AFFINE LIE ALGEBRAS AND CALOGERO SYSTEMS

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1. Introduction

It is known that the Hamiltonian operator of a quantum Calogero system with a potential of the form $1/x^2$ and with a particular value of a parameter is the radial part of a Laplacian on the tangent space $\mathfrak{p}/K$ of a symmetric space P/K . We show that the Hamiltonian operator of a quantum Calogero system with a potential of the form $1/\sin^2 x$ and with a particular value of a parameter can be considered as the radial part of a Laplacian on an affine algebraic version $\overline{\mathfrak{p}}/\overline{K}$ of $\mathfrak{p}/K$.

One dimensional n -body quantum systems given by a Hamiltonian

$$H = -\frac{1}{2}\Delta + g^2 \sum_{i < j} U(x_i - x_j),$$

where Δ is the Laplacian $\sum_{i=1}^n \partial^2/\partial x_i^2$ on $\mathbb{R}^n$, g is a constant, and U is a function of the following types:

$$\begin{aligned} (1) \quad U(x) &= \frac{1}{x^2}, & (2) \quad U(x) &= \frac{1}{\sin^2(x)}, \\ (3) \quad U(x) &= \frac{1}{\sinh^2(x)}, & (4) \quad U(x) &= \wp(x), \end{aligned}$$

are called Calogero systems ($\wp$ is the Weierstrass $\wp$ -function). These and the corresponding classical systems were studied in [M], [CMR], [C], [W], [OP1] and [OP2]. The complete integrability of these systems in both quantum and classical cases was proved using techniques developed in [L] and [SK]. See review papers [OP3] and [OP4].

There is a Lie algebra and group theoretic proof of the complete integrability of systems (1), (2) and (3) (see [OP2], [OP4]). In this paper we are particularly interested in systems (1) and (2). Let $G = SU(n)$ be the compact Lie group consisting of n by n unitary matrices g such that $\det(g) = 1$, and let $\mathfrak{g} = \mathfrak{su}(n)$ be

its Lie algebra consisting of n by n matrices x such that $x + \bar{x} = 0$ and $\text{tr}(x) = 0$. Let $(\bar{\cdot}) : G \rightarrow G$ and $(\bar{\cdot}) : \mathfrak{g} \rightarrow \mathfrak{g}$ be the complex conjugation maps and put $K = \{g \in G \mid \bar{g} = g\} = SO(n)$, $\mathfrak{p} = \{x \in \mathfrak{g} \mid \bar{x} = -x\}$. Then we have $\mathfrak{p}/\text{Ad } K \cong \mathfrak{h}/W$, where $\text{Ad } K$ is the adjoint action of K on $\mathfrak{p}$, $\mathfrak{h}$ = (the diagonal matrices in $\mathfrak{g}$) $\cong \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$, and $W = \mathfrak{S}_n$ is the Weyl group acting on $\mathfrak{h}$ by permuting the entries. The radial part of the Laplacian $\Delta_{\mathfrak{p}}$ on $\mathfrak{p}$ is given by

$$\mathring{\Delta}_{\mathfrak{p}} = j^{-1/2} \circ (\Delta_{\mathfrak{h}} - V) \circ j^{1/2},$$

where

$$j(x) = \prod_{\alpha \in R/\pm} |\langle \alpha, x \rangle|, \quad V(x) = \frac{1}{2} \sum_{\alpha \in R/\pm} \frac{1}{\langle \alpha, x \rangle^2} = \frac{1}{2} \sum_{i < j} \frac{1}{(x_i - x_j)^2},$$

and $R = \{h_i - h_j \mid i \neq j\}$ is the root system of type A_{n-1} (see [H1] and [H2]). So we have

$$H = -\frac{1}{2} (j^{1/2} \circ \mathring{\Delta}_{\mathfrak{p}} \circ j^{-1/2}),$$

with $g^2 = 1/4$. Let $\{\Delta_1 = \Delta_{\mathfrak{p}}, \Delta_2, \dots, \Delta_{n-1}\}$ be a set of generators of the algebra of K -invariant differential operators on $\mathfrak{p}$. The radial parts $\mathring{\Delta}_1, \dots, \mathring{\Delta}_{n-1}$ commute pairwise and are functionally independent. Therefore, $I_k = j^{1/2} \circ \mathring{\Delta}_k \circ j^{-1/2}$ are integrals of the Calogero system (1) and the system is completely integrable.

Now, the theory of affine and loop Lie algebras has remarkable similarities with the theory of finite dimensional simple Lie algebras (see [F]). Therefore, we might formally replace the root system R by the real affine root system $\widehat{R} = \{\alpha + kc \mid \alpha \in R, k \in \mathbb{Z}\}$ of type $\widehat{A}_{n-1}$. Then the compact Lie algebra $\mathfrak{g}$ is replaced by the compact loop algebra $\overline{\mathfrak{g}}^\infty = \{x : \mathbb{R} \rightarrow \mathfrak{g} \text{ (smooth)} \mid x(\tau+1) = x(\tau)\}$, and the subspace $\mathfrak{p}$ is replaced by $\overline{\mathfrak{p}}^\infty = \{x \in \overline{\mathfrak{g}}^\infty \mid \overline{x(\tau)} = -x(-\tau)\}$. The compact group G is replaced by the compact loop group $\overline{G}^\infty = \{g : \mathbb{R} \rightarrow G \text{ (smooth)} \mid g(\tau+1) = g(\tau)\}$, and the subgroup K is replaced by $\overline{K}^\infty = \{g \in \overline{G}^\infty \mid \overline{g(\tau)} = g(-\tau)\}$. We expect an analogy $\overline{\mathfrak{p}}^\infty / \overline{\text{Ad } \overline{K}^\infty} \cong \mathfrak{h} / \widetilde{W}$, where $\widetilde{W}$ is the affine Weyl group, and the radial part of the ‘Laplacian’ $\Delta_{\overline{\mathfrak{p}}}$ on $\overline{\mathfrak{p}}^\infty$ should satisfy

$$\mathring{\Delta}_{\overline{\mathfrak{p}}} = \widehat{j}^{-1/2} \circ (\Delta_{\mathfrak{h}} - \widehat{V}) \circ \widehat{j}^{1/2},$$

where, formally,

$$\widehat{j}(x) = \prod_{\alpha \in \widehat{R}/\pm} |\langle \alpha, x \rangle| \quad \text{and} \quad \widehat{V}(x) = \frac{1}{2} \sum_{\alpha \in \widehat{R}/\pm} \frac{1}{\langle \alpha, x \rangle^2}.$$

We can make sense of the divergent infinite product

$$\widehat{j}(x) = \prod_{\substack{\alpha \in R/\pm \\ k \in \mathbb{Z}}} |\langle \alpha, x \rangle + kc|$$

by using the zeta-regularization method (see [RS]). Namely, we define

$$\widehat{j}(x) = \exp \left(- \frac{d}{ds} \Big|_{s=0} \sum_{\substack{\alpha \in R/\pm \\ k \in \mathbb{Z}}} |\langle \alpha, x \rangle + kc|^{-s} \right),$$

then we have

$$\widehat{j}(x) = \prod_{\alpha \in R/\pm} |\sin \langle \alpha, x \rangle|$$

when $c = \pi$. Also

$$\widehat{V}(x) = \frac{1}{2} \sum_{\substack{\alpha \in R/\pm \\ k \in \mathbb{Z}}} \frac{1}{(\langle \alpha, x \rangle + kc)^2} = \frac{1}{2} \sum_{\alpha \in R/\pm} \frac{1}{\sin^2 \langle \alpha, x \rangle} = \frac{1}{2} \sum_{i < j} \frac{1}{\sin^2(x_i - x_j)}$$

when $c = \pi$, and it is the potential for system (2). We have

$$H = -\frac{1}{2}(\widehat{j}^{1/2} \circ \mathring{\Delta}_{\overline{\mathfrak{p}}} \circ \widehat{j}^{-1/2}),$$

with $g^2 = 1/4$.

The objective of this paper is to show that the above formal arguments can be proved rigorously with appropriate modifications. Although this does not provide a new proof of the complete integrability of the Calogero system (2), this is yet another application of affine Lie algebra theory.

2. Simple and affine Lie algebras

Following [FK] and [S], we show a realization of the complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ consisting of the traceless n by n matrices with the Lie bracket $[x, y] = xy - yx$, and its generalization to the affine Lie algebra.

Definition 2.1. Let $R = \{h_i - h_j \mid i \neq j\} \subset \sum_{i=1}^n \mathbb{Z}h_i$ be the roots of A_{n-1} -type, and let L be the root lattice. We have $R = \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$. Let $\varepsilon : L \times L \rightarrow \{\pm 1\}$

be a bilinear 2-cocycle defined by $\varepsilon(h_i, h_j) = 1$ if $i \leq j$, and $\varepsilon(h_i, h_j) = -1$ if $i > j$. The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is given by

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathbb{C}e^{\alpha},$$

where $\mathfrak{h}_{\mathbb{C}} = L \otimes_{\mathbb{Z}} \mathbb{C}$ is the Cartan subalgebra consisting of the diagonal matrices in $\mathfrak{g}_{\mathbb{C}}$, and e^{α} is defined by $e^{h_i - h_j} = E_{i,j}$ if $i < j$, and $e^{h_i - h_j} = -E_{i,j}$ if $i > j$. ($E_{i,j}$ is the elementary matrix.) The Lie bracket is given by

$$\begin{aligned} [\alpha, \beta] &= 0 \quad (\alpha, \beta \in \mathfrak{h}_{\mathbb{C}}) \\ [\alpha, e^{\beta}] &= \langle \alpha, \beta \rangle e^{\beta} \quad (\alpha \in \mathfrak{h}_{\mathbb{C}}, \beta \in R) \\ [e^{\alpha}, e^{\beta}] &= \begin{cases} -\alpha & (\alpha + \beta = 0) \\ \varepsilon(\alpha, \beta) e^{\alpha + \beta} & (\alpha + \beta \in R) \\ 0 & (\text{otherwise}) \end{cases} \quad (\alpha, \beta \in R). \end{aligned}$$

We define two antilinear involutions $(\bar{})$ and ω on $\mathfrak{g}_{\mathbb{C}}$ by $(\bar{}) =$ the complex conjugation, $\overline{e^{\alpha}} = e^{\alpha}$, $\overline{\alpha} = \alpha$, and $\omega(x) = -\bar{x}$, $\omega(e^{\alpha}) = e^{-\alpha}$, $\omega(\alpha) = -\alpha$ ($\alpha \in R$). The Lie algebra $\mathfrak{g} = \{x \in \mathfrak{g}_{\mathbb{C}} \mid \omega(x) = x\}$ is the compact form of $\mathfrak{g}_{\mathbb{C}}$. We put $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} = \{x \in \mathfrak{g} \mid \bar{x} = x\} = \mathfrak{so}(n)$ and $\mathfrak{p} = \{x \in \mathfrak{g} \mid \bar{x} = -x\}$. We define $\langle x, y \rangle = \text{tr}(xy)$, which is an extension of $\langle , \rangle$ on $\mathfrak{h}_{\mathbb{C}}$ with $\langle \alpha, e^{\beta} \rangle = 0$, and $\langle e^{\alpha}, e^{\beta} \rangle = -\delta_{\alpha + \beta, 0}$.

Definition 2.2. We generalize the above construction to the affine case. $\overline{\mathfrak{g}}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}[z, z^{-1}]$ is a Lie algebra with the Lie bracket $[x \otimes z^n, y \otimes z^m] = [x, y] \otimes z^{n+m}$, and $\widehat{\mathfrak{g}}_{\mathbb{C}} = \overline{\mathfrak{g}}_{\mathbb{C}} \oplus \mathbb{C}c$ is a Lie algebra with the Lie bracket $[x \otimes z^n, y \otimes z^m] = [x, y] \otimes z^{n+m} + \langle x, y \rangle n \delta_{n+m, 0} c$, $[x \otimes z^n, c] = [c, c] = 0$. They are called the affine Lie algebras of $\mathfrak{g}_{\mathbb{C}}$. We put $\overline{\mathfrak{h}}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \otimes \mathbb{C}[z, z^{-1}]$, and $\widehat{\mathfrak{h}}_{\mathbb{C}} = \overline{\mathfrak{h}}_{\mathbb{C}} \oplus \mathbb{C}c$, the Heisenberg algebra of $\mathfrak{h}_{\mathbb{C}}$. Let $\widehat{R} = \{\alpha + kc \mid \alpha \in R, k \in \mathbb{Z}\} \subset \overline{\mathfrak{h}} \oplus \mathbb{C}c$ be the real affine roots. We put $e^{\alpha + kc} = e^{\alpha} \otimes z^k$, and we have an analogue of the simple Lie algebra

$$\overline{\mathfrak{g}}_{\mathbb{C}} = \overline{\mathfrak{h}}_{\mathbb{C}} \oplus \sum_{\alpha \in \widehat{R}} \mathbb{C}e^{\alpha}, \quad \widehat{\mathfrak{g}}_{\mathbb{C}} = \widehat{\mathfrak{h}}_{\mathbb{C}} \oplus \sum_{\alpha \in \widehat{R}} \mathbb{C}e^{\alpha}.$$

We generalize the two antilinear involutions $(\bar{})$ and ω to the affine case. We define $\overline{e^{\alpha + kc}} = e^{\alpha + kc}$, $\overline{\alpha \otimes z^k} = \alpha \otimes z^k$, and $\omega(e^{\alpha + kc}) = e^{-\alpha - kc}$, $\omega(\alpha \otimes z^k) = -\alpha \otimes z^{-k}$ ($\alpha \in R$). The Lie algebra $\overline{\mathfrak{g}} = \{x \in \overline{\mathfrak{g}}_{\mathbb{C}} \mid \omega(x) = x\}$ is called the compact form of $\overline{\mathfrak{g}}_{\mathbb{C}}$. We put $\overline{\mathfrak{g}} = \overline{\mathfrak{k}} \oplus \overline{\mathfrak{p}}$, where $\overline{\mathfrak{k}} = \{x \in \overline{\mathfrak{g}} \mid \bar{x} = x\}$, $\overline{\mathfrak{p}} = \{x \in \overline{\mathfrak{g}} \mid \bar{x} = -x\}$.

3. Orbital theory for $\bar{\mathfrak{g}}$

Let G be the compact Lie group $SU(n)$ whose Lie algebra is $\mathfrak{g} = su(n)$. We fix a length of loops $T > 0$. (In the Introduction, we have set $T = 1$ for simplicity.) The smooth loop group is $\overline{G}^\infty = \{g : \mathbb{R} \rightarrow G \text{ (smooth)} \mid g(\tau + T) = g(\tau)\}$ with the pointwise composition. The smooth loop algebra is $\overline{\mathfrak{g}}^\infty = \{x : \mathbb{R} \rightarrow \mathfrak{g} \text{ (smooth)} \mid x(\tau + T) = x(\tau)\}$ with the pointwise Lie bracket. We regard the affine Lie algebra $\bar{\mathfrak{g}}$ defined in Definition 2.2. as a subalgebra of the loop algebra $\overline{\mathfrak{g}}^\infty$ by setting $z = e^{2\pi i \tau / T}$. Following [F], we show some results on orbital theory for $\bar{\mathfrak{g}}^\infty$.

PROPOSITION 3.1. ([F, §3.1, §3.2]) $\overline{G}^\infty$ acts on $\overline{\mathfrak{g}}^\infty$ with the formula

$$g \cdot x = gxg^{-1} - \frac{dg}{d\tau}g^{-1} \quad (g \in \overline{G}^\infty, x \in \overline{\mathfrak{g}}^\infty).$$

This action is called the adjoint action of $\overline{G}^\infty$ on $\overline{\mathfrak{g}}^\infty$ and is denoted by $\overline{\text{Ad}} \overline{G}^\infty$.

Let $\overline{G}_*^\infty = \{f : [0, T] \rightarrow G \text{ (smooth)} \mid f(0) = I\}$ be the smooth path group. Then $\overline{G}^\infty$ acts on $\overline{G}_*^\infty$ with the formula

$$g \cdot f = g(0)fg^{-1} \quad (g \in \overline{G}^\infty, f \in \overline{G}_*^\infty).$$

This action is called the adjoint action of $\overline{G}^\infty$ on $\overline{G}_*^\infty$ and is denoted by $\overline{\text{Ad}}_* \overline{G}^\infty$. We denote the adjoint action of G on itself by $\text{Ad } G$. There are the following isomorphisms:

$$\begin{array}{ccccc} \overline{\mathfrak{g}}^\infty / \overline{\text{Ad}} \overline{G}^\infty & \cong & \overline{G}_*^\infty / \overline{\text{Ad}}_* \overline{G}^\infty & \cong & \mathfrak{G} / \text{Ad } G \\ x & \longleftrightarrow & f & \longleftrightarrow & c. \end{array}$$

The relations between x, f, c are $df/d\tau = fx$, $c = f(T)$. From c , we can construct x and f by $x = (1/T) \log c$, $f(\tau) = e^{\tau x}$. Thus $\overline{\mathfrak{g}}^\infty / \overline{\text{Ad}} \overline{G}^\infty$ is finite dimensional and every $\overline{G}^\infty$ -orbit in $\overline{\mathfrak{g}}^\infty$ has a constant loop.

The results are known classically as Froquest theory in the theory of differential equations.

4. Laplacian on $\bar{\mathfrak{g}}$

Following [F], we show that the radial part of the ‘Laplacian’ on $\bar{\mathfrak{g}}$ is considered as a Hamiltonian with a constant potential.

Definition 4.1. ([F, §3.3]) Let $\bar{\mathfrak{g}}_*^\infty = \{y : [0, T] \rightarrow \mathfrak{g} \text{ (smooth)} \mid y(0) = 0\}$ be the smooth path algebra. We define the mappings

$$\bar{\mathfrak{g}}^\infty \xrightarrow{s} \bar{\mathfrak{g}}_*^\infty \xrightarrow{i} \bar{G}_*^\infty \xrightarrow{e} G.$$

For $x \in \bar{\mathfrak{g}}^\infty$, we put $(s(x))(\tau) = \int_0^\tau x(\tau) d\tau$. For $y \in \bar{\mathfrak{g}}_*^\infty$, $f = i(y)$ is the unique solution to $df/d\tau = f dy/d\tau$ and $f(0) = I$. For $f \in \bar{G}_*^\infty$, we put $e(f) = f(T)$. Our notation is slightly different from that in [F]. In this paper, Frenkel constructed the following extensions of the above mappings to appropriate completions.

Definition 4.2. ([F, §3.3]) We introduce a norm $|x| = \sup_{0 \leq \tau \leq T} |(s(x))(\tau)|$ and we denote the completion of $\bar{\mathfrak{g}}^\infty$ by $\bar{\mathfrak{g}}^c$ where $\bar{\mathfrak{g}}_*^c = \{y : [0, T] \rightarrow \mathfrak{g} \text{ (continuous)} \mid y(0) = 0\}$ is the completion of $\bar{\mathfrak{g}}_*^\infty$ with respect to the sup-norm. $\bar{G}_*^c = \{f : [0, T] \rightarrow G \text{ (continuous)} \mid f(0) = I\}$ is the completion of $\bar{G}_*^\infty$ with respect to a metric $\rho(f_1, f_2) = \sup_{0 \leq \tau \leq T} \rho_0(f_1(\tau), f_2(\tau))$, where $\rho_0(g_1, g_2)$ is the length of the shortest geodesic between two points g_1 and g_2 of G . We have

$$\bar{\mathfrak{g}}^c \xrightarrow{s} \bar{\mathfrak{g}}_*^c \xrightarrow{i} \bar{G}_*^c \xrightarrow{e} G.$$

Definition 4.3. ([F, §§5.1–5.3, §6.1]) Let $d\mu^t$ be the Gaussian measure on $\bar{\mathfrak{g}}^c$ written formally as

$$d\mu^t(x) = \left\{ \exp \left[-\frac{1}{2t} \frac{1}{T} \int_0^T |x|^2 d\tau \right] \right\} dx.$$

Let dw^t be the Wiener measure on $\bar{\mathfrak{g}}_*^c$ written formally as

$$dw^t(y) = \left\{ \exp \left[-\frac{1}{2t} \frac{1}{T} \int_0^T \left| \frac{dy}{d\tau} \right|^2 d\tau \right] \right\} dy.$$

Let dw^t be the Wiener measure of on $\bar{G}_*^c$ written formally as

$$dw^t(f) = \left\{ \exp \left[-\frac{1}{2t} \frac{1}{T} \int_0^T \left| f^{-1} \frac{df}{d\tau} \right|^2 d\tau \right] \right\} df.$$

For rigorous definitions, see [F]. Let dg be the Haar measure on G . We define the following mappings:

$$L_1(\bar{\mathfrak{g}}^c, d\mu^t) \xrightarrow{S} L_1(\bar{\mathfrak{g}}_*^c, dw^t) \xrightarrow{I} L_1(\bar{G}_*^c, dw^t) \xrightarrow{E_t} L_1(G, \nu_t(g, T)dg).$$

For $F \in L_1(\bar{\mathfrak{g}}^c, d\mu^t)$, S is defined by $(S(F))(y) = F(s^{-1}(y))$. Since $(\bar{\mathfrak{g}}^c, d\mu^t)$ and $(\bar{\mathfrak{g}}_*^c, dw^t)$ are isomorphic as measured spaces, S is an isometry. I is the Ito isometry

defined by $(I(F))(f) = F(i^{-1}(f))$ for $F \in L_1(\overline{\mathfrak{g}}_*^c, dw')$. For $F \in L_1(\overline{G}_*^c, dw')$, we define

$$(E_t(F))(g) = \frac{1}{v_t(g, T)} \int_{f \in \overline{G}_*^c, f(T)=g} F(f) dw'(f),$$

where

$$v_t(g, T) = \int_{f \in \overline{G}_*^c, f(T)=g} dw'(f).$$

THEOREM 4.4. ([F, §§5.1–5.3, §§6.1–6.3])

(1) We have the following commutative diagram:

$$\begin{array}{ccccccc} L_\infty(\overline{\mathfrak{g}}^c) & \xrightarrow{S} & L_\infty(\overline{\mathfrak{g}}_*^c) & \xrightarrow{I} & L_\infty(\overline{G}_*^c) & \xrightarrow{E_{t+s}} & L_\infty(G) \\ \downarrow U_s & & \downarrow U_s & & \downarrow U_s & & \downarrow U_s \\ L_\infty(\overline{\mathfrak{g}}^c) & \xrightarrow{S} & L_\infty(\overline{\mathfrak{g}}_*^c) & \xrightarrow{I} & L_\infty(\overline{G}_*^c) & \xrightarrow{E_t} & L_\infty(G) \end{array}$$

where, U_s ($s \geq 0$) are semigroups on each space defined as follows:

$$\begin{aligned} U_s F(x_0) &= \int_{\overline{\mathfrak{g}}^c} F(x_0 + x) d\mu^s(x) \quad (F \in L_\infty(\overline{\mathfrak{g}}^c)), \\ U_s F(y_0) &= \int_{\overline{\mathfrak{g}}_*^c} F(y_0 + y) dw^s(y) \quad (F \in L_\infty(\overline{\mathfrak{g}}_*^c)), \\ U_s F(f_0) &= \int_{\overline{G}_*^c} F(f f_0) dw^s(f) \quad (F \in L_\infty(\overline{G}_*^c)), \\ U_s F(g_0) &= \int_G F(g g_0) v_s(g, T) dg \quad (F \in L_\infty(G)). \end{aligned}$$

We have $U_s = e^{sT^2 \Delta_G/2}$ on $L_\infty(G)$.

(2) In particular, if we restrict our attention to the $\overline{G}^c$ -invariant functions, then we have

$$\begin{array}{ccc} L_\infty^{\overline{G}^c}(\overline{\mathfrak{g}}^c) & \xrightarrow{\Pi} & L_\infty^G(G) \\ \downarrow U_s & & \downarrow U_s \\ L_\infty^{\overline{G}^c}(\overline{\mathfrak{g}}^c) & \xrightarrow{\Pi} & L_\infty^G(G) \end{array}$$

where $\Pi = E \circ I \circ S$. (E_t does not depend on t .)

THEOREM 4.5. ([B], [HC], [H1], [H2]) Let $\mathbf{T}$ be the maximal torus of G consisting of all diagonal matrices in G , and let $W = \mathfrak{S}_n$ be the Weyl group. Then W acts on

$\mathbf{T}$ by permuting the diagonal components. Let $\tilde{W}$ be the affine Weyl group which acts on $\mathfrak{h}$. Then we have isomorphisms

$$\overline{\mathfrak{g}}^\infty / \overline{\text{Ad}} \overline{G}^\infty \cong G / \text{Ad } G \cong \mathbf{T} / W \cong \mathfrak{h} / \tilde{W},$$

which is an extension of the isomorphisms in Proposition 3.1. $x \in \mathfrak{h}$ is mapped to $e^{Tx} \in \mathbf{T}$, and is regarded as a constant loop in $\overline{\mathfrak{g}}^\infty$. We have

$$\mathring{\Delta}_G = \sigma_T^{-1} \circ (\Delta_{\mathfrak{h}} + \|\rho\|^2) \circ \sigma_T,$$

where

$$\sigma_T(x) = \prod_{\alpha \in R/\pm} 2 \left| \sin \frac{\langle \alpha, Tx \rangle}{2} \right|,$$

and

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha, \quad \|\rho\|^2 = \langle \rho, \rho \rangle = \frac{(n+1)n(n-1)}{12}.$$

THEOREM 4.6. *By an analogy with Euclidean spaces, we define the Laplacian $\Delta_{\overline{\mathfrak{g}}}$ on $\overline{\mathfrak{g}}^c$ by*

$$\frac{1}{2} \Delta_{\overline{\mathfrak{g}}} F(x) = \lim_{s \rightarrow 0} \frac{1}{s} (U_s F(x) - F(x)) \quad (F \in L_\infty(\overline{\mathfrak{g}}^c), x \in \overline{\mathfrak{g}}^c).$$

The radial part of the Laplacian $\mathring{\Delta}_{\overline{\mathfrak{g}}}$ is $\Delta_{\overline{\mathfrak{g}}}$ restricted to the $\overline{G}^c$ -invariant functions. Then we have $T^2 \mathring{\Delta}_G \circ \Pi = \Pi \circ \mathring{\Delta}_{\overline{\mathfrak{g}}}$, and therefore we have

$$\mathring{\Delta}_{\overline{\mathfrak{g}}} = T^2 \sigma_T^{-1} \circ (\Delta_{\mathfrak{h}} + \|\rho\|^2) \circ \sigma_T.$$

So $\mathring{\Delta}_{\overline{\mathfrak{g}}}$ corresponds to a Hamiltonian with a constant potential.

5. Orbital theory for $\bar{\mathfrak{p}}$

In this section, we describe orbital theory for $\bar{\mathfrak{p}}$. Let $(\bar{\cdot}) : G \rightarrow G$ and $(\bar{\cdot}) : \mathfrak{g} \rightarrow \mathfrak{g}$ be the complex conjugation maps. The subgroup $K = \{g \in G \mid \bar{g} = g\} = SO(n)$ has $\mathfrak{k} = \{x \in \mathfrak{g} \mid \bar{x} = x\} = \mathfrak{so}(n)$ as its Lie algebra. Let $P = G/K$ be a symmetric space of compact type. Then $\mathfrak{p} = \{x \in \mathfrak{g} \mid \bar{x} = -x\}$ is the tangent space of P at the origin. We can regard $P = \{gK \mid g \in G\}$ as a subset of G in the following way:

$$P \cong \{g\bar{g}^{-1} \mid g \in G\}, \quad gK \longleftrightarrow g\bar{g}^{-1}.$$

G acts on P from the left i.e. $gK \mapsto hgK$. In other words, $g\bar{g}^{-1} \mapsto h(g\bar{g}^{-1})\bar{h}^{-1}$. In particular, K acts on P by $g\bar{g}^{-1} \mapsto k(g\bar{g}^{-1})k^{-1}$. This action is called the

adjoint action of K on P and is denoted by $\text{Ad } K$. The group K also acts on $\mathfrak{p}$ by $k \cdot x = kxk^{-1}$ ($k \in K, x \in \mathfrak{p}$). This action is called the adjoint action of K on $\mathfrak{p}$ and is denoted by $\text{Ad } K$.

PROPOSITION 5.1. *Let $\overline{G}_{(2)}^\infty = \{g : \mathbb{R} \rightarrow G \text{ (smooth)} \mid g(\tau + 2T) = g(\tau)\}$ be the smooth loop group of double length, and let $\overline{K}^\infty = \{g \in \overline{G}_{(2)}^\infty \mid \overline{g(\tau)} = g(-\tau)\}$ be a subgroup of $\overline{G}_{(2)}^\infty$. Let $\overline{\mathfrak{g}}_{(2)}^\infty = \{x : \mathbb{R} \rightarrow \mathfrak{g} \text{ (smooth)} \mid x(\tau + 2T) = x(\tau)\}$ be the smooth loop algebra of double length, and let $\overline{\mathfrak{p}}^\infty = \{x \in \overline{\mathfrak{g}}_{(2)}^\infty \mid \overline{x(\tau)} = -x(-\tau)\}$ be a subspace of $\overline{\mathfrak{g}}_{(2)}^\infty$. Note that $\overline{K}^\infty$ are not the loops in K , and $\overline{\mathfrak{p}}^\infty$ are not the loops in $\mathfrak{p}$. Also note that $\overline{\mathfrak{p}}$ in Definition 2.2. is a subspace of $\overline{\mathfrak{p}}^\infty$. $\overline{K}^\infty$ acts on $\overline{\mathfrak{p}}^\infty$ with the formula*

$$g \cdot x = gxg^{-1} - \frac{dg}{d\tau}g^{-1} \quad (g \in \overline{K}^\infty, x \in \overline{\mathfrak{p}}^\infty).$$

This action is called the adjoint action of $\overline{K}^\infty$ on $\overline{\mathfrak{p}}^\infty$ and is denoted by $\overline{\text{Ad}} \overline{K}^\infty$. Let $\overline{G}_{(2)}^\infty = \{f : [-T, T] \rightarrow G \text{ (smooth)} \mid f(0) = I\}$ be the smooth path group of double length, and let $\overline{K}_*^\infty = \{f \in \overline{G}_{*(2)}^\infty \mid \overline{f(\tau)} = f(-\tau)\}$ be a subset of $\overline{G}_{*(2)}^\infty$. Then $\overline{K}^\infty$ acts on $\overline{K}_*^\infty$ with the formula*

$$g \cdot f = g(0)fg^{-1} \quad (g \in \overline{K}^\infty, f \in \overline{K}_*^\infty).$$

This action is called the adjoint action of $\overline{K}^\infty$ on $\overline{K}_^\infty$ and is denoted by $\overline{\text{Ad}}_* \overline{K}^\infty$.*

We have the following isomorphisms similar to those in Proposition 3.1:

$$\begin{array}{ccccc} \overline{\mathfrak{p}}^\infty / \overline{\text{Ad}} \overline{K}^\infty & \cong & \overline{K}_*^\infty / \overline{\text{Ad}}_* \overline{K}^\infty & \cong & P / \text{Ad } K \\ x & \longleftrightarrow & f & \longleftrightarrow & c. \end{array}$$

The relations between x , f and c are $df/d\tau = fx$ and $c = f(T)f(-T)^{-1}$. From c , we can construct x and f by $x = (1/2T) \log c$ and $f(\tau) = e^{\tau x}$. So every $\overline{K}^\infty$ -orbit in $\overline{\mathfrak{p}}^\infty$ has a constant loop. Note that we have

$$c = f(T)\overline{f(T)}^{-1} \leftrightarrow f(T)K.$$

6. The Laplacian on $\overline{\mathfrak{p}}$

We construct completions for $\overline{\mathfrak{p}}^\infty$ and we show that the radial part of the Laplacian on $\overline{\mathfrak{p}}$ is considered as a Hamiltonian of the Calogero system (2).

Definition 6.1. Let $\bar{\mathfrak{g}}_{*(2)}^\infty = \{y : [-T, T] \rightarrow \mathfrak{g} \text{ (smooth)} \mid y(0) = 0\}$ be the smooth path algebra of double length, and let $\bar{\mathfrak{k}}_*^\infty = \{y \in \bar{\mathfrak{g}}_{*(2)}^\infty \mid \overline{y(\tau)} = y(-\tau)\}$ be a subspace of $\bar{\mathfrak{g}}_{*(2)}^\infty$. We have the following mappings:

$$\bar{\mathfrak{p}}^\infty \xrightarrow{s} \bar{\mathfrak{k}}_*^\infty \xrightarrow{i} \bar{K}_*^\infty \xrightarrow{e} P$$

which are the restrictions of the mappings in Definition 4.1.

PROPOSITION 6.2. Let $\bar{\mathfrak{g}}_{(2)}^c$, $\bar{\mathfrak{g}}_{*(2)}^c$ and $\bar{G}_{*(2)}^c$ be the completions of $\bar{\mathfrak{g}}_{(2)}^\infty$, $\bar{\mathfrak{g}}_{*(2)}^\infty$ and $\bar{G}_{*(2)}^\infty$. Let $\bar{\mathfrak{p}}^c = \{x \in \bar{\mathfrak{g}}_{(2)}^c \mid \overline{x(\tau)} = -x(-\tau)\}$ be a subspace of $\bar{\mathfrak{g}}_{(2)}^c$. Let $\bar{\mathfrak{k}}_*^c = \{y \in \bar{\mathfrak{g}}_{*(2)}^c \mid \overline{y(\tau)} = y(-\tau)\}$ be a subspace of $\bar{\mathfrak{g}}_{*(2)}^c$. Let $\bar{K}_*^c = \{f \in \bar{G}_{*(2)}^c \mid \overline{f(\tau)} = f(-\tau)\}$ be a subset of $\bar{G}_{*(2)}^c$. We have

$$\bar{\mathfrak{p}}^c \xrightarrow{s} \bar{\mathfrak{k}}_*^c \xrightarrow{i} \bar{K}_*^c \xrightarrow{e} P$$

which are the restrictions of the mappings in Definition 4.2. We have isomorphisms of measured spaces by the restriction $\bar{\mathfrak{k}}_*^c \cong \bar{\mathfrak{g}}_*^c$, $\bar{K}_*^c \cong \bar{G}_*^c$. Moreover, we have the following commutative diagram

$$\begin{array}{ccccccc} \bar{\mathfrak{p}}^c & \xrightarrow{s} & \bar{\mathfrak{k}}_*^c & \xrightarrow{i} & \bar{K}_*^c & \xrightarrow{e} & P \\ \downarrow & & \downarrow \wr & & \downarrow \wr & & \uparrow \pi \\ \bar{\mathfrak{g}}^c & \xrightarrow{s} & \bar{\mathfrak{g}}_*^c & \xrightarrow{i} & \bar{G}_*^c & \xrightarrow{e} & G \end{array}$$

where $\pi(g) = g\bar{g}^{-1}$ is the projection from G to P .

THEOREM 6.3. Let dp be the G -invariant measure on P . We have

$$\int_G F(g) dg = \int_P \left(\int_K F(pk) dk \right) dp \quad (F \in L_1(G, dg)),$$

where dg, dk are the Haar measure on G and K , respectively. Define the averaging operator $M : L_1(G, dg) \rightarrow L_1(P, dp)$ by

$$(MF)(gK) = \int_K F(gk) dk.$$

(1) We have the following commutative diagram:

$$\begin{array}{ccccccccc} L_\infty(\bar{\mathfrak{p}}^c) & \xrightarrow{S} & L_\infty(\bar{\mathfrak{k}}_*^c) \cong L_\infty(\bar{\mathfrak{g}}_*^c) & \xrightarrow{I} & L_\infty(\bar{G}_*^c) & \xrightarrow{E_{I+S}} & L_\infty(G) & \xrightarrow{M} & L_\infty(P) \\ \downarrow U_s & & \downarrow U_s & & \downarrow U_s & & \downarrow U_s & & \downarrow U_s \\ L_\infty(\bar{\mathfrak{p}}^c) & \xrightarrow{S} & L_\infty(\bar{\mathfrak{k}}_*^c) \cong L_\infty(\bar{\mathfrak{g}}_*^c) & \xrightarrow{I} & L_\infty(\bar{G}_*^c) & \xrightarrow{E_I} & L_\infty(G) & \xrightarrow{M} & L_\infty(P) \end{array}$$

where U_s ($s \geq 0$) on $L_\infty(\overline{\mathfrak{p}}^c)$ and $L_\infty(P)$ are semigroups defined as follows:

$$\begin{aligned} U_s F(x_0) &= \int_{\overline{\mathfrak{p}}^c} F(x_0 + x) d\mu^s(x) \quad (F \in L_\infty(\overline{\mathfrak{p}}^c)), \\ U_s F(g_0 K) &= \int_G F(gK) v_s(gK, g_0 K, T) dg \quad (F \in L_\infty(P)), \\ v_s(gK, g_0 K, T) &= \int_K v_s(gkg_0^{-1}, T) dk. \end{aligned}$$

We have $U_s = e^{sT^2 \Delta_P/2}$ on $L_\infty(P)$.

(2) For $\overline{K}^c$ -invariant functions, we have

$$\begin{array}{ccc} L_\infty^{\overline{K}^c}(\overline{\mathfrak{p}}^c) & \xrightarrow{\Pi} & L_\infty^K(P) \\ \downarrow U_s & & \downarrow U_s \\ L_\infty^{\overline{K}^c}(\overline{\mathfrak{p}}^c) & \xrightarrow{\Pi} & L_\infty^K(P). \end{array}$$

where $\Pi = M \circ E \circ I \circ S$. (E_t does not depend on t .)

Proof. The theorem follows from the fact that $M \circ U_s = U_s \circ M$.

THEOREM 6.4. We have isomorphisms

$$\overline{\mathfrak{p}}^\infty / \overline{\text{Ad } \overline{K}^\infty} \cong P / \text{Ad } K \cong \mathbf{T} / W \cong \mathfrak{h} / \tilde{W}$$

which are extensions of the isomorphisms in Proposition 5.1. The radial part $\mathring{\Delta}_P$ is given by

$$\mathring{\Delta}_P = \widehat{j}_T^{-1/2} \circ (\Delta_{\mathfrak{h}} - \widehat{V}_T) \circ \widehat{j}_T^{1/2},$$

where

$$\widehat{j}_T(x) = \prod_{\alpha \in R/\pm} |\sin \langle \alpha, Tx \rangle|$$

and

$$\widehat{V}_T(x) = \frac{1}{2} \sum_{\alpha \in R/\pm} \frac{1}{\sin^2 \langle \alpha, Tx \rangle} + \|\rho\|^2 = \frac{1}{2} \sum_{i < j} \frac{1}{\sin^2 T(x_i - x_j)} + \|\rho\|^2, \quad x \in \mathfrak{h}.$$

Proof. The theorem follows from Proposition 3.11 in Chapter II of [H2] and the addition formula for the cotangent function.

THEOREM 6.5. *We define the Laplacian $\Delta_{\bar{\mathfrak{p}}}$ on $\bar{\mathfrak{p}}^c$ by*

$$\frac{1}{2}\Delta_{\bar{\mathfrak{p}}}F(x) = \lim_{s \rightarrow 0} \frac{1}{s}(U_s F(x) - F(x)) \quad (F \in L_\infty(\bar{\mathfrak{p}}^c), x \in \bar{\mathfrak{p}}^c).$$

The radial part of the Laplacian $\mathring{\Delta}_{\bar{\mathfrak{p}}}$ is $\Delta_{\bar{\mathfrak{p}}}$ restricted to the $\overline{K}^c$ -invariant functions. Then we have $T^2 \mathring{\Delta}_P \circ \Pi = \Pi \circ \mathring{\Delta}_{\bar{\mathfrak{p}}}$, and therefore we have

$$\mathring{\Delta}_{\bar{\mathfrak{p}}} = T^2 \widehat{j}_T^{-1/2} \circ (\Delta_{\mathfrak{h}} - \widehat{V}_T) \circ \widehat{j}_T^{1/2}.$$

So $\mathring{\Delta}_{\bar{\mathfrak{p}}}$ corresponds to the Hamiltonian of the Calogero system (2) when $T = 1$ and $g^2 = 1/4$.

In the finite-dimensional case, we have $\mathfrak{p}/\text{Ad } K \cong \mathfrak{h}/W$, and the radial part of the Laplacian $\Delta_{\mathfrak{p}}$ on $\mathfrak{p}$ is

$$\mathring{\Delta}_{\mathfrak{p}} = j^{-1/2} \circ (\Delta_{\mathfrak{h}} - V) \circ j^{1/2},$$

where

$$j(x) = \prod_{\alpha \in R/\pm} |\langle \alpha, x \rangle|$$

and

$$V(x) = \frac{1}{2} \sum_{\alpha \in R/\pm} \frac{1}{\langle \alpha, x \rangle^2} = \frac{1}{2} \sum_{i < j} \frac{1}{(x_i - x_j)^2}, \quad x \in \mathfrak{h}.$$

(See [H2].) We have $\bar{\mathfrak{p}}^\infty / \overline{\text{Ad } K}^\infty \cong \mathfrak{h}/\widetilde{W}$. We can obtain the above result by taking the limit of $T \rightarrow 0$. Then we have $\bar{\mathfrak{p}}^\infty \rightarrow \mathfrak{p}$ and $\overline{K}^\infty \rightarrow K$ and our potential $T^2 \widehat{V}_T(x) \rightarrow V(x)$ so that we get the results on $\mathfrak{p}/\text{Ad } K$ from $\bar{\mathfrak{p}}^\infty / \overline{\text{Ad } K}^\infty$.

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