ON THE CORANK OF A WAHL MAP ON SMOOTH CURVES WITH A PLANE MODEL

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1. Introduction

1.1. Definition of a Wahl map. For a smooth curve C and its canonical line bundle ω_C , we call the map

$$\begin{aligned}\Phi : \wedge^2 H^0(C, \omega) &\rightarrow H^0(C, \Omega_C^1 \otimes \omega^{\otimes 2}) \\ \sigma \wedge \tau &\mapsto \sigma d\tau - \tau d\sigma\end{aligned}$$

a Wahl map. Wahl [W2] gives an interpretation of $(\text{Coker } \Phi)^*$ in the deformation theory of the cone over the canonical curve in $\mathbb{P}^{g-1}$; he shows that if Φ is surjective then the cone over the canonical line bundle does not smooth to a K-3 surface of degree $2g - 2$ in $\mathbb{P}^g$. The behavior of this map depends only on the geometry of C . One can stratify the moduli space $\mathcal{M}_g$ by the rank of the Wahl map, which is well behaved since the rank is semicontinuous ([CM]). Since good geometric interpretations are given on $\text{Coker } \Phi$ rather than $\text{Im } \Phi$, we usually talk about cork Φ , the corank of the Wahl map, rather than the rank of Φ .

1.2. There is an interesting relationship between the stratification of $\mathcal{M}_g$ by the corank of a Wahl map and other more classical loci, which is cut out by the gonality of the curve. The locus of hyperelliptic curves in $\mathcal{M}_g$ coincides with the locus of curves having cork $\Phi = 3g - 2$ ([W3]), and the locus of trigonal curves in $\mathcal{M}_g$ coincides with the locus of curves having cork $\Phi = g + 5$. ([B])

The following are well known propositions for Wahl maps.

PROPOSITION 1.3. ([B]) *If C is 4-gonal, with $g \geq 7$, then cork $\Phi \geq 9$.*

PROPOSITION 1.4. ([W3]) *If C is a smooth plane curve of degree ≥ 5 , then cork $\Phi = 10$. If a smooth curve C lies on a rational ruled surface $\mathbb{F}_n$ and $g \geq 5$, then cork $\Phi \geq 9$.*

As we have seen Proposition 1.4, the corank of a Wahl map of a smooth curve can be bounded if C lies on some special surface.

On the other hand, every smooth curve admits a plane model which has only ordinary multiple points as its singularities. In that case we can find an explicit basis for the global sections of canonical line bundle as polynomial functions of affine coordinates x, y . Hence we can calculate the corank of a Wahl map explicitly. By Proposition 1.4 we know that if a smooth curve admits a smooth plane model then $\text{cork } \Phi$ is 10. We expect that if a plane model admits one or two singular points then $\text{cork } \Phi$ is almost determined by its singularities.

In this paper we show the following.

THEOREM A. *Let $r \geq 2$ and let C be a smooth curve whose plane model is of degree m and admits one ordinary r -fold point as its only singularity. If $m \geq r + 4$, then the corank of a Wahl map is 9.*

THEOREM B. *Let $r \geq 2$ and let C be a smooth curve whose plane model is of degree m and admits one node and one ordinary r -fold point as its only singularities. Then:*

- (i) *for the case $m \geq r + 6$, the corank of a Wahl map is 8;*
- (ii) *for the case $m = r + 5$, the corank of a Wahl map is 8 if $r = 2$, and 7 if $r \geq 3$;*
- (iii) *for the case $m = r + 4$, the corank of a Wahl map is 9.*

1.5. Notation. When C is a plane curve with only ordinary singularities, we use the same notation C for its smooth model and we will also use a Wahl map for a plane curve C to mean a Wahl map for its smooth model.

2. Proof of Theorem A

2.1. Definition. Let $V(m; r)$ be the set of irreducible plane curves of degree m with one ordinary r -fold point as its only singularity.

2.2. If $C \in V(m; r)$, then we may assume that C takes its singularity at the origin $P(0, 0, 1)$ and C meets the line at infinity at m distinct points.

2.3. The equation of $C \in V(m; r)$ in affine coordinates is given as

$$F(x, y) = \sum_{i+j=r}^m a_{i,j} x^i y^j = F_m + F_{m-1} + \cdots + F_r,$$

where each F_k is a part of a homogeneous polynomial of degree k . By 2.2, F_m and F_r separate to all distinct linear factors. To simplify notation, we write a_i and a'_j

instead of $a_{i,m-i}$ and $a_{j,r-j}$ respectively. The genus of C is

$$g(C) = \frac{(m-1)(m-2)}{2} - \frac{r(r-1)}{2}$$

and the global sections of the canonical line bundle are given as

$$H^0(C, \omega_C) = \left\{ \sum_{r-1 \leq i+j \leq m-3} d_{i,j} x^i y^j \mid d_{i,j} \in \mathbb{C}, i, j \geq 0 \right\}.$$

We choose a basis $\{x^i y^j \mid r-1 \leq i+j \leq m-3, i, j \geq 0\}$ of $H^0(C, \omega_C)$ and consider the image of a Wahl map expressed explicitly with respect to this basis.

LEMMA 2.4. *For $C \in V(m; r)$ with $m-r \geq 3$, $\text{Im } \Phi$ is generated by the following elements of 4 types:*

$$\begin{aligned} G_1 &= \{Y_{i,j} = x^i y^j dy \mid 2r-2 \leq i+j \leq 2m-8, i \geq 0, j \geq 0\} \\ G_2 &= \{X_{i,j} = x^i y^j dx \mid 2r-2 \leq i+j \leq 2m-8, i \geq 0, j \geq 0\} \\ G_3 &= \{\omega_i = x^i y^j (y dx - x dy) \mid i+j = 2m-8, 0 \leq i \leq 2m-8\} \\ G_4 &= \{\tau_i = x^i y^j (y dx - x dy) \mid i+j = 2r-4, 0 \leq i \leq 2r-4\}. \end{aligned}$$

Proof. Consider

$$\begin{aligned} \Phi(x^i y^j \wedge x^k y^l) &= (k-i)x^{k+i-1}y^{l+j}dx - (j-l)x^{k+i}y^{l+j-1}dy, \\ r-1 &\leq i+j, \quad k+l \leq m-3. \end{aligned} \tag{2.4.1}$$

Case 1. If $i+j = k+l = m-3$, then (2.4.1) becomes

$$\Phi(x^i y^j \wedge x^k y^l) = (k-i)x^{k+i-1}y^{2m-7-k-i}(y dx - x dy).$$

Choose $0 \leq i < k \leq m-3$. Then $k-i \neq 0$ and $k+i$ can take every value from 1 to $2m-7$, thus we obtain G_3 which is the set of elements of the highest degree $2m-7$ in $\text{Im } \Phi$.

Case 2. Similar to case 1, if $i+j = k+l = r-1$, then

$$\Phi(x^i y^j \wedge x^k y^l) = (k-i)x^{k+i-1}y^{2r-3-k-i}(y dx - x dy), \quad 0 \leq i < k \leq r-1$$

are elements of G_4 which are of the lowest degree $2r-3$ in $\text{Im } \Phi$.

Case 3. Consider the remaining cases. For any given pair $\{(i, j), (k, l) \mid 1 \leq i + k, j + l\}$, we can choose another pair $\{(i', j'), (k', l') \mid 1 \leq i' + k', j' + l'\}$ satisfying the two conditions

$$\begin{aligned} i + k &= i' + k', & j + l &= j' + l' \\ (k - i) : (j - l) &\neq (k' - i') : (j' - l'). \end{aligned} \quad (2.4.2)$$

In fact, at least one pair satisfies (2.4.2) so we can choose $((i', j'), (k', l'))$ as one of $\{((i+1, j), (k-1, l)), ((i-1, j), (k+1, l)), ((i, j+1), (k, l-1)), ((i, j-1), (k, l+1))\}$.

By (2.4.1), $\alpha \Phi(x^i y^j \wedge x^k y^l) + \beta \Phi(x^{i'} y^{j'} \wedge x^{k'} y^{l'})$ for some $\alpha, \beta \in \mathbb{C}$ can separate each term $x^{k+i-1} y^{j+l} dx$ and $x^{k+i} y^{j+l-1} dy$. We obtain $x^{\alpha-1} y^\beta dx$ and $x^\alpha y^{\beta-1} dy$ for $2r-1 \leq \alpha + \beta \leq 2m-7$, with $\alpha, \beta \geq 1$.

If $i = k = 0$ or $j = l = 0$, we get $x^\alpha dx$ and $y^\alpha dy$ for $2r-2 \leq \alpha \leq 2m-8$. $\square$

2.5. *Notation.* (i) In this section, S_i denotes the linear span of G_i for $1 \leq i \leq 4$.

(ii) $\text{Gen } \Phi$ denotes the union of G_1, G_2, G_3 and G_4 in Lemma 2.4.

(iii) $\# \text{Gen}$ denotes the number of elements in $\text{Gen } \Phi$.

2.6. Thus for $C \in V(m; r)$, $\# \text{Gen} = 4m^2 - 24m - 4r^2 + 8r + 30$. All elements of $\text{Gen } \Phi$ are not linearly independent so we want to find all relations among $\text{Gen } \Phi$; the number of relations will be denoted by $\# \text{Rel}$.

2.7. Let $g_m(t) = F_m(t, 1) = \sum_{k=0}^m a_k t^k$ and consider its resultant $R(m)$. The first $(m-1)$ rows and the last m rows will be denoted by A_i and C_j respectively.

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_m \\ & a_0 & a_1 & \cdots & a_m \\ & & \cdots & \cdots & \cdots \\ & & & a_0 & a_1 & \cdots & a_m \\ a_1 & 2a_2 & \cdots & ma_m \\ & a_1 & 2a_2 & \cdots & ma_m \\ & & \cdots & \cdots & \cdots & \cdots \\ & & & a_1 & 2a_2 & \cdots & ma_m \end{vmatrix} = \begin{vmatrix} A_1 \\ A_2 \\ \vdots \\ A_{m-1} \\ C_1 \\ C_2 \\ \vdots \\ C_m \end{vmatrix}. \quad (2.7.1)$$

By 2.2, $g_m(t)$ has all distinct roots so $R(m)$ is not zero. Thus any matrix of the form $(A_1, A_2, \dots, A_k, C_1, C_2, \dots, C_l)^t$ takes its maximal rank $k+l$ if $k \leq (m-1)$ and $l \leq m$, where the superscript t denotes the transpose of a matrix.

2.8. *Remark.* To calculate $\text{cork } \Phi$, we need to find all relations on $\text{Gen } \Phi$. We have two basic relations: $F = 0$, which gives the relation between x and y , and $dF = 0$, which gives the relation between dx and dy . Consider the following equations (R1), (R2) and (R3) and their linear combinations.

$$\begin{aligned} \text{(R1)} \quad & x^i y^j F dx = 0 \\ \text{(R2)} \quad & x^i y^j F dy = 0 \\ \text{(R3)} \quad & x^i y^j dF = 0. \end{aligned} \tag{2.8.1}$$

We focus on the degree of elements of $\text{Gen } \Phi$. Roughly speaking, we generate some elements of $\text{Gen } \Phi$ step by step by using other elements of $\text{Gen } \Phi$ having the same or lower degree.

2.9. If $m \geq r + 5$, then the maximal difference of degrees of elements of $\text{Im } \Phi$ is $(2m - 7 - (2r - 3) + 1) = (2m - 2r - 3)$ and the maximal difference of degrees of monomials of F is $(m - r + 1)$ so $(2m - 2r - 3) - (m - r + 1) = (m - r - 4) \geq 1$.

LEMMA 2.10. *For $C \in V(m; r)$ with $m \geq r + 5$, $(2m - 12)$ elements of G_3 can be generated by the other elements of $\text{Gen } \Phi$, except elements of G_4 .*

Proof. Consider (2.8.1). Then any relation on $\text{Gen } \Phi$ which contains some ω_i is obtained from the following. Consider

$$(1) \quad x^i y^j F(y dx - x dy) = 0, \quad i + j = m - 8, \quad i \geq 0.$$

For any (k, l) with $r \leq k + l \leq m$, any monomial appearing in $x^{k+i} y^{l+j} F(y dx - x dy)$ has degree between $2r - 3$ and $2m - 6$, so (1) becomes

$$\begin{aligned} 0 &= x^i y^{m-8-i} \left(\sum_{r \leq k+l \leq m} a_{k,l} x^k y^l \right) (y dx - x dy) \\ &= \sum_{0 \leq k \leq m} a_{k, m-k} x^{k+i} y^{2m-8-k-i} (y dx - x dy) + s_1 + s_2 \\ &= \sum_{0 \leq k \leq m} a_k \omega_{k+i} + s_1 + s_2, \quad \exists s_j \in S_j, \quad 0 \leq i \leq m - 8. \end{aligned}$$

Consider

$$(2) \quad x^i y^j (mF dx - x dF) = 0, \quad i + j = m - 7, \quad i \geq 0.$$

We have

$$\begin{aligned}
 0 &= x^i y^{m-7-i} (mF dx - x dF) \\
 &= \sum_{0 \leq k \leq m} (m-k) a_{k,m-k} x^{k+i} y^{2m-7-k-i} dx - (m-k) a_{k,m-k} x^{k+i+1} y^{2m-8-k-i} dy \\
 &\quad + \sum_{r \leq k+l \leq m-1} (m-k) a_{k,l} x^{k+i} y^{l+j} dx + \sum_{r \leq k+l \leq m-1} (-l) a_{k,l} x^{k+i+1} y^{j+l-1} dy \\
 &= \sum_{0 \leq k \leq m} (m-k) a_k \omega_{k+i} + s_1 + s_2, \quad \exists s_j \in S_j.
 \end{aligned}$$

Consider

$$(3) \quad x^i y^j (y dF - mF dy) = 0, \quad i + j = m - 7, \quad i \geq 0.$$

Similarly, we have

$$\begin{aligned}
 0 &= x^i y^{m-7-i} (y dF - mF dy) \\
 &= \sum_{0 \leq k \leq m} k a_k x^{k+i-1} y^{2m-7-k-i} (y dx - x dy) \\
 &\quad + \sum_{r \leq k+l \leq m-1} k a_{k,l} x^{k+i-1} y^{m-7-i+l+1} dx + \sum_{r \leq k+l \leq m-1} (l-m) a_{k,l} x^{k+i} y^{m-7-i+l} dy \\
 &= \sum_{0 \leq k \leq m} k a_k \omega_{k+i-1} + s_1 + s_2, \quad \exists s_j \in S_j.
 \end{aligned}$$

Summing up, we have $2(m-6) + m - 7 = 3m - 19$ equations:

$$\begin{aligned}
 \sum_{0 \leq k \leq m} a_k \omega_{k+i} + s_1 + s_2 &= 0, \quad \exists s_j \in S_j, \quad 0 \leq i \leq m-8 \\
 \sum_{0 \leq k \leq m} (m-k) a_k \omega_{k+i} + s_1 + s_2 &= 0, \quad \exists s_j \in S_j, \quad 0 \leq i \leq m-7 \\
 \sum_{0 \leq k \leq m} k a_k \omega_{k+i-1} + s_1 + s_2 &= 0, \quad \exists s_j \in S_j, \quad 0 \leq i \leq m-7. \quad (2.10.1)
 \end{aligned}$$

Consider the matrix whose i th column consists of coefficients of the ω_{i-1} in

(2.10.1) and denote each row by A_i , B_j and C_j , where $1 \leq i \leq m-7$, $1 \leq j \leq m-6$:

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_m & & & \\ & a_0 & a_1 & \cdots & a_m & & \\ & & \cdots & \cdots & \cdots & & \\ & & & a_0 & a_1 & \cdots & a_m \\ ma_0 & (m-1)a_1 & \cdots & a_{m-1} & & & \\ & ma_0 & (m-1)a_1 & \cdots & a_{m-1} & & \\ & & \cdots & \cdots & \cdots & & \\ & & & ma_0 & (m-1)a_1 & \cdots & a_{m-1} \\ a_1 & 2a_2 & \cdots & ma_m & & & \\ & a_1 & 2a_2 & \cdots & ma_m & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & a_1 & 2a_2 & \cdots & ma_m \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{m-7} \\ B_1 \\ B_2 \\ \vdots \\ B_{m-6} \\ C_1 \\ C_2 \\ \vdots \\ C_{m-6} \end{pmatrix}.$$

We have $B_i + C_{i+1} = mA_i$ for $1 \leq i \leq m-7$ so we drop the first $(m-7)$ rows. Consider the remaining $(2m-12) \times (2m-7)$ matrix. Let's denote it by $M = M_{2m-7}(C)$. If $\text{rank}(M) = s$ it means s elements of G_3 can be generated by other elements of $\text{Gen } \Phi$. By 2.7, $\text{rank}(M) = 2m-12$. $\square$

LEMMA 2.11. For $C \in V(m; r)$ with $m \geq r+5$, $(2r-4)$ elements of G_4 can be generated by other elements of $\text{Gen } \Phi$ except elements of G_3 .

Proof. The proof is similar to that of Lemma 2.10 except that we focus on the lowest degree term τ_i . Consider the equations

- (1) $x^i y^j F(y dx - x dy) = 0, \quad i+j = r-4, \quad i \geq 0$
- (2) $x^i y^j (rF dx - x dF) = 0, \quad i+j = r-3, \quad i \geq 0$
- (3) $x^i y^j (y dF - rF dy) = 0, \quad i+j = r-3, \quad i \geq 0.$

We get the following relations containing elements of G_4 :

$$\begin{aligned} \sum_{0 \leq k \leq r} a'_k \tau_{k+i} + s_1 + s_2 &= 0, \quad \exists s_j \in S_j, \quad 0 \leq i \leq r-4 \\ \sum_{0 \leq k \leq r} (r-k) a'_k \tau_{k+i} + s_1 + s_2 &= 0, \quad \exists s_j \in S_j, \quad 0 \leq i \leq r-3 \quad (2.11.1) \\ \sum_{0 \leq k \leq r} k a'_k \tau_{k+i-1} + s_1 + s_2 &= 0, \quad \exists s_j \in S_j, \quad 0 \leq i \leq r-3. \end{aligned}$$

Consider the matrix whose i th column consists of coefficients of the τ_{i-1} in (2.11.1) and denote each row by A_i , B_j and C_j , where $1 \leq i \leq r-3$, $1 \leq j \leq r-2$:

$$\begin{pmatrix} a'_0 & a'_1 & \cdots & a'_r & & & \\ & a'_0 & a'_1 & \cdots & a'_r & & \\ & & \cdots & \cdots & \cdots & & \\ & & & a'_0 & a'_1 & \cdots & a'_r \\ ra'_0 & (r-1)a'_1 & \cdots & a'_{r-1} & & & \\ & ra'_0 & (r-1)a'_1 & \cdots & a'_{r-1} & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & ra'_0 & (r-1)a'_1 & \cdots & a'_{r-1} \\ a'_1 & 2a'_2 & \cdots & ra'_r & & & \\ & a'_1 & 2a'_2 & \cdots & ra'_r & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & a'_1 & 2a'_2 & \cdots & ra'_r \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{r-3} \\ B_1 \\ B_2 \\ \vdots \\ B_{r-2} \\ C_1 \\ C_2 \\ \vdots \\ C_{r-2} \end{pmatrix}.$$

A similar argument for the rank of this matrix as used in the proof of Lemma 2.10 proves this lemma. $\square$

In Lemmas 2.10 and 2.11 we have found all relations on $\text{Gen } \Phi$ containing elements of $G_3 \cup G_4$.

LEMMA 2.12.

- (i) For $C \in V(m; r)$ with $m \geq r + 6$, $(\sum_{n=r-2}^{m-8} 3n + 4)$ number of elements of $G_1 \cup G_2$ can be generated by other elements of $G_1 \cup G_2$.
- (ii) If $m = r + 5$ or $m = r + 4$, then there is no relation on $\text{Gen } \Phi$ consisting only of elements of $G_1 \cup G_2$.

Proof. (i) For any $r - 2 \leq n \leq m - 8$, consider

$$\begin{aligned} (1) \quad & x^i y^j F dy = 0, \quad i + j = n, \quad i \geq 0 \\ (2) \quad & x^i y^j F dx = 0, \quad i + j = n, \quad i \geq 0 \\ (3) \quad & x^{i+1} y^j dF = 0, \quad i + j = n, \quad i \geq 0 \\ (4) \quad & y^{n+1} dF = 0. \end{aligned} \tag{2.12.1}$$

Because of the range of n , all monomials in (2.12.1) have their degree between

$(2r - 3)$ and $(2m - 7)$. For $0 \leq i \leq n$ we have the relations

$$\begin{aligned}
 (1) \quad & x^i y^{n-i} \left(\sum_{r \leq k+l \leq m} a_{k,l} x^k y^l \right) dy = 0 \\
 & \Rightarrow \sum_{0 \leq k \leq m} a_k Y_{k+i, m+n-k-i} + s_1 = 0, \quad \exists s_1 \in S_1 \\
 (2) \quad & \sum_{0 \leq k \leq m} a_k X_{k+i, m+n-k-i} + s_2 = 0, \quad \exists s_2 \in S_2 \\
 (3) \quad & x^{i+1} y^{n-i} \left(\sum_{r \leq k+l \leq m} k a_{k,l} x^{k-1} y^l dx + \sum_{r \leq k+l \leq m} l a_{k,l} x^k y^{l-1} dy \right) = 0 \\
 & \Rightarrow \sum_{r \leq k+l \leq m} k a_{k,l} x^{k+i} y^{n-i+l} dx + \sum_{r \leq k+l \leq m} l a_{k,l} x^{k+i+1} y^{n-i+l-1} dy = 0 \\
 & \Rightarrow \sum_{0 \leq k \leq m} k a_k X_{k+i, m+n-k-i} + \sum_{0 \leq k \leq m} (m-k) a_k Y_{k+i+1, m+n-k-i-1} \\
 & \quad + s_1 + s_2 = 0, \quad \exists s_j \in S_j \\
 (4) \quad & y^{n+1} (dF) = 0 \\
 & \Rightarrow \sum_{r \leq k+l \leq m} k a_{k,l} x^{k-1} y^{n+1+l} dx + \sum_{r \leq k+l \leq m} l a_{k,l} x^k y^{n+l} dy = 0 \\
 & \Rightarrow \sum_{0 \leq k \leq m} k a_k X_{k-1, m+n-k+1} + \sum_{0 \leq k \leq m} (m-k) a_k Y_{k, m+n-k} \\
 & \quad + s_1 + s_2 = 0, \quad \exists s_j \in S_j.
 \end{aligned}$$

Here s_1 (resp. s_2) is a linear combination of elements of G_1 (resp. G_2) whose degrees are less than $m+n$. Considering all the coefficients of the highest degree ($= m+n$) terms, we have a $(2(n+1) + n+2) \times 2(m+n+1)$ matrix $M_{m+n}(C)$ for each n ,

where $d_i = (m - i)a_i$ and $e_i = ia_i$ with $0 \leq i \leq m$ for short:

$$\left(\begin{array}{cccccccccccccccc} a_0 & a_1 & \cdots & a_m & & & & & & & & & & & & & \\ & a_0 & a_1 & \cdots & a_m & & & & & & & & & & & & \\ & & \cdots & \cdots & \cdots & & & & & & & & & & & & \\ & & & a_0 & a_1 & \cdots & a_m & & & & & & & & & & \\ & & & & & & & a_0 & a_1 & \cdots & a_m & & & & & & \\ & & & & & & & & a_0 & a_1 & \cdots & a_m & & & & & \\ & & & & & & & & \cdots & \cdots & \cdots & & & & & & \\ & & & & & & & & & a_0 & a_1 & \cdots & a_m & & & & \\ & d_0 & d_1 & \cdots & d_{m-1} & & & e_1 & e_2 & \cdots & e_m & & & & & & \\ & & d_0 & d_1 & \cdots & d_{m-1} & & & e_1 & e_2 & \cdots & e_m & & & & & \\ & & & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & & & & & \\ & & & & d_0 & d_1 & \cdots & d_{m-1} & & e_1 & e_2 & \cdots & e_m & & & & \end{array} \right) \begin{array}{l} : (1)i = 0 \\ : (1)i = 1 \\ \vdots \\ : (1)i = n \\ : (2)i = 0 \\ : (2)i = 1 \\ \vdots \\ : (2)i = n \\ : (4) \\ : (3)i = 0 \\ \vdots \\ : (3)i = n \end{array}$$

We claim that for any curve C , the rank of the corresponding matrix $M_{m+n}(C)$ will be maximal. By considering the two matrices $M_{m+n}(C)$ and $M_{2m-7}(C)$, it can be shown easily that if $\text{rank}(M_{m+n}(C))$ is less than the possible maximal value $3n + 4$, then $\text{rank}(M_{2m-7}(C))$ is less than $2m - 12$. Since we show that $M_{2m-7}(C)$ takes its maximal rank, $\text{rank}(M_{m+n}(C))$ is $3n + 4$ for each n .

(ii) When $m \leq r + 5$, there is no relation because of the degree condition. $\square$

In Lemmas 2.10, 2.11 and 2.12 we obtain all possible relations between elements of $\text{Gen } \Phi$.

Proof of Theorem A. For a curve $C \in V(m; r)$ with $m \geq r + 6$,

$$\begin{aligned} \# \text{Rel} &= \left(\sum_{n=r-2}^{m-8} \text{rank}(M_{m+n}) \right) + \text{rank}(M_{2m-7}) + \text{rank}(M_{2r-3}) \\ &= \left(\sum_{n=r-2}^{m-8} (3n + 4) \right) + (2m - 12) + (2r - 4) \\ &= \frac{3m^2 - 3r^2 - 33m + 11r + 78}{2}, \end{aligned}$$

$$\text{cork } \Phi = 5g - 5 - (4m^2 - 24m - 4r^2 + 8r + 30) + \# \text{Rel} = 9.$$

For the case $m = r + 5$, $\# \text{Rel} = (2m - 12) + (2r - 4)$ and $\text{cork } \Phi = 9$ again.

For the case $m = r + 4$, the maximal difference of degrees of elements of $\text{Im } \Phi$ is $(2m - 7 - (2r - 3) + 1) = (2m - 2r - 3) = 5$ and the maximal difference of degrees of monomials of F is $(m - r + 1) = 5$, so $(2m - 2r - 3) = (m - r + 1)$. By the degree condition, any possible relation on $\text{Gen } \Phi$ contains both elements of G_3 and G_4 simultaneously. The equation

$$x^i y^j F(y dx - x dy) = 0, \quad i + j = m - 8 = r - 4, \quad i \geq 0$$

gives relations on $\text{Gen } \Phi$ such as

$$\sum_{0 \leq k \leq m} a_k \omega_{k+i} + \sum_{0 \leq k \leq r} a'_k \tau_{k+i} + s_1 + s_2 = 0, \quad \exists s_j \in S_j, \quad 0 \leq i \leq m - 8 = r - 4.$$

However, except for these, other equations appearing in the proofs of Lemmas 2.10 and 2.11 cannot make relations on $\text{Gen } \Phi$. In fact, consider $x^i y^{m-8-i} (mF dy - y dF) = 0$:

$$\begin{aligned} & x^i y^{m-8-i} (mF dy - y dF) \\ &= x^i y^{r-4-i} (mF dy - y dF) \\ &= x^i y^{r-4-i} \left(\sum_{r \leq k+l \leq m} (m-l) a_{k,l} x^k y^l dy - \sum_{r \leq k+l \leq m} k a_{k,l} x^{k-1} y^{l+1} dx \right) \\ &= x^i y^{r-4-i} \left((m-r) \sum_{k=0}^r a'_k x^k y^l dy - \sum_{k=0}^r k a'_k x^{k-1} y^{r-k} (y dx - x dy) \right) \\ &+ s_1 + s_2 + s_3 = 0, \quad \exists s_j \in S_j. \end{aligned}$$

The remaining lowest degree terms, except elements of G_4 , only disappear when all the a'_k are zeros. This is impossible so $x^i y^{m-8-i} (mF dy - y dF) = 0$ cannot make any relation on $\text{Gen } \Phi$. The same consideration shows that there are no more relations on $\text{Gen } \Phi$ obtained from $x^i y^{r-4-i} (rF dy - y dF) = 0$, $x^i y^{m-8-i} (mF dx - x dF) = 0$ or $x^i y^{r-4-i} (rF dx - x dF) = 0$.

Now $\# \text{Rel} = m - 7$, $\# \text{Gen} = 16m - 66$ and $g = 3m - 9$. Thus

$$\text{cork } \Phi = 5(3m - 9 - 1) - (16m - 66) + (m - 7) = 9.$$

□

2.13. Remark. Let $r \geq 2$ and $m = r + 3$. If C is a smooth curve whose plane model is of degree m and admits one ordinary r -fold point as its only singularity, then $\text{cork } \Phi = g + 5$. In fact, C is a trigonal curve and this result coincides with Brawner's result [B] that every trigonal curve has $\text{cork } \Phi = g + 5$.

Proof of Remark 2.13. The maximal difference of degrees of elements of $\text{Im } \Phi$ is $(2m - 7 - (2r - 3) + 1) = (2m - 2r - 3) = 3$ and the maximal difference of degrees of monomials of F is $(m - r + 1) = 4$, so $(2m - 2r - 3) < (m - r + 1)$.

By the degree condition, any equation obtained from (2.8.1) cannot give any relation on $\text{Gen } \Phi$. So $\# \text{Rel} = 0$, $\# \text{Gen} = 8m - 30$, $g = 2m - 5$ and

$$\text{cork } \Phi = 5(2m - 5 - 1) - (8m - 30) = 2m = g + 5.$$

□

2.14. Remark. Let $r \geq 2$ and $m = r + 2$. If C is a smooth curve whose plane model is of degree m and admits one ordinary r -fold point as its only singularity, then $\text{cork } \Phi = 3g - 2$. In fact, C is a hyperelliptic curve and this result coincides with Wahl's result [W3] that every hyperelliptic curve has $\text{cork } \Phi = 3g - 2$. Its proof is similar to that of 2.13.

3. Proof of Theorem B

3.1. Definition. Let $V(m; r, 2)$ be the set of irreducible plane curves of degree m with one ordinary r -fold point and one node as its only singularity.

3.2. The proof of Theorem B is similar to that of Theorem A. In this section we may assume that $C \in V(m; r, 2)$ takes its r -fold point at $P(0, 0, 1)$, its node at $Q(0, 1, 1)$ and C meets the line at infinity at m distinct points after birational transformation. Then the equation of C in affine coordinates is given as

$$F(x, y) = \sum_{\substack{r \leq i+j \leq m \\ i \geq 2}} a_{ij} x^i y^j + \sum_{l=r-1}^{m-2} b_l x(y-1)y^l + \sum_{l=r}^{m-2} c_l (y-1)^2 y^l. \quad (3.2.1)$$

For convenience, write $a_i = a_{i, m-i}$ ($i \geq 2$), $b = b_{m-2}$, $c = c_{m-2}$, $a'_i = a_{i, r-i}$ ($i \geq 2$) and $b' = b_{r-2}$, $c' = c_{r-2}$. By our assumption, F_m and F_r separate to all distinct linear factors and all of $\sum_{l=r-2}^{m-2} a_{2,l}$, $\sum_{l=r-1}^{m-2} b_l$ and $\sum_{l=r}^{m-2} c_l$ are not zeros.

3.3. Let $C \in V(m; r, 2)$. The genus of C is

$$g(C) = \frac{(m-1)(m-2)}{2} - \frac{r(r-1)}{2} - 1$$

and the global sections of the canonical line bundle are given as

$$H^0(C, \omega_C) = \left\{ \sum_{\substack{r-1 \leq i+j \leq m-3 \\ i \geq 1}} a_{ij} x^i y^j + \sum_{k=r-1}^{m-4} b_k y^k (y-1) \mid a_{ij}, b_k \in \mathbb{C} \right\}.$$

Consider the image of the Wahl map expressed explicitly with respect to this basis.

LEMMA 3.4. *For $C \in V(m; r, 2)$ with $m - r \geq 4$ and $r \geq 3$, $\text{Im } \Phi$ is generated by*

$$\begin{aligned} G_1 &= \{(y-1)^2 y^{2r-2} dy, \dots, (y-1)^2 y^{2m-10} dy, \\ &\quad x(y-1) y^{2r-3} dy, x(y-1) y^{2r-2} dy, \dots, x(y-1) y^{2m-10} dy, \\ &\quad x^2 y^{2r-4} dy, x^2 y^{2r-3} dy, \dots, x^2 y^{2m-10} dy, \\ &\quad x^3 y^{2r-5} dy, x^3 y^{2r-4} dy, \dots, x^3 y^{2m-11} dy, \dots, x^{2m-8} dy\} \\ G_2 &= \{x(y-1) y^{2r-3} dx, x(y-1) y^{2r-2} dx, \dots, x(y-1) y^{2m-10} dx, \\ &\quad x^2 y^{2r-4} dx, x^2 y^{2r-3} dx, \dots, x^2 y^{2m-10} dx, \dots, x^{2m-8} dx\} \\ G_3 &= \{y^{2m-8}((y-1) dx - x dy), xy^{2m-9}((y-1) dx - x dy), \\ &\quad x^2 y^{2m-10}((y-1) dx - x dy), \dots, x^{2m-8}((y-1) dx - x dy)\} \\ &= \{\omega_i = x^i y^j ((y-1) dx - x dy) \mid 0 \leq i \leq 2m-7, i+j = 2m-7\} \\ G_4 &= \{x^2 y^{2r-6} (y dx - x dy), x^3 y^{2r-7} (y dx - x dy), \dots, x^{2r-4} (y dx - x dy)\} \\ &\quad \cup \{y^{2r-4} (y-1)^2 (y dx - x dy), xy^{2r-5} (y-1) (y dx - x dy)\} \\ &= \{\tau_i = x^i y^j (y dx - x dy) \mid i \geq 2, i+j = 2r-4\} \\ &\quad \cup \{\tau_0 = y^{2r-4} (y-1)^2 (y dx - x dy), \tau_1 = -xy^{2r-5} (y-1) (y dx - x dy)\} \\ G_5 &= \{y^{2r-2} ((y-1) dx - x dy), \dots, y^{2m-9} ((y-1) dx - x dy)\} \\ &= \{\sigma_k = y^k ((y-1) dx - x dy) \mid 2r-2 \leq k \leq 2m-9\}. \end{aligned}$$

If $r = 2$, then $G_4 = \{\tau_0\}$ only and all other elements of $\text{Im } \Phi$ remain unchanged.

Proof. The proof is similar to that of Lemma 2.4. □

3.5. *Remark.* We will use the same notation $\text{Gen } \Phi$, $\# \text{Gen}$ and S_i as given in 2.5. Then $\# \text{Gen} = 4m^2 - 24m - 4r^2 + 8r + 25$ if $m - r \geq 4$.

LEMMA 3.6. *Let $C \in V(m; r, 2)$ with $m \geq r + 5$.*

- (i) *For the case $m \geq r + 6$, $(2m - 12)$ elements of G_3 can be generated by other elements of $\text{Gen } \Phi$ except elements of G_4 .*

(ii) For the case $m = r + 5$, $(2m - 13)$ elements of G_3 can be generated by other elements of $\text{Gen } \Phi$ except elements of G_4 .

Proof. (i) Consider (2.8.1). Any relation on $\text{Gen } \Phi$ which contains some ω_i is obtained from the relations

$$\begin{aligned} (1) \quad & x^i y^j F((y-1)dx - xdy) = 0, \quad i+j = m-8, \quad i \geq 0 \\ (2) \quad & x^i y^j (mFdx - x dF) = 0, \quad i+j = m-7, \quad i \geq 0 \\ (3) \quad & x^i y^j ((y-1)dF - mFdy) = 0, \quad i+j = m-7, \quad i \geq 0. \end{aligned} \quad (3.6.1)$$

By the degree assumption that $m \geq r + 6$, any term in (3.6.1) has degree greater than $2r - 3$.

(1) gives

$$\begin{aligned} & x^i y^{m-8-i} F((y-1)dx - xdy) \\ &= \left\{ \sum_{k=2}^m a_k x^{i+k} y^{2m-8-i-k} + b_{m-2} x^{i+1} (y-1) y^{2m-10-i} \right. \\ & \quad \left. + c_{m-2} x^i (y-1)^2 y^{2m-10-i} \right\} ((y-1)dx - xdy) + s, \end{aligned}$$

where s is a linear combination of lower degree terms. If any monomial in s is a multiple of x^2 or $x(y-1)$, then by the degree range it is an element of $S_1 \cup S_2$. Then the remaining lower degree monomials are $y^{m-8}((y-1)dx - xdy) \sum_{l=r}^{m-3} c_l (y-1)^2 y^l$ and these are elements of S_5 . Thus all lower degree terms are elements of $S_1 \cup S_2 \cup S_5$.

Also, (2) gives a similar result to Lemma 2.10, but (3) is different. The equation $x^i y^j (y dF - mF dy) = 0$ give relations on $\text{Gen } \Phi$ if $i \geq 1$ but does not give relations when $i = 0$. Thus we consider (3), $x^i y^j ((y-1)dF - mF dy) = 0$, instead of $x^i y^j (y dF - mF dy) = 0$ and we get similar relations.

We have

$$\begin{aligned} & \sum_{2 \leq k \leq m} a_k \omega_{i+k} + b \omega_{i+1} + c \omega_i = s_1 + s_2 + s_5, \quad \exists s_j \in S_j, \quad 0 \leq i \leq m-8 \\ & \sum_{2 \leq k \leq m} (m-k) a_k \omega_{i+k} + (m-1) b \omega_{i+1} + m c \omega_i = s_1 + s_2 + s_5, \quad \exists s_j \in S_j, \\ & \quad 0 \leq i \leq m-7 \\ & \sum_{2 \leq k \leq m} k a_k \omega_{i+k-1} + b \omega_i = s_1 + s_2 + s_5, \quad \exists s_j \in S_j, \quad 0 \leq i \leq m-7. \end{aligned} \quad (3.6.2)$$

Consider the matrix $M = M_{2m-7}(C)$ whose i th column consists of coefficients of ω_{i-1} for (3.6.2) and denote each row by A_i , B_j and C_j , where $1 \leq i \leq m-7$, $1 \leq j \leq m-6$. It is the same matrix as given in the proof of Lemma 2.10, so $\text{rank}(M) = 2m - 12$ and we obtain the lemma.

(ii) The maximal difference of degrees of elements of $\text{Im } \Phi$ is $(2m - 2r - 3) = 7$ and the maximal difference of degrees of monomials of F is $(m - r + 1) = 6$.

Consider (3.6.1). (2) gives relations on $\text{Gen } \Phi$, but (1) and (3) contains terms of degree $2r - 3$ by the degree condition. Thus consider

$$(1)' \quad x^i y^j F(y dx - x dy) = 0$$

$$(3)' \quad x^i y^j (y dF - mF dy) = 0$$

instead of (1) and (3). It can be shown easily that (1)' induces relations on $\text{Gen } \Phi$ and, for $i \geq 1$, (3)' induces relations on $\text{Gen } \Phi$. However, when $i = 0$, (3)' does not induce relations on $\text{Gen } \Phi$. After summing up all relations, we get the matrix $M = M_{2m-7}(C)$ which is the same as that of the proof of Lemma 2.10 without one row C_1 and $\text{rank}(M) = 2m - 13$. $\square$

LEMMA 3.7. Let $C \in V(m; r, 2)$ with $m \geq r + 4$.

- (i) If $r = 2$, there is no relation containing the lowest degree element τ_0 .
- (ii) If $r \geq 3$ and $m \geq r + 6$, $(2r - 4)$ elements of G_4 can be generated by other elements of $\text{Gen } \Phi$ except elements of G_3 .
- (iii) If $r \geq 3$ and $m = r + 5$, $(2r - 5)$ elements of G_4 can be generated by other elements of $\text{Gen } \Phi$ except elements of G_3 .

Proof. (i) If $r = 2$, G_4 consists of τ_0 only and the lowest degree of monomials of τ_0 is $2r - 3 = 1$. Since every monomial of F has degree larger than or equal to two, $F dx = 0$ or $F dy = 0$ cannot give relations containing τ_0 . Consider $dF = 0$. Let $F_2 = a'_2 x^2 + b'x(y-1)y + c'(y-1)^2 y^2$ be the part of F containing all lowest degree terms of F . Then all lowest degree terms of dF appear in dF_2 . Thus consider dF_2 :

$$\begin{aligned} dF_2 &= a'_2 2x dx + b'\{(y-1)y dx + xy dy + x(y-1) dy\} \\ &\quad + c'\{2(y-1)y^2 dy + 2(y-1)^2 y dy\}. \end{aligned}$$

There is no $x dx$ term and $y dy$ term in τ_0 , so $a'_2 = c' = 0$ if $dF = 0$ gives relations containing τ_0 . However, the remaining $b'((y-1)y dx + xy dy + x(y-1) dy)$ cannot make relations containing τ_0 on $\text{Gen } \Phi$. So there is no relation on $\text{Gen } \Phi$ containing τ_0 .

(ii) This proof is similar to that of Lemma 2.11. Consider the equations

$$\begin{aligned}
 (1) \quad & x^i y^j F(y dx - x dy) = 0, \quad i + j = r - 4, \quad i \geq 0 \\
 (2) \quad & x^i y^j (rF dx - x dF) = 0, \quad i + j = r - 3, \quad i \geq 0 \\
 (3) \quad & x^i y^j (y dF - rF dy) = 0, \quad i + j = r - 3, \quad i \geq 1 \\
 (4) \quad & y^{r-3}(y-1)(y dF - rF dy) = 0, \quad i = 0.
 \end{aligned} \tag{3.7.1}$$

Equations (1), (2) and (3) give the same relations containing elements of G_4 as in the proof of Lemma 2.11. However, $y^{r-3}(rF dy - y dF) = 0$ does not give relations on $\text{Gen } \Phi$:

$$\begin{aligned}
 & y^{r-3}(y dF - rF dy) \\
 &= \sum_{3 \leq k \leq m-1} ka'_k \tau_{k-1} + 2a'_2 \tau_1 - b' \tau_0 \\
 & \quad + 2a'_2 x y^{2r-3} dx + \sum_{r-1 \leq l \leq m-2} (2a_{2,l}) x y^{r-2+l} dx \\
 & \quad + 2b_{r-1} x y^{2r-3} dy + \sum_{r \leq l \leq m-2} (2b_l) x y^{r-2+l} dy \\
 & \quad + \sum_{r \leq l \leq m-2} 2c_l (y-1) y^{r-2+l} dy + s_1 + s_2 + s_5, \quad \exists s_j \in S_j.
 \end{aligned} \tag{3.7.2}$$

If one wants (3.7.2) to be a linear combination of $\text{Gen } \Phi$, then some restriction for the coefficients of F is required. No term in $\text{Gen } \Phi$ contains $(y-1)y^k dy$, so the last line must be a linear combination of $(y-1)^2 y^k dy$ in G_1 . Thus $\sum_{l=r}^{m-2} c_l = 0$ is required. Also, the fourth line of (3.7.2) must be a linear combination of $x(y-1)y^k dy$ in G_1 , so $\sum_{l=r-1}^{m-2} b_l = 0$ is required. Finally, the third line of (3.7.2) must be a linear combination of $x(y-1)y^k dx$ in G_2 , so $\sum_{l=r-2}^{m-2} a_{2,l} = 0$ is also required. But this is impossible by 3.2. Consider (4) $y^{r-3}(y-1)(rF dy - y dF) = 0$ instead of $y^{r-3}(rF dy - y dF) = 0$. Then (4) gives

$$\begin{aligned}
 & \sum_{3 \leq k \leq m-1} ka'_k \tau_{k-1} (y-1) + 2a'_2 \tau_1 (y-1) - b' \tau_0 (y-1) \\
 & \quad + 2a'_2 x y^{2r-3} (y-1) dx + \sum_{r-1 \leq l \leq m-2} (2a_{2,l}) x y^{r-2+l} (y-1) dx \\
 & \quad + 2b_{r-1} x y^{2r-3} (y-1) dy + \sum_{r \leq l \leq m-2} (2b_l) x y^{r-2+l} (y-1) dy \\
 & \quad + \sum_{r \leq l \leq m-2} 2c_l (y-1) y^{r-2+l} (y-1) dy = s_1 + s_2 + s_5, \quad \exists s_j \in S_j
 \end{aligned}$$

and, by the degree condition, it gives a relation on $\text{Gen } \Phi$ as

$$\sum_{k=2}^r k a'_k \tau_{k-1} - b' \tau_0 = s_1 + s_2 + s_5, \quad \exists s_j \in S_j.$$

Summing up, we have $2(r-2) + r - 3 = 3r - 7$ relations on $\text{Gen } \Phi$ containing the lowest degree elements of G_4 and we get a $(3r-7) \times (2r-3)$ matrix $M = M_{2r-3}(C)$ which is similar to the matrix given in the proof of Lemma 2.11 with $a_0 = c'$, $a_1 = -b'$. Thus $\text{rank}(M) = 2r - 4$ and we get the lemma.

(iii) Consider (3.7.1). Then (1), (2) and (3) give relations on $\text{Gen } \Phi$, but (4) cannot give relations on $\text{Gen } \Phi$ by the degree condition $m - 8 = r - 3$. In (4), $y^{m-8}(y-1)(y dF - rF dy) = 0$ contains degree $(2m-7)$ terms which cannot be a linear combination of elements of G_3 unless $b = c = 0$. But this is impossible by the assumption that C meets the line at infinity at m distinct points.

Considering all relations (1), (2) and (3) we have a $(3r-8) \times (2r-3)$ matrix M which is the same matrix as in (ii) except one row is obtained from (4). Thus $\text{rank}(M) = 2r - 5$ and we get the lemma. $\square$

Now we want to find relations on $\text{Gen } \Phi$ which do not contain elements of G_3 and G_4 . For convenience, let's denote

$$\begin{cases} Y_{i,j} = x^i y^j dy & \text{if } i \geq 2 \\ Y_{1,j} = x(y-1)y^{j-1} dy & \text{if } i = 1 \\ Y_{0,j} = (y-1)^2 y^{j-2} dy & \text{if } i = 0 \end{cases}$$

$$\begin{cases} X_{i,j} = x^i y^j dx & \text{if } i \geq 2 \\ X_{1,j} = x(y-1)y^{j-1} dx & \text{if } i = 1 \\ X_{0,j} = y^{j-1}((y-1)dx - x dy) + x(y-1)y^{j-2} dy = \sigma_{j-1} + Y_{1,j-i} & \text{if } i = 0. \end{cases}$$

LEMMA 3.8. *Let $C \in V(m; r, 2)$ with $m \geq r + 6$. Then a number*

$$\left\{ \sum_{n=r-2}^{m-8} (3n+4) \right\} - 1$$

of X_{ij}, Y_{ij} can be generated by other X_{ij}, Y_{ij} .

Proof. For any $r-2 \leq n = i + j \leq m-8$, consider the relations

- (1) $x^i y^j F dx = 0$
- (2) $x^i y^j F dy = 0$
- (3) $x^{i+1} y^j dF = 0$
- (4) $y^{n+1} dF = 0.$

By considering the degree condition and by using the same method as in the proof of Lemma 3.6, we easily obtain the following relations on $\text{Gen } \Phi$ from (1), (2) and (3):

$$\begin{aligned}
 (1) \quad & \sum_{k=2}^m a_k X_{k+i, m+n-i-k} + b X_{i+1, m+n-i-1} + c X_{i, m+n-i} \\
 & = s_1 + s_2 + s_5, \quad \exists s_j \in S_j, \quad 0 \leq i \leq n, \\
 (2) \quad & \sum_{k=2}^m a_k Y_{k+i, m+n-i-k} + b Y_{i+1, m+n-i-1} + c Y_{i, m+n-i} \\
 & = s_1 + s_2 + s_5, \quad \exists s_j \in S_j, \quad 0 \leq i \leq n, \\
 (3) \quad & \sum_{k=2}^m k a_k X_{k+i, m+n-i-k} + b X_{i+1, m+n-i-1} + \sum_{k=2}^m (m-k) a_k Y_{k+i+1, m+n-i-k-1} \\
 & \quad + (m-1) b Y_{i+2, m+n-i-2} + m c Y_{i+1, m+n-i-1} \\
 & = s_1 + s_2 + s_5, \quad \exists s_j \in S_j, \quad 0 \leq i \leq n.
 \end{aligned}$$

In this proof, s_j is a linear combination of elements of G_j whose degrees are less than $m+n$. However, for (4) first consider the $n = r-2$ case, i.e. consider $y^{r-1} dF = 0$:

$$\begin{aligned}
 y^{r-1} dF = & \left\{ \sum_{\substack{r \leq k+l \leq m \\ k \geq 3}} k a_{k,l} x^{k-1} y^{r-1+l} dx + \sum_{k=2}^m l a_{k,l} x^k y^{r+l-2} dy \right. \\
 & + \sum_{l=r-1}^{m-2} l b_l x (y-1) y^{r+l-2} dy + \sum_{l=r}^{m-2} l c_l (y-1)^2 y^{r+l-2} dy \Big\} \\
 & + \left\{ \sum_{\substack{r \leq k+l \leq m \\ k=2}} k a_{k,l} x^{k-1} y^{r-1+l} dx + \sum_{l=r-1}^{m-2} b_l (y-1) y^{r+l-1} dx \right. \\
 & \left. + \sum_{l=r-1}^{m-2} b_l x y^{r+l-1} dy + \sum_{l=r}^{m-2} 2 c_l (y-1) y^{r+l-1} dy \right\}.
 \end{aligned}$$

The first two rows after the equality are nothing but elements of $S_1 \cup S_2$ by the term x^2 , $x(y-1)$ or $(y-1)^2 dy$. However, the last two rows cannot be represented by linear combinations of elements of $\text{Gen } \Phi$. Consider the equation

The last two rows

$$= \left\{ 2 \sum_{l=r-2}^{m-2} a_{2,l} x (y-1) y^{r-1+l} dx + \sum_{l=r-1}^{m-2} b_l (y-1) y^{r+l-1} ((y-1) dx - x dy) \right\}$$

$$+ \sum_{l=r-1}^{m-2} 2b_l x(y-1)y^{r+l-1} dy + \sum_{l=r}^{m-2} 2c_l (y-1)^2 y^{r+l-1} dy \Big\}.$$

If we want the above equation to be a linear combination of $\text{Gen } \Phi$ then, again,

$$\sum_{l=r-1}^{m-2} b_l = \sum_{l=r}^{m-2} c_l = \sum_{l=r-2}^{m-2} a_{2,l} = 0$$

is required. This is impossible by 3.2, and $y^{r-1} dF = 0$ does not induce a relation on $\text{Gen } \Phi$. Similarly, $y^{n+1} dF = 0$ ($r-2 \leq n \leq m-8$) does not induce relations on $\text{Gen } \Phi$. However, if we consider $y^{n-1}(y-1) dF = 0$ ($r-1 \leq n \leq m-8$), then it induces a relation on $\text{Gen } \Phi$ as follows:

$$\begin{aligned} (4) \quad & \sum_{k=2}^m k a_k X_{k-1, m+n-k+1} + b X_{0, m+n} \\ & + \sum_{k=2}^m (m-k) a_k Y_{k, m+n-k} + (m-1) b Y_{1, m+n-1} + m c Y_{0, m+n} \\ & = s_1 + s_2 + s_5, \quad \exists s_j \in S_j, \quad r-1 \leq n \leq m-8. \end{aligned}$$

Thus, for each $r-1 \leq n \leq m-8$, there are $3n+4$ relations on X_{ij}, Y_{ij} whose degrees are less than or equal to $m+n$ and if $n = r-2$, there are $3n+3$ relations containing X_{ij}, Y_{ij} whose degrees are less than or equal to $m+r-2$ and greater than $2r-3$.

Consider the $(3n+4) \times (2m+2n+2)$ matrices $M_{m+n}(C)$ for each $r-1 \leq n \leq m-8$ and the $(3n+3) \times (2n+2m+2)$ matrix $M_{m+n}(C)$ for $n = r-2$. For $r-1 \leq n \leq m-8$, $M_{m+n}(C)$ is the same matrix as appeared in the proof of Lemma 2.12 with $c = a_0$, $b = a_1$. $M_{m+r-2}(C)$ is the same matrix as appeared in the proof of Lemma 2.12 without one row. Thus $\text{rank}(M_{m+n}(C)) = 3n+4$ for $r-1 \leq n \leq m-8$ and $\text{rank}(M_{m+r-2}(C)) = 3(r-2)+3$. $\square$

Proof of Theorem B.

(i) By Lemmas 3.6, 3.7 and 3.8, the number of relations for $\text{Gen } \Phi$ is

$$\begin{aligned} \# \text{Rel} &= (2m-12) + (2r-4) + \left(\sum_{n=r-1}^{m-8} 3n+4 \right) + (3(r-2)+3) \\ &= \frac{1}{2}(3m^2 - 3r^2 - 33m + 11r + 76). \end{aligned}$$

Thus

$$\begin{aligned}
 \text{cork } \Phi &= (5g - 5) + \# \text{Rel} - \# \text{Gen } \Phi \\
 &= \frac{1}{2}(5m^2 - 5r^2 - 15m + 5r - 10) \\
 &\quad + \frac{1}{2}(3m^2 - 3r^2 - 33m + 11r + 76) \\
 &\quad - \frac{1}{2}(8m^2 - 8r^2 - 48m + 16r + 50) \\
 &= 8.
 \end{aligned}$$

(ii) By the degree condition $m - r = 5$, there is no relation containing elements of G_1, G_2 only.

If $r \geq 3$, then by Lemmas 3.6 and 3.7, we have $g = 4m - 15$, $\# \text{Gen} = 24m - 115$ and $\# \text{Rel} = 4m - 28$, so $\text{cork } \Phi = (5g - 5) - \# \text{Gen } \Phi = 7$.

If $r = 2$, then $\# \text{Rel} = 2m - 13 = 1$, so $\text{cork } \Phi = 8$.

(iii) By the degree condition $m - 8 = r - 4$, any possible relation on $\text{Gen } \Phi$ contains both elements of G_3 and G_4 simultaneously. The equations $x^i y^j F(y dx - x dy) = 0$ for $i + j = m - 8 = r - 4$ give relations on $\text{Gen } \Phi$ without difficulty.

Consider the remaining equations $x^i y^j (mF dy - y dF) = 0$ and $x^i y^j (mF dx - x dF) = 0$ for $i + j = m - 7 = r - 3$. We want to know whether the lowest degree terms are in G_4 or not. The lowest degree terms in $x^i y^j (mF dy - y dF) = 0$ for $i + j = r - 3$ are

$$\begin{aligned}
 &mx^i y^{r-3-i} \left\{ \sum_{k=2}^r a_{k,r-k} x^k y^{r-k} + b_{r-1} x(y-1)y^{r-1} + c_r(y-1)^2 y^r \right\} dy \\
 &- x^i y^{r-3-i+1} \left\{ \sum_{k=2}^r k a_{k,r-k} x^{k-1} y^{r-k} dx + \sum_{k=2}^r (r-k) a_{k,r-k} x^k y^{r-k-1} dy \right. \\
 &+ b_{r-1}((y-1)y^{r-1} dx + x y^{r-1} dy + (r-1)x(y-1)y^{r-2} dy) \\
 &\left. + c_r(2(y-1)y^r dy + r(y-1)^2 y^{r-1} dy) \right\}. \tag{3.9}
 \end{aligned}$$

The terms which are not divided by x^{i+1} can be written as

$$c_r x^i y^{r-3-i} \{(m-r)(y-1)^2 y^r dy - 2(y-1)y^{r+1} dy\}. \tag{3.10}$$

If we want (3.10) to be a linear combination of $\text{Gen } \Phi$, then because of the degree $2r - 3$ term $x^i (y-1)^2 y^{2r-i-3} dy$, we need the $x^{i-1} (y-1)^2 y^{2r-i-2} dx$ term;

however, there is no such term in (3.9). Thus what is needed is $c_r = 0$. Similarly, regard the next term

$$b_{r-1}x^i y^{r-3-i} \{(m-r)x(y-1)y^{r-1}dy + (y-1)y^{r-1}(xdy - ydx) - xy^r dy\}.$$

Since $(m-r) \neq 0$, we need $(m-r)b_{r-1}x^i(y-1)y^{2r-i-3}dx$. However, there is no such term, so again $b_{r-1} = 0$. But then P is no longer an ordinary multiple point. Thus $x^i y^{m-7-i}(mF dy - y dF) = 0$ gives no relation on $\text{Gen } \Phi$.

Finally, when we consider $x^i y^{m-7-i}(mF dx - x dF) = 0$, $x^i y^{r-3-i}(rF dy - y dF) = 0$ or $x^i y^{r-3-i}(rF dx - x dF) = 0$ we arrive at the same conclusion.

Thus we have only $m-7$ relations on $\text{Gen } \Phi$ obtained from $x^i y^j F(y dx - x dy) = 0$ for $i+j = m-8$. Thus $g = 3m-10$, $\#\text{Rel} = m-7$, $\#\text{Gen} = 16m-71$ and $\text{cork } \Phi = 9$. $\square$

3.11. Remark. Let $r \geq 2$. For cases $m = r+3$ or $m = r+2$, we arrive at the same conclusion as in Remarks 2.13 or 2.14.

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