (X_1 \times X_2, \mu_{w'})$$

is an H-equivalence and the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_1} & X_1 \times X_2 & \xrightarrow{p_2} & X_2 \\ \downarrow = & & \downarrow h_s & & \downarrow = \\ X_1 & \xrightarrow{i_1} & X_1 \times X_2 & \xrightarrow{p_2} & X_2 \end{array}$$

is homotopy commutative.

Proof. We have

$$w'q + \mu_1(s \times s) = wq + s\mu_2$$

in $[X_2 \times X_2, X_1]$ and since the multiplication by μ_1 is homotopy associative and homotopy commutative, we have

$$\mu_1(\mu_1(1_{X_1} \times s) \times \mu_1(1_{X_1} \times s)) = \mu_1(\mu_1 \times \mu_1(s \times s))T_1$$

in $[X_1 \times X_2 \times X_1 \times X_2, X_1]$, where $T_1 : X_1 \times X_2 \times X_1 \times X_2 \rightarrow X_1 \times X_1 \times X_2 \times X_2$ is a map defined by $T_1(x_1, x_2, y_1, y_2) = (x_1, y_1, x_2, y_2)$.

By definitions of μ_w, h_s , we have

$$\begin{aligned} & h_s \mu_w(x_1, x_2, y_1, y_2) \\ &= (\mu_1(\mu_1(\mu_1(x_1, y_1), wq(x_2, y_2)), s\mu_2(x_2, y_2)), \mu_2(x_2, y_2)) \\ &\simeq (\mu_1(\mu_1(\mu_1(x_1, y_1), w'q(x_2, y_2)), \mu_1(s \times s)(x_2, y_2)), \mu_2(x_2, y_2)) \\ &\simeq (\mu_1(\mu_1(\mu_1(x_1, s(x_2)), \mu_1(y_1, s(y_2))), w'q(x_2, y_2)), \mu_2(x_2, y_2)) \\ &= \mu_{w'}(h_s \times h_s)(x_1, x_2, y_1, y_2), \end{aligned}$$

for $(x_1, x_2, y_1, y_2) \in X_1 \times X_2 \times X_1 \times X_2$. Thus h_s is an H-map.

Let $s' \in [X_2, X_1]$ be an inverse element of s . Then we have

$$\begin{aligned} p_1 h_{s'} h_s &\simeq \mu_1(1_{X_1} \times \mu_1(s, s')) \simeq p_1 \\ p_2 h_{s'} h_s &\simeq p_2 \\ p_1 h_s h_{s'} &\simeq \mu_1(1_{X_1} \times \mu_1(s', s)) \simeq p_1 \\ p_2 h_s h_{s'} &\simeq p_2, \end{aligned}$$

where $p_j : X_1 \times X_2 \rightarrow X_j$ ($j = 1, 2$) are the projections. Thus $h_{s'}$ is a homotopy inverse of h_s . $\square$

Now we may define a map

$$\Psi : [X_2 \wedge X_2, X_1] / \text{Im} HD \longrightarrow CH(X_1, \mu_1; X_2, \mu_2)$$

by

$$\Psi([w]) = [(X_1 \times X_2, \mu_{-w}), i_1, p_2]$$

where $w \in [X_2 \wedge X_2, X_1]$.

THEOREM 4.3. *Let (X_1, μ_1) be a homotopy associative and homotopy commutative H -space and let (X_2, μ_2) be an H -space. Then the set $CH(X_1, \mu_1, X_2, \mu_2)$ is in one-to-one correspondence with the left quotient set $[X_2 \wedge X_2, X_1] / \text{Im} HD$.*

Proof. We shall prove that $\Psi\Phi = \text{the identity}$ and $\Phi\Psi = \text{the identity}$.

For $[(X, \mu), f_1, f_2] \in CH(X_1, \mu_1, X_2, \mu_2)$, we denote

$$\Phi([(X, \mu), f_1, f_2]) = [-\theta],$$

by the definition of Φ , where $f_2 g \simeq 1_{X_2}$ and $f_1(-\theta q) + g\mu_2 = \mu(g \times g)$ are in $[X_2 \times X_2, X]$. Then we define a map

$$h : X_1 \times X_2 \rightarrow X$$

by

$$h(x_1, x_2) = \mu(f_1(x_1), g(x_2))$$

for $(x_1, x_2) \in X_1 \times X_2$.

Since the image of $f_{1*} : [\quad, X_1] \rightarrow [\quad, X]$ is contained in the center of $[\quad, X]$, then $h : (X_1 \times X_2, \mu_{-\theta}) \rightarrow (X, \mu)$ can be shown to be an H -map as follows:

$$h\mu_{-\theta}(x_1, x_2, y_1, y_2) = \mu(f_1\mu_1(-\theta q(x_2, y_2), \mu_1(x_1, y_1)), g\mu_2(x_2, y_2))$$

$$\begin{aligned}
&= \mu(\mu(f_1(-\theta q)(x_2, y_2), f_1\mu_1(x_1, y_1)), g\mu_2(x_2, y_2)) \\
&= \mu(f_1\mu_1(x_1, y_1), \mu(f_1(-\theta q)(x_2, y_2), g\mu_2(x_2, y_2))) \\
&= \mu(f_1\mu_1(x_1, y_1), \mu(g \times g)(x_2, y_2)) \\
&= \mu(\mu(f_1 \times f_1)(x_1, y_1), \mu(g \times g)(x_2, y_2)) \\
&= \mu(h \times h)(x_1, x_2, y_1, y_2).
\end{aligned}$$

Furthermore, we obtain the diagram

$$\begin{array}{ccccc}
X_1 & \xrightarrow{i_1} & X_1 \times X_2 & \xrightarrow{p_2} & X_2 \\
\downarrow = & & \downarrow h & & \downarrow = \\
X_1 & \xrightarrow{f_1} & X & \xrightarrow{f_2} & X_2
\end{array}$$

which is homotopy commutative. Therefore $\Phi\Psi = \text{the identity}$.

We consider a central H-extension

$$(X_1, \mu_1) \xrightarrow{i_1} (X_1 \times X_2, \mu_{-\theta}) \xrightarrow{p_2} (X_2, \mu_2)$$

for $\theta \in [X_2 \wedge X_2, X_1]$. We define

$$g : X_2 \rightarrow X_1 \times X_2$$

by

$$g(x_2) = (*, x_2)$$

for $x_2 \in X_2$. Then, for $(x_2, y_2) \in X_2 \times X_2$,

$$\begin{aligned}
\mu_{-\theta}(g \times g)(x_2, y_2) &= (-\theta q(x_2, y_2), \mu_2(x_2, y_2)) \\
g\mu_2(x_2, y_2) &= (*, \mu_2(x_2, y_2)).
\end{aligned}$$

Thus we have

$$HD(g_2) = i_{1*}(\theta).$$

Therefore, $\Psi\Phi = \text{the identity}$. □

Let $CP(3)$ denote the complex projective 3-space and consider the fiber sequence

$$\Omega S^7 \xrightarrow{\Omega\pi} \Omega CP(3) \xrightarrow{q} S^1 \xrightarrow{\iota} S^7 \xrightarrow{\pi} CP(3).$$

Then ι is null homotopic and $CP(3)$ admits a homotopy action of S^7 along π . By [7], the image of $\Omega\pi_* : [A, \Omega S^7] \rightarrow [A, \Omega CP(3)]$ is contained in the center of

$[A, \Omega CP(3)]$ for any space A . The loop map $\Omega\pi$ and q are H-maps. Thus the fibration

$$\Omega S^7 \xrightarrow{\Omega\pi} \Omega CP(3) \xrightarrow{q} S^1$$

is a central H-extension and $[S^1 \wedge S^1, \Omega S^7] = \{e\}$.

COROLLARY 4.4. ([8]) *The loop space $\Omega CP(3)$ is homotopy commutative.*

Proof. By Theorem 4.3, any central H-extension of ΩS^7 by S^1 is equivalent to the trivial central H-extension. Thus the central H-extension

$$\Omega S^7 \xrightarrow{\Omega\pi} \Omega CP(3) \xrightarrow{q} S^1$$

is equivalent to the trivial central H-extension

$$\Omega S^7 \xrightarrow{i} \Omega S^7 \times S^1 \xrightarrow{p_2} S^1.$$

Since $\Omega S^7 \times S^1$ is homotopy commutative, we obtain that $\Omega CP(3)$ is homotopy commutative. $\square$

Let $K(Z, n)$ be the Eilenberg–MacLane space and let μ_n be a multiplication. $K(Z, n)$ may be considered as an abelian topological group.

Now, we have

$$[K(Z, 2n) \wedge K(Z, 2n), K(Z, 4n)] \cong H^{4n}(K(Z, 2n) \wedge K(Z, 2n); Z) \cong Z,$$

and a generator $w \in [K(Z, 2n) \wedge K(Z, 2n), K(Z, 4n)]$ corresponds to a generator $\iota_{2n} \otimes \iota_{2n} \in H^{4n}(K(Z, 2n) \wedge K(Z, 2n); Z)$, where $\iota_{2n} \in H^{2n}(K(Z, 2n); Z)$ is the fundamental class.

We define a map

$$s : K(Z, 2n) \rightarrow K(Z, 4n)$$

by $s = wq \triangle$. Then we obtain that

$$HD(s)q = s\mu_{2n} - sp_1 - sp_2,$$

where $p_i : K(Z, 4n) \times K(Z, 2n) \rightarrow K(Z, 2in)$ ($i = 1, 2$) are the projections.

One should study $HD(s)^* : H^*(K(Z, 4n); Z) \rightarrow H^*(K(Z, 2n) \wedge K(Z, 2n); Z)$:

$$\begin{aligned} s^*(\iota_{4n}) &= \iota_{2n}^2 \\ HD(s)^*(\iota_{4n}) &= (\iota_{2n} \otimes 1 + 1 \otimes \iota_{2n})^2 - (\iota_{2n} \times 1)^2 - (1 \otimes \iota_{2n})^2 \\ &= 2\iota_{2n} \otimes \iota_{2n}. \end{aligned}$$

So the image of $HD : [K(Z, 2n), K(Z, 4n)] \rightarrow [K(Z, 2n) \wedge K(Z, 2n), K(Z, 4n)]$ is generated by $2w$. Hence, from Theorem 4.3, we have

$$CH(K(Z, 4n), \mu_{4n}; K(Z, 2n), \mu_{2n}) \cong Z_2.$$

In this case, we may identify the set of essentially different multiplications for $K(Z, 4n) \times K(Z, 2n)$ with the set $CH(K(Z, 4n), \mu_{4n}; K(Z, 2n), \mu_{2n})$.

COROLLARY 4.5. ([11]) $K(Z, 4n) \times K(Z, 2n)$ admits only two essentially different multiplications.

COROLLARY 4.6. $K(Z, 2n) \times S^n$ ($n = 1, 3, 7$) admits infinitely many essentially different multiplications

Proof. Since $[S^n \wedge S^n, K(Z, 2n)] \cong Z$ and $[S^n, K(Z, 2n)] = \{e\}$, we have

$$CH(K(Z, 2n), \mu_{2n}; S^n, \mu_{S^n}) \cong Z.$$

Also, we may regard the set of the equivalence classes of the central H-extensions in the same light with the subset of the set of the essentially different multiplications. This completes the proof. $\square$

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