ON THE RANK OF THE HYPERGEOMETRIC SYSTEM $E(n + 1, m + 1; \alpha)$

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0. Introduction

In a previous paper [K2], we showed that the system of hypergeometric differential equations $E(n + 1, m + 1; \alpha)$ admits $\binom{m-1}{n}$ linearly independent hypergeometric integrals as its solutions. However, logically speaking, the system may have another solution which is not written as a linear combination of the integrals. [GKZ] has already given its solutions as power series and the number of its rank in view of the theory of D -modules and toral manifolds. However, in this paper we shall show that the rank of the system is equal to $\binom{m-1}{n}$ in a direct and simple way. Of course a set of all linearly independent solutions of the system is given by the integrals. To prove the statement, we use essentially an identity of determinants of differential operators, which has the analogous form to the Capelli identity.

1. Review of hypergeometric functions of type $(n + 1, m + 1)$

1.1. Hypergeometric functions of type $(n + 1, m + 1)$

We recall the definition of hypergeometric functions of type $(n + 1, m + 1)$ following [G], [GG] and [S]. Let $X_{n+1, m+1}$ denote the space of $(n + 1) \times (m + 1)$ complex matrices

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,m} \\ x_{1,0} & x_{1,1} & \cdots & x_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,0} & x_{n,1} & \cdots & x_{n,m} \end{pmatrix} \in M(n + 1, m + 1, \mathbb{C})$$

such that the rank of x is $n + 1$ and each column vector of x is non-zero.

Let $[v_0 : v_1 : \cdots : v_n]$ be homogeneous coordinates of the n -dimensional complex projective space $\mathbb{P}^n$ and let τ be an n -form on $\mathbb{C}^{n+1}$ defined by

$$\tau := \sum_{i=0}^n (-1)^i v_i dv_0 \wedge \cdots \wedge \widehat{dv_i} \wedge \cdots \wedge dv_n.$$

For a set of complex numbers $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m) \in (\mathbb{C} - \mathbb{Z})^{m+1}$ with the property

$$\sum_{j=0}^m \alpha_j + n + 1 = 0 \quad (1.1)$$

and for a point $x \in X_{n+1, m+1}$, we can regard the n -form

$$\prod_{j=1}^m \left(\sum_{i=0}^n x_{ij} v_i \right)^{\alpha_j} \cdot \tau \quad (1.2)$$

as a many-valued n -form on $\mathbb{P}^n$. For a twisted n -cycle σ associated with the n -form (1.2), we define the *hypergeometric integral* $\Phi(x, \alpha)$ of type $(n+1, m+1)$ as

$$\Phi(x, \alpha) := \int_{\sigma} \prod_{j=1}^m \left(\sum_{i=0}^n x_{ij} v_i \right)^{\alpha_j} \cdot \tau.$$

The above integral is homogeneous under two kinds of group action. $X_{n+1, m+1}$ admits the left $GL(n+1, \mathbb{C})$ -action

$$x \rightarrow g \cdot x \quad (g \in GL(n+1, \mathbb{C}))$$

and the integral $\Phi(x, \alpha)$ changes as

$$\Phi(g \cdot x, \alpha) = (\det g)^{-1} \Phi(x, \alpha).$$

On the other hand, the Cartan subgroup H_{m+1} , which consists of diagonal matrices of $GL(m+1, \mathbb{C})$, acts on $X_{n+1, m+1}$ on the right and $\Phi(x, \alpha)$ transforms as

$$\Phi(xh, \alpha) = \prod_{j=0}^m h_j^{\alpha_j} \cdot \Phi(x, \alpha),$$

where $h = \text{diag}[h_0, \dots, h_m]$.

It is easy to see that the function $\Phi(x, \alpha)$ on $X_{n+1, m+1}$ satisfies the following system $E(n+1, m+1; \alpha)$ of differential equations:

$$\begin{aligned} \sum_{k=0}^m x_{ik} \frac{\partial \Phi}{\partial x_{jk}} &= -\delta_{ij} \Phi, \quad 0 \leq i, j \leq n, \\ \sum_{i=0}^n x_{ik} \frac{\partial \Phi}{\partial x_{ik}} &= \alpha_k \Phi, \quad 0 \leq k \leq m, \\ \frac{\partial^2 \Phi}{\partial x_{ik} \partial x_{jh}} - \frac{\partial^2 \Phi}{\partial x_{ih} \partial x_{jk}} &= 0, \quad 0 \leq i, j \leq n, \quad 0 \leq k, h \leq m. \end{aligned}$$

Suppose that $x \in X_{n+1, m+1}$ satisfies the condition that any $(n+1)$ -minors are non-zero. By using the $GL(n+1, \mathbb{C})$ -action, we can reduce x into the matrix

$$\begin{pmatrix} 1 & & x_{0, n+1} & x_{0, n+2} & \cdots & x_{0, m} \\ & 1 & x_{1, n+1} & x_{1, n+2} & \cdots & x_{1, m} \\ & & \ddots & \vdots & \ddots & \vdots \\ & & & 1 & x_{n, n+1} & x_{n, n+2} & \cdots & x_{n, m} \end{pmatrix} \in M(n+1, m+1, \mathbb{C}).$$

We introduce a non-homogeneous coordinate system $(u_1, \dots, u_n)$ by setting $u_i = v_i/v_0$, $1 \leq i \leq n$, and put

$$\begin{aligned} f_i(u) &= u_i \quad (1 \leq i \leq n), \\ f_j(u) &= x_{0j} + \sum_{i=1}^n x_{ij} u_i \quad (n+1 \leq j \leq m), \\ M &= \mathbb{C}^n - \bigcup_{j=1}^m \{f_j = 0\}. \end{aligned}$$

Then the integral $\Phi(x, \alpha)$ can be written as

$$F(x, \alpha) = \int_{\sigma} \prod_{j=1}^m f_j^{\alpha_j} du_1 \wedge \cdots \wedge du_n,$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{C} - \mathbb{Z})^{m+1}$. We can show that the integral $F(x, \alpha)$ satisfies the following system $E'(n+1, m+1, \alpha)$ of differential equations:

$$\begin{aligned} \sum_{k=n+1}^m x_{ik} \frac{\partial F}{\partial x_{ik}} &= -(1 + \alpha_i)F, \quad 0 \leq i \leq n, \\ \sum_{i=0}^n x_{ik} \frac{\partial F}{\partial x_{ik}} &= \alpha_k F, \quad n+1 \leq k \leq m, \\ \frac{\partial^2 F}{\partial x_{ik} \partial x_{jh}} - \frac{\partial^2 F}{\partial x_{ih} \partial x_{jk}} &= 0, \quad 0 \leq i, j \leq n, \quad n+1 \leq k, h \leq m. \end{aligned}$$

It is known that the system $E(n+1, m+1; \alpha)$ is equivalent to $E'(n+1, m+1, \alpha)$; for the proof see [H, Appendix A].

1.2. The Wronskian of the hypergeometric integrals

In Section 2 we shall use the following fact about the Wronskian of the hypergeometric integrals of type $(n+1, m+1)$. Let ω be the connection form

defined by $\omega := d \log \prod f_j^{\alpha_j}$ and set $\nabla_\omega := d + \omega \wedge$. We denote by $\mathcal{L}_\omega$ the complex local system of solutions of $\nabla_\omega h = 0$, $h \in \mathcal{O}_M$ where $\mathcal{O}_M$ is the sheaf of germs of holomorphic functions on M . Let $\mathcal{L}_\omega^\vee$ be the dual local system to $\mathcal{L}_\omega$ and let $H_n(M, \mathcal{L}_\omega^\vee)$ be the n -dimensional homology group with coefficient in $\mathcal{L}_\omega^\vee$. It is known ([K2]) that $\dim H_n(M, \mathcal{L}_\omega^\vee) = \binom{m-1}{n}$. We choose a basis $\{\sigma_v\}_{1 \leq v \leq \binom{m-1}{n}}$ of the homology group and set

$$F_v(x, \alpha) = \int_{\sigma_v} \prod_{j=1}^m f_j^{\alpha_j} du_1 \wedge \cdots \wedge du_n.$$

Then the following lemma holds.

LEMMA 1. ([K2]) *We assume that $\alpha_j \notin \mathbb{Z}$, $1 \leq j \leq m$ and $\sum_{j=1}^m \alpha_j \notin \mathbb{Z}$. If all minors of x of order $1, \dots, n$ are not zero, then the Wronskian of the hypergeometric integrals is not equal to zero, i.e.*

$$\det \left(\frac{\partial^p F_v}{\partial x_{i_1 k_1} \cdots \partial x_{i_p k_p}} \right) \neq 0,$$

where $1 \leq v \leq \binom{m-1}{n}$, $1 \leq i_1 < \cdots < i_p \leq n$, $n+2 \leq k_1 < \cdots < k_p \leq m$, $0 \leq p \leq n$ and the entries in the determinant are arranged in the lexicographic order of the multi indices $\{i_1 k_1, \dots, i_p k_p\}$, $0 \leq p \leq n$. Hence the $\binom{m-1}{n}$ solutions $F_v(\alpha, x)$ of $E(n+1, m+1; \alpha)$ are linearly independent and we have

$$\text{rank } E(n+1, m+1; \alpha) \geq \binom{m-1}{n}.$$

2. Reduction of differential operators

Throughout this section we use the following notations:

$$\partial_{ij} := \partial / \partial x_{ij}, \quad \vartheta_{ij} := x_{ij} \partial / \partial x_{ij},$$

$$x \binom{i_1 i_2 \cdots i_p}{k_1 k_2 \cdots k_p} = \det \begin{pmatrix} x_{i_1 k_1} & x_{i_1 k_2} & \cdots & x_{i_1 k_p} \\ x_{i_2 k_1} & x_{i_2 k_2} & \cdots & x_{i_2 k_p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_p k_1} & x_{i_p k_2} & \cdots & x_{i_p k_p} \end{pmatrix},$$

$$\vartheta \begin{pmatrix} i_1 i_2 \cdots i_p \\ k_1 k_2 \cdots k_p \end{pmatrix} = \det \begin{pmatrix} \vartheta_{i_1 k_1} & \vartheta_{i_1 k_2} & \cdots & \vartheta_{i_1 k_p} \\ \vartheta_{i_2 k_1} & \vartheta_{i_2 k_2} & \cdots & \vartheta_{i_2 k_p} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{i_p k_1} & \vartheta_{i_p k_2} & \cdots & \vartheta_{i_p k_p} \end{pmatrix}.$$

We begin with the following tableau which represents the system $E(n+1, m+1; \alpha)$ schematically:

$$\vartheta = \begin{pmatrix} \vartheta_{0,n+1} & \vartheta_{0,n+2} & \cdots & \vartheta_{0,m} \\ \vartheta_{1,n+1} & \vartheta_{1,n+2} & \cdots & \vartheta_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{n,n+1} & \vartheta_{n,n+2} & \cdots & \vartheta_{n,m} \end{pmatrix} \quad \begin{array}{l} -(1+\alpha_0) =: -\beta_0 \\ -(1+\alpha_1) =: -\beta_1 \\ \vdots \\ -(1+\alpha_n) =: -\beta_n \end{array}$$

$$\alpha_{n+1} \quad \alpha_{n+2} \quad \cdots \quad \alpha_m$$

This indicates the following equations. From rows

$$\sum_{k=n+1}^m \vartheta_{ik} F = -\beta_i F \quad (0 \leq i \leq n); \quad (2.1)$$

from columns

$$\sum_{i=0}^n \vartheta_{ik} F = \alpha_k F \quad (n+1 \leq k \leq m); \quad (2.2)$$

and from 2×2 minors

$$\partial_{ik} \partial_{jh} F - \partial_{ih} \partial_{jk} F = 0 \quad (0 \leq i, j \leq n, n+1 \leq k, h \leq m), \quad (2.3)$$

where the α_j and β_i are constants satisfying

$$\sum_{i=0}^n \beta_i + \sum_{j=n+1}^m \alpha_j = 0.$$

We set

$$\begin{aligned} I &= \{i_1, \dots, i_p\}, \quad 0 \leq i_1 < \cdots < i_p \leq n, \\ K &= \{k_1, \dots, k_p\}, \quad n+1 \leq k_1 < \cdots < k_p \leq m, \\ L &= \{0, 1, \dots, n\} - I = \{l_0, l_1, \dots, l_{n-p}\}, \quad 0 \leq l_0 < \cdots < l_{n-p} \leq n, \\ H &= \{n+1, \dots, m\} - K = \{h_1, \dots, h_{m-n-p}\}, \\ &\quad n+1 \leq h_1 < \cdots < h_{m-n-p} \leq m. \end{aligned}$$

We show that the rank of the system $E(n+1, m+1; \alpha)$ is less than or equal to $\binom{m-1}{n}$. To this end it suffices to prove the following proposition.

PROPOSITION 2.1. *Let F be a general solution of the hypergeometric system $E(n+1, m+1; \alpha)$, and let $\mathbb{C}(x)$ be the rational function field generated by x_{ij} , $0 \leq i \leq n$, $n+1 \leq j \leq m$.*

(1) *We set*

$$S := \mathbb{C}(x)F + \sum_{p=1}^n \sum_{\substack{I, K \\ |I|=|K|=p}} \mathbb{C}(x) \vartheta \binom{I}{K} F.$$

Then S is closed under the differential operators ∂_{ik} , $0 \leq i \leq n$, $n+1 \leq k \leq m$.

(2) *Every function in S is a linear combination over $\mathbb{C}(x)$ of the functions F and*

$$\vartheta \binom{I}{K} F, \quad \text{where } |I| = |K| = p, p \leq n, I \subset \{1, \dots, n\}, K \subset \{n+2, \dots, m\}.$$

To show Proposition 2.1, we need the following three lemmas.

LEMMA 2.2. (Determinantal formula)

(1) *If $|K| = n+1$, $K = \{k_0, \dots, k_n\}$, then*

$$\vartheta \binom{01 \cdots n}{K} F \in S.$$

(2)

$$\vartheta \binom{01 \cdots n}{k_0 \cdots k_n} F = x \binom{01 \cdots n}{k_0 \cdots k_n} \partial_{0k_0} \cdots \partial_{nk_n} F,$$

and hence

$$\partial_{0k_0} \cdots \partial_{nk_n} F \in S.$$

(3) *For $p \leq n$,*

$$\vartheta \binom{i_1 \cdots i_p}{k_1 \cdots k_p} F = x \binom{i_1 \cdots i_p}{k_1 \cdots k_p} \partial_{i_1 k_1} \cdots \partial_{i_p k_p} F,$$

and hence

$$\partial_{i_1 k_1} \cdots \partial_{i_p k_p} F \in S.$$

Proof. We consider the following matrix:

$$\begin{pmatrix} \vartheta_{0k_0} & \vartheta_{1k_0} & \cdots & \vartheta_{nk_0} \\ \vartheta_{0k_1} & \vartheta_{1k_1} & \cdots & \vartheta_{nk_1} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{0k_n} & \vartheta_{1k_n} & \cdots & \vartheta_{nk_n} \end{pmatrix} \begin{pmatrix} F \\ F \\ \vdots \\ F \end{pmatrix} = \begin{pmatrix} \alpha_{k_0} F \\ \alpha_{k_1} F \\ \vdots \\ \alpha_{k_n} F \end{pmatrix} \quad (2.4)$$

and set

$$\Delta_{ik_j} := (-1)^{i+j} \vartheta \begin{pmatrix} 0 & \cdots & \widehat{i} & \cdots & n \\ k_0 & \cdots & \widehat{k_j} & \cdots & k_n \end{pmatrix}.$$

Multiplying the both sides of (2.4) by the cofactor matrix

$$\begin{pmatrix} \Delta_{0k_0} & \Delta_{0k_1} & \cdots & \Delta_{0k_n} \\ \Delta_{1k_0} & \Delta_{1k_1} & \cdots & \Delta_{1k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{nk_0} & \Delta_{nk_1} & \cdots & \Delta_{nk_n} \end{pmatrix},$$

we get

$$\vartheta \begin{pmatrix} 01 \cdots n \\ K \end{pmatrix} \cdot \begin{pmatrix} F \\ F \\ \vdots \\ F \end{pmatrix} = \begin{pmatrix} (\alpha_{k_0} \Delta_{0k_0} + \alpha_{k_1} \Delta_{0k_1} + \cdots + \alpha_{k_n} \Delta_{0k_n}) F \\ (\alpha_{k_0} \Delta_{1k_0} + \alpha_{k_1} \Delta_{1k_1} + \cdots + \alpha_{k_n} \Delta_{1k_n}) F \\ \vdots \\ (\alpha_{k_0} \Delta_{nk_0} + \alpha_{k_1} \Delta_{nk_1} + \cdots + \alpha_{k_n} \Delta_{nk_n}) F \end{pmatrix}$$

and hence we obtain (1) of our lemma. We have

$$\begin{aligned} \vartheta \begin{pmatrix} 01 \cdots n \\ k_0 \cdots k_n \end{pmatrix} F &= \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) x_{0k_{\sigma(0)}} \cdots x_{nk_{\sigma(n)}} \partial_{0k_{\sigma(0)}} \cdots \partial_{nk_{\sigma(n)}} F \\ &= \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) x_{0k_{\sigma(0)}} \cdots x_{nk_{\sigma(n)}} \partial_{0k_0} \cdots \partial_{nk_n} F \quad (\text{by (2.3)}) \\ &= x \begin{pmatrix} 01 \cdots n \\ k_0 \cdots k_n \end{pmatrix} \partial_{0k_0} \cdots \partial_{nk_n} F, \end{aligned}$$

which shows the first formula in (2) of our lemma. Similarly statement (3) is proved. $\square$

Remark 1. Notice that formulas (2) and (3) in Lemma 2.2 have analogous forms to the Capelli identity ([W, pp. 39–42]).

To compute $\partial_{i_K} \vartheta \begin{pmatrix} l \\ K \end{pmatrix} F$, we use the following two lemmas.

LEMMA 2.3. (Row operation)

$$\vartheta \begin{pmatrix} i_1 i_2 \cdots i_p \\ k_1 k_2 \cdots k_p \end{pmatrix} F = - \sum_{l \in L} \vartheta \begin{pmatrix} l i_2 \cdots i_p \\ k_1 k_2 \cdots k_p \end{pmatrix} F + \sum_{\lambda=1}^p (-1)^{\lambda+1} \alpha_{k_\lambda} \vartheta \begin{pmatrix} i_2 \cdots i_p \\ k_1 \cdots \widehat{k_\lambda} \cdots k_p \end{pmatrix} F.$$

Proof. We can check Lemma 2.3 by using the multilinearity of the determinant and (2.2). $\square$

In the same way, we can show the following.

LEMMA 2.4. (Column operation)

$$\vartheta \begin{pmatrix} i_1 i_2 \cdots i_p \\ k_1 k_2 \cdots k_p \end{pmatrix} F = - \sum_{h \in H} \vartheta \begin{pmatrix} i_1 i_2 \cdots i_p \\ h k_2 \cdots k_p \end{pmatrix} F + \sum_{\lambda=1}^p (-1)^\lambda \beta_{i_\lambda} \vartheta \begin{pmatrix} i_1 \cdots \widehat{i_\lambda} \cdots i_p \\ k_2 \cdots k_p \end{pmatrix} F.$$

Proof of Proposition 2.1. We show the first part of the proposition by induction on $p = |I|$. In the case $p = 0$, it is obvious by the definition of S . We assume that the statement holds for $|I| \leq p - 1$ and we show that it is also true for $|I| = |K| = p$. We divide the proof into three cases.

Case (i) where $\iota \notin I$ and $\kappa \notin K$. By Lemma 2.2(3), we have

$$\begin{aligned} \partial_{\iota\kappa} \vartheta \begin{pmatrix} I \\ K \end{pmatrix} F &= x \begin{pmatrix} I \\ K \end{pmatrix} \partial_{\iota\kappa} \partial_{i_1 k_1} \cdots \partial_{i_p k_p} F \\ &= \frac{x \begin{pmatrix} I \\ K \end{pmatrix}}{x \begin{pmatrix} \iota I \\ \kappa K \end{pmatrix}} \vartheta \begin{pmatrix} \iota I \\ \kappa K \end{pmatrix} F. \end{aligned} \quad (2.5)$$

In the case when $p \leq n - 1$, the statement is trivial; in the case when $p = n$, using Lemma 2.2(1), we see that the last term of (2.5) is in S .

Case (ii) where either $\iota \in I$ or $\kappa \in K$. In the case when ι is in I but κ is not in K , we may assume $\iota = i_1 \in I$ by using a property of the determinant. By Lemma 2.3,

$$\begin{aligned} \partial_{\iota\kappa} \vartheta \begin{pmatrix} i_1 i_2 \cdots i_p \\ k_1 k_2 \cdots k_p \end{pmatrix} F &= - \sum_{l \in L} \partial_{\iota\kappa} \vartheta \begin{pmatrix} l i_2 \cdots i_p \\ k_1 k_2 \cdots k_p \end{pmatrix} F \\ &\quad + \sum_{\lambda=1}^p (-1)^{\lambda+1} \alpha_{k_\lambda} \partial_{\iota\kappa} \vartheta \begin{pmatrix} i_2 \cdots i_p \\ k_1 \cdots \widehat{k_\lambda} \cdots k_p \end{pmatrix} F. \end{aligned} \quad (2.6)$$

Since $\kappa \notin K$ and $l \notin I$, by using the result of case (i), we have that the first term of the right-hand side of (2.6) is in S . The second term is also in S by the induction assumption. Therefore we have $\partial_{\iota\kappa} \vartheta \begin{pmatrix} I \\ K \end{pmatrix} F \in S$. Using Lemma 2.4, we can check the case where $\kappa \in K$, $\iota \notin I$ in the same way.

Case (iii) where $\iota \in I$ and $\kappa \in K$. We may assume that $\iota = i_1$ and $\kappa = k_1$ by using a property of the determinant. Repeated use of Lemmas 2.3 and 2.4 gives the following:

$$\vartheta \begin{pmatrix} I \\ K \end{pmatrix} F = \sum_{l \in L, h \in H} \vartheta \begin{pmatrix} l I' \\ h K' \end{pmatrix} F + \left(\sum_{k \in K} \alpha_k + \sum_{l \in L} \beta_l \right) \vartheta \begin{pmatrix} I' \\ K' \end{pmatrix} F$$

$$\begin{aligned}
& + \sum_{h \in H} \sum_{\mu=2}^p (-1)^\mu \alpha_{k_\mu} \vartheta \left({}^{I'}_{hk_1 \dots \widehat{k_\mu} \dots k_p} \right) F \\
& + \sum_{l \in L} \sum_{\lambda=2}^p (-1)^{\lambda+1} \beta_{i_\lambda} \vartheta \left({}^l_{K'} i_2 \dots \widehat{i_\lambda} \dots i_p \right) F \\
& + \sum_{\mu, \lambda=2}^p (-1)^{\lambda+\mu} \alpha_{k_\mu} \beta_{i_\lambda} \vartheta \left({}^{i_2 \dots \widehat{i_\lambda} \dots i_p}_{k_2 \dots \widehat{k_\mu} \dots k_p} \right) F,
\end{aligned}$$

where $I' := I - \{i_1\}$, $K' := K - \{k_1\}$. By differentiating both sides of (2.7) with respect to $x_{i_1 k_1}$, we see that $\sum \partial_{i_1 k_1} \vartheta \left({}^l_{hK'} i_2 \dots \widehat{i_\lambda} \dots i_p \right) F$ is in S by virtue of the result of case (i). Since the derivatives with respect to $x_{i_1 k_1}$ of the other terms in the right-hand side of (2.7) are sums of minors of order $\leq p-1$, they are in S by the induction assumption. Therefore we have $\partial_{i_1 k_1} \vartheta \left({}^l_K i_2 \dots \widehat{i_\lambda} \dots i_p \right) F \in S$.

Now we prove the second part of the proposition. We consider a minor

$$\vartheta \left({}^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_p} \right)$$

of order p , $p \leq n$. In the case when $i_1 = 0$, Lemma 2.3 gives

$$\vartheta \left({}^{0I'}_K \right) F = - \sum_{l \in L} \vartheta \left({}^l_{K'} i_2 \dots \widehat{i_\lambda} \dots i_p \right) F + \sum_{\mu=1}^p (-1)^{\mu+1} \alpha_{k_\mu} \vartheta \left({}^{I'}_{k_1 \dots \widehat{k_\mu} \dots k_p} \right) F.$$

Since 0 is not included in I' and L , then $\vartheta \left({}^{0I'}_K \right)$ is a linear combination of minors of order $\leq p$ which consists of rows other than the first one. In the case when $n+1 \in K$, we can show the analogous statement by using Lemma 2.4. This completes the proof of Proposition 2.1. $\square$

Remark 2. The condition that $\alpha_j \notin \mathbb{Z}$, $1 \leq j \leq m$ and $\sum_{j=1}^m \alpha_j \notin \mathbb{Z}$, is not assumed in the proof of Proposition 2.1.

Remark 3. In the statement of Proposition 2.1, we can replace $\mathbb{C}(x)$ by $\mathbb{C}[x_{ij}, 1/x \left({}^l_K \right)]$.

THEOREM 2.5. *We suppose that $\alpha_j \notin \mathbb{Z}$, $0 \leq j \leq m$. If all minors of the matrices x of order $1, \dots, n$ are not zero, then the rank of the system $E(n+1, m+1; \alpha)$ is equal to $\binom{m-1}{n}$ and a set of all linearly independent solutions is given by the hypergeometric integrals $\int_{\sigma_v} \prod_{j=1}^m f_j^{\alpha_j} du_1 \wedge \dots \wedge du_n$, where σ_v runs over a basis of $H_n(M, \mathcal{L}_\omega^\vee)$.*

Proof.

$$\begin{aligned}
 & \text{rank } E(n+1, m+1; \alpha) \\
 & \leq \# \left\{ \vartheta \left(\begin{matrix} I \\ K \end{matrix} \right) \middle| |I| = |K| = p, \ 0 \leq p \leq n, \right. \\
 & \quad \left. I \subset \{1, \dots, n\}, \ K \subset \{n+2, \dots, m\} \right\} \\
 & = \sum_{p=0}^n \binom{n}{p} \binom{m-n-1}{p} = \sum_{p=0}^n \binom{n}{n-p} \binom{(m-1)-n}{p} = \binom{m-1}{n}.
 \end{aligned}$$

If $\alpha_j \notin \mathbb{Z}$, $0 \leq j \leq m$, by Lemma 1 the rank of the system $E(n+1, m+1; \alpha)$ is equal to $\binom{m-1}{n}$. $\square$

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