MEAN VALUES AND MEAN-CONVOLUTION PRODUCTS[†]

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(Received 28 June 1995 and revised 18 January 1996)

0. Introduction

Let us consider mean values and mean-convolution products for bounded measurable functions on $\mathbb{R}^m$. These are computed with respect to a weight function, which is non-negative and has the unit mass. Thus, let a weight function $\phi(x)$ be given. If, for a bounded function $f(x)$, the limit

$$M_\phi(f) = \lim_{0 < L \rightarrow +\infty} \frac{1}{L^m} \int_{\mathbb{R}^m} \phi\left(\frac{x-y}{L}\right) f(y) dy \quad (0.1)$$

exists independently of x , then $M_\phi(f)$ is the mean value of f . Similarly, for a pair of bounded functions $f(x)$ and $g(x)$, the limit

$$f \otimes_\phi g(y) = \lim_{0 < L \rightarrow +\infty} \frac{1}{L^m} \int_{\mathbb{R}^m} \phi\left(\frac{x-z}{L}\right) f(y-z) g(z) dz, \quad (0.2)$$

when well defined independently of x , is the mean-convolution product of f and g .

It is classical known fact that, when $m = 1$, mean values and mean-convolution products are well defined for almost periodic functions on the line, independently of the choice of weight functions (see [1, 3, 6, 8, 12, 13, 16, 17]). In this case, we also have

$$M_\phi(f \otimes_\phi g) = M_\phi(f) M_\phi(g). \quad (0.3)$$

However, mean values or mean-convolution products do not always make sense, and even if they do they may depend on weight functions, and may not enjoy property (0.3).

As suggestive examples, consider the functions $c_\pm(x)$ on the line, where

$$c_\pm(x) = \begin{cases} 1, & \pm x > 0, \\ 0, & \pm x < 0. \end{cases}$$

[†] Dedicated to Professor Hikosaburo Komatsu on the occasion of his sixtieth anniversary.

[‡] This work is supported in part by Grant-in-aid for Scientific Research No. (B)05452011, Ministry of Education, Culture and Science, Japanese Government.

For any choice of weight functions $\phi(x)$, their mean values and mean-convolution products make sense. For instance, if $\phi(x)$ is even, $\phi(-x) = \phi(x)$, we have

$$M_\phi(c_\pm) = \frac{1}{2}, \quad c_+ \otimes_\phi c_+ = c_- \otimes_\phi c_- = 0, \quad c_+ \otimes_\phi c_- = \frac{1}{2}.$$

Apparently, property (0.3) is not maintained (see Proposition 1.3 and Examples 4.1, 4.5 for more details).

In this paper we show that the topologies in which the limits (0.1) or (0.2) are taken play a crucial role in the related discussions. We show, in particular, that mean values when well defined are independent of weight functions provided that the limit (0.1) is taken in the strong topology of $L^\infty(\mathbb{R}^m)$ (see Theorem 1.5). This allows us a considerable liberty in computing mean values (see Theorem 1.6 and Proposition 2.4). Unfortunately, this is not the case when the limit is taken pointwise or in the weak* topology (see Proposition 1.3).

There is some interest in functions which have mean values and mean-convolution products, for instance, in relation to asymptotic solutions of quasi-linear systems of partial differential equations (see [4–7, 11, 14]). For the hyperbolic case, when the initial values are restricted to be oscillatory, almost periodic functions are a sufficient means for such an analysis (see [7]). But one might wish to consider a certain type of discontinuous initial data, for which mean values or mean-convolution products still make sense. Then what would be the natural class of functions to work with? Are such functions well adapted as functional analytic means? Related questions also arise in the parabolic or elliptic cases (e.g. stationary random fields, see [6]).

Although the theme is very classical (see [1, 6, 9, 10, 13, 17]), we still have to clarify many basic questions about mean values and mean-convolution products. Here are some such questions:

- (1) What is the condition on functions which have mean values with respect to a given weight function?
- (2) What is the condition for a pair of functions for which the mean-convolution product is well defined with respect to a given weight function?
- (3) How do the mean values or mean-convolution products depend on the choice of weight functions?
- (4) What are the maximal classes of functions for which two elements have the mean-convolution product with respect to a fixed weight function?
- (5) How are mean values and mean-convolution products related, if both make sense?

In addition to these questions, we have to investigate operational properties of mean values or mean-convolution products, including some calculi related to these notions. Here we will also consider the calculus part of mean values and mean-convolution products. We include a discussion on the spaces of functions admitting mean values and their topologies to be furnished therewith (see Section 3).

We then analyze the idea of a mean-convolution product and distinguish some cases in which (0.3) is valid (see Section 4). Commutativity of a mean-convolution product does not necessarily hold, and, in fact, depends on the choice of topologies in which the limit (0.2) is taken (see Proposition 4.1). Similar remarks are valid for associativity of mean-convolution products (see Lemma 6.1).

In Section 5 we observe a kind of Birkoff–Khinchin ergodic property related to admissible functions. We have some results (Lemmas 5.2 and 5.3), but still have much to do in order to establish *the* subclass of admissible functions reasonably related to periodic tilings.

As a consequence of these discussions, while basic questions of the general nature stated above remain rather intact, we propose a certain algebraic object, a mean-convolution algebra, where the mean-convolution product plays the role of commutative multiplication. It presumably stands for a maximal class defined by regrouping the most favorable elements. At present, we are merely content with its roughest property, i.e. a commutative Banach algebra containing constant functions but without a unit (Theorem 6.3 and Proposition 6.4). Though many properties reminiscent of almost periodic functions are shown in such a context, we do not even know how almost periodic functions are related to a mean-convolution algebra in general. Yet suggestive classes of functions, $\mathcal{Q}(\mathbb{R}^m)$, $\mathcal{R}(\mathbb{R}^m)$, are proposed at the end of Section 6, where we also introduce the mean-spectrum, an analogue of the spectrum. Now one might ask whether the elements in the quotient $\mathcal{Q}(\mathbb{R}^m)/\mathcal{R}(\mathbb{R}^m)$ admit spectral syntheses over the mean-spectra, a question which remains open at the moment.

1. Admissibility and mean values

Let $\phi(x)$ be a non-negative, $\phi(x) \geq 0$, and integrable function on $\mathbb{R}^m$ with

$$\int_{\mathbb{R}^m} \phi(x) dx = 1. \quad (1.1)$$

We refer to such a function $\phi(x)$ as a weight function. A standard example of weight functions is the indicator (characteristic) function $c_Q(x)$ of the half-closed

unit cube Q :

$$\phi(x) = c_Q(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q, \end{cases} \quad (1.2)$$

where

$$Q = \overbrace{\left[-\frac{1}{2}, \frac{1}{2}\right] \times \cdots \times \left[-\frac{1}{2}, \frac{1}{2}\right]}^{m \text{ times}}.$$

$c_Q(x)$ will be called the *canonical* weight function.

The Gauss function

$$\phi(x) = e^{-\pi|x|^2}, \quad |x| = \sqrt{x_1^2 + \cdots + x_m^2}, \quad x = (x_1, \dots, x_m), \quad (1.3)$$

is another standard example of a weight function and is infinitely smooth, rapidly decreasing and even, $\phi(-x) = \phi(x)$.

Set

$$\phi_L(x) = \frac{1}{L^m} \phi\left(\frac{x}{L}\right) \quad (1.4)$$

for $L > 0$.

Remark. If ϕ and ψ are weight functions, then so is their convolution product $\phi * \psi$. Furthermore, $(\phi * \psi)_L = \phi_L * \psi_L$.

LEMMA 1.1. Let $f \in \mathbf{L}^\infty(\mathbb{R}^m)$. Consider the convolution products $\phi_L * f(x)$, $L > 0$. Take any $x \in \mathbb{R}^m$. Let $\mathcal{M}_{\phi,x}(f)$ be the set of limit points of $\phi_L * f(x)$, that is, $\mu \in \mathcal{M}_{\phi,x}(f)$ if, for some $L_j \rightarrow +\infty$, $\phi_{L_j} * f(x)$ converges and μ is its limit. Then $\mathcal{M}_{\phi,x}(f)$ is independent of x , and if f is real-valued, then

$$\mathcal{M}_{\phi,x}(f) = [\liminf \phi_L * f(x), \limsup \phi_L * f(x)].$$

Proof. First observe that, for any bounded measurable function $f(x)$, the convolution products $\phi_L * f(x)$, $L \geq L_0$, (with any $L_0 > 0$) are equi-bounded and equi-continuous. In fact, we have

$$\begin{aligned} |\phi_L * f(x)| &\leq \|f\|_\infty = \sup_{x \in \mathbb{R}^m} |f(x)|, \\ |\phi_L * f(x) - \phi_L * f(x')| &\leq \|f\|_\infty \int_{\mathbb{R}^m} \left| \phi\left(\frac{x-x'}{L} + y\right) - \phi(y) \right| dy. \end{aligned} \quad (1.5)$$

Here $\|\cdot\|_p$ denotes the norm of the space $\mathbf{L}^p(\mathbb{R}^m)$, $1 \leq p \leq +\infty$. Thus, on each compact subset K of $\mathbb{R}^m$, there is a sequence $L_j \rightarrow +\infty$ depending on K such

that $\phi_{L_j} * f$ converges uniformly on K as $L_j \rightarrow +\infty$. Moreover, the limit is then constant if K is connected. This proves the first half of Lemma 1.1.

Now if f is real-valued, then for any $a > 0$ and $\ell \geq a$,

$$\inf_{L \geq a} \phi_L * f(x) \leq \phi_\ell * f(x) \leq \sup_{L \geq a} \phi_L * f(x).$$

Thus,

$$\mathcal{M}_{\phi,x}(f) \subset [\liminf \phi_L * f(x), \limsup \phi_L * f(x)].$$

Suppose $\liminf \phi_L * f(x) < \limsup \phi_L * f(x)$ and μ is a point in between. We will show that $\mu \in \mathcal{M}_{\phi,x}(f)$. In fact, we can choose increasing sequences $L_j \rightarrow +\infty$, $L'_j \rightarrow +\infty$, such that $\dots < L_j < L'_j < L_{j+1} < \dots$, $\phi_{L_j} * f(x) < \mu < \phi_{L'_j} * f(x)$, and $\phi_{L_j} * f(x) \rightarrow \liminf \phi_L * f(x)$, $\phi_{L'_j} * f(x) \rightarrow \limsup \phi_L * f(x)$. Since $\phi_L * f(x)$ is a continuous function of $L \in [L_j, L'_j]$, we have $\phi_{L''_j} * f(x) = \mu$, $L_j < L''_j < L'_j$, and thus $\mu \in \mathcal{M}_{\phi,x}(f)$. $\square$

Now the function $f(x)$ is said to be *admissible* with respect to ϕ , in short, ϕ -admissible, if $\phi_L * f$ converges to a constant weakly* in $L^\infty(\mathbb{R}^m)$ as $L \rightarrow +\infty$. The weak* limit is called the *mean value* of f with respect to ϕ , in short, ϕ -mean of f , and denoted $M_\phi(f)$. Thus, f is ϕ -admissible if and only if

$$\lim_{L \rightarrow +\infty} \int_{\mathbb{R}^m} \phi_L * f(x) u(x) dx = M_\phi(f) \int_{\mathbb{R}^m} u(x) dx \quad (1.6)$$

for any $u \in L^1(\mathbb{R}^m)$. Note that $|M_\phi(f)| \leq \|f\|_\infty$ when f is ϕ -admissible.

PROPOSITION 1.2. *The following statements are equivalent:*

- (1) f is ϕ -admissible;
- (2) $\phi_L * f(x)$ converges as $L \rightarrow +\infty$ at each $x \in \mathbb{R}^m$;
- (3) $\phi_L * f(x)$ converges as $L \rightarrow +\infty$ uniformly with respect to x on any compact subset of $\mathbb{R}^m$.

The last two statements are obviously equivalent because of (1.5). That (3) implies (1) is immediate. We show that (2) follows from (1). Fix $x \in \mathbb{R}^m$. Take a smooth function $g(y) \geq 0$ with support in the ball $\|y\| \leq 1$ and $\int_{\mathbb{R}^m} g(y) dy = 1$. Let $\delta > 0$ and consider $g_\delta(y) = \delta^{-m} g(\delta^{-1} y)$. Note

$$\begin{aligned}
& \left| \int_{\mathbb{R}^m} \phi_L * f(y) g_\delta(x-y) dy - \phi_L * f(x) \right| \\
&= \left| \int_{\mathbb{R}^m} \{ \phi_L * f(x-y) - \phi_L * f(x) \} g_\delta(y) dy \right| \\
&\leq \|f\|_\infty \int_{\mathbb{R}^m} g_\delta(y) dy \int_{\mathbb{R}^m} \left| \phi\left(\frac{y}{L} + z\right) - \phi(z) \right| dz.
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Choose $\delta_\epsilon > 0$ so that

$$\int_{\mathbb{R}^m} \left| \phi\left(\frac{y}{L} + z\right) - \phi(z) \right| dz < \frac{\epsilon}{2\|f\|_\infty}, \quad |y| \leq \delta_\epsilon L,$$

and then choose $L_\epsilon > 0$ so that

$$\left| \int_{\mathbb{R}^m} \phi_L * f(y) g_\delta(x-y) dy - M_\phi(f) \right| < \frac{\epsilon}{2}, \quad L \geq L_\epsilon.$$

Then, for $L \geq L_\epsilon$, $|\phi_L * f(x) - M_\phi(f)| < \epsilon$.

Remark. Let $\mathcal{D}$ be a dense subset of $\mathbf{L}^1(\mathbb{R}^m)$. Let $f \in \mathbf{L}^\infty(\mathbb{R}^m)$. Suppose that, for whatever $v \in \mathcal{D}$, f satisfies

$$\lim_{L \rightarrow +\infty} \int_{\mathbb{R}^m} \phi_L * f(x) v(x) dx = \mu \int_{\mathbb{R}^m} v(x) dx$$

with some constant μ . Then f is ϕ -admissible and $M_\phi(f) = \mu$.

On the other hand, if $\phi_L * f$ converges to a constant strongly in $\mathbf{L}^\infty(\mathbb{R}^m)$, then f is said to be *strongly admissible* with respect to ϕ , in short, *strongly ϕ -admissible*. Clearly, strongly ϕ -admissible functions are ϕ -admissible. Thus, if f is strongly ϕ -admissible, then

$$\sup_{x \in \mathbb{R}^m} |\phi_L * f(x) - M_\phi(f)| \rightarrow 0 \quad \text{as } L \rightarrow +\infty. \quad (1.7)$$

However, ϕ -admissible functions are not necessarily strongly ϕ -admissible.

PROPOSITION 1.3. *Suppose $f \in \mathbf{L}^\infty(\mathbb{R})$ satisfies*

$$f(x) \rightarrow f^\pm \quad \text{as } x \rightarrow \pm\infty \quad (1.8)$$

for some constants f^+ and f^- . Then f is ϕ -admissible for any weight function ϕ and

$$M_\phi(f) = f^+ \int_{-\infty}^0 \phi(x) dx + f^- \int_0^{+\infty} \phi(x) dx. \quad (1.9)$$

f is strongly ϕ -admissible if and only if $f^+ = f^-$.

In fact, let $\epsilon > 0$ be arbitrarily small and choose $R > 0$ so that $|f(y) - f^\pm| < \epsilon$ for $\pm y > R$. Then from (1.9) we have

$$\begin{aligned} \phi_L * f(x) - M_\phi(f) &= \int_{x+R}^{+\infty} \frac{1}{L} \phi\left(\frac{y}{L}\right) (f(x-y) - f^-) dy + \int_x^{x+R} \frac{1}{L} \phi\left(\frac{y}{L}\right) (f(x-y) - f^-) dy \\ &\quad + \int_{-\infty}^{x-R} \frac{1}{L} \phi\left(\frac{y}{L}\right) (f(x-y) - f^+) dy + \int_{x-R}^x \frac{1}{L} \phi\left(\frac{y}{L}\right) (f(x-y) - f^+) dy \\ &\quad + \int_0^x \frac{1}{L} \phi\left(\frac{y}{L}\right) (f^+ - f^-) dy. \end{aligned}$$

The first and third terms on the right-hand side of the equality do not exceed ϵ whatever the value of x . Similarly, the second and fourth terms do not exceed $2\|f\|_\infty I(R/L)$. Here

$$I(\delta) = \sup_a \int_a^{a+\delta} \phi(t) dt, \quad \delta > 0.$$

Note $I(\delta)$ is non-increasing in δ and $I(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. When x is fixed, the fifth term tends to zero as $L \rightarrow +\infty$. Now choose

$$\epsilon < \frac{1}{8} |f^+ - f^-| \sup_y \left| \int_0^y \phi(t) dt \right|.$$

if $f^+ \neq f^-$. We then see that

$$\|\phi_L * f(x) - M_\phi(f)\|_\infty > \frac{1}{2} |f^+ - f^-| \sup_y \left| \int_0^y \phi(t) dt \right| > 0$$

for L so large that

$$I\left(\frac{R}{L}\right) < \frac{\epsilon}{2\|f\|_\infty}.$$

(1.9) suggests that the admissibility of a function is sensitive to weight functions; however, this is not always the case.

PROPOSITION 1.4. Assume $m = 1$. Let ϕ be any weight function such that

$$x\phi(x) \rightarrow 0, \quad |x| \rightarrow +\infty, \quad (1.10)$$

and

$$x\phi'(x) \in \mathbf{L}^1(\mathbb{R}). \quad (1.11)$$

Suppose $f \in \mathbf{L}^\infty(\mathbb{R})$ satisfies

$$\lim_{|x| \rightarrow +\infty} \frac{1}{x} \int_0^x f(t) dt = \mu. \quad (1.12)$$

Then f is ϕ -admissible, and $M_\phi(f) = \mu$.

Proof. We show $\lim_{L \rightarrow +\infty} \phi_L * f(0) = \mu$. In fact,

$$\begin{aligned} \phi_L * f(0) &= \left[\phi_L(-y) \int_0^y f(t) dt \right] \Big|_{y=-\infty}^{y=+\infty} + \int_{-\infty}^{+\infty} \phi'_L(-y) \int_0^y f(t) dt dy \\ &= \int_{-\infty}^{+\infty} y \phi'(-y) \left\{ \frac{1}{Ly} \int_0^{Ly} f(t) dt \right\} dy, \end{aligned}$$

where we have employed (1.10). Now we have the result by (1.11) and (1.12) since

$$\int_{-\infty}^{+\infty} y \phi'(-y) dy = \int_{-\infty}^{+\infty} \phi(y) dy = 1.$$

□

Remark. (1.8) does not imply (1.12) unless $f^+ = f^- = \mu$. Note that (1.12) means that

$$(c_\pm^1)_L * f(0) \rightarrow \mu, \quad L \rightarrow +\infty$$

provided that

$$c_\pm^1(x) = \begin{cases} 1, & 0 < \pm x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

That is, f is $c_\pm^1$ -admissible.

Actually, strong admissibility is independent of the choice of weight functions.

THEOREM 1.5. *Let $\phi(x)$ be a weight function. Suppose $f(x)$ is strongly ϕ -admissible. Then $f(x)$ is strongly admissible with any other weight function ψ and its ϕ -mean and ψ -mean coincide:*

$$M_\phi(f) = M_\psi(f).$$

Proof. Let $\epsilon > 0$. Then,

$$\|(\phi_\epsilon * \psi)_L - \psi_L\|_1 = \|\phi_\epsilon * \psi - \psi\|_1 \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus,

$$\begin{aligned}
|f * \psi_L(x) - f * (\phi_\epsilon * \psi)_L(x)| &= |f * \{\psi_L - (\phi_\epsilon * \psi)_L\}(x)| \\
&\leq \|f\|_\infty \|\phi_\epsilon * \psi - \psi\|_1 \\
&\rightarrow 0 \quad \text{as } \epsilon \rightarrow 0
\end{aligned}$$

uniformly with respect to $L > 0$. On the other hand,

$$\begin{aligned}
|f * (\phi_\epsilon * \psi)_L(x) - M_\phi(f)| &= |\{f * (\phi_\epsilon)_L - M_\phi(f)\} * \psi_L(x)| \\
&\leq \|f * (\phi_\epsilon)_L - M_\phi(f)\|_\infty \\
&\rightarrow 0 \quad \text{as } L \rightarrow +\infty
\end{aligned}$$

because of the strong ϕ -admissibility of f . Since

$$\begin{aligned}
f * \psi_L(x) - M_\phi(f) &= \{f * \psi_L(x) - f * (\phi_\epsilon * \psi)_L(x)\} \\
&\quad + \{f * (\phi_\epsilon * \psi)_L(x) - M_\phi(f)\},
\end{aligned}$$

we see f is strongly ψ -admissible and $M_\psi(f) = M_\phi(f)$. $\square$

Thus, for strongly admissible functions f , we will no longer need any reference to weight functions and their mean values will be denoted $M(f)$.

Example 1.1. Let $e^{(k)}$ be the unit m -vector with the only non-vanishing entry 1 as the k th component, $k = 1, \dots, m$. $f(x) \in L^\infty(\mathbb{R}^m)$ is periodic of periods $e^{(1)}, \dots, e^{(m)}$ if $f(x) = f(x + e^{(k)})$, $k = 1, \dots, m$. We denote by $\mathcal{P}_I(\mathbb{R}^m)$ the totality of the functions of periods $e^{(1)}, \dots, e^{(m)}$. Then $f(x) \in \mathcal{P}_I(\mathbb{R}^m)$ is strongly admissible. Moreover, for any $\xi \in \mathbb{R}^m$, $e^{-i\xi \cdot x} f(x)$ is strongly admissible, and

$$M(e^{-i\xi \cdot x} f(x)) = \begin{cases} \int_Q e^{-2\pi i n \cdot x} f(x) dx, & n = \frac{1}{2\pi} \xi \in \mathbb{Z}^m, \\ 0, & \text{otherwise.} \end{cases} \quad (1.13)$$

The verification is immediate by using the canonical weight function (1.2).

We state the following characterization of strong admissibility, which reminds us of the Wiener–Pitt Tauberian theorem (see, e.g. [2, 16]); however, we do not know if any relation can be established.

THEOREM 1.6. *Let $f \in L^\infty(\mathbb{R}^m)$. For each $u \in L^1(\mathbb{R}^m)$, f is strongly admissible with the mean value $M(f)$ if and only if,*

$$\int_{\mathbb{R}^m} u(y) f(x - Ly) dy \rightarrow M(f) \int_{\mathbb{R}^m} u(y) dy \quad (1.14)$$

as $L \rightarrow +\infty$ uniformly with respect to $x \in \mathbb{R}^m$.

Proof. That (1.14) implies strong admissibility is obvious, since a weight function can be taken as u . Now let f be strongly admissible. We show (1.14) for a real valued $u(y)$. Note $u(y) = u_+(y) - u_-(y)$, where

$$u_{\pm}(y) = \max\{\pm u(y), 0\} \in \mathbf{L}^1(\mathbb{R}^m), \quad u_{\pm}(y) \geq 0.$$

Then we have

$$\begin{aligned} \lim_{L \rightarrow +\infty} \int_{\mathbb{R}^m} u_{\pm}(y) f(x - Ly) dy &= \lim_{L \rightarrow +\infty} \frac{1}{L^m} \int_{\mathbb{R}^m} u_{\pm}\left(\frac{x-y}{L}\right) f(y) dy \\ &= M(f) \int_{\mathbb{R}^m} u_{\pm}(y) dy \end{aligned}$$

uniformly with respect to $x \in \mathbb{R}^m$. □

Remark. In particular,

$$|M(f)| \leq \lim_{R \rightarrow +\infty} \operatorname{ess. sup}_{|x| \geq R} |f(x)|$$

for strongly admissible $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$.

ϕ -admissibility is invariant under the operations of translation, dilation and inversion. That is, let

$$\mathcal{T}_y f(x) = f(x - y), \quad y \in \mathbb{R}^m, \quad (1.15)$$

be the translation (by $y \in \mathbb{R}^m$), let

$$\mathcal{D}_d f(x) = f\left(\frac{x}{d}\right), \quad d > 0, \quad (1.16)$$

be the dilation, and let

$$\mathcal{I}f(x) = f(-x) \quad (1.17)$$

be the inversion. Then we have the following.

PROPOSITION 1.7. *Suppose $f(x)$ is ϕ -admissible. Then $\mathcal{T}_y f(x)$ and $\mathcal{D}_d f(x)$ are ϕ -admissible, $\mathcal{I}f(x)$ is $\mathcal{I}\phi$ -admissible, and*

$$M_\phi(f) = M_\phi(\mathcal{T}_y f) = M_\phi(\mathcal{D}_d f) = M_{\mathcal{I}\phi}(\mathcal{I}f).$$

Proof. Immediate from the definition. □

Remark. More generally, for a non-degenerate linear transformation G in $\mathbb{R}^m$, i.e. $G \in \mathbf{GL}(m, \mathbb{R})$, let

$$\mathcal{L}_G f(x) = f(G^{-1}x) \quad \text{for } f(x) \text{ on } \mathbb{R}^m. \quad (1.18)$$

In particular, for a weight function $\phi(x)$, $\mathcal{L}_{G^{-1}}\phi(x)$ is also a weight function provided $|\det G| = 1$. In this case, f is $\mathcal{L}_{G^{-1}}\phi$ -admissible if and only if $\mathcal{L}_G f$ is ϕ -admissible, and then

$$M_\phi(\mathcal{L}_G f) = M_{\mathcal{L}_{G^{-1}}\phi}(f). \quad (1.19)$$

Thus, for any $G \in \mathbf{GL}(m, \mathbb{R})$, $\mathcal{L}_G f$ is strongly admissible for any strongly admissible f and

$$M(\mathcal{L}_G f) = M(f). \quad (1.20)$$

Here we can remove the requirement $|\det G| = 1$ in (1.18) combining with the dilation $\mathcal{D}_{\det G}$.

Example 1.2. Let $\eta^{(1)}, \dots, \eta^{(m)} \in \mathbb{R}^m$ be linearly independent. We say $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$ is periodic of periods $\eta^{(1)}, \dots, \eta^{(m)}$ if

$$f(x) = f(x + \eta^{(k)}), \quad k = 1, \dots, m.$$

Take $H \in \mathbf{GL}(m, \mathbb{R})$ such that $\eta^{(k)} = H e^{(k)}$, $k = 1, \dots, m$. By $\mathcal{P}_H(\mathbb{R}^m)$ we denote the set of the periodic functions $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$ of periods $\eta^{(1)}, \dots, \eta^{(m)}$. We see that $f \in \mathcal{P}_H(\mathbb{R}^m)$ if and only if $\mathcal{L}_{H^{-1}} f \in \mathcal{P}_I(\mathbb{R}^m)$. In particular, if $\det H > 0$, then the image Q_H of the cube Q by the linear transformation H is a half-closed parallelotope and its volume $|Q_H|$ is given by $\det H$. Thus,

$$M(f) = M(\mathcal{L}_{H^{-1}} f) = \frac{1}{\det H} \int_{Q_H} f(x) dx, \quad (1.21)$$

for $\mathcal{L}_{H^{-1}} f(x) = f(Hx)$. Moreover,

$$M(e^{-i\xi \cdot x} f(x)) = \begin{cases} \frac{1}{\det H} \int_{Q_H} e^{-2\pi i {}^t H^{-1} n \cdot x} f(x) dx, & n = \frac{1}{2\pi} {}^t H \xi \in \mathbb{Z}^m, \\ 0, & \text{otherwise,} \end{cases} \quad (1.22)$$

for $f \in \mathcal{P}_H(\mathbb{R}^m)$. Here ${}^t H$ is the transposed matrix of H .

Remark. Suppose

$$\mathbb{Z}^m \cap {}^t H^{-1} \mathbb{Z}^m = \{0\}. \quad (1.23)$$

Comparing Examples 1.1 and 1.2, we see

$$M(e^{-i\xi \cdot x} f(x)) = \begin{cases} M(f), & \xi = 0, \\ 0, & \xi \neq 0, \end{cases} \quad (1.24)$$

for $f \in \mathcal{P}_I(\mathbb{R}^m) \cap \mathcal{P}_H(\mathbb{R}^m)$, hence we have

$$\mathcal{P}_I(\mathbb{R}^m) \cap \mathcal{P}_H(\mathbb{R}^m) = \mathbb{C}, \quad (1.25)$$

by interpreting (1.13), (1.22) and (1.24) in terms of Fourier coefficients.

2. Riemann–Lebesgue properties

Admissibility is somehow related to the Riemann–Lebesgue theorem. We begin by stating a few observations related to strong admissibility.

PROPOSITION 2.1. *Suppose $f \in \mathbf{L}^\infty(\mathbb{R}^m) \cap \mathbf{L}^p(\mathbb{R}^m)$, for some p , $1 \leq p < +\infty$. Then f is strongly admissible and $M(f) = 0$.*

Proof. Let ϕ be the canonical weight function. Since

$$\|\phi_L\|_{p'} = L^{-m/p} \|\phi\|_{p'}, \quad 1 < p' = \frac{p}{p-1} \leq +\infty,$$

we have

$$|\phi_L * f(x)| \leq \|\phi_L\|_{p'} \|f\|_p \rightarrow 0, \quad L \rightarrow +\infty.$$

□

PROPOSITION 2.2. *Let $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$. Then $b(x)f(x)$ is strongly admissible for any bounded measurable function $b(x)$ if and only if $|f(x)|$ is strongly admissible and $M(|f|) = 0$.*

Proof. To see the *if* part, just note

$$\phi_L * (b \cdot f)(x) \leq \phi_L * |b \cdot f|(x) \leq \|b\|_\infty \phi_L * |f|(x)$$

for any weight function ϕ . On the other hand, to show the *only if* part, we first observe the following. Choose

$$b(x) = \text{sign } \overline{f(x)} = \begin{cases} \frac{\overline{f(x)}}{|f(x)|}, & f(x) \neq 0, \\ 0, & f(x) = 0, \end{cases}$$

and we see strong admissibility of $|f(x)|$. This also shows that we may replace $f(x)$ by $|f(x)|$ in the following reasoning. We now show that there would be a bounded function $b(x)$ such that ϕ -admissibility (and hence strong admissibility) of $b(x)|f(x)|$ fails if $M_\phi(|f|) = \delta > 0$ when $\phi(x)$ is the canonical weight function (1.2). Choose an increasing sequence $0 = L_0 < L_1 < \dots < L_n < \dots$, $L_n \rightarrow +\infty$, such that

$$\left(1 - \frac{1}{2^n}\right) \delta \leq \phi_{L_n} * |f|(x) \leq \left(1 + \frac{1}{2^n}\right) \delta, \quad n = 1, 2, \dots,$$

and

$$\begin{aligned} \sum_{j=1}^k \left(1 + \frac{1}{2^{2j-1}}\right) L_{2j-1}^m - \sum_{j=1}^k \left(1 - \frac{1}{2^{2j-2}}\right) L_{2j-2}^m &< \left(\frac{1}{2} - \frac{1}{2^{2k}}\right) L_{2k}^m, \\ \sum_{j=1}^k \left(1 + \frac{1}{2^{2j}}\right) L_{2j}^m - \sum_{j=1}^k \left(1 - \frac{1}{2^{2j-1}}\right) L_{2j-1}^m &< \left(\frac{1}{2} - \frac{1}{2^{2k+1}}\right) L_{2k+1}^m. \end{aligned}$$

Then

$$\phi_{L_1} * |f|(x) > \frac{1}{2} \delta,$$

$$\begin{aligned} \sum_{j=1}^k \left\{ \left(\frac{L_{2j-1}}{L_{2k}}\right)^m \phi_{L_{2j-1}} * |f|(x) - \left(\frac{L_{2j-2}}{L_{2k}}\right)^m \phi_{L_{2j-2}} * |f|(x) \right\} - \phi_{L_{2k}} * |f|(x) \\ < -\frac{1}{2} \delta, \end{aligned}$$

$$\begin{aligned} - \sum_{j=1}^k \left\{ \left(\frac{L_{2j}}{L_{2k+1}}\right)^m \phi_{L_{2j}} * |f|(x) - \left(\frac{L_{2j-1}}{L_{2k+1}}\right)^m \phi_{L_{2j-1}} * |f|(x) \right\} + \phi_{L_{2k+1}} * |f|(x) \\ > \frac{1}{2} \delta. \end{aligned}$$

Now let us set

$$b(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \left\{ \phi\left(\frac{x}{L_n}\right) - \phi\left(\frac{x}{L_{n-1}}\right) \right\}.$$

Thus, since $\phi(x) = c_Q(x)$ the canonical weight function,

$$b(x) = (-1)^{n-1} \quad \text{when} \quad \phi_{L_n}(x) \neq \phi_{L_{n-1}}(x).$$

It then follows that

$$\begin{aligned}\phi_{L_{2k}} * (b \cdot |f|)(0) &< -\frac{1}{2}\delta, \\ \phi_{L_{2k+1}} * (b \cdot |f|)(0) &> \frac{1}{2}\delta,\end{aligned}$$

which is contrary to ϕ -admissibility of $b(x)|f(x)|$. $\square$

Remark. The *if* part is valid when strong admissibility is replaced by ϕ -admissibility for any weight function ϕ . The *only if* part is also valid when strong admissibility is replaced by admissibility with respect to the canonical weight function. The above proof of the *only if* part is based on the idea of my colleague Hiroshi Sato.

For the sake of later convenience, we denote by $\mathcal{N}(\mathbb{R}^m)$ the set of all $f \in \mathbf{L}^\infty(\mathbb{R}^m)$ such that $|f|$ is strongly admissible and $M(|f|) = 0$. We call $\mathcal{N}(\mathbb{R}^m)$ the *null ideal*. This naming will become clear in Section 6. It is clear that

$$\mathcal{P}_H(\mathbb{R}^m) \cap \mathcal{N}(\mathbb{R}^m) = \{0\} \quad \text{for any } H \in \mathbf{GL}(m, \mathbb{R}). \quad (2.1)$$

On the other hand, Proposition 2.1 then means that

$$\mathbf{L}^\infty(\mathbb{R}^m) \cap \mathbf{L}^p(\mathbb{R}^m) \subset \mathcal{N}(\mathbb{R}^m), \quad 1 \leq p < +\infty.$$

However, $f \in \mathcal{N}(\mathbb{R}^m)$ does not imply $|f(x)| \rightarrow 0$ as $|x| \rightarrow +\infty$.

Example 2.1. Let $m = 1$. Consider

$$f(x) = \begin{cases} 1, & n \leq x < n + 2^{-|n|}, \quad n = \pm 1, \pm 2, \dots, \\ 0, & \text{elsewhere.} \end{cases}$$

Then $f \in \mathcal{N}(\mathbb{R})$.

It turns out that this example is quite representative. Let $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$. For any $\epsilon > 0$ and $R > 0$, let

$$e_{\epsilon, R}(f) = \{x \in \mathbb{R}^m; |f(x)| > \epsilon, \quad |x| < R\}.$$

and

$$e_\epsilon(f) = \{x \in \mathbb{R}^m; |f(x)| > \epsilon\} = \bigcup_{R>0} e_{\epsilon, R}(f).$$

Now let $|e_{\epsilon, R}(f)|$ denote the Lebesgue measure of the set $e_{\epsilon, R}(f)$.

THEOREM 2.3. Suppose $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$. Then $f \in \mathcal{N}(\mathbb{R}^m)$ if and only if

$$\lim_{R \rightarrow +\infty} \frac{|e_{\epsilon, R}(f)|}{R^m} = 0 \quad (2.2)$$

for any $\epsilon > 0$.

Proof. Suppose (2.2) holds. Let ϕ be the canonical weight function. Then

$$\begin{aligned} \phi_L * |f|(x) &\leq \|f\|_\infty \int_{e_{\epsilon, L}(f)} \phi_L(x-y) dy + \epsilon \int_{\mathbb{R}^m} \phi_L(x-y) dy \\ &\leq \frac{|e_{\epsilon, L}(f)|}{L^m} \|f\|_\infty + \epsilon, \end{aligned}$$

hence $f \in \mathcal{N}(\mathbb{R}^m)$. On the other hand, if (2.2) fails to hold, then for some $\delta > 0$

$$\limsup_{R \rightarrow +\infty} \frac{|e_{\delta, R}(f)|}{R^m} \geq c > 0.$$

However, this is incompatible with $M(|f|) = 0$. In fact, since then

$$\phi_{2L} * |f|(x) \geq \delta \int_{e_{\delta, L}(f)} \phi_{2L}(x-y) dy,$$

we should have

$$\limsup_{L \rightarrow +\infty} \phi_{2L} * |f|(0) \geq \frac{c\delta}{2^m} > 0.$$

□

Remark. If $|e_\epsilon(f)| < +\infty$ is valid for any $\epsilon > 0$, then (2.2) obviously holds. Example 2.1 corresponds to such a situation. The next example, which is inspired by a conversation with my colleague Setsuo Taniguchi, is more subtle, since the sets $e_\epsilon(f)$ are of infinite measure for small ϵ . In particular, $f \notin \mathbf{L}^p(\mathbb{R})$ for any finite p , $1 \leq p < +\infty$.

Example 2.2. Let a_n , $n = 1, 2, \dots$, and b_n , $n = 1, 2, \dots$, be sequences of positive numbers such that

$$\begin{aligned} a_n &\rightarrow +\infty, \quad \text{as } n \rightarrow +\infty, \\ 0 &< b_n < a_{n+1} - a_n, \quad n = 1, 2, \dots, \\ \sum_{k=1}^n b_k &\rightarrow +\infty, \quad \frac{1}{a_{n+1}} \sum_{k=1}^n b_k \rightarrow 0, \quad n \rightarrow +\infty. \end{aligned}$$

For instance, $a_n = n$, $b_n = 1/(n+1)$. Let

$$f(x) = \begin{cases} 1, & a_n < x < a_n + b_n, \quad n = 1, 2, \dots, \\ 0, & \text{elsewhere.} \end{cases}$$

Then $f(x)$ is an element of $\mathcal{N}(\mathbb{R})$.

Bounded functions yield admissible functions through the convolution with a certain class of absolutely integrable functions. Namely, we have the following.

PROPOSITION 2.4. *Let $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$. If $g(x) \in \mathbf{L}^1(\mathbb{R}^m)$ is a derivative of an absolutely integrable function $h(x)$:*

$$g(x) = \omega \cdot \nabla_x h(x), \quad \omega \in \mathbb{R}^m, \quad \omega \neq 0,$$

then $f * g(x)$ is strongly admissible with the vanishing mean value.

Proof. Note

$$\begin{aligned} (f * g) * \phi_L(x) &= \iint_{\mathbb{R}^m \times \mathbb{R}^m} \phi_L(z) g(x - z - y) f(y) dz dy \\ &= \iint_{\mathbb{R}^m \times \mathbb{R}^m} \omega \cdot \nabla_z \phi_L(z) h(x - z - y) f(y) dz dy \end{aligned}$$

for a smooth weight function $\phi(x)$. After the change of the variables $y' = x - z - y$, $z' = Lz$, we see

$$(f * g) * \phi_L(x) = \frac{1}{L} \iint_{\mathbb{R}^m \times \mathbb{R}^m} (\omega \cdot \nabla_{z'} \phi)(z') h(y') f(x - Lz' - y') dz' dy',$$

which tends to zero uniformly with respect to x as $L \rightarrow \infty$. □

Remark. The above $f * g$ does not necessarily belong to $\mathcal{N}(\mathbb{R}^m)$. For this, see Corollary 5.7 below and the Remark which follows.

Now let $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$. Then since $f(x)$ is interpreted as a tempered distribution, we may talk of its Fourier transform $\hat{f}(\xi)$. Here recall that

$$(\hat{f}, g) = (f, \hat{g}), \quad f \in \mathcal{S}'(\mathbb{R}^m), \quad g \in \mathcal{S}(\mathbb{R}^m),$$

by the duality (coupling) of $\mathcal{S}'(\mathbb{R}^m)$ and $\mathcal{S}(\mathbb{R}^m)$, and

$$\hat{g}(\xi) = \int_{\mathbb{R}^m} e^{-ix \cdot \xi} g(x) dx.$$

We also define $\hat{f}_L(\xi) = (1/L^m)\hat{f}(\xi/L)$, $L > 0$, by

$$(\hat{f}_L(\xi), g(\xi)) = (\hat{f}(\xi), g(L\xi)), \quad g \in \mathcal{S}(\mathbb{R}^m). \quad (2.3)$$

Then Theorem 1.6 is rewritten as follows.

PROPOSITION 2.5. *Let $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$ be strongly admissible. Then*

$$(e^{iy \cdot \xi/L} \hat{f}_L(\xi), g(\xi)) \rightarrow (2\pi)^m M(f)g(0), \quad g \in \mathcal{S}(\mathbb{R}^m), \quad (2.4)$$

uniformly with respect to $y \in \mathbb{R}^m$.

Proof. Note $\hat{g}(-x) \in \mathcal{S}(\mathbb{R}^m)$ and

$$\int_{\mathbb{R}^m} f(y - Lx) \hat{g}(-x) dx = \int_{\mathbb{R}^m} e^{iy \cdot \xi/L} \hat{f}_L(\xi) g(\xi) d\xi = (e^{iy \cdot \xi/L} \hat{f}_L(\xi), g(\xi)).$$

By (1.14), the left-hand side tends to

$$M(f) \int_{\mathbb{R}^m} \hat{g}(-x) dx = M(f)(2\pi)^m g(0)$$

uniformly with respect to y as $L \rightarrow 0$. □

Remark. Uniformness with respect to y is important, since $e^{iy \cdot \xi/L} \delta(\xi) = \delta(\xi)$ pointwise. Here $\delta(\xi)$ is Dirac's delta function, $(\delta(\xi), g(\xi)) = g(0)$.

The following is, in a sense, a converse to the above proposition.

PROPOSITION 2.6. *Let $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$. Suppose*

$$\hat{f}_L(\xi) \rightarrow \mu \delta(\xi), \quad L \rightarrow +\infty, \quad (2.5)$$

in $\mathcal{S}'(\mathbb{R}^m)$. For a weight function $\phi(x)$ belonging to $\mathcal{S}(\mathbb{R}^m)$, $f(x)$ is ϕ -admissible and the ϕ -mean $M_\phi(f)$ is independent of such ϕ 's:

$$M_\phi(f) = \frac{1}{(2\pi)^m} \mu. \quad (2.6)$$

Proof. The assumption means that

$$\int_{\mathbb{R}^m} \frac{1}{L^m} \hat{f}\left(\frac{\xi}{L}\right) g(\xi) d\xi \rightarrow \mu g(0) \quad (2.7)$$

as $L \rightarrow +\infty$ uniformly for $g(\xi)$ from any bounded subset in $\mathcal{S}(\mathbb{R}^m)$. Now let $u(x) \in \mathcal{S}(\mathbb{R}^m)$. Then by Parseval's theorem,

$$\begin{aligned} \int_{\mathbb{R}^m} (f * \phi_L)(x) u(x) dx &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \widehat{f * \phi_L}(\xi) \hat{u}(-\xi) d\xi \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \hat{f}(\xi) \widehat{\phi_L}(\xi) \hat{u}(-\xi) d\xi \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \frac{1}{L^m} \hat{f}\left(\frac{\xi}{L}\right) \hat{\phi}(\xi) \hat{u}\left(-\frac{\xi}{L}\right) d\xi \end{aligned}$$

since $\widehat{\phi_L}(\xi) = \hat{\phi}(L\xi)$. Thus, as $L \rightarrow +\infty$,

$$\begin{aligned} \int_{\mathbb{R}^\infty} (f * \phi_L)(x) u(x) dx &\rightarrow \frac{\mu}{(2\pi)^m} \hat{\phi}(0) \hat{u}(0) \\ &= \frac{\mu}{(2\pi)^m} \int_{\mathbb{R}^m} u(x) dx, \end{aligned}$$

since $\{\hat{\phi}(\xi) \hat{u}(-\xi/L); L \geq L_0 > 0\}$ is bounded in $\mathcal{S}(\mathbb{R}^m)$. Hence (2.6) follows since $\mathcal{S}(\mathbb{R}^m)$ is dense in $L^1(\mathbb{R}^m)$, and the assertion is proved. $\square$

COROLLARY 2.7. Suppose $f \in L^\infty(\mathbb{R}^m)$ satisfies

$$\hat{f}(\xi) = \mu \delta(\xi) + \Delta(\xi),$$

where $\Delta(\xi) \in \mathcal{S}'(\mathbb{R}^m)$ and $0 \notin \text{supp } \Delta$. Then f is ϕ -admissible for any rapidly decreasing weight function ϕ and the ϕ -mean is $(1/(2\pi)^m)\mu$.

Proof. We verify (2.5) or (2.7). In fact, let $g \in \mathcal{S}(\mathbb{R}^m)$. We then have

$$\int_{\mathbb{R}^m} \frac{1}{L^m} \hat{f}\left(\frac{\xi}{L}\right) g(\xi) d\xi = \mu g(0) + (\Delta(\xi), g(L\xi)).$$

Since $0 \notin \text{supp } \Delta$, we have

$$|(\Delta(\xi), g(L\xi))| \leq C \sup_{|\xi| \geq \epsilon} \left\{ (1 + |\xi|)^\lambda \sum_{|\alpha| \leq \rho} |(\partial_\xi^\alpha g)(L\xi)| \right\} \quad (2.8)$$

for some $\epsilon > 0$, λ , ρ being large enough. The right-hand side can be made arbitrarily small as $L \rightarrow +\infty$. $\square$

Remark. If the convergence (2.5) is of order zero, that is (2.7) is valid for $g(\xi)$ such that

$$\sup\{(1 + |\xi|)^N |g(\xi)|\} \leq 1$$

with some $N > 0$, then f is strongly admissible. This is the case if we can take $\rho = 0$ in (2.8) (see [12] and also [8]).

Example 2.3 Let $m = 1$. Suppose $f(x) \in \mathcal{P}_l(\mathbb{R})$, i.e. f is bounded and periodic of period one, $f(x+1) = f(x)$, $-\infty < x < +\infty$. Then

$$\hat{f}(\xi) = \sum_{k \in \mathbb{Z}} c_k \delta(\xi - 2k\pi),$$

with $c_k = 2\pi \int_0^1 e^{-2k\pi ix} f(x) dx$.

Before ending this section, let us discuss on a weak* analogue of the translation convex hull (see, e.g. [8, Ch. VIII, §6]). Let $f \in \mathbf{L}^\infty(\mathbb{R}^m)$. We define $W^*(f)$ by the weak* closure of the set of all the linear combinations of the form

$$\sum_j c_j \mathcal{T}_{y_j} f, \quad \sum_j |c_j| \leq 1, \quad y_j \in \mathbb{R}^m. \quad (2.9)$$

This is an analogue of the *translation convex hull* $W(f)$, which is defined by taking the closure of this set in the strong topology of $\mathbf{L}^\infty(\mathbb{R}^m)$. It is well known that f is almost periodic if and only if $W(f)$ is strongly compact in $\mathbf{L}^\infty(\mathbb{R}^m)$.

$W^*(f)$ contains $W(f)$ but is still bounded, for

$$\|h\|_\infty \leq \|f\|_\infty, \quad h \in W^*(f) \quad (2.10)$$

We have yet another, but very similar, subset related to $f \in \mathbf{L}^\infty(\mathbb{R}^m)$. Let $W_*(f)$ be the weak* closure of the set of functions

$$f * g, \quad g \in \mathbf{L}^1(\mathbb{R}^m), \quad \|g\|_1 \leq 1. \quad (2.11)$$

In fact, these two sets turn out to be the same because of the choice of the topology. We thus call $W^*(f)$ the *weak* translation convex hull* of f .

LEMMA 2.8. $W^*(f)$ coincides with $W_*(f)$.

In fact, to show $W_*(f) \subset W^*(f)$, we choose $g(x) = \phi_\epsilon(x-y)$, $y \in \mathbb{R}^m$, $\epsilon > 0$, where ϕ is a weight function. Then $f * g(x) = f * \phi_\epsilon(x-y) \rightarrow \mathcal{T}_y f(x)$ as $\epsilon \rightarrow 0$

in the weak* topology. Note this convergence is valid in the strong topology only when f is uniformly continuous. On the other hand, to show the converse inclusion, we note that when g and $u \in \mathbf{L}^1(\mathbb{R}^m)$ are smooth and compactly supported, then their convolution is approximated in $\mathbf{L}^1(\mathbb{R}^m)$ by Riemann sums of the form

$$\sum \gamma_j u(x - y_j), \quad y_j \in \mathbb{R}^m,$$

since $g(x)$ is approximated by sums

$$\sum_j \gamma_j c_{I_j}(x),$$

where I_j are small cubic decompositions of $\text{supp } g$ and $y_j \in I_j$.

Therefore, recalling Lemma 1.1, we immediately have the following.

PROPOSITION 2.9. *Let $f \in \mathbf{L}^\infty(\mathbb{R}^m)$. Then*

$$\mathcal{M}_{\phi,x}(f) \subset W^*(f)$$

for any weight function ϕ . In particular, if $f \in \mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$, then its ϕ -mean $M_\phi(f)$ belongs to $W^(f)$. Furthermore, if $W^*(f)$ contains no constants other than zero, then f is ϕ -admissible for any weight function ϕ and $M_\phi(f) = 0$.*

A very closely related notion to the weak* translation convex hull is the weak* spectrum. Let $f \in \mathbf{L}^\infty(\mathbb{R}^m)$. Let $\sigma^*(f)$ be the totality of $\xi \in \mathbb{R}^m$ such that

$$c_\xi e^{ix \cdot \xi} \in W^*(f) \quad \text{for some constant } c_\xi \neq 0. \quad (2.12)$$

The set $\sigma^*(f)$ is called the *weak* spectrum* of f . The general theory then asserts that

$$\sigma^*(f) = \text{supp } \hat{f}, \quad (2.13)$$

where $\hat{f}$ is the Fourier transform of f (see [8]). In particular, the weak* spectrum is a closed subset of $\mathbb{R}^m$.

Then we have a partial converse to Proposition 2.6.

COROLLARY 2.10. *Let $f \in \mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$. If $M_\phi(f) \neq 0$, then $0 \in \text{supp } f$.*

We will again discuss the weak* spectrum in Section 6.

3. The spaces of ϕ -admissible functions

Let ϕ be a weight function on $\mathbb{R}^m$. We denote by $\mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$ and by $\mathbf{L}_{\text{ad}}^\infty(\mathbb{R}^m)$, respectively, the set of all bounded ϕ -admissible functions and the set of strongly admissible functions on $\mathbb{R}^m$. It is evident that

$$\mathbf{L}_{\text{ad}}^\infty(\mathbb{R}^m) \subset \mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m) \subset \mathbf{L}^\infty(\mathbb{R}^m) \quad (3.1)$$

and the ϕ -mean $M_\phi(\cdot)$ is a linear functional defined on $\mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$.

PROPOSITION 3.1. *Suppose $f \in \mathbf{L}^\infty(\mathbb{R}^m)$ and $f_n \in \mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$ (or $\mathbf{L}_{\text{ad}}^\infty(\mathbb{R}^m)$), $n = 1, 2, \dots$, such that $f_n \rightarrow f$ strongly as $n \rightarrow +\infty$ in $\mathbf{L}^\infty(\mathbb{R}^m)$. Then f is ϕ -admissible (or strongly admissible) and $M_\phi(f_n) \rightarrow M_\phi(f)$ (or $M(f_n) \rightarrow M(f)$) as $n \rightarrow +\infty$.*

Proof. We show that for any $\epsilon > 0$, there is an $n(\epsilon) > 0$ large enough so that $|M_\phi(f_n) - M_\phi(f_\ell)| < \epsilon$ for $n, \ell \geq n(\epsilon)$. To see this, we choose $n(\epsilon)$ so that

$$\|f_n - f_\ell\|_\infty < \frac{\epsilon}{3} \quad \text{when } n, \ell > n(\epsilon).$$

In particular,

$$|f_n * \phi_L(x) - f_\ell * \phi_L(x)| \leq \|f_n - f_\ell\|_\infty < \frac{\epsilon}{3}, \quad n, \ell > n(\epsilon).$$

Now let $u(x) \in \mathbf{L}^1(\mathbb{R}^m)$ with $\int_{\mathbb{R}^m} u(x) dx = 1$. Fix any $n > n(\epsilon)$ and choose $L(n, \epsilon)$ so that

$$\left| M_\phi(f_n) - \int_{\mathbb{R}^m} f_n * \phi_L(x) u(x) dx \right| < \frac{\epsilon}{3} \quad \text{for } L > L(n, \epsilon).$$

Thus, we take $n, \ell > n(\epsilon)$ and then $L > L(n, \epsilon)$, $L(\ell, \epsilon)$. We have

$$\begin{aligned} |M_\phi(f_n) - M_\phi(f_\ell)| &\leq \left| M_\phi(f_n) - \int_{\mathbb{R}^m} f_n * \phi_L(x) u(x) dx \right| \\ &\quad + \left| \int_{\mathbb{R}^m} f_n * \phi_L(x) u(x) dx - \int_{\mathbb{R}^m} f_\ell * \phi_L(x) u(x) dx \right| \\ &\quad + \left| \int_{\mathbb{R}^m} f_\ell * \phi_L(x) u(x) dx - M_\phi(f_\ell) \right| < \epsilon. \end{aligned}$$

Now let $\mu = \lim_{n \rightarrow +\infty} M_\phi(f_n)$. Then

$$\left| \int_{\mathbb{R}^m} f * \phi_L(x) u(x) dx - \mu \right|$$

$$\leq \left| \int_{\mathbb{R}^m} f * \phi_L(x) u(x) dx - \int_{\mathbb{R}^m} f_n * \phi_L(x) u(x) dx \right| \\ + \left| \int_{\mathbb{R}^m} f_n * \phi_L(x) u(x) dx - M_\phi(f_n) \right| + |M_\phi(f_n) - \mu| < \epsilon,$$

for n and L sufficiently large. It can be shown by an immediate modification of the above reasoning that f is strongly admissible when the f_n are strongly admissible for n large enough. $\square$

In other words, Proposition 3.1 says that the spaces $\mathbf{L}_{\text{ad}}^\infty(\mathbb{R}^m)$ and $\mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$ are closed subspaces of $\mathbf{L}^\infty(\mathbb{R}^m)$ and the functional $M_\phi(\cdot)$ is bounded on them.

Example 3.1. Let $m = 1$ and let $f(x)$ be an almost periodic function on the real line. Then $f(x)$ is strongly admissible, because it is then a uniform limit of a sequence of periodic functions.

As another application, we have the following curious results.

PROPOSITION 3.2. Let $f, g \in \mathbf{L}^2(\mathbb{R}^m)$ and

$$h(x) = \int_{\mathbb{R}^m} |f(x+y) - g(y)|^2 dy.$$

Then $e^{-ix \cdot \xi} h(x)$, $\xi \in \mathbb{R}^m$, are strongly admissible and

$$M(e^{-ix \cdot \xi} h(x)) = \begin{cases} \|f\|_2^2 + \|g\|_2^2 = \int_{\mathbb{R}^m} \{|f(y)|^2 + |g(y)|^2\} dy, & \xi = 0, \\ 0, & \xi \neq 0. \end{cases}$$

In fact, take $f_n, g_n \in \mathcal{S}(\mathbb{R}^m)$ so that $\|f_n - f\|_2 \rightarrow 0$ and $\|g_n - g\|_2 \rightarrow 0$ as $n \rightarrow +\infty$. Then $\|h_n - h\|_\infty \rightarrow 0$ as $n \rightarrow +\infty$ provided that

$$h_n(x) = \int_{\mathbb{R}^m} |f_n(x+y) - g_n(y)|^2 dy.$$

Note $|h(x)| \leq 2\|f\|_2^2 + 2\|g\|_2^2$ and $|h_n(x)| \leq 2\|f_n\|_2^2 + 2\|g_n\|_2^2 \leq 2\|f\|_2^2 + 2\|g\|_2^2 + 1$ for n large enough. Then

$$h_n(x) = \|f_n\|_2^2 + \|g_n\|_2^2 - 2 \int_{\mathbb{R}^m} \operatorname{Re}\{f_n(x+y) \overline{g_n(y)}\} dy.$$

Now

$$\left| \int_{\mathbb{R}^m} \operatorname{Re}\{f_n(x+y) \overline{g_n(y)}\} dy \right| \leq \int_{\mathbb{R}^m} |f_n(x+y)| |g_n(y)| dy \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty$$

since f_n and $g_n \in \mathcal{S}(\mathbb{R}^m)$.

We pick up some details from the the above proof.

LEMMA 3.3. *Let $f_n \in \mathcal{N}(\mathbb{R}^m)$ and $f \in \mathbf{L}^\infty(\mathbb{R}^m)$. Suppose $\|f_n - f\|_\infty \rightarrow 0$ as $n \rightarrow +\infty$. Then $f \in \mathcal{N}(\mathbb{R}^m)$. That is, $\mathcal{N}(\mathbb{R}^m)$ is a closed subspace of $\mathbf{L}^\infty(\mathbb{R}^m)$.*

In fact, we have

$$\| |f_n| - |f| \|_\infty \leq \|f_n - f\|_\infty,$$

and we then appeal to Proposition 3.1.

COROLLARY 3.4. *Let $f \in \mathbf{L}^p(\mathbb{R}^m)$ and $g \in \mathbf{L}^{p'}(\mathbb{R}^m)$, $1 < p < +\infty$, $p' = p/(p-1)$. Then*

$$\ell(x) = \int_{\mathbb{R}^m} f(x-y)g(y)dy \in \mathcal{N}(\mathbb{R}^m).$$

Remark. Thus, $h(x)$ in Proposition 3.3 is constant modulo $\mathcal{N}(\mathbb{R}^m)$. Compare with Proposition 1.3.

However, Proposition 3.1 is no longer valid if we replace the strong topology by the weak* topology. For instance, $M_\phi(\cdot)$ is not necessarily continuous with respect to the weak* topology of $\mathbf{L}^\infty(\mathbb{R}^m)$.

Example 3.1. Let $m = 1$ and let $\phi(x)$ be any weight function. Let

$$c_n(x) = \begin{cases} 1, & |x| < n, \\ 0, & |x| \geq n, \end{cases} \quad n = 1, 2, \dots$$

Then the sequence c_n weakly* converges to the constant function $c(x) = 1$ while $M_\phi(c_n) = 0 \neq 1 = M_\phi(c)$.

LEMMA 3.5. *Let $g \in \mathbf{L}^1(\mathbb{R}^m)$. Suppose $f \in \mathbf{L}^\infty(\mathbb{R}^m)$ is ϕ -admissible (or strongly admissible). Then their convolution product $g * f(x)$ is ϕ -admissible (or strongly admissible), and*

$$M_\phi(g * f) = \int_{\mathbb{R}^m} g(x)dx M_\phi(f).$$

Furthermore, $g * f(x)$ is bounded and uniformly continuous with respect to $x \in \mathbb{R}^m$.

Proof. This is an application of Fubini's theorem. Moreover, let $u \in \mathbf{L}^1(\mathbb{R}^m)$, then note

$$\int_{\mathbb{R}^m} (g * f * \phi_L)(x)u(x)dx = \iint_{\mathbb{R}^m \times \mathbb{R}^m} (f * \phi_L)(y)g(x-y)u(x)dx dy \quad (3.2)$$

and that $\int_{\mathbb{R}^m} g(x-y)u(x)dx$ is absolutely integrable (with respect to y) and its integral is the product of those of g and u . $\square$

Remark. Compare with (1.7). Since

$$\int_{\mathbb{R}^m} f * \phi_L(x)u(x)dx = (f * \mathcal{I}u) * \phi_L(0), \quad u \in \mathbf{L}^1(\mathbb{R}^m),$$

we see, by taking $u(x) = \mathcal{T}_z \mathcal{I}v(x) = v(z-x)$, $z \in \mathbb{R}^m$, $\int_{\mathbb{R}^m} v(x)dx = 1$, that

$$(f * v) * \phi_L(z) \rightarrow M_\phi(f), \quad L \rightarrow +\infty,$$

at each z , and thus uniformly with respect to z on any compact subset of $\mathbb{R}^m$. Yet $f * v$ is not necessarily strongly ϕ -admissible unless f itself is. Again, $f(x)$ discussed in Proposition 1.3, together with a suitably chosen v , serves as a counter-example.

We have, however, a certain completeness result concerning ϕ -admissibility.

Let $\mathbf{C}_{\text{unif}}(\mathbb{R}^m)$ be the space of the bounded and uniformly continuous functions on $\mathbb{R}^m$ furnished with the topology induced from $\mathbf{L}^\infty(\mathbb{R}^m)$. Now let us denote by $\mathbf{C}_{\text{unif}, \phi\text{-ad}}(\mathbb{R}^m)$ the closed subspace of $\mathbf{C}_{\text{unif}}(\mathbb{R}^m)$ consisting of ϕ -admissible functions. We furnish the space $\mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$ with the weakest locally-convex topology that renders continuous all the mappings

$$\mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m) \ni f(x) \longrightarrow g * f(x) \in \mathbf{C}_{\text{unif}, \phi\text{-ad}}(\mathbb{R}^m), \quad g \in \mathbf{L}^1(\mathbb{R}^m) \quad (3.3)$$

LEMMA 3.6. *Let $f(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$ be such that $f * g(x)$ is ϕ -admissible for any $g(x) \in \mathbf{L}^1(\mathbb{R}^m)$ with $\int_{\mathbb{R}^m} g(x)dx = 1$ and that the ϕ -mean is independent of $g(x)$: $M_\phi(f * g) = \mu$. Then $f(x)$ itself is ϕ -admissible and $M_\phi(f) = \mu$.*

Proof. Choose a weight function as $g(x)$ in (3.3). Let $g_\delta(x) = \delta^{-m}g(x/\delta)$, $\delta > 0$. Then $f * g_\delta$ is ϕ -admissible and $M_\phi(f * g_\delta) = \mu$. Now let $\epsilon > 0$ be arbitrary, and for $u(x) \in \mathbf{L}^1(\mathbb{R}^m)$, choose $\delta > 0$, which depends on u , so that

$$\|\mathcal{I}g_\delta * u - u\|_1 < \frac{\epsilon}{2\|f\|_\infty}$$

(recall (1.17)) and then $L > 0$ so that

$$\left| \int_{\mathbb{R}^m} (f * g_\delta) * \phi_L(x)u(x)dx - \mu \int_{\mathbb{R}^m} u(x)dx \right| < \frac{\epsilon}{3}.$$

For this choice of $L > 0$, by observing (3.2) we conclude

$$\left| \int_{\mathbb{R}^m} f * \phi_L(x) u(x) dx - \mu \int_{\mathbb{R}^m} u(x) dx \right| < \epsilon,$$

as has been claimed. $\square$

THEOREM 3.7. *The space $\mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$ topologized as above is sequentially complete and the mapping $M_\phi(\cdot)$ is continuous on it.*

Proof. Let $f_n \in \mathbf{L}_{\phi\text{-ad}}(\mathbb{R}^m)$, $n = 1, 2, \dots$, satisfy

$$|g * f_n(x) - g * f_\ell(x)| \rightarrow 0$$

as $n, \ell \rightarrow +\infty$ uniformly with respect to $x \in \mathbb{R}^m$ for any $g \in \mathbf{L}^1(\mathbb{R}^m)$. Then we see, in particular, that $M_\phi(f_n)$ converges as $n \rightarrow +\infty$. We also see that f_n weakly* converges by the Banach–Steinhaus theorem. Therefore, if $f \in \mathbf{L}^\infty(\mathbb{R}^m)$ is the weak* limit of the sequence f_n , then $g * f_n(x) \rightarrow g * f(x)$ in $\mathbf{C}_{\text{unif}}(\mathbb{R}^m)$. It follows that $g * f$ is ϕ -admissible, and that

$$M_\phi(g * f) = \lim_{n \rightarrow +\infty} M(g * f_n) = \int_{\mathbb{R}^m} g(x) dx \lim_{n \rightarrow +\infty} M_\phi(f_n).$$

Hence, f is ϕ -admissible and

$$M_\phi(f) = \lim_{n \rightarrow +\infty} M_\phi(f_n).$$

$\square$

The following also involves a limiting procedure.

LEMMA 3.8. *Suppose $f(x, t) \in \mathbf{L}^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ is ϕ -admissible for each fixed $t \in \mathbb{R}^n$. Then, for any $g(t) \in \mathbf{L}^1(\mathbb{R}^n)$, $F(x) = \int_{\mathbb{R}^n} f(x, t) g(t) dt$ is ϕ -admissible, and*

$$M_\phi(F) = \int_{\mathbb{R}^n} M_\phi(f(\cdot, t)) g(t) dt. \quad (3.4)$$

The same conclusion is valid when ϕ -admissibility is replaced by strong admissibility.

Proof. Note $F \in \mathbf{L}^\infty(\mathbb{R}^m)$ with $\|F\|_\infty \leq \|f\|_\infty \|g\|_1$. Let $x \in \mathbb{R}^m$ be fixed. We have

$$\int_{\mathbb{R}^m} \phi_L(x - y) F(y) dy = \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^m} \phi_L(x - y) f(y, t) dy \right\} g(t) dt$$

by Fubini's theorem. Since the right-hand side converges as $L \rightarrow +\infty$ by the ϕ -admissibility of $f(\cdot, t)$ and Lebesgue's dominated convergence theorem, we see that F is ϕ -admissible and (3.4) holds. $\square$

4. Mean-convolution product

Let $f, g \in \mathbf{L}^\infty(\mathbb{R}^m)$ and let $\phi(x)$ be a weight function. Suppose $f(y-x)g(x) = \mathcal{IT}_{-y}f(x) \cdot g(x)$ is ϕ -admissible for any $y \in \mathbb{R}^m$. Recall $\mathcal{T}$ stands for the translation and $\mathcal{I}$ for the inversion (see (1.15) and (1.17)). We call a pair of functions such as f and g a ϕ -intimate pair. We will see shortly that the order of f and g should be respected in general. Their mean-convolution product with respect to ϕ , in short, ϕ -mean-convolution product, $f \otimes_\phi g(y)$, is then defined as the ϕ -mean $M_\phi(\mathcal{IT}_{-y}f \cdot g)$:

$$f \otimes_\phi g(y) = M_\phi(\mathcal{IT}_{-y}f \cdot g). \quad (4.1)$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^m} f \otimes_\phi g(y) u(x) dx &= \lim_{L \rightarrow +\infty} \int_{\mathbb{R}^m} \phi_L * (\mathcal{IT}_{-y}f \cdot g)(x) u(x) dx \\ &= \lim_{L \rightarrow +\infty} \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^m} \phi_L(x-z) f(y-z) g(z) dz \right\} u(x) dx \end{aligned} \quad (4.2)$$

for any $u(x) \in \mathbf{L}^1(\mathbb{R}^m)$. For each fixed y , the pointwise convergence at x of $\phi_L * (\mathcal{IT}_{-y}f \cdot g)(x)$ as $L \rightarrow +\infty$ suffices in defining the mean-convolution product in the spirit of Lemma 1.2. We may speak of a *strongly intimate pair* if, for each fixed $y \in \mathbb{R}^m$,

$$\sup_x |\phi_L * (\mathcal{IT}_{-y}f \cdot g)(x) - f \otimes_\phi g(y)| \rightarrow 0, \quad L \rightarrow +\infty. \quad (4.3)$$

Here recall that we no longer need any reference to weight functions because of Theorem 1.5. Thus, for such a case, we will henceforth simply write $f \otimes g$ and call it a mean-convolution product of f and g . Theorem 1.6 then implies that f and g constitute a strongly intimate pair if and only if, for each $u \in \mathbf{L}^1(\mathbb{R}^m)$,

$$\int_{\mathbb{R}^m} f(y-x+Lz)g(x-Lz)u(z) dz \rightarrow (f \otimes g)(y) \int_{\mathbb{R}^m} u(z) dz \quad (4.4)$$

as $L \rightarrow +\infty$ uniformly with respect to $x \in \mathbb{R}^m$.

It is clear that

$$|f \otimes_\phi g(y)| \leq \|f\|_\infty \|g\|_\infty \quad (4.5)$$

for a ϕ -intimate pair f and g .

PROPOSITION 4.1. *Let f and $g \in \mathbf{L}^\infty(\mathbb{R}^m)$ be a ϕ -intimate pair for a weight function ϕ . Suppose either that ϕ is even, $\phi(-x) = \phi(x)$, or that f and g are a strongly intimate pair. Then we have*

$$f \otimes_\phi g(y) = g \otimes_\phi f(y). \quad (4.6)$$

In fact, note that

$$\mathcal{I}T_{-y}f \cdot g = \mathcal{I}(T_{-y}f \cdot \mathcal{I}g) = \mathcal{I}T_{-y}(f \cdot \mathcal{I}T_{-y}g).$$

Hence (4.6) follows from Proposition 1.7. $\square$

Example 4.1. Let $m = 1$. If $f, g \in \mathbf{L}^\infty(\mathbb{R})$ satisfy

$$\lim_{x \rightarrow \pm\infty} f(x) = f^\pm, \quad \lim_{x \rightarrow \pm\infty} g(x) = g^\pm,$$

for some constants $f^\pm, g^\pm$, then f and g are an intimate pair for any weight function ϕ :

$$f \otimes_\phi g(y) = f^- g^+ \int_{-\infty}^0 \phi(x) dx + f^+ g^- \int_0^{+\infty} \phi(x) dx$$

(see Proposition 1.3). Thus, (4.6) holds if and only if

$$(f^+ g^- - f^- g^+) \left(\int_0^{+\infty} \phi(x) dx - \int_{-\infty}^0 \phi(x) dx \right) = 0.$$

On the other hand, if f, g and f, h are ϕ -intimate pairs, then so is f and any linear combination of g and h , and

$$f \otimes_\phi (ag + bh) = af \otimes_\phi g + bf \otimes_\phi h, \quad a, b \in \mathbb{C}. \quad (4.7)$$

Example 4.2. Any pair of periodic functions (of the same period) is a standard example of a strongly intimate pair:

$$(f \otimes g)(y) = \int_Q f(y-x)g(x) dx \quad (4.8)$$

for f and $g \in \mathcal{P}_I(\mathbb{R}^m)$. In this case, the mean-convolution product also belongs to $\mathcal{P}_I(\mathbb{R}^m)$.

Example 4.3. Wiener [14] investigated the mean value

$$\lim_{L \rightarrow +\infty} \frac{1}{2L} \int_{-L}^L f(y+z) \bar{f}(z) dz$$

for $f \in \mathbf{L}^\infty(\mathbb{R})$. This mean value, when well defined, coincides with the c_Q -mean-convolution product:

$$(\mathcal{I}f \otimes_{c_Q} \bar{f})(-y) = \lim_{L \rightarrow +\infty} \int_{\mathbb{R}} c_{Q,L}(x-z) f(y+z) \bar{f}(z) dz,$$

the limit being taken pointwise with respect to x , $-\infty < x < +\infty$. Recall c_Q is the canonical weight function (1.4) and that $\mathcal{I}f(t) = f(-t)$ is the inversion (1.17). Thus, in this case, $\mathcal{I}f$ and $\bar{f}$ constitute a c_Q -intimate pair in our terminology. Weight functions other than c_Q might be chosen, while the limits could also be taken in other topologies, such as the strong topology in $\mathbf{L}^\infty(\mathbb{R})$.

It is evident that a constant function $g(x) = k$ and any ϕ -admissible function $f(x)$ constitute a ϕ -intimate pair, and $f \otimes_\phi g = kM_\phi(f)$. However, if we demand too much, the result turns out rather trivial.

LEMMA 4.2. *Let $f(x)$ be strongly admissible. If, for any $g(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$, $f(x)$ and $g(x)$ constitute a strongly intimate pair, then $f \in \mathcal{N}(\mathbb{R}^m)$, and $f \otimes g = 0$.*

Proof. Immediate from Proposition 2.2. □

COROLLARY 4.3. *Let $f(x) \in \mathbf{L}^p(\mathbb{R}^m) \cap \mathbf{L}^\infty(\mathbb{R}^m)$ for some $1 \leq p < +\infty$. Then, for any $g(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$, $f(x)$ and $g(x)$ constitute a strongly intimate pair and $f \otimes g = 0$.*

Proof. This follows from Proposition 2.1. □

LEMMA 4.4. *Let ϕ and ψ be weight functions. If*

$$\lim_{L \rightarrow +\infty} \sup_{x,y} |\phi_L * (\mathcal{I}T_{-y} f \cdot g)(x) - f \otimes_\phi g(y)| = 0,$$

then

$$\lim_{L \rightarrow +\infty} \sup_{x,y} |\psi_L * (\mathcal{I}T_{-y} f \cdot g)(x) - f \otimes_\psi g(y)| = 0.$$

In fact, this can be seen as in the proof of Theorem 1.5. Therefore, we may call f, g a *uniformly intimate pair* when the convergence (4.3) or, equivalently, (4.4) is uniform with respect to $y \in \mathbb{R}^m$. A pair of periodic functions are, as a pair, not only strongly intimate but also uniformly intimate.

Remark. $f \otimes g(y)$ is continuous in y for a uniformly intimate pair f, g .

In this respect, the following lemma is worth noting.

LEMMA 4.5. *Let $f_n, g_n \in \mathbf{L}^\infty(\mathbb{R}^m)$, $n = 1, 2, \dots$, and $f, g \in \mathbf{L}^\infty(\mathbb{R}^m)$ be such that $\|f_n - f\|_\infty \rightarrow 0$ and $\|g_n - g\|_\infty \rightarrow 0$ as $n \rightarrow +\infty$. Let ϕ be a weight function. Suppose that for each n , f_n and g_n constitute a ϕ -intimate (or strongly or uniformly intimate) pair. Then so do f and g .*

In fact, note that

$$\begin{aligned} & \|\mathcal{IT}_{-y}f_n \cdot g_n - \mathcal{IT}_{-y}f \cdot g\|_\infty \\ & \leq \|(\mathcal{IT}_{-y}f_n - \mathcal{IT}_{-y}f) \cdot g_n\|_\infty + \|\mathcal{IT}_{-y}f \cdot (g_n - g)\|_\infty \\ & \leq \|f_n - f\|_\infty \|g_n\|_\infty + \|f\|_\infty \|g_n - g\|_\infty \\ & \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$. Here the convergence is uniform with respect to y . It remains to apply Proposition 3.1.

Here is another instance of uniformly intimate pairs.

LEMMA 4.6. *Let $f \in \mathcal{N}(\mathbb{R}^m)$. Then f and any $g \in \mathbf{L}^\infty(\mathbb{R}^m)$ constitute a uniformly admissible pair and $f \otimes g = 0$.*

In fact, for any weight function ϕ , we have

$$\left| \int_{\mathbb{R}^m} \phi_L(x - z) f(y - z) g(z) dz \right| \leq \|g\|_\infty \|\phi_L * |f|\|_\infty \rightarrow 0$$

as $L \rightarrow +\infty$.

We now show that a uniformly intimate pair behaves neatly.

PROPOSITION 4.7. *Suppose that $f(x)$ and $g(x)$ are a uniformly intimate pair. If f is admissible with respect to a weight function ψ (or strongly admissible), then $f \otimes g$ is ψ -admissible (or strongly admissible). Furthermore, if g is ϕ -admissible for a weight function ϕ , then*

$$M_\psi(f \otimes g) = M_\psi(f) M_\phi(g). \quad (4.9)$$

Proof. Let $z \in \mathbb{R}^m$. We show that

$$\psi_L * (f \otimes g)(z) - \psi_{L'} * (f \otimes g)(z) \rightarrow 0, \quad L, L' \rightarrow +\infty.$$

Let

$$F_R(x, y) = \int_{\mathbb{R}^m} \phi_R(x - t) f(y - t) g(t) dt, \quad R > 0.$$

for a weight function ϕ . Here, for later convenience, we take ϕ just as a working weight function. Then $|F_R(x, y)| \leq \|f\|_\infty \|g\|_\infty$, and

$$\sup_{x, y} |f \otimes g(y) - F_R(x, y)| \rightarrow 0, \quad R \rightarrow +\infty.$$

Now note that

$$\begin{aligned} & \psi_L * (f \otimes g)(z) - \psi_{L'} * (f \otimes g)(z) \\ &= \psi_L * \{f \otimes g - F_R(x, \cdot)\}(z) - \psi_{L'} * \{f \otimes g - F_R(x, \cdot)\}(z) \\ & \quad + (\psi_L - \psi_{L'}) * F_R(x, \cdot)(z). \end{aligned}$$

Here the first and second terms on the right-hand side can be arbitrarily small as $R \rightarrow +\infty$, independent of x and of L or L' . As for the third term, observe that

$$\begin{aligned} & (\psi_L - \psi_{L'}) * F_R(x, \cdot)(z) \\ &= \int_{\mathbb{R}^m} \phi_R(x - \zeta) g(\zeta) \{\psi_L * f(z - \zeta) - \psi_{L'} * f(z - \zeta)\} d\zeta. \end{aligned}$$

Thus, for fixed $R > 0$, this term tends to zero as $L, L' \rightarrow +\infty$ by Lebesgue's dominated convergence theorem provided that f is ψ -admissible. If f is strongly admissible, then this convergence is uniform with respect to x (and z). To see (4.9), note that

$$\begin{aligned} & \psi_L * (f \otimes g)(z) - M_\psi(f) M_\phi(g) \\ &= \psi_L * (f \otimes g - F_R(x, \cdot))(z) + M_\psi(f) \left\{ \int_{\mathbb{R}^m} \phi_R(x - \zeta) g(\zeta) d\zeta - M_\phi(g) \right\} \\ & \quad + \int_{\mathbb{R}^m} \phi_R(x - \zeta) g(\zeta) \{\psi_L * f(z - \zeta) - M_\psi(f)\} d\zeta \end{aligned} \quad (4.10)$$

for any x . We thus obtain (4.9) by first choosing R large enough and then making $L \rightarrow +\infty$. $\square$

Example 4.4. Let $f, g, f_1, g_1 \in \mathbf{L}^2(\mathbb{R}^m)$. Following Proposition 3.2, define

$$\begin{aligned} h(x) &= \int_{\mathbb{R}^m} |f(x + y) - g(y)|^2 dy, \\ h_1(x) &= \int_{\mathbb{R}^m} |f_1(x + y) - g_1(y)|^2 dy. \end{aligned}$$

Then $h(x)$ and $h_1(x)$ constitute a uniformly intimate pair, and

$$h \otimes h_1(y) = (\|f\|_2^2 + \|g\|_2^2)(\|f_1\|_2^2 + \|g_1\|_2^2)$$

so that (4.9) certainly holds.

Remark. Examples violating (4.9) will be given below (see Example 4.5).

COROLLARY 4.8. *Let $f(x)$ and $g(x)$ be a uniformly intimate pair. Suppose f is strongly admissible. If $M(f) = 0$, then $M(f \otimes g) = 0$. If $M(f) \neq 0$, then g is strongly admissible.*

In fact, this follows from the last part of the proof of Proposition 4.7. To handle the case $M(f) \neq 0$, replace $M_\phi(g)$ by $\mu = M(f \otimes g)/M(f)$ in (4.10). The left-hand side of (4.10) then tends to zero as $L \rightarrow +\infty$ uniformly with respect to z . The first term on the right-hand side tends to zero as $R \rightarrow +\infty$ uniformly with respect to L and z , and so does the third term as $L \rightarrow +\infty$ uniformly with respect to R and x . It then follows that g is strongly admissible.

When f and g are a strongly intimate pair, we still have

$$\psi_L * \{f \otimes g - F_R(x, \cdot)\}(z) \rightarrow 0$$

as $R \rightarrow +\infty$, but the convergence depends on L . On the other hand, if f is strongly admissible, then

$$(\psi_L - \psi_{L'}) * F_R(x, \cdot)(z) \rightarrow 0$$

as $L, L' \rightarrow +\infty$ independent of $R > 0$ large enough for any weight function ψ . Similar remarks are valid in handling (4.9). Therefore, we have the following corollary.

COROLLARY 4.9. *Suppose $f(x)$ and $g(x)$ are a strongly intimate pair. If f is strongly admissible, then $f \otimes g$ is admissible with respect to any weight function ψ . Furthermore, if g is admissible with respect to a weight function ϕ , then (4.9) is valid.*

The following result suggests what is seen from a somewhat different direction.

PROPOSITION 4.10. *Suppose both $f(x)$ and $g(x) \in \mathbf{L}^\infty(\mathbb{R}^m)$ are strongly admissible. If $f(x)$ and $g(x)$ constitute a ϕ -intimate pair with respect to a weight function ϕ , then their ϕ -mean-convolution product $f \otimes_\phi g$ is admissible with respect to any weight function ψ , and a version of (4.9) holds:*

$$M_\psi(f \otimes_\phi g) = M(f)M(g).$$

Proof. Let $u(x), v(x) \in \mathbf{L}^1(\mathbb{R}^m)$. We have

$$\begin{aligned} & \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \psi_L * f \otimes_\phi g(z) v(z) dz u(x) dx - M(f)M(g) \int_{\mathbb{R}^m} v(z) dz \int_{\mathbb{R}^m} u(x) dx \\ &= - \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \psi_L(z-y) \int_{\mathbb{R}^m} [F_R(x, y) - f \otimes_\phi g(y)](z) u(x) dx dy v(z) dz \\ & \quad + \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \{\psi_L * F_R(x, \cdot)(z) - M(f)M(g)\} u(x) v(z) dx dz. \end{aligned} \quad (4.11)$$

Here $F_R(x, y)$ is as in the proof of Proposition 4.7. Since f and g are strongly admissible, we have

$$\begin{aligned} & |\psi_L * F_R(x, \cdot) - M(f)M(g)| \\ & \leq \|\psi_L * f - M(f)\|_\infty \|g\|_\infty + |M(f)| \|\phi_R * g - M(g)\|_\infty, \end{aligned}$$

for

$$\psi_L * F_R(x, \cdot)(z) = \int_{\mathbb{R}^m} (\psi_L * f)(z-t) \phi_R(x-t) g(t) dt.$$

On the other hand, by Lebesgue's dominated convergence theorem, the first term on the right-hand side of (4.11) tends to zero as $R \rightarrow +\infty$ for each fixed L because f and g constitute a ϕ -intimate pair. Now, for any $\epsilon > 0$, choose $L > 0$ so that

$$\|\psi_L * f - M(f)\|_\infty < \frac{1}{2\|g\|_\infty} \epsilon,$$

and then let $R \rightarrow +\infty$. Then the modulus of the left-hand side of (4.11) becomes smaller than

$$\epsilon \int |u(x)| dx \int |v(x)| dx.$$

□

We now check how the ordinary convolution product and the ϕ -mean convolution product are related. We will later see that the ϕ -mean convolution product is separately sequentially continuous in the topology given by (3.2).

LEMMA 4.11. *Let $f, g \in \mathbf{L}^\infty(\mathbb{R}^m)$ be a ϕ -intimate (or strongly intimate or uniformly intimate) pair. Then, for any $u, v \in \mathbf{L}^1(\mathbb{R}^m)$, $f * u$ and $g * v$ constitute a ϕ -intimate (or strongly intimate or uniformly intimate) pair, and*

$$(f * u) \otimes_\phi (g * v) = u * ((f \otimes_\phi g) * v) = (u * (f \otimes_\phi g)) * v.$$

Proof. Let t and $y \in \mathbb{R}^m$ be fixed. The discussions below can also be uniformly valid with respect to t or y according to the initial premise on the degree of intimacy between f and g . Now

$$\begin{aligned} & \int_{\mathbb{R}^m} \phi_L(t-z)(f * u)(y-z)(g * v)(z) dz \\ &= \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^m} \phi_L(t-z) f(y-z-x) g(z-x') dz \right\} u(x) dx \right\} v(x') dx' \end{aligned}$$

by Fubini's theorem. The right-hand side converges to

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f \otimes_\phi g(y-x-x') u(x) v(x') dx dx'$$

by Lebesgue's dominated convergence theorem. Thus, $f * u$ and $g * v$ constitute a ϕ -intimate (or strongly intimate or uniformly intimate) pair. The remaining equalities are obvious. $\square$

In a similar way, we have the following corollary.

COROLLARY 4.12. *Let $f, g \in \mathbf{L}^\infty(\mathbb{R}^m)$ be a ϕ -intimate (or strongly intimate or uniformly intimate) pair. Suppose $h \in \mathbf{L}^1(\mathbb{R}^m)$. Then $f * h$ and g constitute a ϕ -intimate (or strongly intimate or uniformly intimate) pair, and*

$$(f * h) \otimes_\phi g = h * (f \otimes_\phi g)$$

holds.

LEMMA 4.13. *Suppose $f, g \in \mathbf{L}^\infty(\mathbb{R}^m)$ constitute a ϕ -intimate (or strongly intimate) pair. Then, for any $h \in \mathbf{L}^1(\mathbb{R}^m)$, $(h * \mathcal{I}f) \cdot g$ and $(h * \mathcal{I}g) \cdot f$ are ϕ -admissible (or strongly admissible), and*

$$\begin{aligned} M_\phi((h * \mathcal{I}f) \cdot g) &= M_\phi((h * \mathcal{I}g) \cdot f) \\ &= \int_{\mathbb{R}^m} f \otimes_\phi g(y) h(y) dy. \end{aligned} \quad (4.12)$$

Proof. Let $x \in \mathbb{R}^m$. Then

$$\begin{aligned} \phi_L * (h * \mathcal{I}f \cdot g)(x) &= \int_{\mathbb{R}^m} \phi_L(x-z) \left\{ \int_{\mathbb{R}^m} h(y) f(y-z) dy \right\} g(z) dz \\ &= \int_{\mathbb{R}^m} h(y) \left\{ \int_{\mathbb{R}^m} \phi_L(x-z) f(y-z) g(z) dz \right\} dy \end{aligned}$$

by Fubini's theorem. When $L \rightarrow +\infty$, the extreme right-hand side converges to $\int_{\mathbb{R}^m} h(y) f \otimes_{\phi} g(y) dy$ by Lebesgue's dominated convergence theorem since f and g are a ϕ -intimate pair. $\phi_L * (h * \mathcal{I}g \cdot f)$ is handled analogously. $\square$

The following assertion is slightly stronger than Proposition 4.7 and Corollary 4.9.

PROPOSITION 4.14. *Let $f, g \in \mathbf{L}^{\infty}(\mathbb{R}^m)$ be a ϕ -intimate pair. Suppose, furthermore, that f is strongly admissible and g is ϕ -admissible. Then $f \otimes_{\phi} g$ is admissible with respect to any weight function ψ , and (4.9) holds.*

Proof. Let $h \in \mathbf{L}^1(\mathbb{R}^m)$. For $t \in \mathbb{R}^m$, we have

$$\begin{aligned} M_{\phi}((\mathcal{T}_{-t}h * \mathcal{I}g) \cdot f) &= M_{\phi}(h * \mathcal{I}g \cdot \mathcal{T}_t f) \\ &= \int_{\mathbb{R}^m} h(y) \mathcal{T}_t(f \otimes_{\phi} g)(y) dy \end{aligned}$$

since

$$\mathcal{T}_t f \otimes_{\phi} g = f \otimes_{\phi} \mathcal{T}_t g = \mathcal{T}_t(f \otimes_{\phi} g). \quad (4.13)$$

By Lemma 4.10, we then have

$$M_{\phi}(h * \mathcal{I}g \cdot \psi_L * f) = \int_{\mathbb{R}^m} h(y) \psi_L * (f \otimes_{\phi} g)(y) dy$$

for any weight function ψ . Since f is strongly admissible and g is ϕ -admissible, we have

$$M_{\phi}(h * \mathcal{I}g \cdot \psi_L * f) \rightarrow M_{\phi}(g) M_{\psi}(f) \int_{\mathbb{R}^m} h(y) dy$$

as $L \rightarrow +\infty$. Thus, $f \otimes_{\phi} g$ is ψ -admissible and (4.9) holds. $\square$

Even if $\phi = \psi$, (4.9) does not always hold when neither f nor g are strongly admissible.

Example 4.5. Let $m = 1$. We take $f, g \in \mathbf{L}^{\infty}(\mathbb{R})$ as given in Example 4.1. Let ϕ be a weight function and let

$$\lambda = \int_0^{+\infty} \phi(x) dx.$$

Then, by Proposition 1.3 and Example 4.1, we have

$$M_{\phi}(f) M_{\phi}(g) - M_{\phi}(f \otimes_{\phi} g) = (f^+ - f^-) \{(\lambda - 1)^2 g^+ - \lambda^2 g^-\}.$$

In particular, we see that (4.9) does not hold in general even if $\lambda = 1/2$.

We now reformulate observation (4.13) given in the proof of Proposition 4.14.

COROLLARY 4.15. *Suppose $f, g \in \mathbf{L}^\infty(\mathbb{R}^m)$ constitute a ϕ -intimate pair. If g is periodic, then the ϕ -mean convolution product $f \otimes_\phi g$ is also periodic with the same period as g . In particular, if g is almost periodic, then so is $f \otimes_\phi g$.*

Note that the ordinary convolution product $u * g$, $u \in \mathbf{L}^1(\mathbb{R}^m)$, also enjoys these properties.

On the other hand, when $m = 1$, $f \otimes g$ is necessarily constant for periodic f and g unless their periods are commensurable. A higher-dimensional analogue also holds, as we will discuss in detail later.

To end this section, we give a completeness result about intimacy in the topology given by (3.3).

THEOREM 4.16. *Let $f_n \in \mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$, $n = 1, 2, \dots$, and let $g \in \mathbf{L}^\infty(\mathbb{R}^m)$. Suppose each f_n and g constitute a ϕ -intimate pair. If f_n converges to $f \in \mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$ as $n \rightarrow +\infty$ in the topology given by (3.3), then f and g also constitute a ϕ -intimate pair, and $f_n \otimes_\phi g$ converges to $f \otimes_\phi g$ weakly* as $n \rightarrow +\infty$ in $\mathbf{L}^\infty(\mathbb{R}^m)$.*

Proof. Let $u \in \mathbf{L}^1(\mathbb{R}^m)$. Then

$$\int_{\mathbb{R}^m} f_n \otimes_\phi g(y) u(y) dy = M_\phi(u * \mathcal{I} f_n \cdot g).$$

Since $u * \mathcal{I} f_n \rightarrow u * \mathcal{I} f$ in the strong topology of $\mathbf{C}_{\text{unif}}(\mathbb{R}^m)$, then $u * \mathcal{I} f_n \cdot g \rightarrow u * \mathcal{I} f \cdot g$ in $\mathbf{L}^\infty(\mathbb{R}^m)$. Now apply Proposition 3.1. $\square$

5. Ergodic properties

Recall the totality of bounded measurable functions $\mathcal{P}_H(\mathbb{R}^m)$ of m linearly independent periods $\eta^{(1)}, \dots, \eta^{(m)} \in \mathbb{R}^m$ as discussed in Example 1.2. In particular, $H \in \mathbf{GL}(m, \mathbb{R})$ such that $\eta^{(k)} = H e^{(k)}$, $k = 1, \dots, m$, where we assume that $\det H > 0$. The parallelotope Q_H is the image of the cube Q by the linear transformation H :

$$Q_H = \left\{ x = \sum_{j=1}^m t_j \eta^{(j)}; -\frac{1}{2} \leq t_j < \frac{1}{2}, j = 1, \dots, m \right\}.$$

Its volume is $|Q_H| = \det H$.

Now let $\mathbb{Z}_H^m$ be the m -lattice generated by $\eta^{(1)}, \dots, \eta^{(m)}$:

$$\mathbb{Z}_H^m = \left\{ \alpha = \sum_{j=1}^m \alpha_j \eta^{(j)}; \alpha_j \in \mathbb{Z}, j = 1, \dots, m \right\}.$$

Then

$$Q_H + \alpha = \{x + \alpha; x \in Q_H\}, \quad \alpha \in \mathbb{Z}_H^m,$$

define a tiling (tessellation) over $\mathbb{R}^m$ since

$$(Q_H + \alpha) \cap (Q_H + \alpha') = \emptyset, \quad \alpha \neq \alpha', \quad \alpha, \alpha' \in \mathbb{Z}_H^m,$$

and

$$\bigcup_{\alpha \in \mathbb{Z}_H^m} (Q_H + \alpha) = \mathbb{R}^m.$$

For $N = 0, 1, 2, \dots$, let N_H^* be the set of $(2N + 1)^m$ lattice points

$$\alpha = \sum_{j=1}^m \alpha_j \eta^{(j)} \in \mathbb{Z}_H^m, \quad -N \leq \alpha_j \leq N, \quad j = 1, \dots, m.$$

Let

$$Q_{H,N}^* = \bigcup_{\alpha \in N_H^*} (Q_H + \alpha).$$

Its volume is then

$$|Q_{H,N}^*| = (2N + 1)^m |Q_H| = (2N + 1)^m \det H.$$

Let us consider how a canonical mapping P_H is defined onto $\mathcal{P}_H(\mathbb{R}^m)$ from a subset of $\mathbf{L}^\infty(\mathbb{R}^m)$. A natural tool should be an ergodic theorem of the Birkoff–Khinchin type (see [2, 16]). So let $f \in \mathbf{L}^\infty(\mathbb{R}^m)$ be given, and define

$$f_{H,N}(x) = \frac{1}{(2N + 1)^m} \sum_{\alpha \in N_H^*} f(x + \alpha), \quad x \in \mathbb{R}^m. \quad (5.1)$$

We are interested in the class of functions $f \in \mathbf{L}^\infty(\mathbb{R}^m)$ for which there exists an $f_H \in \mathcal{P}_H(\mathbb{R}^m)$ such that

$$\lim_{N \rightarrow +\infty} \sup_{y \in \mathbb{R}^m} \left| \int_{Q_H} f_{H,N}(y - x) g(x) dx - \int_{Q_H} f_H(y - x) g(x) dx \right| = 0 \quad (5.2)$$

for any $g \in \mathcal{P}_H(\mathbb{R}^m)$. Thus, (5.2) may be rewritten as

$$\lim_{N \rightarrow +\infty} \sup_{\alpha \in \mathbb{Z}_H^m} \sup_{y \in Q_H} \left| \int_{Q_H + \alpha} f_{H,N}(y-x)g(x) dx - \int_{Q_H} f_H(y-x)g(x) dx \right| = 0.$$

Remark. Since (5.2) implies

$$\left| \int_{Q_H} f_H(y-x)g(x) dx \right| \leq \|f\|_\infty \|g\|_{L^1(Q_H)},$$

the choice of g in (5.2) can be extended to any locally integrable function in $\mathbb{R}^m$ which is periodic of periods $\eta^{(1)}, \dots, \eta^{(m)}$.

We denote the totality of such $f \in L^\infty(\mathbb{R}^m)$ that satisfy (5.2) by $\mathcal{Q}_H(\mathbb{R}^m)$. In other words, $f \in \mathcal{Q}_H(\mathbb{R}^m)$ if and only if $f_{H,N}(y-x)$ converges to $f_H(y-x)$ weakly in $L^1(Q_H)$ uniformly with respect to y . That such a weak limit, if it exists, is always periodic is rather easy to see, but its uniform existence with respect to the translations is not at all trivial.

We first note the following.

PROPOSITION 5.1. $\mathcal{Q}_H(\mathbb{R}^m)$ is closed in $L^\infty(\mathbb{R}^m)$.

In fact, let $f_n \in \mathcal{Q}_H(\mathbb{R}^m)$ strongly converge to a function f in $L^\infty(\mathbb{R}^m)$. Thus, each f_n satisfies (5.2) with $f_{H,N}$ and f_H replaced by $f_{n,H,N}$ and some $f_{n,H} \in \mathcal{P}_H(\mathbb{R}^m)$, respectively. Obviously, $f_{n,H,N}$ converges strongly to $f_{H,N}$ in $L^\infty(\mathbb{R}^m)$ as $n \rightarrow +\infty$ so that $f_{n,H}$ weakly converges to a function $f_H \in L^1(Q_H)$. Now by the above remark, we see that f_H is in fact bounded and hence $f \in \mathcal{Q}_H(\mathbb{R}^m)$.

Along the line of Yosida's proof of the mean ergodic theorem (see [16]), we observe the following.

LEMMA 5.2. *Let $f \in L^\infty(\mathbb{R}^m)$. Then $f_{H,N}(x)$, $N = 0, 1, 2, \dots$, are bounded in $L^2_{\text{loc}}(\mathbb{R}^m)$. In particular, there is a subsequence $f_{H,N'}(x)$ which converges weakly to an element $f_*(x) \in L^2_{\text{loc}}(\mathbb{R}^m)$. $f_*(x)$ is periodic of periods $\eta^{(1)}, \dots, \eta^{(m)}$ in the sense that $T_H^{(k)} f_* = f_*$ for $k = 1, \dots, m$.*

Before verifying this lemma, we introduce several changes of notation for later convenience. Let us write the translations

$$\mathcal{T}_{-\eta^{(k)}} = \mathcal{T}_H^{(k)}, \quad k = 1, \dots, m,$$

and

$$\mathcal{T}_H^\alpha = (\mathcal{T}_H^{(1)})^{\alpha_1} \cdots (\mathcal{T}_H^{(m)})^{\alpha_m}$$

for $\alpha = \alpha_1 \eta^{(1)} + \cdots + \alpha_m \eta^{(m)} \in \mathbb{Z}_H^m$. Then (5.1) can be written as

$$f_{H,N}(x) = \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} \mathcal{T}_H^\alpha f(x).$$

Note that, for any N ,

$$\begin{aligned} \mathcal{T}_H^{(k)} f_{H,N} - f_{H,N} &= \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} \mathcal{T}_H^\alpha (\mathcal{T}_H^{(k)} - I) f \\ &= \frac{1}{(2N+1)^m} \left\{ \sum_{\substack{\alpha \in N_H^* \\ \alpha_k = N}} \mathcal{T}_H^\alpha \mathcal{T}_H^{(k)} f - \sum_{\substack{\alpha \in N_H^* \\ \alpha_k = -N}} \mathcal{T}_H^\alpha f \right\} \end{aligned} \quad (5.3)$$

for each $k = 1, \dots, m$. Therefore,

$$|\mathcal{T}_H^{(k)} f_{H,N}(x) - f_{H,N}(x)| \leq \frac{2}{2N+1} \|f\|_\infty.$$

It follows that

$$\left\| \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} \mathcal{T}_H^\alpha h \right\|_\infty \leq \frac{2}{2N+1} (\|h_1\|_\infty + \cdots + \|h_m\|_\infty), \quad (5.4)$$

for

$$\begin{aligned} h &= (I - \mathcal{T}_H^{(1)})h_1 + \cdots + (I - \mathcal{T}_H^{(m)})h_m \in (I - \mathcal{T}_H^{(1)})\mathbf{L}^\infty(\mathbb{R}^m) + \cdots \\ &\quad + (I - \mathcal{T}_H^{(m)})\mathbf{L}^\infty(\mathbb{R}^m). \end{aligned}$$

On the other hand, observe that

$$\begin{aligned} I - \mathcal{T}_H^\alpha &= (I - \mathcal{T}_H^{(1)})(I + \cdots + (\mathcal{T}_H^{(1)})^{\alpha_1-1})\mathcal{T}_H^{\alpha'} + (I - \mathcal{T}_H^{(2)})(I + \cdots \\ &\quad + (\mathcal{T}_H^{(2)})^{\alpha_2-1})\mathcal{T}_H^{\alpha''} + \cdots + (I - \mathcal{T}_H^{(m)})(I + \cdots + (\mathcal{T}_H^{(m)})^{\alpha_m-1}), \end{aligned}$$

where $\alpha = (\alpha_1, \alpha') = \cdots = (\cdots, \alpha_m)$. Thus, for each fixed N ,

$$\begin{aligned} f(x) - \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} f(x + \alpha) \\ \in (I - \mathcal{T}_H^{(1)})\mathbf{L}^\infty(\mathbb{R}^m) + \cdots + (I - \mathcal{T}_H^{(m)})\mathbf{L}^\infty(\mathbb{R}^m). \end{aligned} \quad (5.5)$$

Now we are ready for a proof of Lemma 5.2. In fact, let Ω be a bounded measurable subset of $\mathbb{R}^m$. Then for all $N = 0, 1, 2, \dots$, we have

$$|f_{H,N}(x)| \leq \|f\|_\infty, \quad \text{a.e. } x \in \Omega,$$

$$\int_{\Omega} |f_{H,N}(x)| dx \leq |\Omega| \|f\|_\infty,$$

hence

$$\sqrt{\int_{\Omega} |f_{H,N}(x)|^2 dx} \leq |\Omega|^{1/2} \|f\|_\infty. \quad (5.6)$$

Therefore, $f_{H,N}(x)$, $N = 0, 1, 2, \dots$, are bounded in each $\mathbf{L}^2(\Omega)$. A weakly convergent subsequence $f_{H,N'}(x)$ in $\mathbf{L}_{\text{loc}}^2(\mathbb{R}^m)$ is then chosen by the diagonal argument. Let $f_*(x)$ be the weak limit of $f_{H,N'}(x)$. Since, for each k ,

$$\begin{aligned} \int \mathcal{T}_H^{(k)} f_{H,N'}(x) g(x) dx &= \int f_{H,N'}(x) (\mathcal{T}_H^{(k)})^{-1} g(x) dx \\ &= \int f_{H,N'}(x) g(x) + \int (\mathcal{T}_H^{(k)} - I) f_{H,N'}(x) g(x) dx \end{aligned}$$

for any compactly supported $g(x)$, we obtain

$$\int f_*(x) (\mathcal{T}_H^{(k)})^{-1} g(x) dx = \int f_*(x) g(x) dx$$

by letting $N' \rightarrow +\infty$. In other words, $\mathcal{T}_H^{(k)} f_* = f_*$, $k = 1, \dots, m$.

Remark. If $g \in \mathbf{L}^p(\mathbb{R}^m)$ for some p , $1 \leq p \leq +\infty$, then we have

$$g_{H,N} = \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} \mathcal{T}_H^\alpha g \in \mathbf{L}^p(\mathbb{R}^m)$$

and

$$\|g_{H,N}\|_p \leq \|g\|_p$$

instead of (5.6).

In some cases, the sequence $f_{H,N}(x)$ itself converges and $f_*(x)$ can be explicitly computed. First of all, let us verify how the above observations work along the classical line.

LEMMA 5.3. *Let $f \in \mathcal{P}_I(\mathbb{R}^m)$. Then $f \in \mathcal{Q}_H(\mathbb{R}^m)$ and the sequence $f_{H,N}(x)$ converges to an element $f_H(x) \in \mathcal{P}_H(\mathbb{R}^m) \cap \mathcal{P}_I(\mathbb{R}^m)$ almost everywhere as $N \rightarrow +\infty$. Recall that $\mathcal{P}_H(\mathbb{R}^m) \cap \mathcal{P}_I(\mathbb{R}^m) = \mathbb{C}$ if $\mathbb{Z}^m \cap \mathbb{Z}_H^m = \{0\}$.*

In fact, let $f_*(x)$ be that of Lemma 5.2. We claim that $f_*(x)$ is what we should call $f_H(x)$. Note that, for any N ,

$$f_{H,N}(x) - f_*(x) = \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} T_H^\alpha (f - f_*)(x),$$

while

$$f - f_* = \text{weak} \lim_{N' \rightarrow +\infty} \left\{ f - \frac{1}{(2N'+1)^m} \sum_{\alpha \in N_H'^*} T_H^\alpha f \right\}$$

belongs to the closure of $(I - T_H^{(1)})\mathcal{P}_I(\mathbb{R}^m) + \cdots + (I - T_H^{(m)})\mathcal{P}_I(\mathbb{R}^m)$ in $L^2(\Omega)$ for any bounded $\Omega \subset \mathbb{R}^m$. Thus, we may consider everything in the frame of $L^2(Q)$ by means of the tiling $\mathbb{R}^m = \bigcup_{\alpha \in \mathbb{Z}^m} (Q + \alpha)$. In particular,

$$\int_Q |T_H^\alpha g(x)|^2 dx \leq 2^m \int_Q |g(x)|^2 dx$$

for each $g \in \mathcal{P}_I(\mathbb{R}^m)$ and for any $\alpha \in \mathbb{Z}_H^m$. Now for an arbitrarily small $\epsilon > 0$, we have $h \in (I - T_H^{(1)})\mathcal{P}_I(\mathbb{R}^m) + \cdots + (I - T_H^{(m)})\mathcal{P}_I(\mathbb{R}^m)$ such that

$$\int_Q |f - f_* - h|^2 dx < \epsilon^2.$$

Hence, for any N ,

$$\int_Q \left| \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} T_H^\alpha (f - f_* - h) \right|^2 dx \leq 2^m \epsilon^2.$$

On the other hand, (5.4) in this context reads as

$$\int_Q \left| \frac{1}{(2N+1)^m} \sum_{\alpha \in N_H^*} T_H^\alpha \right|^2 dx \leq \frac{C}{(2N+1)^2}$$

for some constant $C > 0$. Therefore,

$$\int_Q |f_{H,N}(x) - f_*(x)|^2 dx \rightarrow 0, \quad N \rightarrow +\infty.$$

Since a subsequence of $f_{H,N}(x)$ then converges to $f_*(x)$ almost everywhere, we see that $f_* \in \mathbf{L}^\infty(\mathbb{R}^m)$ and

$$\|f_*\|_\infty \leq \|f\|_\infty.$$

The Birkoff–Khinchin ergodic theorem now completes the proof of the lemma. $\square$

Remark. Note that, for any bounded Ω , we have

$$\int_{\Omega} |f_{H,N}(y-x) - f_H(y-x)| dx \rightarrow 0, \quad f \in \mathcal{P}_I(\mathbb{R}^m), \quad (5.7)$$

as $N \rightarrow +\infty$ uniformly with respect to $y \in \mathbb{R}^m$. In particular, we may take $\Omega = Q_H$ and (5.2) certainly holds.

Here is another instance which suggests the role of the topology.

Example 5.1. Let $f(x) \in \mathbf{L}^\infty(\mathbb{R})$ be that of Proposition 1.3. That is,

$$f(x) \rightarrow f^\pm \quad \text{as } x \rightarrow \pm\infty$$

(recall (1.8)). Then, for any $y \in \mathbb{R}$ and $0 \leq x \leq 1$,

$$f_N(x+y) = \frac{1}{2N+1} \sum_{k=-N}^N f(x+y+k) \rightarrow \frac{1}{2}(f^+ + f^-)$$

as $N \rightarrow +\infty$. In other words, $f_*(x) = \frac{1}{2}(f^+ + f^-)$. However, unless $f^+ = f^-$, the convergence is not uniform with respect to y . In fact, let $\epsilon > 0$ be arbitrary. Let $R_\epsilon \in \mathbb{N}$ be large enough so that

$$\begin{aligned} |f(z) - f^+| &< \epsilon, & z > R_\epsilon, \\ |f(z) - f^-| &< \epsilon, & z < -R_\epsilon. \end{aligned}$$

Thus, for $x \in [0, 1]$ and $N > R_\epsilon - x - y$, and $[y]$ denoting the largest integer not exceeding y , we have

$$\begin{aligned} &f_N(x+y) - \frac{1}{2}(f^+ + f^-) \\ &= \frac{1}{2N+1} \sum_{k=R_\epsilon-[y]}^N (f(x+y+k) - f^+) \\ &\quad + \frac{1}{2N+1} \sum_{k=-N}^{-R_\epsilon-[y]-2} (f(x+y+k) - f^-) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2N+1} \sum_{k=-R_\epsilon-[y]-1}^{R_\epsilon-[y]-1} f(x+y+k) + \frac{1}{2N+1} \sum_{k=R_\epsilon-[y]}^N f^+ - \frac{1}{2} f^+ \\
& + \frac{1}{2N+1} \sum_{k=-N}^{-R_\epsilon-[y]-2} f^- - \frac{1}{2} f^-,
\end{aligned}$$

hence

$$\begin{aligned}
& \left| f_N(x+y) - \frac{1}{2}(f^+ + f^-) - \frac{[y]+1}{2N+1}(f^+ - f^-) \right| \\
& \leq 2\epsilon + \frac{2R_\epsilon}{2N+1} \|f\|_\infty + \frac{2R_\epsilon+1}{2N+1} \left| \frac{f^+ + f^-}{2} \right|.
\end{aligned}$$

Now let us show the significance of (5.2). Let

$$K = 2\pi^t H^{-1}.$$

Here t indicates the transposed matrix. Observe that

$$\frac{1}{\sqrt{|Q_H|}} e_\beta, \quad e_\beta(x) = e^{i\beta \cdot x}, \quad \beta \in \mathbb{Z}_K^m,$$

constitute an orthonormal basis of $L^2(Q_H)$. Note that $e_\beta(x)$, $\beta \in \mathbb{Z}_K^m$, are also periodic of periods $\alpha \in \mathbb{Z}_H^m$, for

$$\frac{1}{2\pi} \alpha \cdot \beta \in \mathbb{Z} \quad \text{when} \quad \alpha \in \mathbb{Z}_H^m, \quad \beta \in \mathbb{Z}_K^m.$$

PROPOSITION 5.4. *In order that $f \in \mathbf{L}_{\text{ad}}^\infty(\mathbb{R}^m)$ constitute a uniformly intimate pair with any $g \in \mathcal{P}_H(\mathbb{R}^m)$, it is necessary and sufficient that (5.2) holds. In this case,*

$$f_H(x) = \sum_{\beta \in \mathbb{Z}_K^m} M(e_{-\beta} \cdot f) e_\beta(x), \quad (5.8)$$

which converges almost everywhere.

Proof. Let $g(x) \in \mathcal{P}_H(\mathbb{R}^m)$. Then

$$\frac{1}{|Q_H|} \int_{Q_H} f_{H,N}(y-z) g(z) dz = \frac{1}{|Q_{H,N}^*|} \int_{Q_{H,N}^*} f(y-z) g(z) dz. \quad (5.9)$$

Therefore, the left-hand side converges as $N \rightarrow +\infty$ uniformly with respect to y if (5.2) holds. Then f and g constitute a uniformly intimate pair, and

$$f \otimes g(y) = \frac{1}{|Q_H|} \int_{Q_H} f_H(y-z)g(z) dz. \quad (5.10)$$

On the other hand, if f and g constitute a uniformly intimate pair, then the right-hand side of (5.9) converges to $f \otimes g(y)$ as $N \rightarrow +\infty$ uniformly with respect to y . Thus, $f_{H,N}(y-x)$ weakly converges to an $f_H(y-x)$ in $L^1(Q_H)$. Now recall (5.6). Then the above limiting procedure implies that we may allow $L^1(Q_H)$ and $f_{H,N}(y-x)$ to weakly* converge in $L^\infty(Q_H)$. This limit necessarily coincides almost everywhere with $f_H(y-x)$ so that we have $f_H(x) \in L^\infty(Q_H)$. We then extend $f_H(x)$ by using the tiling $Q_H + \alpha$, $\alpha \in \mathbb{Z}_H^m$. To see (5.8), recall (5.6) with $\Omega = Q_H$. Therefore, $f_{H,N}$ weakly converges in $L^1(Q_H)$. Now choose $g(x) = e_\beta(x)$, $\beta \in \mathbb{Z}_K^m$, in (5.9). We then get

$$\frac{1}{|Q_H|} \int_{Q_H} f_H(z) e^{-i\beta \cdot z} dz = M(e_{-\beta} \cdot f), \quad (5.11)$$

and (5.8) follows in $L^2(Q_H)$. $\square$

(5.2) actually serves as a criterion of strong admissibility.

COROLLARY 5.5. *Let $f \in L^\infty(\mathbb{R}^m)$. Suppose $f_{H,N}(x)$ satisfies (5.2). Then f is strongly admissible. That is, $Q_H(\mathbb{R}^m) \subset L_{ad}^\infty(\mathbb{R}^m)$.*

In fact, this follows from Corollary 4.8.

Proposition 5.4 may admit several interpretations. Let us remark on its relation to Poisson's summation formula. Let $c^{H,N}(x)$ be a smooth function on $\mathbb{R}^m$ such that

$$c^{H,N}(x) = \begin{cases} 1, & x \in Q_{H,N}^*, \\ 0, & x \notin Q_{H,N+1}^*, \end{cases} \quad (5.12)$$

so that $c^{H,N}(x) = c^{I,N}(H^{-1}x)$. In particular, for $x \in Q_H$,

$$c^{H,N}(x + \alpha) = \begin{cases} 1, & \alpha \in N_H^*, \\ 0, & \alpha \notin (N+1)_H^*. \end{cases}$$

Poisson's summation formula now yields

$$\sum_{\alpha \in \mathbb{Z}_H^m} c^{H,N}(x + \alpha) f(x + \alpha) = \frac{1}{(2\pi)^m} \frac{1}{|Q_H|} \sum_{\beta \in \mathbb{Z}_K^m} \int_{\mathbb{R}^m} \widehat{c^{H,N}}(\xi) \hat{f}(\beta - \xi) d\xi \cdot e^{i\beta \cdot x}$$

in $L^2(Q_H)$. If $x \in Q_H$, then

$$\begin{aligned} f_{H,N}(x) &= -\frac{1}{(2N+1)^m} \sum_{\alpha \in (N+1)_H^* \setminus N_H^*} c^{H,N}(x+\alpha) f(x+\alpha) \\ &\quad + \frac{1}{(2\pi)^m} \sum_{\beta \in \mathbb{Z}_K^m} \frac{1}{(2N+1)^m |Q_H|} \int_{\mathbb{R}^m} \widehat{c^{H,N}}(\xi) \hat{f}(\beta - \xi) d\xi \cdot e^{i\beta \cdot x}. \end{aligned} \quad (5.13)$$

Since $(2N+3)^m - (2N+1)^m \leq 2m(2N+3)^{m-1}$, the first term on the right-hand side is of order $1/N$ uniformly with respect to x and thus converges to zero as $N \rightarrow +\infty$. Now

$$\begin{aligned} &\frac{1}{(2\pi)^m} \frac{1}{(2N+1)^m |Q_H|} \int_{\mathbb{R}^m} \widehat{c^{H,N}}(\xi) \hat{f}(\beta - \xi) d\xi \\ &= \frac{1}{|G_{H,N}^*|} \int_{Q_{H,N}^*} e^{-i\beta \cdot x} f(x) dx \\ &\quad + \frac{1}{(2N+1)^m \det H} \int_{Q_{H,N+1}^* \setminus Q_{H,N}^*} c^{H,N}(x) e^{-i\beta \cdot x} f(x) dx, \end{aligned}$$

where the second term is of order $1/N$ uniformly with respect to β and the first term tends to $M(e_{-\beta} \cdot f)$ provided that the $e_{-\beta} \cdot f$ are strongly admissible. Thus, we regain essentially (5.8) to which (5.13) weakly converges in $L^2(Q_H)$ with the required uniformness as regards the translation.

Noting the boundedness of $f_{H,N}(x)$, we thus obtain the following.

PROPOSITION 5.6. *Let $f \in L^\infty(\mathbb{R}^m)$ be strongly admissible. Suppose that $e_{-\beta} \cdot f$, $\beta \in \mathbb{Z}_K^m$, are all strongly admissible. Then $f \in Q_H(\mathbb{R}^m)$ and constitutes a uniformly intimate pair with any $g \in \mathcal{P}_H(\mathbb{R}^m)$.*

6. Mean-convolution algebras

Consider a closed subset of $L_{\text{ad}}^\infty(\mathbb{R}^m)$ such that any pair consisting of its elements are uniformly intimate and their mean-convolution product again belongs to this set. Such a set forms an algebra by the ordinary sum of functions and the mean-convolution product together with the scalar multiplication. In fact, this is a consequence of the following observation.

LEMMA 6.1. *Let $f(x), g(x), h(x) \in L^\infty(\mathbb{R}^m)$. Suppose that f and h are strongly admissible, pairs f, g and g, h are uniformly intimate, and pairs $f, g \otimes h$ and $f \otimes g, h$*

are strongly intimate. Then

$$f \otimes (g \otimes h)(y) = (f \otimes g) \otimes h(y), \quad y \in \mathbb{R}^m. \quad (6.1)$$

Proof. Let y be fixed. We show that

$$F(z, z'; y) = f(y - z')g(z' - z)h(z) \in L^\infty(\mathbb{R}^m \times \mathbb{R}^m)$$

is strongly admissible, and its mean value is given by (6.1). Note that $\phi(z)\psi(z')$ is a weight function if $\phi(x), \psi(x) \in L^1(\mathbb{R}^m)$ are weight functions. Note that

$$\begin{aligned} & \iint_{\mathbb{R}^m \times \mathbb{R}^m} \phi_L(t - z)\psi_L(t' - z')F(z, z'; y) dz dz' - f \otimes (g \otimes h)(y) \\ &= \left\{ \int_{\mathbb{R}^m} \psi_L(t' - z')f(y - z')(g \otimes h)(z') dz' - f \otimes (g \otimes h)(y) \right\} \\ & \quad + \int_{\mathbb{R}^m} \psi_L(t' - z')f(y - z') \left\{ \int_{\mathbb{R}^m} \phi_L(t - z)g(z' - z)h(z) dz - g \otimes h(z') \right\} dz'. \end{aligned}$$

The first term on the right-hand side tends to zero as $L \rightarrow +\infty$ uniformly with respect to t' since f and $g \otimes h$ are strongly intimate. The second term tends to zero as $L \rightarrow +\infty$ uniformly with respect to t, t' since g and h are uniformly intimate and f is strongly admissible. This shows strong admissibility of $F(z, z'; y)$ and its mean value is $f \otimes (g \otimes h)$. Now choose $\phi(z)\psi(z - z')$ as a new weight function. We then get the mean value $(f \otimes g) \otimes h(y)$ by following a similar argument to the above. $\square$

Remark. The proof of Lemma 6.1 shows that if f and $g \otimes h$ constitute a uniformly intimate pair, then $F(z, z'; y)$ is strongly admissible uniformly with respect to y .

COROLLARY 6.2. *Let $f, g, h \in L^\infty(\mathbb{R}^m)$. Suppose f is strongly admissible, pairs g, h and f, g are uniformly intimate, and $f, g \otimes h$ constitute a uniformly intimate pair. Then $f \otimes g$ and h constitute a uniformly intimate pair and (6.1) holds.*

Proof. By the proof of Lemma 6.1, we know $F(z, z'; y)$ is strongly admissible uniformly with respect to y . Now take $L, L' > 0$ large enough. Let $\phi(x), \psi(x)$ be weight functions. Then

$$\begin{aligned} & \int_{\mathbb{R}^m} \phi_L(x - z)(f \otimes g)(y - z)h(z) dz - \int_{\mathbb{R}^m} \phi_{L'}(x - z)(f \otimes g)(y - z)h(z) dz \\ &= \iint_{\mathbb{R}^m \times \mathbb{R}^m} \phi_L(x - z)\psi_L(x' - z')f(y - z - z')g(z')h(z) dz dz' \end{aligned}$$

$$\begin{aligned}
& - \int \int_{\mathbb{R}^m \times \mathbb{R}^m} \phi_{L'}(x-z) \psi_{L'}(x'-z') f(y-z-z') g(z') h(z) dz dz' \\
& + \int_{\mathbb{R}^m} \phi_L(x-z) h(z) \\
& \times \left\{ f \otimes g(y-z) - \int_{\mathbb{R}^m} \psi_L(x'-z') f(y-z-z') g(z') dz' \right\} dz \\
& - \int_{\mathbb{R}^m} \phi_{L'}(x-z) h(z) \\
& \times \left\{ f \otimes g(y-z) - \int_{\mathbb{R}^m} \psi_{L'}(x'-z') f(y-z-z') g(z') dz' \right\} dz.
\end{aligned}$$

Here the last two terms on the right-hand side tend to zero as $L, L' \rightarrow 0$ uniformly with respect to y . On the other hand, since

$$f(y-z-z') g(z') h(z) = F(z, z+z'; y),$$

the right-hand side in total tends to zero as $L, L' \rightarrow +\infty$ uniformly with respect to y . The fact that (6.1) holds is then obvious. $\square$

That such a set is a closed subset of the Banach space $\mathbf{L}^\infty(\mathbb{R}^m)$ is due to Proposition 3.1 and Lemma 4.5. Algebraic properties are ensured by Proposition 4.7, Lemma 5.2 and (4.7). Also recall (4.5).

Here are some standard examples.

Example 6.1. Consider $\mathcal{P}_H(\mathbb{R}^m)$ with $\det H > 0$ (recall Example 1.2). Then, as we have already seen, $\mathcal{P}_H(\mathbb{R}^m)$ is a closed subset of $\mathbf{L}^\infty(\mathbb{R}^m)$ with the required algebraic property. In particular,

$$\begin{aligned}
M(f) &= \frac{1}{|Q_H|} \int_{Q_H} f(x) dx, \\
f \otimes g(x) &= \frac{1}{|Q_H|} \int_{Q_H} f(x-y) g(y) dy
\end{aligned}$$

for $f, g \in \mathcal{P}_H(\mathbb{R}^m)$. Here Q_H is the parallelotope with the volume $|Q_H| = \det H$ (recall Example 1.2).

Example 6.2. The null ideal $\mathcal{N}(\mathbb{R}^m)$ also satisfies all the requirements. See Theorem 2.3, Lemma 3.3 and Lemma 4.6. In particular, any mean-convolution product containing a factor from this ideal always vanishes.

The totality of such algebras are partially ordered by the set theoretic inclusion. Thus, by Zorn's lemma, we may talk of a maximal set which we call a *mean-convolution algebra*. Summarizing what we have so far been discussing, we state the following.

THEOREM 6.3. *A mean-convolution algebra is a commutative Banach algebra with a multiplicative continuous functional $M(\cdot)$.*

At present we have the roughest of ideas about a mean-convolution algebra.

PROPOSITION 6.4. *A mean-convolution algebra contains the constant functions and the null ideal but does not have a unit.*

In fact, constant functions belong to a mean-convolution algebra due to the latter's maximality. Similarly, because of Lemma 4.6, it contains $\mathcal{N}(\mathbb{R}^m)$. But then $f \otimes g = 0$ for whatever g so that no units can be included.

On the other hand, in spite of our requirement of maximality, we have no knowledge about how many mean-convolution algebras do exist in $L^\infty(\mathbb{R}^m)$, or whether any mean-convolution algebra contains almost periodic functions. However, there is *at least* one which contains all the almost periodic functions. Corollary 4.15 then implies that the almost periodic functions form a closed ideal in such a mean-convolution algebra. In the mean time, we make the following observations.

LEMMA 6.5. *Let*

$$e_\xi(x) = e^{ix \cdot \xi}, \quad \xi \in \mathbb{R}^m.$$

Let $f \in L^\infty(\mathbb{R}^m)$ be strongly admissible. Then the following conditions are equivalent:

- (1) *e_ξ and f constitute a strongly intimate pair;*
- (2) *e_ξ and f constitute a uniformly intimate pair;*
- (3) *$e_{-\xi} \cdot f$ is strongly admissible.*

In this situation, we have

$$e_\xi \otimes f = M(e_{-\xi} \cdot f) e_\xi. \quad (6.2)$$

Proof. Let ϕ be a weight function. Note that

$$\begin{aligned} & \left| \int_{\mathbb{R}^m} \phi_L(x-z) e_\xi(y-z) f(z) dz - \int_{\mathbb{R}^m} \phi_{L'}(x-z) e_\xi(y-z) f(z) dz \right| \\ &= \left| \int_{\mathbb{R}^m} \phi_L(x-z) (e_{-\xi} \cdot f)(z) dz - \int_{\mathbb{R}^m} \phi_{L'}(x-z) (e_{-\xi} \cdot f)(z) dz \right| \end{aligned}$$

for $L, L' > 0$ since $|e^{iy \cdot \xi}| = 1$. Equivalence of (1)–(3) is now obvious and (6.2) immediately follows. $\square$

Remark. Recall Examples 1.1 and 1.2 for such instances. Also see Proposition 5.4. We see e_ξ and e_η are uniformly intimate, and

$$e_\xi \otimes e_\eta = \begin{cases} 0, & \xi \neq \eta, \\ e_\xi, & \xi = \eta. \end{cases} \quad (6.3)$$

In passing, we also recall

$$\|e_\xi - e_\eta\|_\infty = \begin{cases} 2, & \xi \neq \eta, \\ 0, & \xi = \eta. \end{cases} \quad (6.4)$$

Note, however, that e_ξ is weak* continuous in ξ .

COROLLARY 6.6. *Let $f, g \in \mathbf{L}_{\text{ad}}^\infty(\mathbb{R}^m)$ be a uniformly intimate pair. Furthermore, suppose the uniform intimacy of pairs f, e_ξ and g, e_ξ . Then $f \otimes g$ and e_ξ constitute a uniformly intimate pair, and*

$$M(e_{-\xi} \cdot f \otimes g) = M(e_{-\xi} \cdot f)M(e_{-\xi} \cdot g). \quad (6.5)$$

In fact, by the associative law,

$$(f \otimes g) \otimes e_\xi = f \otimes (g \otimes e_\xi) = M(e_{-\xi} \cdot f)M(e_{-\xi} \cdot g)e_\xi,$$

and the left-hand side reduces to $M(e_{-\xi} \cdot f \otimes g)e_\xi$.

Remark. For a mean-convolution algebra which contains all the almost periodic functions, we can introduce a family of multiplicative functionals $M^\xi, \xi \in \mathbb{R}^m$, by $M^\xi(f) = M(e_{-\xi} \cdot f)$.

COROLLARY 6.7. *Suppose $e_{-\xi} \cdot f$ is strongly admissible. Then for any $g \in \mathbf{L}^1(\mathbb{R}^m)$, $e_{-\xi} \cdot (f * g)$ is also strongly admissible, and*

$$M(e_{-\xi} \cdot (f * g)) = \hat{g}(\xi)M(e_{-\xi} \cdot f). \quad (6.6)$$

Here

$$\hat{g}(\xi) = \int_{\mathbb{R}^m} e^{-ix \cdot \xi} g(x) dx$$

is the Fourier transform.

Proof. Immediate. $\square$

Remark. If g is that of Proposition 2.4, then

$$M(e_{-\xi} \cdot f * g) = i\omega \cdot \xi M(e_{-\xi} \cdot f) \hat{h}(\xi).$$

Let $\mathcal{Q}(\mathbb{R}^m)$ be the totality of the functions $f \in \mathbf{L}^\infty(\mathbb{R}^m)$ such that

$$e_{-\xi} \cdot f \in \mathbf{L}_{\text{ad}}^\infty(\mathbb{R}^m) \quad \text{for any } \xi \in \mathbb{R}^m.$$

Here recall that $e_{-\xi}(x) = e^{-i\xi \cdot x}$, $x \in \mathbb{R}^m$. Obviously, $\mathcal{Q}(\mathbb{R}^m)$ is a closed subset of $\mathbf{L}^\infty(\mathbb{R}^m)$. Actually, Corollary 5.5, Proposition 5.6 and Lemma 6.5 imply the following.

PROPOSITION 6.8. *We have*

$$\mathcal{Q}(\mathbb{R}^m) = \bigcap_{H \in \mathbf{GL}(m, \mathbb{R})} \mathcal{Q}_H(\mathbb{R}^m). \quad (6.7)$$

$\mathcal{Q}(\mathbb{R}^m)$ is invariant under translation, dilation, inversion and any non-degenerate linear transformation, i.e.

$$\mathcal{T}_y f, \mathcal{D}_d f, \mathcal{I} f \text{ and } \mathcal{L}_G f \in \mathcal{Q}(\mathbb{R}^m)$$

for $f \in \mathcal{Q}(\mathbb{R}^m)$, where $y \in \mathbb{R}^m$, $d > 0$ and $G \in \mathbf{GL}(m, \mathbb{R})$ (see (1.15)–(1.18)).

$\mathcal{Q}(\mathbb{R}^m)$ is not very large. To see this, let us introduce the mean spectrum $\sigma^\mu(f)$ of $f \in \mathcal{Q}(\mathbb{R}^m)$ by the closure of

$$\xi \in \mathbb{R}^m, \quad M(e_{-\xi} \cdot f) \neq 0. \quad (6.8)$$

Recall the weak* spectrum $\sigma^*(f)$ (see (2.12)). Note that $ce_\xi \in W^*(f)$ if and only if $c \in W^*(e_{-\xi} \cdot f)$. Thus, $\xi \in \sigma^*(f)$ if and only if $0 \in \sigma^*(e_{-\xi} \cdot f)$. By Proposition 2.9, $0 \in \sigma^*(f)$ if $M_\phi(f) \neq 0$ for some ϕ , and $0 \notin \sigma^*(f)$ and $f \in \mathbf{L}_{\phi\text{-ad}}^\infty(\mathbb{R}^m)$ imply $M(f) = 0$. However, $0 \in \sigma^*(f)$ does not necessarily imply $M(f) \neq 0$; $M(f) = 0$ does not necessarily imply $0 \notin \sigma^*(f)$. In any event, we have $M(e_{-\xi} \cdot f)e_\xi \in W^*(f)$ and

$$\sigma^\mu(f) \subset \sigma^*(f) \quad (6.9)$$

for $f \in \mathcal{Q}(\mathbb{R}^m)$.

PROPOSITION 6.9. *Let $f(x) \in \mathcal{Q}(\mathbb{R}^m)$. Then its mean spectrum $\sigma^\mu(f)$ is a null set. Furthermore, when f and $g(x) \in \mathcal{Q}(\mathbb{R}^m)$ are uniformly intimate, then*

$$\sigma^\mu(f) \cap \sigma^\mu(g) = \sigma^\mu(f \otimes g). \quad (6.10)$$

In fact, Lemma 6.5 implies that

$$\int_{\mathbb{R}^m} M(e_{-\xi} \cdot f) v(\xi) d\xi = M(\hat{v} \cdot f) = 0$$

for any $v(\xi) \in \mathcal{S}(\mathbb{R}^m)$. (6.10) is an immediate consequence of (6.8) and (6.5).

Our real concern is in fact to know whether $\mathcal{Q}(\mathbb{R}^m)$ is a mean-convolution algebra; for the time being we only have fragmentary observations. For instance, we have the following.

PROPOSITION 6.10. *Any mean-convolution algebra within $\mathcal{Q}(\mathbb{R}^m)$ contains all the bounded periodic functions and hence, in particular, the almost periodic functions.*

In fact, this follows from Lemma 5.3, Corollary 6.6 and Proposition 6.4 together with the maximality of a mean-convolution algebra.

A reformulation of Propositions 5.4 and 5.6 in the present context now reads as follows.

THEOREM 6.11. *Let $\mathcal{M}$ be a mean-convolution algebra which contains all the bounded periodic functions. Then $\mathcal{M} \subset \mathcal{Q}(\mathbb{R}^m)$. For any $H \in \mathbf{GL}(m, \mathbb{R})$, the mapping*

$$\mathcal{M} \ni f \mapsto f_H \in \mathcal{P}_H(\mathbb{R}^m) \quad (6.11)$$

is a homomorphism such that

$$\|f_H\|_{H,\infty} \leq \|f\|_H, \quad (6.12)$$

$$(f \otimes g)_H = f_H \otimes g_H, \quad (6.13)$$

$$M(f) = M(f_H) \quad (6.14)$$

for $f, g \in \mathcal{M}$. Here $\|\cdot\|_{H,\infty}$ is the norm of $\mathbf{L}^\infty(Q_H)$.

On the other hand, we are still unable to determine the subset of $\mathcal{Q}(\mathbb{R}^m)$ consisting of those f with the empty mean spectra, $\sigma^\mu(f) = \emptyset$. Call this subset $\mathcal{R}(\mathbb{R}^m)$.

LEMMA 6.12. *$\mathcal{R}(\mathbb{R}^m)$ is a closed subset of $\mathcal{Q}(\mathbb{R}^m)$ and contains $\mathcal{N}(\mathbb{R}^m)$. For any mean-convolution algebra $\mathcal{M}$ in $\mathcal{Q}(\mathbb{R}^m)$, $\mathcal{R}(\mathbb{R}^m) \cap \mathcal{M}$ is an ideal.*

The last statement follows from Corollary 6.6.

Remark. (6.10) implies that $f \otimes g \in \mathcal{R}(\mathbb{R}^m)$ if $\sigma^\mu(f) \cap \sigma^\mu(g) = \emptyset$ and the mean-convolution product makes sense.

The importance of $\mathcal{R}(\mathbb{R}^m)$ is clear by the following fact.

PROPOSITION 6.13. *Let μ be a non-trivial multiplicative linear functional on a mean-convolution algebra $\mathcal{M}$ contained in $\mathcal{Q}(\mathbb{R}^m)$. Then there is a $\xi_\mu \in \mathbb{R}^m$ such that*

$$\mu(e_\xi) = \begin{cases} 1, & \xi = \xi_\mu, \\ 0, & \xi \neq \xi_\mu, \end{cases} \quad (6.15)$$

and

$$\mu(e_\xi \otimes f) = \mu(e_\xi)M(e_{-\xi} \cdot f), \quad f \in \mathcal{M}. \quad (6.16)$$

In fact, (6.15) follows from (6.3) and (6.16) follows from (6.2).

Remark. When $\mu = M^\xi$, $\xi_\mu = \xi$ (see the remark following Corollary 6.6).

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