0. Introduction

By self-similar measures we mean solutions of the functional equation

$$\mu = \sum_{j=1}^{m} p_j \mu \circ w_j^{-1},$$

where $W = (w_1, \ldots, w_m)$ is a set of (continuous) transformations of a topological space $X$ and $P = (p_1, \ldots, p_m)$ is a positive probability vector. The triple $(X, W, P)$ is called a system, and a probability measure satisfying the equation is called an invariant measure of the system. In this paper we are interested in the invariant measures. These measures constitute a convex set, but perhaps empty. We shall study the existence and the uniqueness of the invariant measures, and see that the extremal points of this convex set is described by some ergodicity and that extremal measure are often unidimensional. In some special cases, we shall give a multifractal analysis of the unique invariant measure. Assume throughout that $(X, d)$ is a locally compact and complete metric space.

A compact set $A$ verifying $\bigcup_{j=1}^{m} w_j(A) = A$ is said to be self-similar. If we have only $\bigcup_{j=1}^{m} w_j(A) \subset A$, we say that $A$ is quasi-self-similar.

This paper is in three sections. In Section 1, we begin with a simple criterion for the existence of invariant measures with compact support, which says that there is such an invariant measure iff there is a quasi-self-similar set. At the same time, we see that the compact support of such a measure is self-similar. At the second step, we introduce a notion of ergodicity and deduce an ergodic theorem from the Chacon–Ornstein theorem. An invariant measure is said to be ergodic if it is trivial on invariant sets defined by $B = w_j^{-1}(B)$ for $1 \leq j \leq m$. Our ergodic theorem is
concerned with the $\mu$-a.e convergence of $N^{-1} \sum_{n=0}^{N} Q^{n} f$, where

$$Qf = \sum_{j=1}^{m} p_{j} f \circ w_{j},$$

$\mu$ is invariant and $f \in L^{1}(\mu)$. The limit is measurable relative to the $\sigma$-field, which is the completion with respect to $\mu$ of all invariant sets just defined. If $\mu$ is ergodic, the limit is a constant, equal to the integral of $f$ with respect to $\mu$. As a consequence of this ergodic theorem, it is proved that any two ergodic invariant measures are either mutually singular or identical and that ergodic measures are just extremal points of the convex set of invariant measures. At the end of this section, it is proved that a weak contraction condition on $w_{j}$ implies the existence and the uniqueness of invariant measures and that $Q^{n} f$ converges pointwise and uniformly on compact sets to a constant for every continuous function $f$. Recall that a transformation $w$ of a metric space is weak contractive if $\Omega_{w}(s) < s$, where $\Omega_{w}$ is the modulus of continuity of $w$.

In Section 2, we discuss the dimension of an ergodic measure. A measure $\mu$ on $X$ is of dimension $\alpha$ (we also say $\alpha$-dimensional) provided $\mu$-a.e.

$$\liminf_{r \to \infty} \frac{\log \mu(B_{r}(x))}{\log r} = \alpha,$$

where $B_{r}(x)$ is the ball of radius $r$ centered at $x$. Under the condition that $\dim w_{j}(B) \geq \dim B$ (dim $B$ means the Hausdorff dimension), it is proved that every ergodic invariant measure is $\alpha$-dimensional for some $\alpha \geq 0$. Some examples, such as Bernoulli convolutions, electrostatic equilibrium measures on Julia sets etc, are discussed at the end of this section.

The last section concerns the multifractal analysis of two measures: the invariant measure of a symbolic system and the invariant measure of a system consisting of contractive similarities of $\mathbb{R}^{d}$ with a unique self-similar set which is totally disconnected. Recall that multifractal analysis of a measure $\mu$ studies the singularity points of $\mu$ defined by $E_{\beta} = \{x \in X : D(\mu, x) = \beta\}$ ($\beta \in \mathbb{R}$) where

$$D(\mu, x) = \lim_{r \to \infty} \frac{\log \mu(B_{r}(x))}{\log r}.$$  

The aim is to estimate $\dim E_{\beta}$. For the two examples just mentioned, an exact formula is proved.
The first systematic account of the system we study here was that of [31] and [20], though similar ideas were certainly around earlier. The authors of these works, as well as the authors of [18, 19], considered contractions and weak contractions. They were interested mainly in self-similar sets other than invariant measures. The interest changed in [3, 4] to invariant measures where some sufficient conditions were found for existence and uniqueness of invariant measures. Our introduction of ergodicity permits us to further recognize the set of invariant measures in the general case. Our proof of sufficiency of weak contraction condition for existence and uniqueness is simple.

Notice that if \( W \) reduces to a single transformation, we obtain the classical ergodic theory. It was believed ([12]) that for an ergodic measure \( \mu, D(\mu, x) \) is a constant \( \mu \)-a.e. if the limit \( D(\mu, x) \) exists \( \mu \)-a.e. However, this is not the case ([11]). So, to ensure the \( \alpha \)-dimensionality of an ergodic invariant measure, a condition like \( \dim w_j(B) \geq \dim B, \) or some other, cannot be removed.

The formalism of Gibbs measures was often used to analyze multifractal properties of measures ([8, 14, 30]). Also, the Chernoff theorem for large derivation was involved. Our multifractal analysis of self-similar measures avoids the Gibbs measures as well as the Chernoff theorem. What we use is simply the well known Billingsley theorem and the strong law of large numbers.

1. Ergodicity of self-similar measures

1.1. Definitions of invariance and ergodicity

Let \( X \) be a locally compact and \( \sigma \)-compact Hausdorff space, let \( W = (w_1, \ldots, w_m) \) be a set of \( m \) Borel measurable transformations on \( X \) and let \( P = (p_1, \ldots, p_m) \) be a probability vector such that \( p_i > 0 \) and \( \sum p_i = 1. \) Let \( C_b(X) \) denote the Banach space of bounded continuous functions on \( X \) whose dual space is, by the Riesz representation theorem, the space \( M_b(X) \) of bounded regular Borel measures ([28]). The triple \( (X, W, P) \) will be referred as a weighted dynamical system or simply a system.

We define an operator by

\[
Qf = \sum_{j=1}^{m} p_j f \circ w_j
\]

for measurable \( f. \) Assume throughout that \( Q \) maps \( C_b(X) \) into \( C_b(X). \) Its adjoint
operator, denoted again by $Q : M_b(X) \rightarrow M_b(X)$, can be written as

$$\mu Q = \sum_{j=1}^{m} p_j \mu \circ w_j^{-1}.$$  

We are interested in invariant (probability) measures $\mu$, also called self-similar measures, which by definition are fixed points $\mu$ of $Q$, i.e. $\mu Q = \mu$. A Borel set $B$ is called invariant if it is invariant under all $w_j$, i.e. for $1 \leq j \leq m$ we have $B = w_j^{-1}(B)$. Clearly, all invariant sets constitute a $\sigma$-field, which will be denoted by $\mathcal{J}$. An invariant measure $\mu$ is said to be ergodic if $\mu$ is trivial on $\mathcal{J}$.

The previously defined operator $Q$ is actually a Markov operator. Let $\mathcal{B}$ be the $\sigma$-field of Borel sets of $X$. The corresponding Markov transition probability defined on $X \times \mathcal{B}$ is

$$N(x, B) = \sum_{j=1}^{m} p_j \mathbf{1}_B \circ w_j(x).$$

We can then write

$$Qf(\cdot) = \int f(y)N(\cdot, dy)$$

$$\mu Q(\cdot) = \int \mu(dx)N(x, \cdot).$$

Notice that $Q$ acts from the left on functions and from the right on measures. Notice also that $Qf$ can be defined for larger classes of functions $f$ (for example, positive or bounded) and we have the dual relation

$$\langle \mu Q, f \rangle = \langle \mu, Qf \rangle,$$

where $\langle \mu, f \rangle$ denotes the integral of $f$ with respect to $\mu$. We point out that in the following we shall use $\langle f, g \rangle_\mu$ to denote the integral with respect to a measure $\mu$ of the product $fg$ of two functions.

We can allow the $p_j$ to be functions of $x$. Under the condition that the $p_j$ are strictly positive, most of the results hold again (the sufficiency of weak contraction for uniqueness is an exception). We restrict our discussion to the constant case only, for simplicity of presentation.

### 1.2. Existence of invariant measures

We begin with a treatment of the compact case where there is always some invariant measure, and then continue to the general case where some sufficient conditions will
be given.

Suppose \( X \) is a compact Hausdorff space. There is at least one invariant measure; this is well known ([22]). In order to prove this fact, we have only to apply the Schauder–Tychonoff theorem ([28]) to \( Q \) which acts affinely and continuously on the compact convex set of probability measures on \( X \). In reality, if \( X \) is a compact Hausdorff space, the set of all invariant measures is again a non-empty compact convex set and then has some extremal points by the Krein–Milman theorem ([28]). In what follows, after proving an ergodic theorem, we shall see that the extremal points are exactly the ergodic measures. So in this special case, there are not only invariant measures but also ergodic measures.

Return to the case where \( X \) is a locally compact and \( \sigma \)-compact Hausdorff space. In general, trivial examples to illustrate the case where invariant measures exist but not ergodic measures, and even where there are no invariant measures, are easy to find. Here we give a necessary and sufficient condition for the existence of invariant measures with compact support.

**Theorem 1.** Suppose \( X \) is a locally compact and \( \sigma \)-compact Hausdorff space. There exists an invariant measure with compact support iff there exists a compact subset \( S \) such that \( \bigcup_{j=1}^{m} w_j(S) \subset S \). Moreover, suppose that \( w_j \) maps closed sets onto closed sets and that \( \mu \) is an invariant measure with support \( S \); then we have \( S = \bigcup_{j=1}^{m} w_j(S) \).

**Proof.** Suppose \( S \) is a compact set such that \( \bigcup_{j=1}^{m} w_j(S) \subset S \). That is to say \( w_j(S) \subset S \) for all \( 1 \leq j \leq m \). Then the restriction of \( w_j \) on \( S \) is a transformation of \( S \), and \( Q \) maps \( C(S) \) into \( C(S) \). In fact, by the Urysohn lemma ([28]), every continuous function on \( S \) can be extended as a continuous function on \( X \). Let \( \phi \) be such an extension of \( f \). It is easy to see that \( Q\phi \) is an extension of \( Qf \). By the known result cited above, there exists an invariant measure \( \mu \) concentrated on \( S \) such that for \( f : S \to \mathbb{R}^+ \),

\[
\int_S f \, d\mu = \sum_{j=1}^{m} p_j \int_S f \circ w_j \, d\mu.
\]

We extend \( \mu \) onto \( X \) by defining \( \mu(S^c) = 0 \). \( \mu \) will be a desired invariant probability measure with support \( S \). Conversely, suppose \( \mu \) is an invariant probability measure with compact support \( S \). Again, by the Urysohn lemma, there is a sequence of continuous functions \( \phi_n \) such that \( 0 \leq \phi_n \leq 1 \) and \( \phi_n \downarrow 1_S \). By the invariance of \( \mu \), we have

\[
\int (\phi_n - Q\phi_n) \, d\mu = 0,
\]
which implies \( \phi_n = Q\phi_n \) on \( S \). Then \( 1_S = Q1_S \). So, \( w_j(S) \subset S \) (1 \( \leq j \leq m \)). The first assertion is thus proved.

For an invariant measure \( \mu \) with compact support \( S \), we have seen \( \bigcup_{j=1}^{m} w_j(S) \subset S \). Suppose we have a strict inclusion. Then there exists \( y \in S \) but \( y \notin \bigcup_{j=1}^{m} w_j(S) \), i.e. \( y \in \bigcap w_j(S)^c \). This intersection is open, we denote it by \( U \). Now we have

\[
\begin{align*}
\text{then } w_j^{-1}(U) \subset w_j^{-1}(w_j(S)^c) = (w_j^{-1}w_j(S))^c \subset S^c
\end{align*}
\]

This contradiction proves \( S = \bigcup_{j=1}^{m} w_j(S) \). \( \square \)

A compact set \( S \) such that \( S = \bigcup_{j=1}^{m} w_j(S) \) is called a self-similar set. Theorem 1 guarantees the existence of self-similar sets by showing the existence of invariant measures with compact support. If such a self-similar set is unique, we say it is the attractor of the system \( W \). When the \( w_j \) are contractions, or weak contractions, there is an attractor \( S \). There are several concrete ways to express \( S \). For example,

\[
S = \text{closure} \left( \bigcup_{n \geq 1: 1 \leq j_1, \ldots, j_n \leq m} \text{Fix}(w_{j_1} \circ \cdots \circ w_{j_n}) \right),
\]

which is the Williams formula ([31], see also [19]). Another example is

\[
S = \lim_{n \to \infty} W^n(A)
\]

for any compact set \( A \), where \( W(A) \) is defined by

\[
W(A) = w_1(A) \bigcup \cdots \bigcup w_m(A).
\]

We call this expression the Hutchinson formula ([20], see also [19]). The last limit is taken under the Hausdorff metric defined on the space of all compact sets of \( X \) ([131]).

There are some well known necessary and sufficient conditions, more or less explicit, for the existence of (finite) invariant measure for a Markov operator ([21]). We interpret this into our context: there exists a finite invariant measure if and only if there exists a finite measure \( \mu \) such that \( \mu Q \ll \mu \) and for any \( A \) with \( \mu(A) > 0 \) we have

\[
\lim_{N \to \infty} \sup \frac{1}{N} \sum_{n=1}^{N} \sum_{j_1, \ldots, j_n} p_{j_1} \cdots p_{j_n} \mu \circ w_{j_1}^{-1} \circ \cdots \circ w_{j_n}^{-1}(A) > 0.
\]
1.3. Invariant sets

First, we give some descriptions of the invariant sets defined in Section 1.1, which we shall call universal invariant sets, and then construct a universal invariant set from a given Borel set. Second, for an invariant measure \( \mu \), we construct two positive contractions on \( L^1(\mu) \). As in the general theory of contractions on \( L^1 \) \((127)\), we shall have two \( \sigma \)-fields of invariant sets associated respectively to our two contractions. We shall see that these two \( \sigma \)-fields are the same. We denote it by \( \mathcal{J}_\mu \). It will be evident that \( \mathcal{J} \subset \mathcal{J}_\mu \). Moreover, we shall prove in the next subsection that \( \mathcal{J}_\mu \) is the completion of \( \mathcal{J} \) with respect to \( \mu \). A set in \( \mathcal{J}_\mu \) will be said to be \( \mu \)-invariant. This work is preparation for the ergodic theorem we shall state in the next subsection.

**Proposition 1.** A Borel set \( B \) is universally invariant iff one of the following is verified:

(a) \[ Q1_B = 1_B; \]
(b) \[ B = \bigcap_{j=1}^m w_j^{-1}(B); \]
(c) \[ B = \bigcup_{j=1}^m w_j^{-1}(B). \]

**Proof.** Evidently, (a) \( \Leftrightarrow \) (b) and (c) \( \Leftrightarrow \) \( Q1_B = 1_B \). As \( Q1 = 1 \), we then have (a) \( \Leftrightarrow \) \( Q1_B = 1_B \). Thus (a), (b) and (c) are equivalent and are clearly implied by the invariance of \( B \). Suppose now (b) and then (c) holds. We have for any \( j \) that

\[ w_j^{-1}(B) \subset \bigcup_{j=1}^m w_j^{-1}(B) = B = \bigcap_{j=1}^m w_j^{-1}(B) \subset w_j^{-1}(B). \]

So, \( B \) is invariant. \( \square \)

**Proposition 2.** Let \( B \) be a Borel set. The limit set

\[ \tilde{B} = \limsup_{n \to \infty} \bigcup_{j_1, \ldots, j_k} w_{j_1}^{-1} \circ \cdots \circ w_{j_k}^{-1}(B) \]

is a universal invariant set.

**Proof.** We shall apply criterion (c) of Proposition 1 to \( \tilde{B} \). First, we remark that for a finite number of sequences of sets \( \{A_{j,n}\}_{n \geq 1} \) \((1 \leq j \leq m)\), we have

\[ \bigcup_{j=1}^m \limsup_{n \to \infty} A_{j,n} = \limsup_{n \to \infty} \bigcup_{j=1}^m A_{j,n}. \]
Now, let
\[ A_{j,n} = w_j^{-1}\left(\bigcup_{j_1, \ldots, j_n} w_{j_1}^{-1} \circ \cdots \circ w_{j_n}^{-1}(B)\right). \]

We have
\[ w_j^{-1}(\tilde{B}) = \limsup_{n \to \infty} A_{j,n}. \]

Then, by the above remark,
\[
\bigcup_{j=1}^m w_j^{-1}(\tilde{B}) = \limsup_{n \to \infty} \bigcup_{j=1}^m A_{j,n} = \limsup_{n \to \infty} \bigcup_{j_1, j_2, \ldots, j_n} w_{j_1}^{-1} \circ w_{j_2}^{-1} \circ \cdots \circ w_{j_n}^{-1}(B) = \tilde{B}.
\]

We remark that for an infinite number of sequences of sets \(\{A_{j,n}\}\), we have only
\[
\bigcup_{j=1}^\infty \limsup_{n \to \infty} A_{j,n} \subset \limsup_{n \to \infty} \bigcup_{j=1}^\infty A_{j,n}.
\]

A counterexample is \(A_{j,n} = [0, j]\) for \(1 \leq n \leq j\) and \(A_{j,n} = \emptyset\) otherwise. Then we have
\[
\bigcup_{j=1}^\infty \limsup_{n \to \infty} A_{j,n} = \emptyset, \quad \limsup_{n \to \infty} \bigcup_{j=1}^\infty A_{j,n} = \mathbb{R}.
\]

So, the above argument does not work for infinite systems which consist of infinitely many transformations \(w_j\).

Now construct our contractions. There is no question that there is always some finite measure \(\mu\) such that \(\mu Q \ll \mu\). This property is equivalent to \(\nu \ll \mu\) implying \(\nu Q \ll \mu\). That is to say, \(Q\) acting from the right induces an operator on \(L^1(\mu)\) defined by
\[
Sf = \frac{d[(f \mu)Q]}{d\mu} \quad (f \in L^1(\mu)).
\]

For these properties and the following proposition, we can refer to [27].

PROPOSITION 3. Suppose \(\mu Q \ll \mu\). The operator \(S\) maps \(L^1(\mu)\) into \(L^1(\mu)\) and is positive and contractive (i.e. of norm \(\leq 1\)), whose adjoint operator \(S^* : L^\infty(\mu) \to L^\infty(\mu)\) coincides with the restriction of \(Q\) on \(L^\infty(\mu)\), acting from the right.
In particular, $S : L^1(\mu) \to L^1(\mu)$ is well defined when $\mu$ is invariant. For such an invariant probability measure $\mu$, we now consider another contraction. For each $1 \leq j \leq m$, $\mu \circ w_j^{-1} \ll \mu$, we have

$$0 \leq \frac{d\mu \circ w_j^{-1}}{d\mu} \leq \frac{1}{p_j} \quad \mu\text{-a.e.}$$

because

$$\sum_{j=1}^{m} p_j \frac{d\mu \circ w_j^{-1}}{d\mu} = 1 \quad \mu\text{-a.e.}$$

Consequently, $Q$ acting from the left is well defined on $L^1(\mu)$. We denote this restriction of $Q$ on $L^1(\mu)$ by $T$ and we have the following proposition.

**Proposition 4.** Suppose $\mu$ is invariant. The operator

$$Tf = \sum_{j=1}^{m} p_j f \circ w_j$$

maps $L^1(\mu)$ into $L^1(\mu)$ and is a positive contraction. Its adjoint operator $T^* : L^\infty(\mu) \to L^\infty(\mu)$ is also a positive contraction defined by

$$T^*g = \sum_{j=1}^{m} p_j \frac{d[(g\mu) \circ w_j^{-1}]}{d\mu}.$$

**Proof.** $T$ is a contraction because

$$\langle Tf, \mu \rangle = \langle Qf, \mu \rangle = \langle f, \mu Q \rangle = \langle f, \mu \rangle.$$

We verify the formula for $T^*$: for $f \in L^1(\mu)$:

$$\begin{align*}
\langle T^*g, f \rangle_\mu &= \langle g, Tf \rangle_\mu \\
&= \sum_{j=1}^{m} p_j \int gf \circ w_j \, d\mu \\
&= \sum_{j=1}^{m} p_j \int f \, \frac{d[(g\mu) \circ w_j^{-1}]}{d\mu} \, d\mu \\
&= \left\langle \sum_{j=1}^{m} p_j \frac{d[(g\mu) \circ w_j^{-1}]}{d\mu}, f \right\rangle_\mu.
\end{align*}$$

□
$T : L^1(\mu) \to L^1(\mu)$ is contractive even if $T$ is an arbitrary Markov operator. However, $T^*$ is not so clear.

**Proposition 5.** Suppose $\mu$ is an invariant probability measure. Let $A$ be a Borel set. We have $T^*1_A = 1_A \mu$-a.e. iff $S^*1_A = 1_A \mu$-a.e.

**Proof.** Suppose $T^*1_A = 1_A \mu$-a.e. For any Borel set $B$, by virtue of the formula of $T^*$ we have

$$\mu \left( A \cap B \right) = \langle T^*1_A, 1_B \rangle_\mu$$

$$= \sum_{j=1}^{m} p_j \int_B d[(1_A \mu) \circ w_j^{-1}] \, d\mu$$

$$= \sum_{j=1}^{m} p_j \mu \left( A \cap w_j^{-1}(B) \right).$$

Suppose $\mu(A) > 0$. Choose $B = A$ in the preceding equality. We have

$$1 = \sum_{j=1}^{m} p_j \mu \left( A \cap w_j^{-1}(A) \right) / \mu(A),$$

which implies that $\mu \left( A \cap w_j^{-1}(A) \right) = \mu(A)$ for $1 \leq j \leq m$. As $A \cap w_j^{-1}(A) \subset A$, we have $1_A \cdot 1_A \circ w_j = 1_A \mu$-a.e. So $1_A \circ w_j = 1_A \mu$-a.e. on $A$. Then

$$S^*1_A = \sum_{j=1}^{m} p_j 1_A \circ w_j = 1_A \mu$$-a.e. on $A$.

If $\mu(A^c) = 0$, we have proved $S^*1_A = 1_A \mu$-a.e. If $\mu(A^c) > 0$, as $T^*1_{A^c} = 1_{A^c}$ $\mu$-a.e., we can also prove that $S^*1_{A^c} = 1_{A^c}$ $\mu$-a.e. on $A^c$. This implies $S^*1_A = 1_A \mu$-a.e. on $A^c$ because $S^*1 = 1$. Thus $S^*1_A = 1_A \mu$-a.e.

Now suppose that $S^*1_A = 1_A \mu$-a.e. As $S^* = Q$, this means $1_A = 1_A \circ w_j \mu$-a.e. for $1 \leq j \leq m$. For $f \in L^1(\mu)$, consider

$$\langle T^*1_A, f \rangle_\mu = \int f \sum_{j=1}^{m} p_j \frac{d[(1_A \mu) \circ w_j^{-1}]}{d\mu} \, d\mu$$

$$= \sum_{j=1}^{m} p_j \int f \circ w_j 1_A \, d\mu.$$
\[
\begin{align*}
T^*1_A &= \sum_{j=1}^m p_j f \circ w_j 1_A \circ w_j d\mu \\
&= \langle Q(f1_A), \mu \rangle \\
&= \langle 1_A, f1_A \rangle.
\end{align*}
\]

So \( T^*1_A = 1_A \) \( \mu \)-a.e. \( \square \)

1.4. Ergodic theorem

Recall that we have two positive contractions \( T \) and \( S \) on \( L^1(\mu) \) if \( \mu \) is an invariant measure. \( T \) is the natural restriction of \( Q \) acting from the left on \( L^1(\mu) \) and defined by

\[
Tf = \sum p_j f \circ w_j,
\]

and \( S \) is the natural restriction of \( Q \) acting from the right on \( L^1(\mu) \) and defined by

\[
Sf = \frac{d[(f \mu)Q]}{d\mu}.
\]

As \( T1 = 1 \) and \( S1 = 1 \), both \( T \) and \( S \) are conservative contractions to which we can apply the Chacon–Ornstein theorem ([10], see also [24], [27]). Also recall that a Borel set is called invariant if \( Q1_A = 1_A \) everywhere. Such an invariant set has been referred to as the universal invariant set because it does not depend upon the measure \( \mu \). These sets constitute a \( \sigma \)-field \( \mathcal{J} \). Now, for the invariant measure \( \mu \), we introduce

\[
\mathcal{J}_\mu = \{ A : T^*1_A = 1_A \ \mu-\text{a.e.} \} = \{ A : S^*1_A = 1_A \ \mu-\text{a.e.} \}
\]

(see Proposition 5). It is well known that \( \mathcal{J}_\mu \) is a \( \sigma \)-field ([27, p. 125]).

It is evident that \( \mathcal{J} \subset \mathcal{J}_\mu \). We complete \( \mathcal{J} \) with respect to \( \mu \). Now, we shall see that \( \mathcal{J}_\mu \) is just the completion of \( \mathcal{J} \). Consequently, the ergodicity of \( \mu \) can be described by the triviality of \( \mu \) on \( \mathcal{J}_\mu \).

**Proposition 6.** Let \( \mu \) be an invariant probability measure. For every \( A \in \mathcal{J}_\mu \), there is an \( \tilde{A} \in \mathcal{J} \) such that \( \mu(A \triangle \tilde{A}) = 0 \).

**Proof.** We take for \( \tilde{A} \) the set defined in Proposition 2. \( A \in \mathcal{J}_\mu \) means \( Q1_A = 1_A \) \( \mu \)-a.e. That is

\[
\sum_{j=1}^m p_j 1_A \circ w_j = 1_A \ \mu\text{-a.e.}
\]
As $p_j > 0$, we have $1_A \circ w_j = 1_A \mu$-a.e. So
\[ \mu(w_j^{-1}(A) \triangle A) = 0 \quad (1 \leq j \leq m). \]
Now observe that $\mu \circ w_k^{-1} \ll \mu \ (1 \leq k \leq m)$. We then have
\[ \mu(w_k^{-1} \circ w_j^{-1}(A) \triangle w_k^{-1}(A)) = 0. \]
By induction, we have
\[ \mu(w_{j_1}^{-1} \circ \cdots \circ w_{j_n}^{-1}(A) \triangle A) = 0 \quad (1 \leq j_1, \ldots, j_n \leq m), \]
which implies $\mu(\tilde{A} \triangle A) = 0$. \hfill \Box

Here is an interpretation of the Chacon–Ornstein theorem into our context.

**Theorem 2.** Let $\mu$ be an invariant measure. For any $f \in L^1(\mu)$ and $g \in L^1_+(\mu)$, we have
\[
\lim_{n \to \infty} \frac{f + Tf + \cdots + T^n f}{g + Tg + \cdots + T^n g} = \frac{E_\mu(f|\mathcal{J}_\mu)}{E_\mu(g|\mathcal{J}_\mu)}.
\]
\[
\lim_{n \to \infty} \frac{f + Sf + \cdots + S^n f}{g + Sg + \cdots + S^n g} = \frac{E_\mu(f|\mathcal{J}_\mu)}{E_\mu(g|\mathcal{J}_\mu)}.
\]
$\mu$-a.e. on \[ \left\{ \sum T^n g > 0 \right\} = \left\{ \sum S^n g > 0 \right\} = \{E_\mu(g|\mathcal{J}_\mu) > 0\}. \]

For $g \equiv 1$, we obtain the ergodic theorem: $\mu$-a.e.
\[
\lim_{n \to \infty} \frac{f + Tf + \cdots + T^n f}{n} = E_\mu(f|\mathcal{J}_\mu).
\]
\[
\lim_{n \to \infty} \frac{f + Sf + \cdots + S^n f}{n} = E_\mu(f|\mathcal{J}_\mu).
\]

We have seen that $T^*$ and $S^*$ define the same $\mu$-invariant sets; but $T^*$ and $S^*$ are different as operators. Also, $T$ and $S$ are different. To see this, we consider the special case where we have only one transformation $w$. Moreover, we suppose $w$ is bijective. If $\mu$ is an invariant measure, i.e. $\mu = \mu \circ w^{-1}$, we have
\[ Tf = f \circ w, \quad T^* g = g \circ w^{-1}, \]
\[ Sf = f \circ w^{-1}, \quad S^* g = g \circ w. \]
It suffices to verify for $T^*$ and $S$. The verifications are straightforward using the definitions.
1.5. The convex of invariant measures

We shall prove a dichotomous property of ergodic measures: as a consequence of the ergodic theorem, two ergodic measures are either mutually singular or identical. Consequently, we shall see that the ergodic measures are just extremal points of the convex set of all invariant probability measures.

**Theorem 3.** If $\mu_1$ and $\mu_2$ are two ergodic probability measures, then either $\mu_1 = \mu_2$ or $\mu_1 \perp \mu_2$. A measure $\mu$ is ergodic if and only if it is an extremal point of the convex set of invariant measures.

**Proof.** Suppose $\mu_1 \neq \mu_2$. There is an $f \in C_b(X)$ such that $\langle \mu_1, f \rangle \neq \langle \mu_2, f \rangle$. Let

$$A = \left\{ x \in X : \lim_{n \to \infty} \frac{f + Qf + \cdots + Q^n f}{n} = \langle \mu_1, f \rangle \right\}.$$

By the ergodic theorem, we have $\mu_1(A) = 1$ and $\mu_2(A) = 0$. Thus we prove the first assertion.

Let $\mu$ be an ergodic measure. Suppose it is not extremal, i.e. $\mu = \alpha \mu_1 + (1 - \alpha) \mu_2$ for some $0 < \alpha < 1$ and two distinguished invariant measures $\mu_1$ and $\mu_2$. If $B \in \mathcal{F}$, $\mu(B) = 0$ or 1 since $\mu$ is ergodic. Consequently, $\mu_1(B) = \mu_2(B) = 0$ or 1 since $0 < \alpha < 1$. That means $\mu_1$ and $\mu_2$ are ergodic; then $\mu_1 \perp \mu_2$. Take $A$ as defined above and then define $\tilde{A}$ as in the Proposition 2. We have $\tilde{A} \in \mathcal{F}$, $\mu_1(\tilde{A}) = 1$ and $\mu_2(\tilde{A}) = 0$. So $\mu(\tilde{A}) = \alpha > 0$ then $1 > \alpha = \mu(\tilde{A}) = 1$ by ergodicity. This is a contradiction. Conversely, let $\mu$ be an extremal point. Suppose $\mu$ is not ergodic. Let $A$ be an invariant set such that $0 < \mu(A) < 1$. Define

$$\mu_1 = \frac{1_A \mu}{\mu(A)} \quad \mu_2 = \frac{1_{\tilde{A}} \mu}{1 - \mu(A)}.$$

We then have for $\alpha = \mu(A)$,

$$\mu = \alpha \mu_1 + (1 - \alpha) \mu_2.$$

To end the proof, we have only to observe that $\mu_1$ and $\mu_2$ are invariant. In fact, for $f \in C_b(X)$,

$$\langle \mu_1 Q, f \rangle = \frac{1}{\mu(A)} \langle 1_A \mu, Qf \rangle = \frac{1}{\mu(A)} \langle \mu, Q(1_A f) \rangle = \langle \mu_1, f \rangle.$$
The same applies for \( \mu_2 \).

\[ \Box \]

1.6. *The uniqueness of invariant measures*

We shall prove that weak contractivity is a sufficient condition for the uniqueness of invariant measures in the case where \((X, d)\) is a complete metric space, and prove that in this case the unique invariant measure is attractive and strongly mixing in some sense that we shall make precise.

The following known criterion ([22]) will be useful to us when \( X \) is compact. Denote

\[ \sigma_n f(x) = \frac{1}{n} \sum_{j=0}^{n-1} Q^j f(x). \]

Suppose \( X \) is a compact metric space. The following are equivalent:

(i) there is a unique invariant measure;

(ii) for every \( f \) in \( C(X) \), \( \sigma_n f \) converges uniformly to a constant;

(iii) for every \( f \) in \( C(X) \), \( \sigma_n f \) converges pointwise to a constant.

Let \( \omega_j \) be the modulus of continuity \( w_j \). For \( s > 0 \), let

\[ \Omega_j(s) = \lim_{t \to s^+} \omega_j(t). \]

\( \Omega_j(s) \) is non-decreasing and right continuous. For \( r \geq 0 \), let

\[ H_n(r) = \sup_{1 \leq j_1, \ldots, j_n \leq m} \Omega_{j_1} \circ \cdots \circ \Omega_{j_n}(r). \]

For \( x \in X \), let \( O_n(x) \) be the set of all points \( w_{j_1} \circ \cdots \circ w_{j_n}(x) \) \( (1 \leq j_1, \ldots, j_n \leq m) \).

**Proposition 7.** Let \( X \) be a metric space. Suppose \( H_n(r) \to 0 \) \((n \to +\infty)\) for any \( r > 0 \). Then there is at most one invariant measure with compact support.

**Proof.** We can assume that \( X \) is compact. In fact, suppose \( \mu_1 \) and \( \mu_2 \) are two invariant measures with compact supports \( S_1 \) and \( S_2 \). We have seen that

\[ S_1 = \bigcup w_j(S_1), \quad S_2 = \bigcup w_j(S_2). \]

Let \( S = S_1 \cup S_2 \), which is compact. We have

\[ S = \bigcup w_j(S). \]
Ergodicity, unidimensionality and multifractality of self-similar measures

So we can restrict our discussion to the compact set $S$ and we have only to show that there is at most one invariant measure on $S$.

As $X$ is assumed compact, we can use the criterion cited above. In reality, we can prove that $Q^n f$ converges pointwise to a constant. Writing

$$Q^n f(x) = \sum_{j_1, \ldots, j_n} p_{j_1} \cdots p_{j_n} f \circ w_{j_1} \circ \cdots \circ w_{j_n}(x),$$

and denoting $y = w_{j_{n+1}} \circ \cdots \circ w_{j_{n+p}}(x)$ for $p \geq 1$, we can express

$$Q^{n+p} f(x) - Q^n f(x) = \sum_{j_1, \ldots, j_n} p_{j_1} \cdots p_{j_n} \times \sum_{j_{n+1}, \ldots, j_{n+p}} p_{j_{n+1}} \cdots p_{j_{n+p}} [f(w_{j_1} \circ \cdots \circ w_{j_n}(y)) - f(w_{j_1} \circ \cdots \circ w_{j_n}(x))].$$

However,

$$|f(w_{j_1} \circ \cdots \circ w_{j_n}(y)) - f(w_{j_1} \circ \cdots \circ w_{j_n}(x))| \leq \omega_f(H_n(d(y, x))),$$

where $\omega_f$ is the modulus of continuity of $f$. Finally, we have

$$|Q^{n+p} f(x) - Q^n f(x)| \leq \Omega_f \left( H_n \left( \sup_{z \in O_n(x)} d(z, x) \right) \right).$$

$X$ being compact, then $r = \sup_{z \in O_n(x)} d(z, x) < +\infty$. The last inequality implies that $Q^n f(x)$ is a Cauchy sequence that converges. In the same way, we can obtain

$$|Q^n f(y) - Q^n f(x)| \leq \omega_f(H_n(d(y, x))),$$

which implies that the limit of $Q^n f(y)$ is that of $Q^n f(x)$. \hfill \square

Now we consider a particular situation. A map $w : X \to X$ is called a contraction (resp. weak contraction) if $\Omega_w(s) \leq \delta s$ for some $\delta < 1$ (resp. $\Omega_w(s) < s$), where $\Omega_w$ is defined as $\Omega_j$.

**Theorem 4.** Let $X$ be a complete metric space. Suppose $w_j$ are weak contractions of $X$. Then there is a unique invariant measure $\mu$. The measure $\mu$ is of compact support, attractive in that $vQ^n$ tends weakly to $\mu$ for any probability measure $v$, and strongly mixing in that

$$\lim_{n \to \infty} \langle f, Q^n g \rangle_\mu = \langle f, 1 \rangle_\mu \langle g, 1 \rangle_\mu, \text{ for any } f, g \in C_b(X).$$
Proof. The existence can be proved by using Theorem 1 and a result in [18, 19] which ensures the existence of a compact set $S$ such that $S = \bigcup_{j=1}^{m} w_j(S)$. For the uniqueness, we shall firstly prove $H_\varepsilon(r) \to 0$ which implies the uniqueness of invariant measures with compact support. The modulus of continuity $\Omega_j$, being increasing and continuous on the right, can be modified to obtain an increasing and continuous function $\Omega_j^*$ such that

$$\Omega_j(s) \leq \Omega_j^*(s) < s \ (\forall s > 0).$$

So, without loss of generality, we can assume the continuity for every $\Omega_j \ (1 \leq j \leq m)$. Let $r > 0$ and $\varepsilon > 0$ be fixed. On the compact interval $[\varepsilon, r]$, every function $\Omega_j(s)/s < 1$. Therefore, we can find a $\delta < 1$ such that

$$\Omega_j(s) \leq \delta s \ (\forall s \in [\varepsilon, r], \forall j).$$

Let $K$ be the least integer $k$ such that $\delta^k r < \varepsilon$. We have, for $n \geq K$,

$$\Omega_{j_1} \circ \cdots \circ \Omega_{j_k}(r) \leq \Omega_{j_1} \circ \cdots \circ \Omega_{j_{k-1}}(\delta^k r) \leq \delta^k r < \varepsilon.$$

Next, let $\mu$ be the unique invariant measure with compact support and let $\nu$ be an invariant measure. We are going to show that $\nu = \mu$. Let $f \in C_b(X)$. As in proving Proposition 7, we can prove that $Q^n f$ converges (pointwise and uniform on compact sets) to a constant. The only point to observe is the boundedness of the sequence $\sup_{x \in F} \sup_{x \in G} d(z, x) \ (p \geq 1)$. For this, it suffices to see that $\bigcup_{x \in F} O_p(x) = W^p(F)$, which tends to the attractor of the system if $F$ is compact (see the Hutchinson formula in Section 1.2.). The convergence of $Q^n f$ and the Lebesgue dominated convergence theorem imply that the limit is $(f, 1)_{\nu}$. Also, the limit must be $(f, 1)_{\mu}$. Thus we have $(f, 1)_{\nu} = (f, 1)_{\mu}$ and finish the proof of uniqueness.

The attractive property and the mixing property are also consequences of the convergence of $Q^n f$ and the Lebesgue dominated convergence theorem. \qed

If $W$ consists of contractions (resp. weak contractions), $(X, W, P)$ is called a hyperbolic system (resp. a weak hyperbolic system). We know that a weak hyperbolic system admits a unique invariant measure whose support is the unique compact self-similar set which is called the attractor of the system.

The same was proved in [4] if $p_j$ are continuous positive functions stochastically away from zero whose modulus of continuity verify the Dini condition, and if $w_j$
are average-contractions in that
\[ \sum_{j=1}^{m} p_j(x) d^q(w_j(x), w_j(y)) \leq r d^q(x, y), \]
for some \( r < 1 \) and \( q > 0 \). Clearly, with more general conditions, this result cannot include the just proven Theorem 4, since there are systems of weak contractions which are not average contractions (see Example 2 in Section 2.3.). The hyperbolic systems were treated in [3] via symbolic systems.

2. Unidimensionality of ergodic measures

2.1. Unidimensionality

Suppose \((X, d)\) is a metric space, where we can define Hausdorff dimensions for subsets in \( X \) ([13, 23]). A Borel measure \( \mu \) defined on \( X \) is said to be \( \alpha \)-dimensional \((\alpha \geq 0)\) provided that \( \mu \) is supported by a Borel set of Hausdorff dimension \( \alpha \) and \( \mu(B) = 0 \) for every Borel set \( B \) of Hausdorff dimension strictly smaller than \( \alpha \). If \( X \) is a locally compact metric space, then the Hausdorff dimension is equal to the capacity dimension (an unpublished result of Kaufman, and a partial result is contained in [1] if \( X \) is of homogeneous type). It was proved in [14] that \( \mu \) is \( \alpha \)-dimensional if and only if
\[ \liminf_{n \to 0} \frac{\log \mu(B_r(x))}{\log r} = \alpha \quad \mu \text{-a.e.,} \]
where \( B_r(x) \) is the ball of radius \( r \) about \( x \). A map \( w : X \to X \) is said to be dimensionally expansive if \( \dim w(A) \geq \dim A \) for every Borel set \( A \) of \( X \). The system \( W \) is said to be dimensionally expansive if every transformation in \( W \) is dimensionally expansive.

The following theorem was given in [15].

**THEOREM 5.** Suppose that the system \( W \) is dimensionally expansive. Then every ergodic measure is \( \alpha \)-dimensional for some \( \alpha \geq 0 \).

**Proof.** We shall prove \( \mu \) is \( \alpha \)-dimensional with
\[ \alpha = \sup\{\beta \geq 0 : \dim B < \beta \Rightarrow \mu(B) = 0\} \]
By the definition of $\alpha$, it is clear that $\mu(B) = 0$ if $\dim B < \alpha$. We have only to construct a Borel set which supports $\mu$ and is of dimension $\alpha$. For each $n \geq 1$, we choose a Borel set $A$ such that

$$\mu(A) > 0, \quad \alpha \leq \dim A < \alpha + 1/n.$$ 

Construct $\tilde{A}$ from $A$ as was done in Proposition 2. Since $\mu$ is invariant, we have for any Borel set $B$ the inequality

$$\mu \left( \bigcup_j w_j^{-1}(B) \right) \geq \max_j \mu(w_j^{-1}(B)) \geq \mu(B).$$

Using the Fatou lemma and this inequality, we then have

$$\mu(\tilde{A}) \geq \limsup_{n \to \infty} \mu \left( \bigcup_{j_1, \ldots, j_n} w_{j_1}^{-1} \circ \cdots \circ w_{j_n}^{-1}(A) \right) \geq \mu(A) > 0.$$ 

But $\tilde{A}$ is invariant and $\mu$ is ergodic, so $\mu(\tilde{A}) = 1$. Concerning the dimension of $\tilde{A}$, we have

$$\dim \tilde{A} \leq \sup \max_{n, j_1, \ldots, j_n} \dim w_{j_1}^{-1} \circ \cdots \circ w_{j_n}^{-1}(A)$$

by the $\sigma$-stability of Hausdorff dimension. We can even assert that $\dim \tilde{A} \leq \dim A$ because $W$ is dimensionally expansive, since a map $w$ is dimensionally expansive if and only if $\dim w^{-1}(B) \leq \dim B$ for every Borel set $B$. Thus, for $\tilde{A}$ we have

$$\mu(\tilde{A}) = 1, \quad \alpha \leq \dim \tilde{A} \leq \alpha + \frac{1}{n}.$$ 

This $\tilde{A}$ depends upon $n$. The intersection of all these $\tilde{A}$ ($n$ varying) is a desired Borel set.

\begin{proof}

2.2. **Condition for $\dim w(A) \geq \dim A$**

Let $(X, d)$ and $(Y, \delta)$ be two metric spaces. A map $w$ from $X$ into $Y$ is of *finite type* if there exists a positive $c$ and an integer $N$ such that the preimage under $w$ of a ball in $Y$ of radius $r$ can be covered by at most $N$ balls in $X$ of radius $cr$. It is of *quasi-finite type* if there exists a closed set $S$ in $Y$ whose preimage is of null dimension and its restriction on the complementary $w^{-1}(S^c)$ is of finite type for every $\epsilon > 0$, where $S_\epsilon$ consists of all $y \in Y$ such that $\delta(y, S) < \epsilon$. Roughly speaking, we would like to avoid the locally constant maps.
PROPOSITION 8. If \( w : X \to Y \) is of quasi finite type, it is dimensionally expansive.

Proof. It suffices to prove \( \dim w^{-1}(B) \leq \dim B \) for each \( B \subseteq Y \). To this end, it suffices to show for \( \epsilon > 0 \) that we have

\[
\dim w^{-1} \left( B \cap S^c_\epsilon \right) \leq \dim (B \cap S^c_\epsilon)
\]

for

\[
w^{-1}(B) = w^{-1} \left( B \cap S \right) \bigcup_{\epsilon \to 0} \lim w^{-1} \left( B \cap S^c_\epsilon \right).
\]

Let \( \alpha = \dim (B \cap S^c_\epsilon) \). Let \( \eta > 0 \) and \( r > 0 \) with \( 0 < r < \epsilon/4 \). Choose a covering of \( B \cap S^c_\epsilon \) by balls \( \{B_n\} \) of radius smaller than \( r \) such that

\[
B \cap S^c_\epsilon \subseteq \bigcup_n B_n, \quad \sum_n (\text{diam } B_n)^{\alpha + \eta} < \delta.
\]

We can assume that \( B_n \cap S_{\epsilon/4} = \emptyset \). According to the expansiveness of \( w \), for every \( n \) we can find some balls \( A_{n,j} \) \((1 \leq j \leq N)\) in \( X \) such that

\[
w^{-1}(B_n) \subseteq \bigcup_j A_{n,j}, \quad \text{diam } A_{n,j} \leq c \text{ diam } B_n.
\]

All balls \( A_{n,j} \) \((n \geq 1, 1 \leq j \leq N)\) of radius smaller than \( cr \) form a covering of \( w^{-1}(B \cap S^c_\epsilon) \) and

\[
\sum_n \sum_j (\text{diam } A_{n,j})^{\alpha + \eta} \leq Nc^{\alpha + \eta} \sum_n (\text{diam } B_n)^{\alpha + \eta} \leq N\delta^{\alpha + \eta}.
\]

Thus we prove \( \dim B \cap S^c_\epsilon \leq \alpha + \eta \).

\[
\Box
\]

2.3. Examples

Example 1. Denote the torus by \( T = \mathbb{R}/\mathbb{Z} \). Let \( w_j(x) = x + \alpha_j \) \((\mod 1)\) for \( 1 \leq j \leq m \). If one of the \( \alpha_j \) is irrational, there is a unique invariant measure for the system \((T, W, P)\). If the \( \alpha_j \) are all rationals and \( \alpha_j = a_j/b_j \) are irreducible fractions, a measure \( \mu \) is invariant iff \( \text{supp } \mu \subseteq q \mathbb{Z} \), where \( q \) is the least common multiple of \( b_j \). Notice that such a measure is one which is \((1/q)\)-invariant, i.e. \( \mu(E) = \mu(E + 1/q) \) for a Borel set \( E \).

To see these, it suffices to calculate Fourier coefficients of an invariant measure using its defining equation. This gives

\[
1 = \sum_{j=1}^m p_j e^{2\pi i a_j n} \quad (\forall n),
\]
which implies \( \alpha_{jn} = 0 \pmod{Z} \).

**Example 2.** Take \( X = [0, 1] \). Define \( h_\lambda : X \rightarrow X \) by \( h_\lambda(x) = x/(1 + \lambda x) \) for \( \lambda > 0 \). It is clear that each \( h_\lambda \), whose modulus of continuity is its self, is a weak contraction and any two \( h_\lambda \) and \( h_\mu \) verify \( h_\lambda \circ h_\mu = h_{\lambda + \mu} \). Take \( W = (w_1, \ldots, w_m) \) with \( w_j = h_{\lambda_j} \), where \( \lambda_j \) are fixed positives. For any \( P \), the system \((X, W, P)\) is weak hyperbolic. Its attractor is \([0]\) (by the Williams formula for \( h_\lambda \), it has a unique fixed point) and its invariant measure is the Dirac measure \( \delta_0 \). We point out that \( w_j \) are not average-contractive.

**Example 3.** Let \( w_j : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be affine transformations defined by \( w_j(x) = A_jx + b_j \) \( (1 \leq j \leq m) \). Suppose that \( s_j = \|A_j\| < 1 \) (\( \|A_j\| \) being the spectral norm of \( A_j \)). The system \((\mathbb{R}^d, W, P)\) has a unique invariant measure \( \mu \).

If \( A_j = A \) \( (1 \leq j \leq m) \), the measure \( \mu \) is the distribution of the sum of random series

\[
\sum_{n=0}^{\infty} A^\xi_n,
\]

where \( \xi_n \) are i.i.d. variables such that \( \xi_n \) take values \( b_j \) with probability \( p_j \). To see this, we calculate the Fourier transform of \( \mu \) and the characteristic function of the random sum. Both give rise to the same function

\[
\widehat{\mu}(u) = \prod_{n=0}^{\infty} \sum_{j=1}^{m} p_j e^{(u,A^nb_j)}.
\]

More particular is the one-dimensional case, where \( m = 2 \), \( A = \rho \) \((0 \leq \rho < 1)\), \( b_1 = 0 \), \( b_2 = 1 - \rho \) and \( p_1 = p_2 = 1/2 \). The support of the invariant measure is contained in \([-1/(1-\rho), 1/(1-\rho)]\) and the Fourier transform is

\[
\widehat{\mu}(u) = \prod_{n=0}^{\infty} \cos \rho^nu.
\]

All these measures are unidimensional. But the exact dimensions have been calculated in some special cases only. If the \( w_j \) are similarities (compositions of dilatation, rotation and translation) and satisfy an open set condition, then the dimension of \( \mu \) is proved ([29]) to be

\[
\alpha = \frac{\sum_{j=1}^{m} p_j \log p_j}{\sum_{j=1}^{m} p_j \log s_j}.
\]
In [17], a little more is proved: \( \mu \)-a.e
\[
\lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} = \alpha.
\]

We shall prove the last equality in another way in the next section.

Little is known if the open condition is taken off. Even in the case of dimension one, it has been seen that the arithmetic property of \( \rho \) plays an essential role; the best known is the golden number \( \rho = (\sqrt{5} - 1)/2 \) ([16, 25, 26]).

**Example 4.** Let \( \overline{C} = C \cup \{\infty\} \) be the Riemann sphere with spherical metric. Consider the dynamical system \( (\overline{C}, f_\lambda) \), where \( f_\lambda(z) = z^2 - \lambda \) with parameter \( \lambda \in \overline{C} \). Denote the two branches of \( f_\lambda \) by
\[
w_1 = \sqrt{z + \lambda}, \quad w_2 = -\sqrt{z + \lambda}.
\]

\( w_1 \) and \( w_2 \) are not continuous on \( \overline{C} \). However, it is easy to verify that if \( p_1 = p_2 = 1/2 \), the associated operator \( Q \) preserves continuous functions. The system \( (\overline{C}, (w_1, w_2), (1/2, 1/2)) \) is related to the Julia set of \( f_\lambda \).

Let \( J_\lambda \) be the Julia set of \( f_\lambda \). As \( f_\lambda(J_\lambda) = J_\lambda = f_\lambda^{-1}(J_\lambda) \), we can consider the system \( (J_\lambda, (w_1, w_2), (1/2, 1/2)) \). Since \( J_\lambda \) is compact, there is an invariant measure. It can be proved from [7, p. 143] that there is only one invariant measure which is the electrostatic equilibrium measure of \( J_\lambda \). This measure is then unidimensional.

3. The multifractal property

Let \( \mu \) be a measure defined on a metric space. The multifractal analysis of \( \mu \) consists of estimating the size, measured by Hausdorff dimension, of the set of singularity points of order \( \beta \ (\in \mathbb{R}) \) defined by
\[
E_\beta = \{ x : D(\mu, x) = \beta \},
\]
where
\[
D(\mu, x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r}
\]
and \( B_r(x) \) denotes the ball of center \( x \) and of radius \( r \). There is a well known theory on Gibbs measures ([6]) which can provide an exact estimation of \( \dim E_\beta \) in many cases ([14, 8]). However, in the case we are going to study, we shall give a direct method without using this theory.
For a weak hyperbolic system \((X, W, P)\), we can associate a symbolic system \((\Sigma, U, P)\) which will be defined later. A sequence of symbols is considered as an address or a code of a point in the attractor \(A\) of \((X, W, P)\). This gives rise to a continuous map from the code space \(\Sigma\) onto the attractor \(A\). The unique invariant measure of \((X, W, P)\) is the image under this map of the unique invariant measure of \((\Sigma, U, P)\). For the later, it is easy to calculate its dimension and to give its multifractal analysis. Using the mentioned map, we can transfer these results to the former if the attractor is totally disconnected. The use of the symbolic system appeared in [2, 3, 19, 20].

3.1. The symbolic system

For \(m \geq 2\), let \(\Sigma = \{1, 2, \ldots, m\}^\mathbb{N}\) be the set of infinite sequences of \(m\) symbols \(\{1, 2, \ldots, m\}\). A point \(\sigma \in \Sigma\) will be written as \(\sigma = \sigma_1\sigma_2\cdots\). For \(\sigma \in \Sigma\) and \(\tau \in \Sigma\), we define

\[
n(\sigma, \tau) = \inf\{j \geq 1 ; \sigma_j \neq \tau_j\}
\]

(with the convention \(\inf \emptyset = +\infty\)). Then we define a metric on \(\Sigma\) by

\[
d(\sigma, \tau) = e^{-n(\sigma, \tau)+1}.
\]

There are many other equivalent metrics. However, the present metric is the most convenient for us. Equipped with this metric, \((\Sigma, d)\) becomes a compact metric space. We denote by \(I_n(\sigma)\) the closed ball centered at \(\sigma\) and of radius \(e^{-n}\), i.e.

\[
I_n(\sigma) = \{\tau \in \Sigma \colon n(\sigma, \tau) > n\} = \{\tau \in \Sigma \colon \tau_1 = \sigma_1, \ldots, \tau_n = \sigma_n\}.
\]

Now we define, for \(1 \leq j \leq m\), \(u_j : \Sigma \to \Sigma\) by \(u_j(\sigma) = j\sigma\). Denote \(U = (u_1, u_2, \ldots, u_m)\). These maps are contractions and we have

\[
d(u_j(\sigma), u_j(\tau)) = \frac{1}{e}d(\sigma, \tau).
\]

For a probability vector \(P = (p_1, p_2, \ldots, p_m)\) with \(p_j > 0\), the system \((\Sigma, U, P)\), called a symbolic system, admits a unique invariant measure. We denote it by \(\nu\). It is actually the infinite product of the probability defined on \(\{1, 2, \ldots, m\}\) which takes \(j\) with probability \(p_j\). Precisely,

\[
\nu(I_n(\sigma)) = p_{\sigma_1}p_{\sigma_2}\cdots p_{\sigma_n}.
\]
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Every weak hyperbolic system \((X, \mathcal{W}, P)\) can be associated to the symbolic system \((\Sigma, U, P)\) by a map defined in the following proposition whose first assertion can be found in [19], and whose second assertion is a consequence of the first one.

**Proposition 9.** Let \((X, \mathcal{W}, P)\) be a weak hyperbolic system with the attractor \(A\) and the invariant measure \(\mu\).

(a) The limit
\[
\Phi(\sigma) = \lim_{n \to \infty} u_{\sigma_0} \circ u_{\sigma_2} \circ \cdots \circ u_{\sigma_n}(x)
\]
exists and is independent of \(x \in X\). The map \(\Phi : \Sigma \to A\) is continuous and onto, and satisfies
\[
u_j \circ \Phi = \Phi \circ \nu_j
\]
for \(1 \leq j \leq m\).

(b) The measure \(\mu\) is the image under \(\Phi\) of the measure \(\nu\). So, for a bounded Borel function \(f\) on \(X\),
\[
\int_X f \, d\mu = \int_{\Sigma} f \circ \Phi \, d\nu.
\]

Here is a complete multifractal analysis of the invariant measure of the symbolic system. The analysis is based on the strong law of large numbers and the Billingsley theorem ([51]) which asserts that a Borel set \(E\) is of Hausdorff dimension \(\alpha\) provided there is a Borel measure \(\mu\) such that \(\mu(E) > 0\) and \(D(\mu, x) = \alpha\) for \(x \in E\). Another form of the Billingsley theorem can be found in [13].

The case where \(p_1 = p_2 = \cdots = p_m = 1/m\) being trivial, we assume that
\[
p_1 \geq p_2 \geq \cdots \geq p_m, \quad p_1 > p_m.
\]
To study the size of the set of singularity points of order \(\beta\) \((\beta \in \mathbb{R})\),
\[
E_\beta = \{\sigma \in \Sigma : D(\nu, \sigma) = \beta\},
\]
we introduce the function defined by
\[
h(\alpha) = -\log \sum_{j=1}^{m} p_j^\alpha.
\]
Denoting \(p(\alpha) = \sum_{j=1}^{m} p_j^\alpha\), we have
\[
h'(\alpha) = -\frac{1}{p(\alpha)} \sum_{j=1}^{m} p_j^\alpha \log p_j.
\]
\[ h''(\alpha) = \frac{1}{p(\alpha)^2} \left[ \left( \sum_{j=1}^{m} p_j^\alpha \log p_j \right)^2 - p(\alpha) \sum_{j=1}^{m} p_j^\alpha \log^2 p_j \right]. \]

Lemma 1 gives some properties of these functions.

**LEMMA 1.** Suppose \( p_1 \geq p_2 \geq \cdots \geq p_m, \ p_1 > p_m. \) Let \( \beta_{\text{min}} = -\log p_1 \) and \( \beta_{\text{max}} = -\log p_m. \)

1. \( h : \mathbb{R} \rightarrow \mathbb{R} \) is bijective, strictly increasing and strictly concave.
2. \( h' : \mathbb{R} \rightarrow (\beta_{\text{min}}, \beta_{\text{max}}) \) is bijective and strictly decreasing.
3. Let \( f(\beta) = \inf_{a \in \mathbb{R}} \{ a\beta - h(\alpha) \}. \) Then \( f \) is finitely defined and strictly concave in \((\beta_{\text{min}}, \beta_{\text{max}}). \) We have

\[ f(h'(\alpha)) = zh'(\alpha) - h(\alpha) \]

\[ f(h'(\pm \infty)) = \log m', \quad f(h'(-\infty)) = \log m'', \]

where \( m' = \operatorname{Card}\{j : p_j = p_1\} \) and \( m'' = \operatorname{Card}\{j : p_j = p_m\}. \)

**Proof.** Notice that \( h' > 0 \) and \( h'' < 0 \) by using the Hölder inequality. Notice also that \( h(+\infty) = +\infty \) and \( h(-\infty) = -\infty. \) Then (1) follows.

Writing

\[
\begin{align*}
h'(\alpha) &= -\frac{\sum_{j=1}^{m} (p_j/p_1)^{\alpha} \log p_j}{\sum_{j=1}^{m} (p_j/p_1)^{\alpha}} \\
h'(\alpha) &= -\frac{\sum_{j=1}^{m} (p_j/p_m)^{\alpha} \log p_j}{\sum_{j=1}^{m} (p_j/p_m)^{\alpha}},
\end{align*}
\]

we obtain \( h'(\pm \infty) = \beta_{\text{min}} \) and \( h'(-\infty) = \beta_{\text{max}}. \) Thus we have proved (2).

With \( f \) being the Legendre transform of \( h \) which is strictly concave, \( f \) can be explicitly calculated as stated. For finishing the proof of (3), we observe the following equivalents of \( h \) at infinity:

\[ h(\alpha) = -\alpha \beta_{\text{min}} - \log m' + o(1) \quad (\alpha \to +\infty) \]

\[ h(\alpha) = -\alpha \beta_{\text{max}} - \log m'' + o(1) \quad (\alpha \to -\infty) \]

and that of \( h' \)

\[
\begin{align*}
h'(\alpha) - \beta_{\text{min}} &\sim \frac{\operatorname{Card}\{j : p_j = p'_{\text{max}}\}}{\operatorname{Card}\{j : p_j = p_{\text{max}}\}} \left( \frac{p'_{\text{max}}}{p_{\text{max}}} \right)^{\alpha} \log \frac{p'_{\text{max}}}{p_{\text{max}}} \quad (\alpha \to +\infty) \\
h'(\alpha) - \beta_{\text{max}} &\sim \frac{\operatorname{Card}\{j : p_j = p'_{\text{min}}\}}{\operatorname{Card}\{j : p_j = p_{\text{min}}\}} \left( \frac{p'_{\text{min}}}{p_{\text{min}}} \right)^{\alpha} \log \frac{p'_{\text{min}}}{p_{\text{min}}} \quad (\alpha \to -\infty),
\end{align*}
\]
where \( p_{\min} = p_m \) and \( p'_{\min} \) is the second smallest value of the \( p_j \), \( p_{\max} = p_1 \) and \( p'_{\max} \) is the second greatest value of the \( p_j \). It follows that

\[
\lim_{\alpha \to +\infty} (\alpha h'(\alpha) - h(\alpha)) = -\log m', \quad \lim_{\alpha \to -\infty} (\alpha h'(\alpha) - h(\alpha)) = -\log m''.
\]

\( \square \)

**Theorem 6.** Keep the assumption and the notation as above. Let \( \nu \) be the invariant measure of \((\Sigma, U, P)\). We have

\[
\dim \nu = - \sum_{j=1}^{m} p_j \log p_j.
\]

For \( \beta \in [\beta_{\min}, \beta_{\max}] \), we have

\[
\dim E_{\beta} = \inf_{\alpha \in \mathbb{R}} \{ \alpha \beta - h(\alpha) \}.
\]

For \( \beta \notin [\beta_{\min}, \beta_{\max}] \), we have \( E_{\beta} = \emptyset \).

**Proof.** For \( j \geq 1 \), define \( \xi_j : \Sigma \to \{1, 2, \ldots, m\} \) by \( \xi_j(\sigma) = \sigma_j \). Then \( \{\xi_j\} \) is a sequence of i.i.d. random variables on the probability space \((\Sigma, \nu)\). Thus, by the strong law of large numbers, we have \( \nu \)-a.e.

\[
\lim_{n \to \infty} \frac{\log \nu(I_n(\sigma))}{n} = \sum_{j=1}^{m} p_j \log p_j.
\]

Notice that \(-n\) is the logarithm of the radius of \( I_n(\sigma) \). The last limit gives us the dimension of \( \nu \).

Now, for \( \alpha \in \mathbb{R} \), let \( p(\alpha) = \sum_{j=1}^{m} p_j^\alpha \). The vector \( P_\alpha = (p_1^\alpha / p(\alpha), \ldots, p_m^\alpha / p(\alpha)) \) is a positive probability vector. Denote by \( \nu_\alpha \) the invariant measure of the system \((\Sigma, U, P_\alpha)\). Apply the dimension formula for \( \nu \) to \( \nu_\alpha \); we obtain

\[
\dim \nu_\alpha = - \frac{\alpha}{p(\alpha)} \sum_{j=1}^{m} p_j^\alpha \log p_j + \log p(\alpha) = \alpha h'(\alpha) - h(\alpha).
\]

According to the definitions of \( \nu \) and \( \nu_\alpha \) as infinite products, it is easy to get the relation between \( D(\nu_\alpha, \sigma) \) and \( D(\nu, \sigma) \):

\[
D(\nu_\alpha, \sigma) = \alpha D(\nu, \sigma) - h(\alpha).
\]
Suppose $\alpha \neq 0$. By virtue of the preceding relation, the singularity points set $E_{h'(\alpha)}$ defined by $\eta$ can be determined by $\nu_\alpha$ as

$$E_{h'(\alpha)} = \{ \sigma \in \Sigma : D(\nu_\alpha, \sigma) = \alpha h'(\alpha) - h(\alpha) \}.$$ 

The dimension formula for $\nu_\alpha$ given above indicates that $E_{h'(\alpha)}$ is a Borel support of $\nu_\alpha$, i.e. $\nu_\alpha(E_{h'(\alpha)}) = 1$. So, by the Billingsley theorem,

$$\dim E_{h'(\alpha)} = \alpha h'(\alpha) - h(\alpha).$$

Suppose $\alpha = 0$. We have $h(0) = -\log m$, $h'(0) = -\frac{1}{m} \sum_{j=1}^{m} \log p_j$ and $\nu_0$ is the Haar measure on $\Sigma$. It is clear that $\nu_0(E_{h'(0)}) = 1$. So $\dim E_{h'(0)} = \log m = \log m'$. Thus, by Lemma 1, the result for $\beta \in (\beta_{\min}, \beta_{\max})$ is proved.

We are going to show $\dim E_{\beta_{\min}} = \log m'$. In fact, $\dim E_{\beta_{\min}} \geq \log m'$ because $E_{\beta_{\min}}$ contains the Cantor set defined by $\{ \sigma : \sigma_j = p_1, \forall j \geq 0 \}$ which is of dimension $\log m'$. On the other hand, given $\epsilon > 0$ and $\alpha > 0$, for a point $\sigma \in E_{\beta_{\min}}$, since $\beta_{\min} < h'(\alpha)$, we have for large $n$

$$e^{-nh'(\alpha)} < \prod_{j=1}^{n} p_{\sigma_j}.$$ 

Consider the collection of cylinders determined by all $\sigma_1 \cdots \sigma_n$ which satisfy the last inequality. This collection is a Vitali covering of $E_{\beta_{\min}}$. For any $\delta > 0$, we can extract a countable disjoint subcollection of disjoint cylinders $\{ I_n(\sigma) \}$ with diam $I_n(\sigma) < \delta$ such that $\{ I_{n-2}(\sigma) \}$ is a covering of $E_{\beta_{\min}}$. However, since $e^{-h(\alpha)} = p(\alpha)$,

$$\sum_{\sigma} (\text{diam } I_n(\sigma))^{\alpha h'(\alpha) - h(\alpha)} \leq C \sum_{\sigma} \prod_{j=1}^{n} \frac{p_{\sigma_j}^\alpha}{p(\alpha)} \leq C \nu_\alpha(\Sigma) < \infty.$$ 

It follows that $\dim E_{\beta_{\min}} \leq \alpha h'(\alpha) - h(\alpha)$. By Lemma 1, $\dim E_{\beta_{\min}} \leq f(+\infty) = \log m'$. In the same way we can prove $\dim E_{\beta_{\max}} = \log m''$.

The last assertion of the theorem is evident because

$$(\min p_j)^n \leq \nu(I_n(\sigma)) \leq (\max p_j)^n.$$ 

3.2. Self-similar measures on $R^d$

We now consider some measures that we have described in Example 2 in Section 2.3. Recall that

$$w_j(x) = s_j R_j(x) + b_j,$$
where $0 \leq s_j < 1$, $R_j$ are rotations of $R^d$ and $b_j$ are vectors in $R^d$. The $w_j$ are called similarities of $R^d$.

**Proposition 10.** Let $w_j$ be similarities of $R^d$. Suppose the attractor $A$ of $(R^d, W, P)$ satisfies $w_j(A) \cap w_k(A) = \emptyset$ for $j \neq k$. Then there is a constant $c > 0$ such that

$$c^{-1}s_{\sigma_1} \cdots s_{\sigma_{k(s, r)}} \leq \|\Phi(\sigma) - \Phi(\tau)\| \leq cs_{\sigma_1} \cdots s_{\sigma_{k(s, r)}},$$

where $\Phi$ is the map in Proposition 9. In particular, if $s_1 = \cdots = s_m = s$, we have

$$\|\Phi(\sigma) - \Phi(\tau)\| \approx d(\sigma, \tau)^{-\log s}.$$

**Proof.** Under the hypothesis, $\Phi$ is a bijection from $\Sigma$ onto $A$. By the definition of $\Phi$, we have

$$\Phi(\sigma) = w_{\sigma_1} \circ \cdots \circ w_{\sigma_n}(z_1)$$

$$\Phi(\tau) = w_{\sigma_1} \circ \cdots \circ w_{\sigma_n}(z_2),$$

where $n = n(\sigma, \tau)$ and

$$z_1 = \lim_{p \to \infty} w_{\sigma_{n+1}} \circ \cdots \circ w_{\sigma_{n+p}}(z)$$

$$z_2 = \lim_{p \to \infty} w_{\tau_{n+1}} \circ \cdots \circ w_{\tau_{n+p}}(z)$$

for some $z \in A$. As $z_1$ and $z_2$ belong to different parts $\{w_j(A)\}$ of $A$, we have

$$\delta \leq \|z_1 - z_2\| \leq \Delta,$$

where $\delta = \inf_{j \neq k} d(w_j(A), w_k(A))$ and $\Delta$ is the diameter of $A$. Since $w_j$ are similarities, we then have the desired inequalities with $C = \max(\Delta, \delta^{-1})$. \hfill \Box

In the special case where $s_1 = \cdots = s_m = s$, $\Phi$ is bi-Lipschitzian. Then, for any set $E \subset \Sigma$, we have

$$\dim \Phi(E) = \frac{1}{-\log s} \dim E$$

$$D(\mu, x) = \frac{1}{-\log s} D(\nu, \Phi^{-1}(x)).$$

With these in mind, the results on $\nu$ can be translated to $\mu$ as follows:

$$\dim \mu = \frac{1}{\log s} \sum_{j=1}^{m} p_j \log p_j$$

$$\dim E_{h(\alpha)} = \inf_{\alpha} [\alpha h'(\alpha) - h(\alpha)].$$
where

\[ h(\alpha) = \frac{1}{\log s} \log \sum_{j=1}^{m} p_j^{\alpha}. \]

We point out that, formally, the result in the symbolic case corresponds to \( s_1 = s_2 = \ldots = s_m = 1/e \). Without the condition \( s_1 = \ldots = s_m \), the map \( \Phi \) is no longer bi-Lipschitzian as it is in the symbolic case.

We introduce a function \( h : \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[ \sum_{j=1}^{m} p_j^{\alpha} s_j^{\alpha h(\alpha)} = 1. \]

Clearly, \( h \) is well defined, strictly increasing and differentiable. Simple calculus gives

\[
h'(\alpha) = \frac{\sum_{j=1}^{m} p_j^{\alpha} s_j^{\alpha h(\alpha)} \log p_j}{\sum_{j=1}^{m} p_j^{\alpha} s_j^{\alpha h(\alpha)} \log s_j},
\]

\[
h''(\alpha) = \frac{\sum_{j=1}^{m} p_j^{\alpha} s_j^{\alpha h(\alpha)} (\log p_j - h'(\alpha) \log s_j)^2}{\sum_{j=1}^{m} p_j^{\alpha} s_j^{\alpha h(\alpha)} \log s_j}.
\]

Notice that \( h''(\alpha) \leq 0 \). \( h \) is concave. Suppose \( h''(\alpha_0) = 0 \) for some \( \alpha_0 \in \mathbb{R} \). This means

\[
\frac{\log p_1}{\log s_1} = \ldots = \frac{\log p_m}{\log s_m} = h'(\alpha_0),
\]

i.e. \( p_j = s_j^{h'(\alpha_0)} \) (1 \( \leq j \leq m \)). This is equivalent to \( h(\alpha) = D(\alpha - 1) \), where \( D \) is determined by

\[
\sum_{j=1}^{m} s_j^D = 1.
\]

**Lemma 2.** Let \( \rho_j = \log p_j / \log s_j \). Suppose \( \beta_{\min} = \rho_1 \leq \ldots \leq \rho_m = \beta_{\max} \) and \( \rho_1 < \rho_m \).

1. \( h : \mathbb{R} \rightarrow \mathbb{R} \) is bijective, strictly increasing and strictly concave.
2. \( h' : \mathbb{R} \rightarrow (\beta_{\min}, \beta_{\max}) \) is bijective and strictly decreasing.
3. Let \( f(\beta) = \inf_{\alpha \in \mathbb{R}} \{ \alpha \beta - h(\alpha) \} \). Then \( f \) is finitely defined and strictly concave in \( (\beta_{\min}, \beta_{\max}) \). We have

\[
f(h'(\alpha)) = \alpha h'(\alpha) - h(\alpha)
\]

\[
f(h'(+\infty)) = D' \quad f(h'(-\infty)) = D''
\]
where $D'$ and $D''$ are respectively determined by

$$
\sum_{j: \rho_j = \rho_1} s_j^{D'}, \sum_{j: \rho_j = \rho_m} s_j^{D''} = 1.
$$

Proof. The same as the proof of Lemma 1.

THEOREM 7. Suppose that the $w_j$ are similarities of $R^d$ and that the attractor $A$ of $(R^d, W, P)$ satisfies $w_j(A) \bigcap w_k(A) = \emptyset$ for $j \neq k$. Let $\mu$ be the invariant measure of $(R^d, W, P)$. If $\rho_1 \leq \cdots \leq \rho_m$ and $\rho_1 < \rho_m$, we have

$$
dim \mu = \frac{\sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log s_j};
$$

for $\beta \in [\beta_{\min}, \beta_{\max}]$, we have

$$
dim E_\beta = \inf_{\alpha \in \mathbb{R}} \{ \alpha \beta - h(\alpha) \};
$$

and for $\beta \notin [\beta_{\min}, \beta_{\max}]$, $E_\beta = \emptyset$. If $\rho_1 = \cdots = \rho_m$, we have $D(\mu, x) = D$ for every $x \in A$ and $D(\mu, x) = +\infty$ for every $x \notin A$.

Proof. Suppose $\rho_1 < \rho_m$. Let

$$
F_\beta = \{ \sigma \in \Sigma : D(\mu, \Phi(\sigma)) = \beta \}.
$$

Clearly $E_\beta = \Phi(F_\beta)$. We shall work with $F_\beta$ instead of $E_\beta$. For $\sigma$ fixed, by Proposition 9, we have

$$
\mu(B_r(\Phi(\sigma))) = \nu[\tau : \|\Phi(\tau) - \Phi(\sigma)\| < r].
$$

By Proposition 10, we have

$$
\nu(I_{\overline{\theta}(\sigma, r)}(\sigma)) \leq \mu(B_r(\Phi(\sigma))) \leq \nu(I_{\underline{\theta}(\sigma, r)}(\sigma)),
$$

where

$$
\overline{\theta}(\sigma, r) = \inf\{ n : s_{\sigma_1} \cdots s_{\sigma_n} \leq cr \}
$$

$$
\underline{\theta}(\sigma, r) = \sup\{ n : s_{\sigma_1} \cdots s_{\sigma_n} \geq c^{-1}r \}.
$$

Notice that

$$
0 \leq \overline{\theta}(\sigma, r) - \underline{\theta}(\sigma, r) = O(1),
$$
where $O(1)$ does not depend upon either $\sigma$ or $r$, from which we deduce that $\mu$ as well as $\nu$ is a double measure in that $\mu(B_r(x)) = O(\mu(B_r(x)))$. Both $n(\sigma, r)$ and $\vec{n}(\sigma, r)$ tend to infinity when $r$ tends to zero. So
\[
\liminf_{r \to 0} \frac{\log \nu(I_{\vec{n}(\sigma, r)}(\sigma))}{\log r} \leq \liminf_{r \to 0} \frac{\log \mu(B_r(\Phi(\sigma)))}{\log r} \\
\leq \limsup_{r \to 0} \frac{\log \mu(B_r(\Phi(\sigma)))}{\log r} \leq \limsup_{r \to 0} \frac{\log \nu(I_{\vec{n}(\sigma, r)}(\sigma))}{\log r}.
\]

We know that $\nu$-a.e.
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log s_{nk} = \sum_{j=1}^{m} p_j \log s_j \\
\lim_{n \to \infty} \frac{\log \nu(I_n(\sigma))}{n} = \sum_{j=1}^{m} p_j \log p_j.
\]

Consequently, we have $\nu$-a.e.
\[
\lim_{r \to 0} \frac{\log r}{n(\sigma, r)} = \lim_{r \to 0} \frac{\log r}{\vec{n}(\sigma, r)} = \sum_{j=1}^{m} p_j \log s_j \\
\lim_{r \to 0} \frac{\log \nu(I_{\vec{n}(\sigma, r)}(\sigma))}{\log r} = \lim_{r \to 0} \frac{\log \nu(I_{\vec{n}(\sigma, r)}(\sigma))}{\log r} = \sum_{j=1}^{m} p_j \log p_j = \sum_{j=1}^{m} p_j \log s_j.
\]

So, $\nu$-a.e.
\[
D(\mu, \Phi(\sigma)) = \frac{\sum_{j=1}^{m} p_j \log p_j}{\sum_{j=1}^{m} p_j \log s_j}.
\]

By Proposition 9, $\mu$-a.e.
\[
D(\mu, x) = \frac{\sum_{j=1}^{m} p_j \log p_j}{\sum_{j=1}^{m} p_j \log s_j},
\]

from which we get the dimension of $\mu$.

Now let $\nu_\alpha$ be the invariant measure of $(\Sigma, U, P_\alpha)$, where $P_\alpha = (p_1^{a_1 s_1^{-h(\alpha)}}, \ldots, p_m^{a_m s_m^{-h(\alpha)}})$, and let $\mu_\alpha$ be its image under $\Phi$ which is the invariant measure of $(X, W, P_\alpha)$. In the same manner as above, we can assert that $\nu_\alpha$-a.e.
\[
\lim_{r \to 0} \frac{\log r}{n(\sigma, r)} = \lim_{r \to 0} \frac{\log r}{\vec{n}(\sigma, r)} = \sum_{j=1}^{m} p_j^{a_j s_j^{-h(\alpha)}} \log s_j.
\]
Let
\[ R = \left\{ \sigma \in \Sigma : \lim_{n \to 0} \frac{\log r}{n(\sigma, r)} = \lim_{n \to 0} \frac{\log r}{n(\sigma, r)} = \ell(\sigma) \neq 0 \right\}. \]

We can think of the points of \( \Phi(R) \) as 'regular' points in the attractor \( \Lambda \). We have just seen that \( v_\alpha(R) = 1 \) and then \( \mu_\alpha(\Phi(R)) = 1 \). Therefore, for \( \sigma \in R \), we have the following relation between \( D(\mu_\alpha, \Phi(\sigma)) \) and \( D(\mu, \Phi(\sigma)) \): \( v \)-a.e.
\[ D(\mu_\alpha, \Phi(\sigma)) = \alpha D(\mu, \Phi(\sigma)) - h(\alpha). \]

In fact,
\[ \lim_{r \to 0} \frac{\log \mu(\Phi(\sigma))}{\log r} = \frac{1}{\ell(\sigma)} \lim_{n \to \infty} \frac{\log v(\Phi(\sigma))}{n} = \frac{1}{\ell(\sigma)} \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{m} \log p_{\sigma_j}, \]

and \( v \)-a.e.
\[ \lim_{r \to 0} \frac{\log \mu_\alpha(\Phi(\sigma))}{\log r} = \frac{1}{\ell(\sigma)} \lim_{n \to \infty} \frac{\log v_\alpha(\Phi(\sigma))}{n} = \frac{1}{\ell(\sigma)} \left( \alpha \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{m} \log p_{\sigma_j} - h(\alpha)\ell(\sigma) \right). \]

Notice that the existence of one of these limits implies the existence of others if \( \sigma \in R \) and \( n \) represents \( n(\sigma, r) \) or \( n(\sigma, r) \).

We also know \( v_\alpha \)-a.e.
\[ \ell(\sigma) = \sum_{j=1}^{m} p_{\sigma_j}^{\alpha} s_j^{-h(\alpha)} \log s_j. \]

Using this and the relation between \( D(\mu_\alpha, \Phi(\alpha)) \) and \( \log p_{\sigma_j} \) as above, we have \( v_\alpha \)-a.e.
\[ D(\mu_\alpha, \Phi(\sigma)) = \alpha h'(\alpha) - h(\alpha). \]

Then \( \mu_\alpha \)-a.e.
\[ D(\mu_\alpha, x) = \alpha h'(\alpha) - h(\alpha). \]

Suppose \( \alpha \neq 0 \) (\( \alpha = 0 \) is a trivial case). Using the relation between \( D(\mu_\alpha, \Phi(\sigma)) \) and \( D(\mu, \Phi(\sigma)) \), we see that for \( x \in \Phi(R) \cap E_{h'(\alpha)} \),
\[ D(\mu_\alpha, x) = \alpha h'(\alpha) - h(\alpha). \]
By the Billingsley theorem, it follows that
\[
\dim E_{h'(\alpha)} \geq \dim \Phi(R) \bigcap E_{h'(\alpha)} = \alpha h'(\alpha) - h(\alpha).
\]

In order to obtain the inverse inequality, observe that for a given \( \epsilon > 0 \) and for every \( x \in E_{h'(\alpha)} \),
\[
r^{h'(\alpha - \epsilon)} < \mu(B_r(x)) < r^{h'(\alpha + \epsilon)}
\]
for \( r \) sufficiently small. The collection of balls \( B_r(x) \) satisfying the last inequalities is then a Vitali covering of \( E_{h'(\alpha)} \). We can extract, for any \( \delta > 0 \), a subcollection of disjoint balls with radii smaller than \( \delta \) such that all balls with the same centers as the disjoint one, but with radii four times greater, form a \( \delta \)-covering of \( E_{h'(\alpha)} \). Let \( B_r(x) \) be a ball in the subcollection. We can find a unique \( \sigma \in \Sigma \) such that \( x = \Phi(\sigma) \). The last inequalities imply
\[
r^{h'(\alpha - \epsilon)} < \nu(I_{\Phi(\sigma)}(\sigma)) \leq M \nu(I_{\Phi(\sigma)}(\sigma)) < Mr^{h'(\alpha + \epsilon)},
\]
where \( M \) is a constant (we knew that \( \nu \) is double). Suppose \( \alpha - \epsilon > 0 \). The first inequality implies
\[
\mu_{\alpha - \epsilon}(B_r(x)) \geq \nu_{\alpha - \epsilon}(I_{\Phi(\sigma)}(\sigma)) = \prod_{j=1}^{n(\sigma)} \frac{a_j}{a_j} \prod_{j=1}^{n(\sigma)} \frac{a_j}{a_j} \geq C \prod_{j=1}^{n(\sigma)} \frac{a_j}{a_j} \geq C' \prod_{j=1}^{n(\sigma)} \frac{a_j}{a_j} - h(\alpha - \epsilon),
\]
where \( C \) and \( C' \) are absolute constants. The same can also be obtained if \( \alpha - \epsilon < 0 \). It follows that \( \dim E_{h'(\alpha)} \leq (\alpha - \epsilon) h'(\alpha - \epsilon) - h(\alpha - \epsilon) \) for \( \epsilon > 0 \). So \( \dim E_{h'(\alpha)} \leq \alpha h'(\alpha) - h(\alpha) \).

We have just proved the stated result for \( \beta \in (\beta_{\min}, \beta_{\max}) \). If \( \beta \) is equal to one of the extremal points of this interval, the very same formula holds, and this can be proved in the same manner as in the proof of the symbolic case.

Suppose \( \rho_1 = \cdots = \rho_m = D \). We have \( p_j = s_j^D \) for every \( 1 \leq j \leq m \). In virtue of the comparison between \( \mu \) and \( \nu \) given at the beginning of the proof, we can obtain for \( \sigma \in \Sigma \) that
\[
c^{-1} r^D \leq \mu(B_r(\Phi(\sigma))) \leq cr^D,
\]
where $c$ is that involved in the definitions of $\tilde{r}(\sigma, r)$ and $\tilde{r}(\sigma, r)$. This is stronger than what we stated in the theorem for this case. \hfill \Box

The dimension formula in this theorem was proved in [17] under exactly the same condition but from a different direction. In [29], it was also proved under a slightly weaker condition that there is a non-empty open set $U$ such that $w_j(U) \subset U$, $w_j(U)$ are disjoint and the intersection $w_j(U) \bigcap w_k(U)$ is of null measure. Theorem 7 was also proved in [9] in a different way. However, there is a common point which is the use of the measures $\mu_\alpha$. Our inspiration came from [14].

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