COMPUTATION OF THE MONODROMY OF THE GENERALIZED HYPERGEOMETRIC FUNCTION

$${}_pF_{p-1}(a_1, \dots, a_p; b_2, \dots, b_p; z)$$

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1. Introduction

Let $p_1, \dots, p_n \in \mathbb{C}[t_1, \dots, t_k]$ be distinct irreducible polynomials which contain a parameter z , let D be the divisor in $\mathbb{C}^k$ defined by $\prod_{j=1}^n p_j = 0$, let $\{A_1, \dots, A_n\}$ be a set of $m \times m$ matrices of holomorphic functions on $T = \mathbb{C}^k - D$, and let Ω^p be the sheaf of holomorphic p -forms on T .

We suppose that the covariant derivation

$$\nabla : \Gamma(U, \Omega^p) \otimes \mathbb{C}^m \ni \tau \mapsto d\tau - \sum_{j=1}^n A_j \frac{dp_j}{p_j} \wedge \tau \in \Gamma(U, \Omega^{p+1}) \otimes \mathbb{C}^m$$

is integrable, i.e. $\nabla \nabla = 0$ for any open set U in T . Let $\mathcal{L} = \text{Ker } \nabla$ and let $\mathcal{L}'$ be the dual sheaf of $\mathcal{L}$. Then $\mathcal{L}$ is a locally constant sheaf of rank m on T and we have the twisted homology group $H_k(T, \mathcal{L})$. We define the bilinear form

$$H_k(T, \mathcal{L}) \times H^k(T, \mathcal{L}^\vee) \longrightarrow \mathbb{C}$$

by integration (e.g. [2]).

When we fix a base of the homology and a base of the cohomology, we can identify the bilinear form with a function $f(z)$ as explained later. We further assume that z moves in an open subset N of $\mathbb{C}$ and denote the fundamental group of N by $\pi_1(N, z_0)$ with a base point z_0 . Then the smooth deformation of $H_k(T, \mathcal{L})$ by continuation along loops of $\pi_1(N, z_0)$ induces a representation

$$\pi_1(N, z_0) \longrightarrow \text{Aut}(H_k(T, \mathcal{L})).$$

We call this representation the monodromy representation of the function $f(z)$.

If $p_1, \dots, p_n$ are linear forms, then the divisor D defines a hyperplane arrangement in $\mathbb{C}^k$. Assume that the rank of $\mathcal{L}$ is equal to one. Then the functions

defined above by $\mathcal{L}$ are called hypergeometric functions of type $(k+1, n+1)$ and have been studied by many authors. For a generic arrangement of hyperplanes, the monodromy group of $H_k(T, \mathcal{L})$ is computed by Matsumoto *et al* in [9]; however, the method of [9] does not work when D is not generic and even when there exists non-linear p_j .

The purpose of this paper is to give a new method of computing the monodromy group for degenerate arrangements of hyperplanes and to give an explicit computation of the monodromy group of ${}_pF_{p-1}$ by using our method.

In [8], A. H. M. Levelt first computed generators of the monodromy group of the generalized hypergeometric differential equation. In [10], Okubo obtained generators as a special case of his general method of constructing monodromy groups of hypergeometric systems. Their methods use properties of the differential equation to compute monodromy matrices of solutions. On the other hand, our method is based on the integral representations of ${}_pF_{p-1}$ and we can explicitly compute generators of the monodromy group corresponding to twisted cycles; thus, this method is geometric and the author thinks that our method is more natural.

We explain the idea of the method. By use of the function

$$g(t_k, z) = \int_{\Delta''} \prod_{j=1}^n p_j^{\beta_j} dt_1 \wedge \cdots \wedge dt_{k-1},$$

we can rewrite the multiple integral to the single integral as follows:

$$f(z) = \int_{\Delta} \prod_{j=1}^n p_j^{\beta_j} dt_1 \wedge \cdots \wedge dt_k = \int_{\Delta'} g(t_k, z) dt_k.$$

We regard the second integral as a bilinear form on the product of a twisted homology group and a twisted cohomology group.

Let $\mathcal{H} = \{(t_1, \dots, t_k) \in \mathbb{C}^k \mid t_1 = \cdots = t_k\}$, which is a line in $\mathbb{C}^k$. The intersection $D' = \mathcal{H} \cap D$ is a divisor in $\mathcal{H} \simeq \mathbb{C}$. Then there exist linear forms $p'_i(t)$ so that D' is the divisor determined by $\prod_i p'_i(t) = 0$. We define

$$\nabla' : \tau' \longmapsto d\tau' - \sum_i B_i \frac{dp'_i}{p'_i} \wedge \tau'$$

and get the locally constant sheaf $\mathcal{L}' = \text{Ker } \nabla'$ of rank $(k-1)$ on $T' = \mathbb{C} - D'$, where we choose matrices B_j such that $g(t, z)$ is regarded as a section of $\mathcal{L}'$. Then the function $f(z)$ is a bilinear form on $H_1(T', \mathcal{L}') \times H^1(T', \mathcal{L}'^\vee)$. Since

T' is 1-dimensional, the arrangement D' is generic and we can easily compute the monodromy group of the twisted homology group $H_1(T', \mathcal{L}')$ if we know the behavior of $g(t, z)$. On the other hand, since there exists an injection

$$H_k(T, \mathcal{L}) \longrightarrow H_1(T', \mathcal{L}'),$$

the monodromy group of $H_k(T, \mathcal{L})$ is embedded into the monodromy group of $H_1(T', \mathcal{L}')$. Thus we can compute the monodromy group of the twisted homology group $H_k(T, \mathcal{L})$. We emphasize that it is important that we choose a basis of $H_1(T', \mathcal{L}')$ corresponding to the basis of $H_k(T, \mathcal{L})$ determined by the compact chambers of the arrangement D .

We remark that an integral whose kernel is written by a Gauss hypergeometric function was treated by Takano [11] to compute the monodromy group of Appell's F_4 and that, in Takayama [12], a similar method is used to compute the intersection number of twisted homology groups. We also note that n -fold integrals of Selberg type have single integral representations whose kernels are written by $(n-1)$ -fold integrals of Selberg type. Aomoto [1] utilized this fact to study Gauss–Manin connections of Selberg type integrals.

We next explain how to apply the method to the generalized hypergeometric function defined as

$${}_pF_{p-1}(a_1, \dots, a_p; b_2, \dots, b_p; z) = \sum_{n=0}^{\infty} \frac{(a_1; n)(a_2; n) \cdots (a_p; n)}{(b_2; n)(b_3; n) \cdots (b_p; n)n!} z^n.$$

It is a solution of the differential equation

$$\left\{ \prod_{i=1}^p \left(z \frac{d}{dz} + b_i - 1 \right) - z \prod_{i=1}^p \left(z \frac{d}{dz} + a_i \right) \right\} y = 0, \quad (1.1)$$

where we put $b_1 = 1$ for simplicity. Let $t_p = z$ and define linear forms $q_1, \dots, q_p$ in $\mathbb{C}[t_1, \dots, t_{p-1}]$ as follows:

$$q_j = \begin{cases} 1 - t_1 & (j = 1), \\ t_j - t_{j-1} & (j = 2, \dots, p). \end{cases}$$

Then, as is well known, ${}_pF_{p-1}(z)$ is equal to

$$\prod_{l=2}^p \frac{\Gamma(b_l)}{\Gamma(a_l)\Gamma(b_l - a_l)} z^{1-b_p} \int_{\Delta} \prod_{i=1}^{p-1} t_i^{a_{i+1}-b_i} \prod_{i=1}^p q_i^{b_i-a_i-1} dt_1 \wedge \cdots \wedge dt_{p-1}, \quad (1.2)$$

where the domain of integration Δ is the compact chamber enclosed by the hyperplanes $\{q_1 = 0, \dots, q_p = 0\}$. Relative to the covariant derivation

$$\nabla : \tau \mapsto d\tau - \left(\sum_{i=1}^{p-1} (a_{i+1} - b_i) \frac{dt_i}{t_i} + \sum_{i=1}^p (b_i - a_i - 1) \frac{dq_i}{q_i} \right) \wedge \tau,$$

we have the twisted homology group $H_{p-1}(T, \text{Ker } \nabla)$. Since the degenerate hyperplane arrangement D in $\mathbb{R}^{p-1}$, which is defined by $\prod_{j=1}^{p-1} t_j \prod_{j=1}^p q_j = 0$, has p compact chambers, the dimension of $H_{p-1}(T, \text{Ker } \nabla)$ is equal to p .

The monodromy of the solutions of the generalized hypergeometric differential equation (1.1) is well known and has been studied extensively (see e.g. [3]). However, all the known methods utilize the differential equation (1.1) instead of the integral representation (1.2). It seems more natural to compute the monodromy group with respect to the p compact chambers. In order to compute the monodromy group of the twisted cycles corresponding to the p compact chambers, we translate the multiple integral into a single integral as follows:

$$\begin{aligned} & z^{1-b_p} \int_{\Delta} \prod_{i=1}^{p-1} t_i^{a_{i+1}-b_i} \prod_{i=1}^p q_i^{b_i-a_i-1} dt_1 \wedge \dots \wedge dt_{p-1} \\ &= z^{1-b_p} \int_0^z t_{p-1}^{a_p-1} q_p^{b_p-a_p-1} \\ & \quad \times \left(t_{p-1}^{1-b_{p-1}} \int_{\Delta''(t_{p-1})} \prod_{i=1}^{p-2} t_i^{a_{i+1}-b_i} \prod_{i=1}^{p-1} q_i^{b_i-a_i-1} dt_1 \wedge \dots \wedge dt_{p-2} \right) dt_{p-1}. \end{aligned}$$

Rewriting this, we get

$$\begin{aligned} & \frac{\Gamma(a_p)\Gamma(b_p-a_p)}{\Gamma(b_p)} {}_pF_{p-1}(a_1, \dots, a_p; b_2, \dots, b_p; z) \\ &= z^{-b_p} \int_0^z t^{a_p} (z-t)^{b_p-a_p} {}_{p-1}F_{p-2}(a_1, \dots, a_{p-1}; b_2, \dots, b_{p-1}; t) \frac{z dt}{t(z-t)}. \end{aligned}$$

We apply the general discussion above to this integral representation. The intersection $D \cap \{t_1 = \dots = t_{p-2}\}$ is identified with three points 0, z , and 1 in $\mathbb{C}$ and the covariant derivation ∇' on $T' = \mathbb{C} \setminus \{0, 1, z\}$ is written in the form

$$\nabla' : \tau' \mapsto d\tau' - \left(M_1 \frac{dt}{t} + M_2 \frac{dt}{t-1} + M_3 \frac{dt}{t-z} \right) \wedge \tau';$$

the 1-form is the Pfaffian form of the differential equation (1.1) for ${}_{p-1}F_{p-2}$, where M_1 , M_2 , and M_3 are certain $(p-1) \times (p-1)$ matrices of numbers. Then we obtain the locally constant sheaf $\mathcal{L}' = \text{Ker } \nabla'$ and the twisted homology group $H_1(T', \mathcal{L}')$. Thus, if we can choose a basis of $H_1(T', \mathcal{L}')$ which corresponds to a basis of $H_{p-1}(T, \mathcal{L})$ defined by compact chambers of D , we can compute the monodromy group of the generalized hypergeometric function. We explain the method used to construct the basis in Section 4 for the special cases when $p = 2, 3$, and in Theorem 5.1 for the general case. We explicitly compute the monodromy group relative to $H_1(T', \mathcal{L}')$ in Section 5.

2. Construction of bases of a twisted homology group

We start by reviewing the theory of twisted homology necessary to compute the monodromy group of ${}_pF_{p-1}$. For the proof, refer to [2].

Let T be an ℓ -dimensional complex manifold, let Ω_T^p be the sheaf of holomorphic p -forms on T , let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on T , and let Ω be a $(m \times m)$ -matrix-valued holomorphic 1-forms on T . We define the covariant derivation

$$\nabla : \Gamma(U, \Omega_T^p) \otimes \mathbb{C}^m \ni \tau \longmapsto d\tau - \Omega \wedge \tau \in \Gamma(U, \Omega_T^{p+1}) \otimes \mathbb{C}^m$$

and assume $\nabla \nabla = 0$. We denote by $\text{Ker } \nabla$ the sheaf associated to the presheaf

$$\Gamma(U, \text{Ker } \nabla) = \{f \in \Gamma(U, \mathcal{O}) \otimes \mathbb{C}^m \mid \nabla f = 0\}$$

and the restriction map

$$\rho_{UW} : \Gamma(U, \text{Ker } \nabla) \ni f \longmapsto f \in \Gamma(W, \text{Ker } \nabla).$$

The sheaf $\text{Ker } \nabla$ is a locally constant sheaf of rank m on T .

We suppose that a smooth triangulation K of T is given. For an oriented p -simplex Δ of K and an element $u_\Delta \in \lim_{\Delta \subset U} \Gamma(U, \text{Ker } \nabla)$, we denote by $\Delta \otimes u_\Delta$ the formal pair of Δ and u_Δ , and denote by $C_p(T, \text{Ker } \nabla)$ the space of formal sums of $\Delta \otimes u_\Delta$. We define a linear mapping

$$\partial_\nabla : C_p(T, \text{Ker } \nabla) \longrightarrow C_{p-1}(T, \text{Ker } \nabla)$$

by

$$\partial_\nabla(\Delta \otimes u_\Delta) = (\partial \Delta) \otimes (u_\Delta|_{\partial \Delta}),$$

where ∂ is the topological boundary operator. Clearly, $\partial_{\nabla}\partial_{\nabla} = 0$ and $\{C.(T, \text{Ker } \nabla), \partial_{\nabla}\}$ is a chain complex. The associated homology is denoted by $H_p(T, \text{Ker } \nabla)$.

Let $p_1, \dots, p_n$ be linear forms in $\mathbb{C}^\ell$ and let D be a hyperplane arrangement defined by $\prod_{i=1}^n p_i = 0$ in $\mathbb{C}^\ell$. We suppose that the hyperplane arrangement D is generic. Then, for a derivation $\nabla : \tau \mapsto d\tau - \sum_{i=1}^n \alpha_i (dp_i/p_i) \wedge \tau$, the dimension of $H_n(\mathbb{C}^\ell - D, \text{Ker } \nabla)$ is equal to $\binom{n-1}{\ell}$ under a certain condition on the parameters $\alpha_1, \dots, \alpha_n$ (see e.g. Kita [5, Theorem 2]). We prove a proposition similar to this theorem later, when the rank is more than one and the dimension is equal to one.

Now let x_i ($1 \leq i \leq n$) be distinct n points in $\mathbb{C}$ and set $T = \mathbb{C} \setminus \{x_1, \dots, x_n\}$. We denote by S^*f the analytic continuation of a section f of $\text{Ker } \nabla$ along any loop S inside T .

THEOREM 2.1. *Let S_i be a loop around the point x_i for each i . If there exists S_{i_0} such that $\det(S_{i_0}^* - \text{id}) \neq 0$, then $\dim_{\mathbb{C}} H_1(T, \text{Ker } \nabla) = m(n-1)$.*

Proof. Without loss of generality, we can assume that $\det(S_1^* - \text{id}) \neq 0$. Let P_i be the base point of the loop S_i and let $\overline{P_1 P_i}$ be the oriented segment from P_1 to P_i in T . We define the triangulation K of

$$\tilde{T} = \left(\bigcup_{i=1}^n S_i \right) \cup \left(\bigcup_{i=2}^n \overline{P_1 P_i} \right)$$

by choosing two consecutive points Q_i and R_i for each S_i as in Figure 1 and by regarding S_i as the sum of three simplices $\overline{P_i Q_i}$, $\overline{Q_i R_i}$, and $\overline{R_i P_i}$. Since $\tilde{T}$ is a deformation retract of T , $H_1(\tilde{T}, \text{Ker } \nabla) \simeq H_1(T, \text{Ker } \nabla)$. Let U be a simply connected open set in T which contains $\bigcup_{i=2}^n \overline{P_1 P_i} \cup \bigcup_{i=1}^n \overline{P_i Q_i}$. Any bounded cycle σ of $C_1(K, \text{Ker } \nabla)$ is written as

$$\sigma = \sum_{i=1}^n (\overline{P_i Q_i} \otimes v_i + \overline{Q_i R_i} \otimes v'_i + \overline{R_i P_i} \otimes v''_i) + \sum_{i=2}^n \overline{P_1 P_i} \otimes w_i,$$

where $\{w_i, v_i\}$ is a set of sections of $\text{Ker } \nabla$ on U , $\{v'_i\}$ is a set of sections on a neighborhood U'_i of the path $\overline{P_i Q_i}$, and $\{v''_i\}$ is a set of sections on a neighborhood U''_i of the path $\overline{Q_i R_i}$. Since

$$\partial_{\nabla} \sigma = \sum_{i=1}^n (Q_i \otimes (v_i - v'_i) + R_i \otimes (v'_i - v''_i))$$

$$\begin{aligned}
& + \sum_{i=2}^n P_i \otimes (v_i'' - v_i + w_i) + P_1 \otimes \left(v_1'' - v_1 - \sum_{i=2}^n w_i \right) \\
& = 0,
\end{aligned}$$

v_i' is the analytic continuation of v_i along the path $\overline{P_i Q_i}$ starting from P_i , and v_i'' is the analytic continuation of v_i' along the path $\overline{Q_i R_i}$ starting from Q_i ; that is, $v_i'' = S_i^* v_i$. Moreover, it holds that $w_i = -(S_i^* - \text{id})v_i$ for each $i = 2, \dots, n$ and that

$$v_1 = (S_1^* - \text{id})^{-1} \left(\sum_{i=2}^n w_i \right)$$

by the assumption of the theorem. Thus, any bounded cycle c depends only on a set $\{v_2, \dots, v_n\}$. Since $\dim_{\mathbb{C}} \Gamma(U, \text{Ker } \nabla) = m$, it holds that $\dim_{\mathbb{C}} H_1(K, \text{Ker } \nabla) = m(n-1)$. $\square$

FIGURE 1. $\tilde{T}$ and U .

The base given in the proof of Theorem 2.1 may contain cycles in the form

$$S_1 \otimes R_1 f + \overline{P_1 P_i} \otimes f + S_i \otimes R_2 f \quad ((S_i^* - \text{id})^n f = 0), \quad (2.1)$$

where R_1 and R_2 are constant matrices and f is a section of $\Gamma(U, \text{Ker } \nabla)$. We cannot compute monodromy groups in the case when there are cycles in the form of (2.1) by using our method explained in latter sections of this paper. So, we add technical conditions to construct a base which contains no cycles in the form of (2.1).

Under the assumption of Theorem 2.1, we assume the condition that, for each i such that $\det(S_i^* - \text{id}) = 0$, there exists a decomposition $\Gamma(U, \text{Ker } \nabla) = \Gamma'_i \oplus \Gamma''_i$ so that $\det(S_i^*|_{\Gamma'_i} - \text{id}|_{\Gamma'_i}) \neq 0$ and that $S_i^*|_{\Gamma''_i} = \text{id}|_{\Gamma''_i}$. Now assume that $\det(S_1^* - \text{id}) \neq 0$ as in the proof of Theorem 2.1 and construct a set $\{\sigma_{ij}\}$ of the bounded cycles as follows. From the proof of Theorem 2.1, any bounded cycle depends only on a set $\{v_2, \dots, v_n\}$. If $\det(S_i^* - \text{id}) \neq 0$, then we take a basis $\{u_{ij} \mid 1 \leq j \leq m\}$ of $\Gamma(U, \text{Ker } \nabla)$ and define $v_i^{(j)} = -(S_i^* - \text{id})^{-1}u_{ij}$. For each set

$$\{0, \dots, v_i^{(j)}, \dots, 0\},$$

we define a cycle σ_{ij} , that is,

$$\sigma_{ij} = S_1 \otimes (S_1^* - \text{id})^{-1}u_{ij} + \overline{P_1 P_i} \otimes u_{ij} - S_i \otimes (S_i^* - \text{id})^{-1}u_{ij} \quad (1 \leq j \leq m). \quad (2.2)$$

If $\det(S_i^* - \text{id}) = 0$, then let $\{u_{ij} \mid 1 \leq j \leq \dim_{\mathbb{C}} \Gamma'_i\}$ be a basis of Γ'_i and let $\{u_{ij} \mid \dim_{\mathbb{C}} \Gamma'_i < j \leq m\}$ be a basis of Γ''_i . We define

$$v_i^{(j)} = \begin{cases} -(S_i^*|_{\Gamma'_i} - \text{id}|_{\Gamma'_i})^{-1}u_{ij} & (1 \leq j \leq \dim_{\mathbb{C}} \Gamma'_i), \\ u_{ij} & (\dim_{\mathbb{C}} \Gamma'_i < j \leq m). \end{cases}$$

Then, for each set $\{0, \dots, v_i^{(j)}, \dots, 0\}$, we define a cycle σ_{ij} , that is,

$$\begin{aligned} \sigma_{ij} &= S_1 \otimes (S_1^* - \text{id})^{-1}u_{ij} + \overline{P_1 P_i} \otimes u_{ij} \\ &\quad - S_i \otimes (S_i^*|_{\Gamma'_i} - \text{id}|_{\Gamma'_i})^{-1}u_{ij} \quad (1 \leq j \leq \dim_{\mathbb{C}} \Gamma'_i), \end{aligned} \quad (2.3)$$

$$\sigma_{ij} = S_i \otimes u_{ij} \quad (\dim_{\mathbb{C}} \Gamma'_i < j \leq m). \quad (2.4)$$

From the proof of Theorem 2.1, clearly, the set $\{\sigma_{ij} \mid 2 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of $H_1(T, \text{Ker } \nabla)$. Thus, we have the following theorem.

THEOREM 2.2. *Suppose that $\det(S_1^* - \text{id}) \neq 0$ and that, if $\det(S_i^* - \text{id}) = 0$ for some i , there exists a decomposition $\Gamma(U, \text{Ker } \nabla) = \Gamma'_i \oplus \Gamma''_i$ so that $\det(S_i^*|_{\Gamma'_i} - \text{id}|_{\Gamma'_i}) \neq 0$ and that $S_i^*|_{\Gamma''_i} = \text{id}|_{\Gamma''_i}$. Then $\{\sigma_{ij} \mid 2 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of $H_1(T, \text{Ker } \nabla)$.*

We give a few remarks. First, due to Theorem 2.2, the homology group $H_1(T, \text{Ker } \nabla)$ has a natural decomposition. Let $\text{Pr}(H_1(T, \text{Ker } \nabla))$ be the subspace of $H_1(T, \text{Ker } \nabla)$ spanned by σ_{ij} of (2.2) and (2.3), and let $\text{Deg}(H_1(T, \text{Ker } \nabla))$ be the subspace spanned by σ_{ij} of (2.4). Then $H_1(T, \text{Ker } \nabla) = \text{Pr}(H_1(T, \text{Ker } \nabla)) \oplus \text{Deg}(H_1(T, \text{Ker } \nabla))$. We call $\text{Pr}(H_1(T, \text{Ker } \nabla))$ the *primary part* and $\text{Deg}(H_1(T, \text{Ker } \nabla))$ the *degenerate part*.

Second, we recall the notion of regularization. Let (x_1, x_j) be the oriented polygonal arc $\overline{x_1 P_1 P_i x_j}$ which includes neither end points. The analytic continuation of $u \in \Gamma(U, \text{Ker } \nabla)$ starting up from P_1 along $\overline{P_1 P_i x_j}$ and along $\overline{P_1 x_1}$ are also sections of $\text{Ker } \nabla$; we denote these by u again. We denote by $H_p^{lf}(T, \text{Ker } \nabla)$ the homology group associated to the complex of the spaces $C_p^{lf}(T, \text{Ker } \nabla)$ consisting of local finite formal sums of $\Delta \otimes u$ and, under Theorem 2.2, define a linear map reg from $H_1^{lf}(T, \text{Ker } \nabla)$ to $H_1(T, \text{Ker } \nabla)$ by

$$\text{reg}((x_1, x_j) \otimes u_{ij}) = \sigma_{ij} \quad \text{and} \quad \text{reg}(S_i \otimes u_{ij}) = \sigma_{ij},$$

where $\{\sigma_{ij}\}$ is the basis of $H_1(T, \text{Ker } \nabla)$ as above. The map is an isomorphism of $\mathbb{C}$ -vector spaces and is called the regularization (see e.g. [2]). Moreover, the twisted cycle $\sigma = \text{reg}((x_1, x_j) \otimes u)$ depends on the base point as follows. Let P'_k be a point on the path (x_1, x_j) and assume that there exists $v \in \Gamma(U, \text{Ker } \nabla)$ such that $(S'_k - \text{id})v = u$, where S'_k is a loop around x_k with the base point P'_k . Then there clearly exists a decomposition of cycles as

$$\text{reg}((x_1, x_j) \otimes u) = \text{reg}((x_1, x_k) \otimes u) + \text{reg}((x_k, x_j) \otimes u),$$

where (x_k, x_j) is the polygonal arc $\overline{x_k P'_k P_j x_j}$.

Finally, we define the pairing of cycles and differential forms. Let $u_\Delta = {}^t(u_1, \dots, u_m)$ be a section of $\text{Ker } \nabla$ and let $\omega = {}^t(\omega_1, \dots, \omega_m)$ be a m -dimensional vector-valued holomorphic 1-form on T . Suppose $\Delta \otimes u_\Delta \in H_1(T, \text{Ker } \nabla)$. The pairing $(\Delta \otimes u_\Delta, \omega)$ is defined by the integration

$$(\Delta \otimes u_\Delta, \omega) = \int_\Delta \sum_{i=1}^m u_i \omega_i.$$

The pairing is used in later sections.

3. Smooth deformation

We call

$$X_n = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{(x_1, \dots, x_n) \mid x_i - x_j = 0\}$$

the pure braid space of dimension n . The fibre of the canonical projection

$$p : X_{n+1} \ni (x, t) = (x_1, \dots, x_n, t) \mapsto (x_1, \dots, x_n) = x \in X_n,$$

is denoted by $T(x)$

We give a $(m \times m)$ -matrix-valued holomorphic 1-form $\Omega(x_1, \dots, x_n, t)$ on X_{n+1} such that $\nabla_{x,t} \nabla_{x,t} = 0$, where $\nabla_{x,t} u = d_{x,t} u - \Omega \wedge u$ and $d_{x,t}$ is the exterior differentiation with respect to both $x_1, \dots, x_n$, and t . Let $\mathcal{S} = \text{Ker } \nabla_{x,t}$. The pull-back $\Omega_x^*(t) = \iota^* \Omega(x, t)$ by the natural inclusion $\iota : T(x) \hookrightarrow X_{n+1}$ is a $(m \times m)$ -matrix-valued holomorphic 1-form on $T(x)$. Put $\mathcal{L}_x = \text{Ker } \nabla_x$, where $\nabla_x u = d_t u - \Omega_x^*(t) \wedge u$. Then we have $\mathcal{S}|_{T(x)} = \mathcal{L}_x$.

Let $\pi_1(X_n, a)$ be the fundamental group of X_n with a base point a . If we can smoothly deform a base of $H_1(T(x), \mathcal{L}_x)$ along an element γ of $\pi_1(X_n, a)$, then we get the monodromy representation of $\pi_1(X_n, a)$:

$$\pi_1(X_n, a) \ni \gamma \mapsto \gamma^* \in \text{Aut}(H_1(T(x), \mathcal{L}_x)).$$

In the actual computation, we need to define rigorously the notion of smooth deformation. Although we defined twisted homology groups based on a smooth triangulation of the base space, it is more convenient to regard twisted cycles as singular chains when defining smooth deformation. We say that a mapping Δ is a singular simplex in the fibre $T(x)$ if $\Delta : [0, 1] \rightarrow T(x)$ is a C^∞ mapping. Let f be a germ of $\mathcal{L}_x$ at $\Delta(0)$. We write the formal pair of Δ and f by $\Delta \otimes f$ and call it the singular chain.

Suppose that K is a triangulation of the image $\Delta([0, 1])$ and that $f_i \in \Gamma(U_i, \mathcal{L}_x)$ is an analytic continuation of f along the image $\Delta([0, 1])$ in $T(x)$, where U_i is a neighborhood of a simplex Δ'_i of K . We identify the singular chain $\Delta \otimes f(t)$ with the chain $\sum_{\Delta'_i} \Delta'_i \otimes f_i(t)$.

We next define the smooth deformation of paths and the continuation of sections of a locally constant sheaf.

Definition 3.1. Suppose that a path $\gamma(s) : [0, 1] \rightarrow X_n$ and a singular simplex $\Delta_0(u)$ in $T(\gamma(0))$ are given. A singular simplex $\Delta_1(u)$ in $T(\gamma(1))$ is called a *smooth*

deformation of $\Delta_0(u)$ along γ if there exists a smooth mapping $F : [0, 1] \times [0, 1] \ni (s, u) \mapsto F(s, u) \in X_{n+1}$ satisfying the following conditions:

- (1) $F(s, \cdot) : [0, 1] \rightarrow T(\gamma(s)) \subset \mathbb{C}$ is a singular chain in the fibre $T(\gamma(s))$ for all $0 \leq s \leq 1$;
- (2) $\Delta_0(u) = F(0, u)$ and $\Delta_1(u) = F(1, u)$.

We denote $F(s, u)$ by $\Delta_s(u)$.

Definition 3.2. Let $\gamma(s)$ be a path in X_n and let Δ_1 be a smooth deformation of Δ_0 along γ . Put $\xi(s) = (\gamma(s), \Delta_s(0))$, i.e. ξ is a lift of γ to a path in X_{n+1} . Suppose that f_i ($i = 0, 1$) is a germ of $\mathcal{L}_{\gamma(i)}$ at $\Delta_i(0)$. f_1 is called the *continuation* of f_0 along ξ if, over a simply connected open set $U \subset X_{n+1}$ which contains ξ , there exists $g(s, t) \in \Gamma(U, \mathcal{S})$ which satisfies the condition that, for $i = 0$ and $i = 1$, f_i is equal to the germ $[g(s, t)|_{T(\gamma(i))}]$ of $\mathcal{L}_{\gamma(i)}$ at $\Delta_i(0)$, where $g(s, t)|_{T(\gamma(i))}$ is a section of $\mathcal{L}_{\gamma(i)}$ in a neighborhood of $\Delta_i(0)$ in $T(\gamma(i))$.

FIGURE 2. The lift ξ of γ .

We can now define the smooth deformation of $\Delta_s(u) \otimes f_s(t)$.

Definition 3.3. Let $\gamma(s)$ be a path in X_n and Δ_1 a smooth deformation of Δ_0 along γ . $\Delta_1 \otimes f_1$ is called the *smooth deformation* of $\Delta_0 \otimes f_0$ along γ if f_1 is the continuation of f_0 along the path $\xi(s) = (\gamma(s), \Delta_s(0))$ in X_{n+1} .

The finite composition of smooth deformations defined above is also called a smooth deformation.

The monodromy group acts on the basis $\{v_1, v_2\}$ of $H_1(T(t), \text{Ker } \nabla)$ as

$$\begin{aligned}\gamma_1^* : (v_1, v_2) &\longmapsto (v_1, v_2) \frac{1}{c_6} \begin{pmatrix} c_6 & 0 \\ c_5 c_6 - 1 & 1 \end{pmatrix}, \\ \gamma_2^* : (v_1, v_2) &\longmapsto (v_1, v_2) \begin{pmatrix} c_2 c_5 & -(c_2 - 1) \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Here, $\gamma_1(s)$ is the closed curve around the point 0 in the positive direction and $\gamma_2(s)$ is the closed curve around the point 1 in the positive direction as shown in Figure 4.

FIGURE 4. $\gamma_1, \gamma_2 \in \pi_1(\mathbb{C} \setminus \{0, 1, t\})$.

Put

$$\begin{aligned}\omega_1 &= \frac{ds}{s} - \frac{d(t-s)}{t-s} = \frac{t ds}{s(t-s)}, \\ \omega_2 &= -(\alpha_4 t + (\alpha_6 + 1)s) \frac{t ds}{s(t-s)^2}.\end{aligned}$$

Then we have

$$t \frac{d}{dt} \left(t^{\alpha_6} \int u \omega_1 \right) = t^{\alpha_6} \int u \omega_2.$$

We defined $(v_j, \omega_k) = \int_{v_j} u \omega_k$ in Section 2. To define the covariant derivation for the case ${}_3F_2$ in the next example, we rely on the fact that the pairing $t^{\alpha_6} \begin{pmatrix} (v_j, \omega_1) \\ (v_j, \omega_2) \end{pmatrix}$ satisfies the formula

$$\begin{aligned}& d_t \left\{ t^{\alpha_6} \begin{pmatrix} (v_j, \omega_1) \\ (v_j, \omega_2) \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 0 & 1 \\ 0 & -\alpha_6 - 1 \end{pmatrix} \frac{dt}{t} + \begin{pmatrix} 0 & 0 \\ -\alpha_1 \alpha_4 & \alpha_2 + \alpha_5 - 2 \end{pmatrix} \frac{d(1-t)}{1-t} \right\} \\ &\quad \times \left\{ t^{\alpha_6} \begin{pmatrix} (v_j, \omega_1) \\ (v_j, \omega_2) \end{pmatrix} \right\}.\end{aligned}$$

The formula is induced by the differential equation (1.1).

$$c_{3j-2}c_{3j-1}c_{3j} = 1.$$

We consider the 1-dimensional subspace $X'_3 = \{(0, 1, z) \mid z \neq 0, 1\}$ of the braid space X_3 and the projection

$$X_4 \supset X'_4 = \{(0, 1, z, t) \mid z, t \neq 0, 1, z \neq t\} \xrightarrow{\pi} X'_3 \subset X_3.$$

Then the function $t^{\alpha_{3p-2}}(z-t)^{\alpha_{3p-1}} {}_{p-1}F_{p-2}(a; b; t)$ defines a locally constant sheaf $\mathcal{S}$ of rank $(p-1)$ on X'_4 and we regard the generalized hypergeometric function ${}_pF_{p-1}$ as a pairing of an element of $H_1(\pi^{-1}(z), \mathcal{S}_z)$ and a vector-valued 1-form on $\pi^{-1}(z)$.

Thus, we can consider twisted cycles on the 1-dimensional space with a locally constant sheaf of rank $(p-1)$ instead of twisted cycles on $(p-1)$ -dimensional space with a locally constant sheaf of rank one, and we can compute monodromy groups by utilizing the general arguments given in Section 3.

In this section, we consider two cases ${}_2F_1$ and ${}_3F_2$ to clarify our idea. The computation in the general case will be given in Section 5.

Example 4.1 We review the computation of the monodromy for a Gauss hypergeometric function that has the integral representation

$${}_2F_1(a_1, a_2; b_2; t) = \frac{\Gamma(\alpha_4 + \alpha_5)}{\Gamma(\alpha_4)\Gamma(\alpha_5)} t^{\alpha_6} \int_0^t s^{\alpha_4} (t-s)^{\alpha_5} (1-s)^{\alpha_2-1} \frac{t ds}{s(t-s)}.$$

We define a locally constant sheaf which corresponds to this integration as follows. We put the holomorphic 1-form

$$\Omega = \alpha_4 \frac{ds}{s} + \alpha_5 \frac{d(t-s)}{t-s} + (\alpha_2 - 1) \frac{d(1-s)}{1-s}$$

over $T(t) = \mathbb{C} \setminus \{0, t, 1\}$ and define the covariant derivation

$$\nabla : \tau \longmapsto d\tau - \Omega \wedge \tau.$$

Suppose that α_2 , α_4 , and α_5 are not integers. Let $U = \{s \in \mathbb{C} \mid \text{Im } s < 0\}$ be the lower half-plane and let u be a branch of $s^{\alpha_4}(t-s)^{\alpha_5}(1-s)^{\alpha_1}$ in U . We should notice that $s^{\alpha_4}(t-s)^{\alpha_5}(1-s)^{\alpha_1}$ is multi-valued in a neighborhood of t . Therefore, we identify t with the point x_1 of Theorem 2.2. Let $v_1 = \text{reg}((t, 1) \otimes u)$ and $v_2 = \text{reg}((0, t) \otimes u)$. Then $\{v_1, v_2\}$ is the basis of $H_1(T(t), \text{Ker } \nabla)$ which corresponds to the compact chambers of the arrangement $\{s \mid s(t-s)(1-s) = 0\}$ in $\mathbb{R}$.

Example 3.4. Put $n = 2$ and regard each configuration of 2 points in $\mathbb{C}$ as a point in X_2 . Let γ be a path in X_2 such that the point x_2 encircles the fixed point x_1 in the positive direction. Suppose that Δ is a singular chain that encircles the point x_2 in the positive direction. In this case, the smooth deformation of Δ along the path γ is again Δ . Let us consider the lift ξ of γ in the (x_1, x_2, t) -space X_3 . ξ is the path such that the points t and x_2 encircle the point x_1 in the positive direction. Consider the twisted chain $\Delta \otimes \text{Log}(t - x_1)$. Since the analytic continuation of the function $\text{Log}(t - x_1)$ along the path ξ is $\text{Log}(t - x_1) + 2\pi\sqrt{-1}$, the smooth deformation of $\Delta \otimes \text{Log}(t - x_1)$ is $\Delta \otimes \text{Log}(t - x_1) + \Delta \otimes 2\pi\sqrt{-1}$.

FIGURE 3. $\Delta \otimes \text{Log}(t - x_1)$ and γ .

4. The generalized hypergeometric function ${}_pF_{p-1}$

The generalized hypergeometric function

$${}_pF_{p-1}(a_1, \dots, a_p; b_2, \dots, b_p; z) = \sum_{n=0}^{\infty} \frac{(a_1; n)(a_2; n) \cdots (a_p; n)}{(b_2; n)(b_3; n) \cdots (b_p; n)n!} z^n$$

has a single integral representation

$$\begin{aligned} & \frac{\Gamma(\alpha_{3p-2})\Gamma(\alpha_{3p-1})}{\Gamma(\alpha_{3p-2} + \alpha_{3p-1})} {}_pF_{p-1}(a_1, \dots, a_p; b_2, \dots, b_p; z) \\ &= z^{\alpha_{3p}} \int_0^z t^{\alpha_{3p-2}} (z-t)^{\alpha_{3p-1}} {}_{p-1}F_{p-2}(a_1, \dots, a_{p-1}; b_2, \dots, b_{p-1}; t) \frac{z dt}{t(z-t)}, \end{aligned}$$

where we use the notation $\alpha_{3j-2} = a_j$, $\alpha_{3j-1} = b_j - a_j$, $\alpha_{3j} = -b_j$ and put $b_1 = 1$ for simplicity. Let $c_i = \exp(2\pi\sqrt{-1}\alpha_i)$. Since $\alpha_{3j-2} + \alpha_{3j-1} + \alpha_{3j} = 0$, we have

Example 4.2. As we have seen, ${}_3F_2$ has an integral representation

$$\frac{\Gamma(\alpha_7)\Gamma(\alpha_8)}{\Gamma(\alpha_7 + \alpha_8)} {}_3F_2(a_1, a_2, a_3; b_2, b_3; z) = z^{\alpha_9} \int_0^z t^{\alpha_7} (z-t)^{\alpha_8} {}_2F_1(a_1, a_2; b_2; t) \frac{z dt}{t(z-t)}.$$

We put the matrix-valued holomorphic 1-form

$$\begin{aligned} \Omega = & \begin{pmatrix} 0 & 1 \\ 0 & -\alpha_6 - 1 \end{pmatrix} \frac{dt}{t} + \begin{pmatrix} 0 & 0 \\ -\alpha_1 \alpha_4 & \alpha_2 + \alpha_5 - 2 \end{pmatrix} \frac{d(1-t)}{1-t} \\ & + \begin{pmatrix} \alpha_7 & 0 \\ 0 & \alpha_7 \end{pmatrix} \frac{dt}{t} + \begin{pmatrix} \alpha_8 & 0 \\ 0 & \alpha_8 \end{pmatrix} \frac{d(z-t)}{z-t} \end{aligned}$$

over $T(z) = \mathbb{C} \setminus \{0, z, 1\}$ and define the covariant derivation

$$\nabla : \tau \longmapsto d\tau - \Omega \wedge \tau.$$

Suppose that α_7 , $\alpha_7 - \alpha_6$, α_8 , and $\alpha_2 + \alpha_5$ are not integers. For the pairing $\binom{(v_j, \omega_1)}{(v_j, \omega_2)}$ of Example 4.1, put

$$q_j = t^{\alpha_6 + \alpha_7} (z-t)^{\alpha_8} \begin{pmatrix} (v_j, \omega_1) \\ (v_j, \omega_2) \end{pmatrix};$$

it is clear that $\nabla q_j = 0$.

Now we consider the arrangement $D = \{st(z-t)(1-s)(t-s) = 0\}$. By the definition of v_1 , the twisted cycle $\text{reg}((z, 1) \otimes q_1)$ corresponds to $\Delta^{(1)} \otimes t^{\alpha_7} (z-t)^{\alpha_8} u(t, s)$. Here $\Delta^{(1)}$ is the compact chamber in D enclosed by the hyperplanes $\{z-t=0, t-s=0, 1-s=0\}$. In the same way, let $\Delta^{(2)}$ be the compact chamber enclosed by the hyperplanes $\{t=0, z-t=0, t-s=0, 1-s=0\}$ and let $\Delta^{(3)}$ be the compact chamber enclosed by the hyperplanes $\{z-t=0, t-s=0, s=0, t=0\}$. Then $\text{reg}((0, z) \otimes q_j)$ corresponds to $\Delta^{(j+1)} \otimes t^{\alpha_7} (z-t)^{\alpha_8} u(t, s)$ ($j = 1, 2$) (Figure 5).

Define four bounded twisted cycles:

$$\begin{aligned} \sigma_1 &= \text{reg}((z, 1) \otimes q_1), & \sigma_2 &= \text{reg}((0, z) \otimes q_1), \\ \sigma_3 &= \text{reg}((0, z) \otimes q_2), & \sigma_4 &= S_\varepsilon^1(1) \otimes \{(c_2 - 1)q_1 - (c_2 c_5 - 1)q_2\}, \end{aligned}$$

where $S_\varepsilon^1(1)$ is the circle of radius ε with center 1. The set $\{\sigma_1, \dots, \sigma_4\}$ is a basis of $H_1(T(z), \text{Ker } \nabla)$ by Theorem 2.2. Moreover, we notice that $\{\sigma_1, \sigma_2, \sigma_3\}$ spans the primary part of $H_1(T(z), \text{Ker } \nabla)$ and corresponds to the compact chambers in D .

FIGURE 5. The arrangement $D = \{st(z-t)(t-s)(1-s) = 0\}$. FIGURE 6. $\{\sigma_1, \dots, \sigma_4\}$.

For the same $\gamma_1, \gamma_2 \in \pi_1(\mathbb{C} \setminus \{0, 1\}, z)$ as in Example 4.1, the monodromy group acts on the primary part as

$$\gamma_1^* : (\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, \sigma_2, \sigma_3) \frac{1}{c_9} \begin{pmatrix} c_9 & 0 & 0 \\ c_8 c_9 - c_6 & c_6 & 0 \\ 1 - c_5 c_6 & c_5 c_6 - 1 & 1 \end{pmatrix},$$

$$\gamma_2^* : (\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, \sigma_2, \sigma_3) \begin{pmatrix} c_2 c_5 c_8 & 1 - c_2 c_5 & c_2 - 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This fact in a more general setting will be proved by using induction in the next section.

5. The monodromy group of ${}_pF_{p-1}$

We put the $((p-1) \times (p-1))$ -matrix-valued holomorphic 1-form

$$\begin{aligned} \Omega = & \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 0 & -B_1 & \cdots & -B_{p-2} \end{pmatrix} \frac{dt}{t} + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ A_0 & \cdots & A_{p-2} \end{pmatrix} \frac{d(1-t)}{1-t} \\ & + \begin{pmatrix} \alpha_{3p-2} & & & \\ & \alpha_{3p-2} & & \\ & & \ddots & \\ & & & \alpha_{3p-2} \end{pmatrix} \frac{dt}{t} \\ & + \begin{pmatrix} \alpha_{3p-1} & & & \\ & \alpha_{3p-1} & & \\ & & \ddots & \\ & & & \alpha_{3p-1} \end{pmatrix} \frac{d(z-t)}{z-t} \end{aligned}$$

over $T(z) = \mathbb{C} \setminus \{0, z, 1\}$, where A_i and B_i are defined as follows:

$$\begin{aligned} A_0 + A_1 X + \cdots + A_{p-2} X^{p-2} &= X \prod_{k=2}^{p-1} (X + b_k - 1) - \prod_{k=1}^{p-1} (X + a_k), \\ B_1 X + \cdots + B_{p-2} X^{p-2} + X^{p-1} &= X \prod_{k=2}^{p-1} (X + b_k - 1). \end{aligned}$$

Relative to the covariant derivation $\nabla : \tau \mapsto d\tau - \Omega \wedge \tau$, $\text{Ker } \nabla$ is a locally constant sheaf on $T(z)$. Moreover, we introduce the notation

$$\begin{aligned} S_{ij} &= \begin{cases} (-1)^{i+j} \left(c_{3p-3i+3} + \frac{\delta_{ij} - 1}{c_{3p-3i+4}} \right) & (j \leq i \leq p), \\ 0 & (\text{otherwise}), \end{cases} \\ T_i &= (-1)^{i+1} \left(\prod_{l=0}^{p-i} c_{3l+2} - 1 \right), \end{aligned}$$

where δ_{ij} is Kronecker's delta, and define a $p \times p$ lower-triangular matrix M_1 and a $p \times p$ upper-triangular matrix M_2 as follows:

$$M_1 = \begin{pmatrix} S_{11} & & \mathbf{0} \\ \vdots & \ddots & \\ S_{p,1} & \cdots & S_{p,p} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 \end{pmatrix} + \begin{pmatrix} T_1 & \cdots & T_p \\ & \mathbf{0} & \end{pmatrix}.$$

THEOREM 5.1. *Suppose that*

$$\alpha_{3k-1}, \alpha_{3k-2} - \alpha_{3j}, \sum_{i=1}^k \alpha_{3i-1} \notin \mathbb{Z} \quad \text{for all } 1 \leq j \leq k < p. \quad (5.1)$$

Let $A = H_1(\mathbb{C} \setminus \{0, z, 1\}, \text{Ker } \nabla)$. Then $\dim_{\mathbb{C}} A = 2p - 2$. The twisted homology group A can be decomposed as $A = \text{Pr}(A) \oplus \text{Deg}(A)$, where both $\text{Pr}(A)$ and $\text{Deg}(A)$ are invariant under the action of the monodromy group. Then the same loops $\gamma_1, \gamma_2 \in \pi_1(\mathbb{C} \setminus \{0, 1\}, z)$ as in Example 4.1 act on $\text{Pr}(A)$ as

$$\begin{aligned} \gamma_1^* : (v_1, \dots, v_p) &\longmapsto (v_1, \dots, v_p) \frac{1}{c_{3p}} M_1, \\ \gamma_2^* : (v_1, \dots, v_p) &\longmapsto (v_1, \dots, v_p) M_2. \end{aligned}$$

Proof. It is clear that each part of A is invariant under the action of the monodromy group. In order to find the monodromy matrix on $\text{Pr}(A)$, we argue by induction on p . In order to distinguish the twisted homology group A on each step of induction, we denote by $A(p)$ the twisted homology group A with parameter p and use similar notation for S_{ij} , T_i , M_i , and ∇ . We will prove $\dim_{\mathbb{C}} A(p+1) = 2p$ and construct a basis of $A(p+1)$ from a basis of $A(p) = H_1(\mathbb{C} \setminus \{0, z, 1\}, \text{Ker } \nabla(p))$.

For the initial step, we assume $p = 2$. In this case, the theorem is equivalent to Example 4.1, that is,

$$\begin{aligned} \begin{pmatrix} c_6 & 0 \\ c_5 c_6 - 1 & 1 \end{pmatrix} &= \begin{pmatrix} S_{11}(2) & 0 \\ S_{21}(2) & S_{22}(2) \end{pmatrix}, \\ \begin{pmatrix} c_2 c_5 & -(c_2 - 1) \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 + T_1(2) & T_2(2) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Next we assume that the assertion is true for p and proceed to $p+1$. The assumption implies that

$$\begin{aligned} \gamma_1^* : (v_1, \dots, v_p) &\longmapsto (v_1, \dots, v_p) \frac{1}{c_{3p}} M_1(p), \\ \gamma_2^* : (v_1, \dots, v_p) &\longmapsto (v_1, \dots, v_p) M_2(p). \end{aligned}$$

We can take p -dimensional vector-valued rational 1-forms $\omega_1, \dots, \omega_p$ with the poles at 0, z , and 1 such that $\nabla(p+1)q_j = 0$, where

$$q_j = t^{\alpha_{3p} + \alpha_{3p+1}} (z - t)^{\alpha_{3p+2} - 1} \begin{pmatrix} (v_j, \omega_1) \\ \vdots \\ (v_j, \omega_p) \end{pmatrix}, \quad (j = 1, \dots, p).$$

In fact, we construct ω_k 's by the condition

$$\omega_1 = \begin{pmatrix} \frac{z dt}{t(z-t)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$z \frac{d}{dz} (z^{\alpha_{3p}} (v_j, \omega_k)) = z^{\alpha_{3p}} (v_j, \omega_{k+1}) \quad (1 \leq j \leq p, 1 \leq k < p).$$

The explicit forms of ω_k are very complicated and not written down here. However, we should note that for any p -dimensional vector ω of rational 1-forms with poles at 0, z , and 1, the action of the monodromy group on (v_j, ω) is decided only by v_j and not by ω .

With this chain of 1-forms, we check the assumption of Theorem 2.1. Let $S^1(0)$, $S^1(1)$, and $S^1(z)$ be loops around the points 0, 1, and z , respectively. Since $S^1(1)^* : (q_1, \dots, q_p) \mapsto c_{3p+2}(q_1, \dots, q_p)$, we have $\det(S^1(1)^* - \text{id}) \neq 0$ by the assumption (5.1). Hence, the assumption of Theorem 2.1 is satisfied and we get $\dim_{\mathbb{C}} A(p+1) = 2p$.

Since $S^1(0)^* : (q_1, \dots, q_p) \mapsto (q_1, \dots, q_p) c_{3p+1} M_1(p)$, the eigenvalues of $S^1(0)^*$ are $\{c_{3k+3} c_{3p+1} \mid 1 \leq k < p\}$. By the assumption (5.1), $c_{3k+3} c_{3p+1} \neq 1$ ($1 \leq k < p$), that is, $\det(S^1(0)^* - \text{id}) \neq 0$. Moreover, since $S^1(z)^* : (q_1, \dots, q_p) \mapsto (q_1, \dots, q_p) M_2(p)$, the eigenvalues of $M_2(p)$ are $\{1 + T_1(p), 1, \dots, 1\}$. Thus $\det(S^1(z)^* - \text{id}) = 0$. On the other hand, since $T_1(p) \neq 0$ by (5.1), we can see that the matrix $M_2(p)$ is diagonalizable. Therefore, the assumption of Theorem 2.2 is satisfied.

We identify $\{q_i\}$ with a set of sections of $\text{Ker } \nabla(p+1)$ on the lower half-plane $\{t \in \mathbb{C} \mid \text{Im } t < 0\}$ and define $2p$ bounded cycles as

$$\sigma_1 = \text{reg}((z, 1) \otimes q_1(t)),$$

$$\begin{aligned}\sigma_{j+1} &= \text{reg}((0, z) \otimes q_j(t)) \quad (j = 1, \dots, p), \\ \sigma_{p+k} &= S^1(z) \otimes \{T_k(p)q_1(t) - T_1(p)q_k(t)\} \quad (k = 2, \dots, p),\end{aligned}$$

where $T_k(p)q_1 - T_1(p)q_k$ are distinct eigenvectors of $M_2(p)$ with eigenvalue 1. The set of these cycles is a basis of $A(p+1)$ by Theorem 2.2. Note that $\text{Pr}(A(p+1))$ is spanned by $\sigma_1, \dots, \sigma_{p+1}$, and $\text{Deg}(A(p+1))$ by $\sigma_{p+2}, \dots, \sigma_{2p}$. As we have seen, $\sigma_1, \dots, \sigma_{p+1}$ correspond to compact chambers of the arrangement D .

First, we will trace the action of γ_1^* on $\sigma_1, \dots, \sigma_{p+1}$. Recalling the process of decomposing cycles by referring to Section 2, we obtain the action

$$\begin{aligned}\gamma_1^* : \text{reg}((z, 1) \otimes q_1) &\longmapsto \text{reg}((z, 1) \otimes q_1) + \text{reg}((0, z) \otimes (c_{3p+2}q_1)) \\ &\quad + \text{reg}\left((z, 0) \otimes \left(c_{3p+2}c_{3p+1}c_{3p} \frac{1}{c_{3p}} \sum_{i=1}^p S_{i,1}(p)q_i\right)\right) \\ &= \sigma_1 + \frac{1}{c_{3p+3}c_{3p+1}}\sigma_2 - \frac{1}{c_{3p+3}} \sum_{j=2}^{p+1} S_{j-1,1}(p)\sigma_j \\ &= \frac{1}{c_{3p+3}} \left\{ c_{3p+3}\sigma_1 + \left(\frac{1}{c_{3p+1}} - S_{11}(p) \right) \sigma_2 \right. \\ &\quad \left. + \sum_{j=3}^{p+1} (-S_{j-1,1}(p))\sigma_j \right\} \\ &= \frac{1}{c_{3p+3}} \sum_{j=1}^{p+1} S_{j,1}(p+1)\sigma_j,\end{aligned}$$

where we have used the identity

$$S_{j,1}(p+1) = \begin{cases} c_{3p+3} & (j = 1), \\ \frac{1}{c_{3p+1}} - S_{11}(p) & (j = 2), \\ -S_{j-1,1}(p) & (j = 3, \dots, p+1). \end{cases}$$

Similarly, by tracing the smooth deformation of the path in detail, we see that

$$\begin{aligned}\gamma_1^* : \text{reg}((0, z) \otimes q_j) &\longmapsto \text{reg}\left((0, z) \otimes \left\{ c_{3p+1}c_{3p} \frac{1}{c_{3p}} \sum_{i=1}^p S_{i,j}(p)(c_{3p+2}q_i) \right\}\right) \\ &= \frac{1}{c_{3p+3}} \sum_{i=1}^p S_{i,j}(p)\sigma_{i+1}\end{aligned}$$

$$= \frac{1}{c_{3p+3}} \sum_{i=1}^p S_{i+1,j+1}(p+1) \sigma_{i+1},$$

where we used the identity $S_{ij}(p) = S_{i+1,j+1}(p+1)$. Thus, γ_1^* acts on $\text{Pr}(A)$ as

$$\begin{aligned} (\sigma_1, \dots, \sigma_{p+1}) &\longmapsto (\sigma_1, \dots, \sigma_{p+1}) \frac{1}{c_{3p+3}} \\ &\times \begin{pmatrix} S_{11}(p+1) & & & & 0 \\ S_{21}(p+1) & S_{22}(p+1) & & & \\ \vdots & & \ddots & & \\ S_{p+1,1}(p+1) & S_{p+1,2}(p+1) & \cdots & S_{p+1,p+1}(p+1) \end{pmatrix} \\ &= (\sigma_1, \dots, \sigma_{p+1}) \frac{1}{c_{3p+3}} M_1(p+1). \end{aligned}$$

FIGURE 7. The smooth deformation of σ_1 under the action of γ_1^* .

FIGURE 8. The smooth deformation of σ_{j+1} under the action of γ_1^* .

Second, we will trace the action of γ_2^* on $\sigma_1, \dots, \sigma_{p+1}$. In a similar way to as above, as shown in Figure 9, by tracing the smooth deformation of the path in detail, we see that

$$\gamma_2^* : \text{reg}((z, 1) \otimes q_1) \longmapsto \text{reg}((z, 1) \otimes \{(1 + T_1(p))(c_{3p+2}q_1)\}) = (1 + T_1(p+1))\sigma_1$$

FIGURE 9. The smooth deformation of σ_1 under the action of γ_2^* .

by using $(1 + T_1(p))c_{3p+2} = 1 + T_1(p+1)$.

Moreover, as in Figure 10, we see that

$$\begin{aligned}
 \gamma_2^* : \text{reg}((0, z) \otimes q_j) &\longmapsto \text{reg}((0, z) \otimes q_j) + \text{reg}((z, 1) \otimes q_j) \\
 &\quad + \text{reg}((1, z) \otimes (T_j(p)q_1 + q_j)) \\
 &= \sigma_{j+1} - T_j(p)\sigma_1 \\
 &= \sigma_{j+1} + T_{i+1}(p+1)\sigma_1,
 \end{aligned}$$

where we used the relation $-T_i(p) = T_{i+1}(p+1)$.

FIGURE 10. The smooth deformation of σ_{j+1} under the action of γ_2^* .

Therefore, this means that γ_2^* acts on $\text{Pr}(A)$ as

$$\begin{aligned}
 (\sigma_1, \dots, \sigma_{p+1}) &\longmapsto (\sigma_1, \dots, \sigma_{p+1}) \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right. \\
 &\quad \left. + \begin{pmatrix} T_1(p+1) & \cdots & T_{p+1}(p+1) \\ & 0 & \end{pmatrix} \right\} \\
 &= (\sigma_1, \dots, \sigma_{p+1})M_2(p+1).
 \end{aligned}$$

Thus, Theorem 5.1 is proved. $\square$

COROLLARY 5.2. *The monodromy matrices of solutions of the differential equation*

$$\left\{ z \frac{d}{dz} \prod_{k=2}^p \left(z \frac{d}{dz} + b_k - 1 \right) - z \prod_{k=1}^p \left(z \frac{d}{dz} + a_k \right) \right\} y = 0$$

which correspond to the compact chambers of the hyperplane arrangement

$$\left\{ (1 - t_1)(z - t_{p-1}) \prod_{i=1}^{p-2} (t_{i+1} - t_i) \prod_{i=1}^{p-1} t_i = 0 \right\}$$

are given by the matrices $M_1(p)$ and $M_2(p)$.

Example 5.3. We give a table of monodromy representations of $p = 2, 3$, and 4.

$p = 2$:

$$M_1 = \begin{pmatrix} c_6 & 0 \\ c_5 c_6 - 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} c_2 c_5 & 1 - c_2 \\ 0 & 1 \end{pmatrix}.$$

$p = 3$:

$$M_1 = \begin{pmatrix} c_9 & 0 & 0 \\ c_8 c_9 - c_6 & c_6 & 0 \\ 1 - c_5 c_6 & c_5 c_6 - 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} c_2 c_5 c_8 & 1 - c_2 c_5 & c_2 - 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$p = 4$:

$$M_1 = \begin{pmatrix} c_{12} & 0 & 0 & 0 \\ c_{11} c_{12} - c_9 & c_9 & 0 & 0 \\ c_6 - c_8 c_9 & c_8 c_9 - c_6 & c_6 & 0 \\ c_5 c_6 - 1 & 1 - c_5 c_6 & c_5 c_6 - 1 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} c_2 c_5 c_8 c_{11} & 1 - c_2 c_5 c_8 & c_2 c_5 - 1 & 1 - c_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We remark that

$$c_{3i} = \exp(-2\pi\sqrt{-1}b_i), \quad c_{3i-1} = \exp(2\pi\sqrt{-1}(b_i - a_i)),$$

and

$$c_{3i-2} = c_{3i-1}^{-1} c_{3i}^{-1} = \exp(2\pi\sqrt{-1}a_i).$$

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