REPRESENTATION FORMULA, NET FORCE AND ENERGY RELATION TO THE STATIONARY NAVIER–STOKES EQUATIONS IN 3-DIMENSIONAL EXTERIOR DOMAINS†

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0. Introduction

Let Ω be an exterior domain in $\mathbb{R}^3$, a domain having a compact complement $\mathbb{R}^3 \setminus \Omega$, with smooth boundary $\partial\Omega$. Consider the boundary-value problem for the stationary Navier–Stokes equations in Ω :

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = \operatorname{div} F & \text{in } x \in \Omega, \\ \operatorname{div} u = 0 & \text{in } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (\text{N-S})$$

where $u = u(x) = (u^1(x), u^2(x), u^3(x))$ and $p = p(x)$ denote the unknown velocity vector and pressure of the fluid at the point $x = (x_1, x_2, x_3) \in \Omega$ respectively, while $F = F(x) = \{F_{ij}(x)\}_{i,j=1,2,3}$ is a given 3×3 tensor with $\operatorname{div} F$ denoting the external force. Here

$$\operatorname{div} F = \left(\sum_{j=1}^3 \frac{\partial F_{j1}}{\partial x_j}, \sum_{j=1}^3 \frac{\partial F_{j2}}{\partial x_j}, \sum_{j=1}^3 \frac{\partial F_{j3}}{\partial x_j} \right).$$

For every $F \in L^2(\Omega)$ there exists at least one weak solution u in the class D of functions with finite Dirichlet integral

$$\int_{\Omega} |\nabla u(x)|^2 dx < \infty. \quad (\text{D})$$

(See Leray [22], Fujita [10] and Finn [7, 9].)

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The first purpose of this paper is to exhibit a representation formula for *arbitrary weak* solutions of (N–S) in class D. In the series of famous papers [4, 6, 7, 9], Finn showed the representation of smooth solutions u in class D with the volume potential $\int_{\Omega} E(x-y)u \cdot \nabla u(y) dy$ of the nonlinear term, where E denotes the fundamental tensor of the Stokes equations. It had been unknown whether or not we might integrate by parts to get the expression $-\int_{\Omega} \nabla E(x-y)u \otimes u(y) dy$ for arbitrary weak solutions u in class D. In the case of inhomogeneous data at infinity such as $u(x) \rightarrow u_{\infty} (\in \mathbb{R}^3) \neq 0$ as $|x| \rightarrow \infty$, the corresponding expression to ours was proved by Finn [5, 8] with E replaced by the fundamental tensor of Oseen's equations. Since, outside of the parabolic wake region behind $\partial\Omega$, the solution of Oseen's equations decays more rapidly than that of the Stokes equations, one can integrate by parts and take the limit of the surface integral on spheres with large radius. Recently, making use of the truncated fundamental tensor E^R introduced by Fujita [10], Galdi [13] proved the above expression in question of weak solutions $u \in D$ with the second derivatives $\nabla^2 u \in L'_{\text{loc}}(\bar{\Omega})$, $1 < r < \infty$, even for homogeneous data at infinity. He established uniform estimates for R of the truncated tensor E^R and showed that the remainder term of the volume integral arising due to truncation vanishes as $R \rightarrow \infty$. We shall prove our representation directly by taking the limit of the surface integral on the large sphere in the same way as Finn [5, 8]. Our formula holds for all weak solutions u in class D; we do not need to impose either regularity on $\nabla^2 u$ or any decay for F except for $F \in L^2(\Omega)$ under which existence of weak solutions was obtained without any restriction.

As an application, we shall show that if u is a weak solution of (N–S) in class D with

$$u \in L^3(\Omega), \quad (0.1)$$

then the net force exerted on $\partial\Omega$ is equal to zero:

$$\int_{\partial\Omega} (T(u, p) + F) \cdot \nu dS = 0, \quad (0.2)$$

where $T(u, p) = \{T_{ij}(u, p)\}_{i,j=1,2,3}$ with $T_{ij}(u, p) = \partial u^i / \partial x^j + \partial u^j / \partial x^i - \delta_{ij} p$ denotes the stress tensor and ν is the unit outer normal to $\partial\Omega$. Similar results were obtained under the stronger hypothesis such that $\nabla u \in L^{3/2}(\Omega)$ by Kozono and Sohr [18], Borchers and Miyakawa [2] and Miyakawa [24]. In the case of inhomogeneous data $u_{\infty} \neq 0$ at infinity with $F \equiv 0$, Finn [8] introduced the class PR of physically reasonable solutions and showed that $u \in \text{PR}$ decays like $|u(x) - u_{\infty}| = o(|x|^{-1})$ as $|x| \rightarrow \infty$ if and only if the net force condition (0.2) is satisfied. On account of

the temperate decay properties of $E(x)$ at infinity, however, it seems to be difficult to show that in class D the net force condition (0.2) implies (0.1). On the other hand, using the class $L^{p,\infty}(\Omega)$ of weak L^p -space over Ω , Kozono and Yamazaki [19] constructed a solution u of (N–S) with

$$u \in L^{3,\infty}(\Omega), \quad \nabla u \in L^{3/2,\infty}(\Omega)$$

for small $F \in L^{3/2,\infty}(\Omega)$. See also Galdi and Padula [11], Galdi and Simader [12] and Novotny and Padula [25]. We shall show that if the weak solution u in class D is small in $L^{3,\infty}$, then the net force condition (0.2) yields (0.1) together with $\nabla u \in L^{3/2}(\Omega)$.

For an arbitrary weak solution u in class D, it is not known whether or not the energy identity

$$\int_{\Omega} |\nabla u|^2 dx = - \int_{\Omega} F \cdot \nabla u dx \quad (0.3)$$

holds. The validity of the energy identity plays an important role for uniqueness criteria of weak solutions (see [17, 20, 23]). The second purpose of this paper is to find a class of solutions in which the energy identity (0.3) is fulfilled. Galdi [13] proved that (0.3) holds for all weak solutions u in class D with $u \in L^4(\Omega)$. Investigating behaviour of the pressure p associated with the weak solution u , we shall obtain a larger class in D with $u \in L^{9/2}(\Omega)$. Our method is based on a simple lemma (see Lemma 2.1) and is different from Galdi's [13].

1. Results

Before stating our results, we introduce some function spaces and then give a definition of weak solutions. Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \phi^2, \phi^3)$ with compact support in Ω , such that $\operatorname{div} \phi = 0$. $\hat{H}_{0,\sigma}^{1,r}$ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the homogeneous norm $\|\nabla \phi\|_r$; $\|\cdot\|_r$ is the L^r -norm and $(\cdot, \cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r -space over Ω , $1 < r < \infty$. Since Ω is an exterior domain, the space $\hat{H}_{0,\sigma}^{1,r}$ is strictly larger than $H_{0,\sigma}^{1,r}$ which is the closure of $C_{0,\sigma}^\infty$ with respect to the usual $H^{1,r}$ -norm $\|\phi\|_{H^{1,r}} = \|\nabla \phi\|_r + \|\phi\|_r$. For concrete characterization of the space $\hat{H}_{0,\sigma}^{1,r}$, see, for example, Kozono and Sohr [16, 17]. $L^{r,\infty}$ is the weak L^r -space over Ω whose norm is denoted by $\|\cdot\|_{r,\infty}$ (see, for example, Bergh and Löfström [1]).

Our definition of a weak solution of (N–S) is as follows.

Definition 1. Let F be in L^2 . A measurable function u on Ω is called a *weak solution* of (N–S) if:

- (i) $u \in \hat{H}_{0,\sigma}^{1,2}$,
- (ii) $(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) = -(F, \nabla \phi)$ for all $\phi \in C_{0,\sigma}^\infty$.

For the existence of weak solutions, see Leray [22], Fujita [10], Finn [7], Ladyzhenskaya [21] and Temam [27].

Concerning the pressure p associated with the weak solution, we have the result of Kozono and Sohr [17, Theorem 3].

PROPOSITION 1. *Let $F \in L^2$. Suppose that u is a weak solution of (N–S). Then there is a scalar function p in $L^2 + L^3$, unique up to an additive constant, such that*

$$(\nabla u, \nabla \psi) + (u \cdot \nabla u, \psi) - (p, \operatorname{div} \psi) = -(F, \nabla \psi) \quad \text{for all } \psi \in C_0^\infty. \quad (1.1)$$

In what follows, we shall take the pressure p as above associated with the weak solution u of (N–S).

Our result on the representation formula now reads as follows.

THEOREM 1. *Let $F \in L^2$. Then every weak solution u of (N–S) admits the representation*

$$\begin{aligned} u(x) = & \int_{\Omega} \nabla E(x-y)(-u \otimes u(y) + F(y)) dy \\ & + \int_{\partial\Omega} E(x-\xi)(T(u, p)(\xi) + F(\xi)) \cdot \nu_\xi dS_\xi \end{aligned} \quad (1.2)$$

for almost all $x \in \Omega$, where $E(x) = \{E_{ij}(x)\}_{i,j=1,2,3}$ denotes the fundamental tensor of the Stokes equations defined by

$$E_{ij}(x) = \frac{1}{8\pi} \left(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3} \right),$$

$T(u, p) = \{T_{ij}(u, p)\}_{i,j=1,2,3}$ is the stress tensor defined by

$$T_{ij}(u, p) = \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right) - \delta_{ij} p,$$

and ν_ξ is the unit outer normal to $\xi \in \partial\Omega$.

Remark 1.

1. Since we deal with arbitrary weak solutions u , the trace $(T(u, p) + F) \cdot \nu|_{\partial\Omega}$ should be understood in a generalized sense. Actually, by Proposition 1 and the generalized Stokes formula (see Temam [27, Chapter I, Theorem 2]), we have

$$(T(u, p) + F) \cdot \nu|_{\partial\Omega} \in H^{-1/2,2}(\partial\Omega) \equiv H^{1/2,2}(\partial\Omega)^*.$$

Hence, the second term of the right-hand side of (1.2) may be written as

$$\langle E(x - \cdot), (T(u, p) + F) \cdot \nu \rangle_{\partial\Omega},$$

- where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{1/2,2}(\partial\Omega)$ and $H^{-1/2,2}(\partial\Omega)$.
2. Galdi [13, IX, Theorem 6.2] showed the same representation as (1.2) with $\operatorname{div} F$ replaced by $f \in L^q \cap L^s_{\operatorname{loc}}(\bar{\Omega})$ for $1 < q < 3/2$, $1 < s < \infty$. He assumed that the second derivatives $\nabla^2 u$ of solutions u are in $L^r_{\operatorname{loc}}(\bar{\Omega})$ for some $1 < r < \infty$. Instead of $E(x - y)$ itself, he made use of the truncated fundamental tensor $E^R(x - y)$ which coincides with $E(x - y)$ for $|x - y| \leq R$, and proved that the volume integral in (1.2) of the remainder term arising by truncation yields no contribution as $R \rightarrow \infty$; the method is, however, different from ours.

We shall next investigate the relation between decay properties at infinity of weak solutions and the net force exerted on $\partial\Omega$.

THEOREM 2.

- (1) Let $F \in L^2 \cap L^{3/2}$. Suppose that u is a weak solution of (N–S). If $u \in L^3$, then the net force exerted on $\partial\Omega$ is equal to zero:

$$\int_{\partial\Omega} (T(u, p) + F) \cdot \nu \, dS = 0. \quad (1.3)$$

- (2) Conversely, for every $1 \leq q \leq 3/2$, there is a constant $\lambda = \lambda(q)$ such that, if $F \in L^2 \cap L^q$ and if u is a weak solution of (N–S) in $L^{3,\infty}$ with (1.3) and additionally

$$\|u\|_{L^{3,\infty}(B_R^c)} \leq \lambda \quad \text{for some } R > 0 \text{ with } B_R \equiv \{|x| < R\} \supset \partial\Omega, \quad (1.4)$$

then we have

$$u \in L^{3q/(3-q)} \text{ with } \nabla u \in L^q \quad \text{provided } 1 < q \leq \frac{3}{2}, \quad (1.5)$$

$$u \in L^{3/2,\infty} \text{ with } \nabla u \in L^{1,\infty} \quad \text{provided } q = 1. \quad (1.6)$$

Instead of (1.4), we can restate the above converse assertion (2) in terms of F .

COROLLARY 1. *For every $1 \leq q \leq 3/2$, there is a constant $\mu = \mu(q)$ such that if $F \in L^2 \cap L^q$ satisfies $\|F\|_{3/2, \infty} \leq \mu$, then every weak solution u of (N-S) with (1.3) and the energy inequality*

$$\int_{\Omega} |\nabla u|^2 dx \leq - \int_{\Omega} F \cdot \nabla u dx \quad (1.7)$$

has the additional properties (1.5) and (1.6).

Remark 2.

1. As we remarked after Theorem 1, (1.3) should be understood as

$$\langle c, (T(u, p) + F) \cdot \nu \rangle_{\partial\Omega} = 0 \quad \text{for all constant vectors } c \in \mathbb{R}^3,$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{1/2,2}(\partial\Omega)$ and $H^{-1/2,2}(\partial\Omega)$.

2. Kozono and Sohr [18, Theorem C] and Borchers and Miyakawa [2, Theorem 2.4] gave a result similar to Theorem 2(1) under the stronger assumption that $\nabla u \in L^{3/2}$. Here, we do not need to impose any decay on ∇u other than one on u itself. Note that $u \in \hat{H}_{0,\sigma}^{1,2}$ with $\nabla u \in L^{3/2}$ yields $u \in L^3$.
3. Miyakawa [24, Theorem C] treated smooth solutions u in the class $\sup_{x \in \Omega} |x| |u(x)| + \sup_{x \in \Omega} |x|^2 |\nabla u(x)| < \infty$. He obtained a result similar to the above Theorem 2(2) under the assumption that $u(x)$ can be approximated by a sequence of functions in $C_{0,\sigma}^\infty$ with respect to the norm $\|\nabla u\|_{3/2, \infty}$.
4. In the case $u(x) \rightarrow u_\infty (\in \mathbb{R}^3) \neq 0$ as $|x| \rightarrow \infty$ with $F \equiv 0$, Finn considered u of class PR (physically reasonable) satisfying $|u(x) - u_\infty| = O(|x|^{-1/2-\varepsilon})$, $\varepsilon > 0$ as $|x| \rightarrow \infty$. In [8, Theorem 5.8], he showed that every PR solution u decays like $|u(x) - u_\infty| = o(|x|^{-1})$ as $|x| \rightarrow \infty$ if and only if (1.3) is fulfilled. The main difference between the homogeneous ($u_\infty = 0$) and inhomogeneous ($u_\infty \neq 0$) cases stems from behaviour at infinity of the volume integral in the representation formula (1.2). In the inhomogeneous case with $F \equiv 0$, if u is a PR solution, then $\int_{\Omega} \nabla \chi(x-y) u \otimes u(y) dy = O(|x|^{-3/2-\varepsilon})$ holds for some $\varepsilon > 0$ as $|x| \rightarrow \infty$, where χ denotes the fundamental tensor of the Oseen equations. (See Finn [5, Theorem 8].) In our homogeneous case, to obtain better decay of the volume integral in (1.2) than that of given solutions u , we need to consider a class such as $u(x) = O(|x|^{-1-\varepsilon})$ as $|x| \rightarrow \infty$ or such as $u \in L^p$ for $1 < p < 3$.

We are next concerned with the class of weak solutions satisfying the energy identity. It is not known whether or not every weak solution fulfills even the energy inequality (1.7).

THEOREM 3. *Let $F \in L^2$. Suppose that u is a weak solution of (N–S). If $u \in L^{9/2}$, then the energy identity*

$$\int_{\Omega} |\nabla u|^2 dx = - \int_{\Omega} F \cdot \nabla u dx \quad (1.8)$$

holds.

Remark 3.

1. Galdi [13, IX, Theorem 2.1] gave the same result for a smaller class of weak solutions u of (N–S) with u in L^4 . Note that $\hat{H}_{0,\sigma}^{1,2} \cap L^4 \subset \hat{H}_{0,\sigma}^{1,2} \cap L^{9/2}$.
2. It can be easily seen that Theorem 3 also holds for the whole space $\mathbb{R}^3$. Hence, an immediate consequence of (1.8) is a theorem of Liouville type: if $u \in L^{9/2}(\mathbb{R}^3)$ with $\nabla u \in L^2(\mathbb{R}^3)$ satisfies (N–S) with $F \equiv 0$ in the sense of distributions in $\mathbb{R}^3$, then it follows that $u \equiv 0$. See also Galdi [13, IX, Theorem 9.7].

2. Preliminaries

In what follows we shall denote by C various constants. In particular, we shall denote by $C = C(*, \dots, *)$ the constants which depend only on the quantities appearing in parenthesis. The following lemmas are elementary, but will play an important role for the proof of our theorems.

LEMMA 2.1. *Let $f \in L^q$ for $1 \leq q \leq 3/2$. Then there is a sequence $\{R_m\}_{m=1}^{\infty}$ with $2^m < R_m < 2^{m+1}$ ($m = 1, \dots$) such that*

$$\int_{|\xi|=R_m} |f(\xi)| dS_{\xi} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad (2.1)$$

where dS_{ξ} denotes the surface element of the sphere $\{\xi \mid |\xi| = R_m\}$.

The same conclusion is also true for $f \in L^1 + L^{3/2}$.

Proof. We shall first show that there is a sequence $\{R_m\}_{m=1}^{\infty}$ with $2^m < R_m < 2^{m+1}$ ($m = 1, \dots$) such that

$$2^m \int_{|\xi|=R_m} |f(\xi)|^q dS_{\xi} \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (2.2)$$

Set $G(r) \equiv \int_{|\xi|=r} |f(\xi)|^q dS_\xi$. Since $f \in L^q$, we have

$$\lim_{m \rightarrow \infty} \int_{2^m}^{2^{m+1}} G(r) dr = 0. \quad (2.3)$$

Suppose that there does not exist such a sequence $\{R_m\}_{m=1}^\infty$ such that (2.2) holds. Then there is a positive constant ε_0 and a subsequence $\{2^{m(j)}\}_{j=1}^\infty$ of $\{2^m\}_{m=1}^\infty$ such that

$$2^{m(j)} G(r) \geq \varepsilon_0 \quad \text{for all } r \text{ with } 2^{m(j)} < r < 2^{m(j)+1}, j = 1, \dots$$

Hence, it follows from the mean-value property that

$$\int_{2^{m(j)}}^{2^{m(j)+1}} G(r) dr \geq 2^{-m(j)} \varepsilon_0 \int_{2^{m(j)}}^{2^{m(j)+1}} dr = \varepsilon_0$$

for all $j = 1, \dots$, which contradicts (2.3).

By the Hölder inequality, we have

$$\begin{aligned} \int_{|\xi|=R_m} |f(\xi)| dS_\xi &\leq \left(\int_{|\xi|=R_m} |f(\xi)|^q dS_\xi \right)^{1/q} \left(\int_{|\xi|=R_m} dS_\xi \right)^{1-1/q} \\ &\leq (4\pi R_m^2)^{1-1/q} 2^{-m/q} \left(2^m \int_{|\xi|=R_m} |f(\xi)|^q dS_\xi \right)^{1/q} \\ &\leq (4\pi)^{1-1/q} 2^{m(2-3/q)} \left(2^m \int_{|\xi|=R_m} |f(\xi)|^q dS_\xi \right)^{1/q} \end{aligned} \quad (2.4)$$

for all $m = 1, \dots$. Since $2 - 3/q \leq 0$, from (2.2) and (2.4) we obtain (2.1).

Next we consider the case

$$f = f_1 + f_2, \quad \text{where } f_1 \in L^1, f_2 \in L^{3/2}.$$

Since the function $|f_1| + |f_2|^{3/2}$ is summable on Ω , similar to (2.2) we have that

$$\lim_{m \rightarrow \infty} 2^m \int_{|\xi|=R_m} (|f_1(\xi)| + |f_2(\xi)|^{3/2}) dS_\xi = 0$$

for a sequence $\{R_m\}_{m=1}^\infty$ with $2^m < R_m < 2^{m+1}$ ($m = 1, \dots$). Hence, in the same way as (2.4) we get

$$\begin{aligned}
& \int_{|\xi|=R_m} |f(\xi)| dS_\xi \\
& \leq \int_{|\xi|=R_m} (|f_1(\xi)| + |f_2(\xi)|) dS_\xi \\
& \leq 2^{-m} \left(2^m \int_{|\xi|=R_m} |f_1(\xi)| dS_\xi \right) + (4\pi)^{1/3} \left(2^m \int_{|\xi|=R_m} |f_2(\xi)|^{3/2} dS_\xi \right)^{2/3} \\
& \rightarrow 0 \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

which yields the desired result. $\square$

LEMMA 2.2.

- (1) Let $1 \leq \alpha < 3$ and let $f \in L^q$ for $1 \leq q \leq 3/(\alpha - 1)$. For every $R > 0$ with $B_R \equiv \{x \in \mathbb{R}^3; |x| < R\} \supset \partial\Omega$, there is a sequence $\{R_m\}_{m=1}^\infty$ with $2R < R_1 < R_2 < \dots < R_m < \dots \rightarrow \infty$ such that

$$\sup_{x \in \Omega_R} \left(\int_{|\xi|=R_m} \frac{|f(\xi)|}{|x - \xi|^{3-\alpha}} dS_\xi \right) \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad (2.5)$$

where $\Omega_R \equiv \Omega \cap B_R$.

- (2) Let $f \in L^1 + L^3$ and let $g \in L^1 + L^r$ for $1 \leq r < \infty$. Then for every $R > 0$ with $B_R \supset \partial\Omega$, there is a sequence $\{R_m\}_{m=1}^\infty$ with $2R \leq R_1 < R_2 < \dots < R_m < \dots \rightarrow \infty$ such that

$$\sup_{x \in \Omega_R} \int_{|\xi|=R_m} \frac{|f(\xi)|}{|x - \xi|} dS_\xi + \sup_{x \in \Omega_R} \int_{|\xi|=R_m} \frac{|g(\xi)|}{|x - \xi|^2} dS_\xi \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (2.6)$$

Proof. (1) Let us take an integer m_0 so large that $2^{m_0-1} > R$. In the same way as (2.2), there is a sequence $\{R_m\}_{m=1}^\infty$ with $2^{m_0+m} < R_m < 2^{m_0+m+1}$ such that

$$2^{m_0+m} \int_{|\xi|=R_m} |f(\xi)|^q dS_\xi \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (2.7)$$

Let $x \in \Omega_R$. Since $R_m/2 > |x|$ for all $m = 1, \dots$, we have

$$|x - \xi| \geq |\xi| - |x| = R_m - |x| > \frac{1}{2}R_m \quad \text{for all } x \in \Omega_R \text{ and all } |\xi| = R_m, m = 1, \dots$$

Hence it follows from the Hölder inequality that

$$\begin{aligned}
& \int_{|\xi|=R_m} \frac{|f(\xi)|}{|x-\xi|^{3-\alpha}} dS_\xi \\
& \leq 2^{3-\alpha} R_m^{\alpha-3} \left(\int_{|\xi|=R_m} |f(\xi)|^q dS_\xi \right)^{1/q} \left(\int_{|\xi|=R_m} dS_\xi \right)^{1-1/q} \\
& \leq 2^{3-\alpha} R_m^{\alpha-3} \left(2^{m_0+m} \int_{|\xi|=R_m} |f(\xi)|^q dS_\xi \right)^{1/q} 2^{-(m_0+m)/q} (4\pi R_m^2)^{1-1/q} \\
& \leq C 2^{m(\alpha-1-3/q)} \left(2^{m_0+m} \int_{|\xi|=R_m} |f(\xi)|^q dS_\xi \right)^{1/q} \tag{2.8}
\end{aligned}$$

for all $x \in \Omega_R$ and all $m = 1, \dots$ with $C = C(m_0, \alpha, q)$. Since $\alpha - 1 - 3/q \leq 0$, from (2.7) and (2.8) we obtain (2.5).

(2) By assumption, f and g can be written as

$$f = f_1 + f_2 \quad \text{for } f_1 \in L^1, f_2 \in L^3,$$

and

$$g = g_1 + g_2 \quad \text{for } g_1 \in L^1, g_2 \in L^r \ (1 \leq r < \infty)$$

respectively. Since the function $|f_1| + |f_2|^3 + |g_1| + |g_2|^r$ is summable on Ω , similar to (2.7), we can take a sequence $\{R_m\}_{m=1}^\infty$ with $2^{m_0+m} < R_m < 2^{m_0+m+1}$ so that

$$2^{m_0+m} \int_{|\xi|=R_m} (|f_1(\xi)| + |f_2(\xi)|^3 + |g_1(\xi)| + |g_2(\xi)|^r) dS_\xi \rightarrow 0 \quad \text{as } m \rightarrow \infty. \tag{2.9}$$

Hence, in the same way as (2.8) with $\alpha = 1, 2$, it follows that

$$\begin{aligned}
& \int_{|\xi|=R_m} \left(\frac{|f(\xi)|}{|x-\xi|} + \frac{|g(\xi)|}{|x-\xi|^2} \right) dS_\xi \\
& \leq \int_{|\xi|=R_m} \frac{|f_1(\xi)| + |f_2(\xi)|}{|x-\xi|} dS_\xi + \int_{|\xi|=R_m} \frac{|g_1(\xi)| + |g_2(\xi)|}{|x-\xi|^2} dS_\xi \\
& \leq C \left(2^{-2m} \left(2^{m_0+m} \int_{|\xi|=R_m} |f_1(\xi)| dS_\xi \right) + \left(2^{m_0+m} \int_{|\xi|=R_m} |f_2(\xi)|^3 dS_\xi \right)^{1/3} \right) \\
& \quad + C \left(2^{-3m} \left(2^{m_0+m} \int_{|\xi|=R_m} |g_1(\xi)| dS_\xi \right) \right. \\
& \quad \left. + 2^{-3m/r} \left(2^{m_0+m} \int_{|\xi|=R_m} |g_2(\xi)|^r dS_\xi \right)^{1/r} \right)
\end{aligned}$$

for all $x \in \Omega_R$ and all $m = 1, \dots$ with $C = C(m_0, r)$. Now the desired result (2.6) follows from the above estimate and (2.9). $\square$

3. Proof of the Theorems

3.1. Representation formula: proof of Theorem 1

Before proving Theorem 1, we shall consider the representation formula of *weak* solutions in *bounded* domains.

Let $\mathcal{D} \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\mathcal{D}$. Define an auxiliary space $E(\mathcal{D}) \equiv \{u \in L^2(\mathcal{D}); \operatorname{div} u \in L^2(\mathcal{D})\}$. It is known by Temam [27, Chapter I, Theorem 1.2] that $u \cdot \nu \in H^{-1/2,2}(\partial\mathcal{D}) \equiv H^{1/2,2}(\partial\mathcal{D})^*$ with

$$\|u \cdot \nu\|_{H^{-1/2,2}(\partial\mathcal{D})} \leq C(\|u\|_{L^2(\mathcal{D})} + \|\operatorname{div} u\|_{L^2(\mathcal{D})}) \quad \text{for all } u \in E(\mathcal{D}) \quad (3.1)$$

with $C = C(\mathcal{D})$, where ν denotes the unit outer normal to $\partial\mathcal{D}$. Furthermore, the generalized Stokes formula

$$\int_{\mathcal{D}} \nabla p \cdot u \, dx + \int_{\mathcal{D}} p \operatorname{div} u \, dx = \langle p, u \cdot \nu \rangle_{\partial\mathcal{D}} \quad \text{for all } p \in H^{1,2}(\mathcal{D}) \text{ and all } u \in E(\mathcal{D}) \quad (3.2)$$

holds, where $\langle \cdot, \cdot \rangle_{\partial\mathcal{D}}$ denotes the duality pairing between $H^{1/2,2}(\partial\mathcal{D})$ and $H^{-1/2,2}(\partial\mathcal{D})$. The following representation formula is well known for smooth solutions of the Stokes equations in $\mathcal{D}$. Since we shall treat general *weak* solutions of (N–S), the corresponding formula for weak solutions is necessary.

LEMMA 3.1. *Let $F \in L^2(\mathcal{D})$ and let $\phi \in H^{1/2,2}(\partial\mathcal{D})$ with $\int_{\partial\mathcal{D}} \phi \cdot \nu \, dS = 0$. Suppose that $u \in H^{1,2}(\mathcal{D})$ and $p \in L^2(\mathcal{D})$. Assume that the pair $\{u, p\}$ satisfies*

$$\begin{cases} -\Delta u + \nabla p = \operatorname{div} F & \text{in the sense of distributions in } \mathcal{D}, \\ \operatorname{div} u = 0 & \text{in } \mathcal{D}, \\ u = \phi & \text{on } \partial\mathcal{D}. \end{cases} \quad (3.3)$$

Then we have

$$(T(u, p) + F) \cdot \nu \in H^{-1/2,2}(\partial\mathcal{D}) \quad (3.4)$$

and u admits the representation

$$\begin{aligned} u(x) &= \int_{\mathcal{D}} \nabla E(x - y) F(y) \, dy \\ &\quad - \int_{\partial\mathcal{D}} T(E, e)(x - \xi) \phi(\xi) \cdot \nu_{\xi} \, dS_{\xi} + \langle E(x - \cdot), (T(u, p) + F) \cdot \nu \rangle_{\partial\mathcal{D}} \end{aligned} \quad (3.5)$$

for almost all $x \in \mathcal{D}$, where $e(x) = (e_1(x), e_2(x), e_3(x))$ is the fundamental vector for the pressure of the Stokes equations defined by $e_j(x) = x_j/4\pi|x|^3$ ($j = 1, 2, 3$).

Proof. By (3.3) we have $T(u, p) + F \in L^2(\mathcal{D})$ with $\operatorname{div}(T(u, p) + F) = 0$ in the sense of distributions in $\mathcal{D}$. Hence (3.4) follows from (3.1).

To prove (3.5), let us take sequences $\{F_j\}_{j=1}^\infty$ in $C_0^\infty(\mathcal{D})$ and $\{\phi_j\}_{j=1}^\infty$ in $C^\infty(\partial\mathcal{D})$ with $\int_{\partial\mathcal{D}} \phi_j \cdot \nu \, dS = 0$ ($j = 1, \dots$) so that

$$F_j \rightarrow F \text{ in } L^2(\mathcal{D}), \quad \phi_j \rightarrow \phi \text{ in } H^{1/2,2}(\partial\mathcal{D}) \quad (3.6)$$

as $j \rightarrow \infty$. It follows from Cattabriga [3] (see also Kozono and Sohr [15, Proposition 2.10]) that for each $j = 1, \dots$, there is a unique pair $\{u_j, p_j\}$ of smooth solutions for the equations

$$\begin{cases} -\Delta u_j + \nabla p_j = \operatorname{div} F_j & \text{in } \mathcal{D}, \\ \operatorname{div} u_j = 0 & \text{in } \mathcal{D}, \\ u_j = \phi_j & \text{on } \partial\mathcal{D}. \end{cases} \quad (3.7)$$

By the *a priori* estimate in $H^{1,2}$ for solutions of the Stokes equations [3, 15], it follows that

$$\|u_j - u\|_{H^{1,2}(\mathcal{D})} + \|p_j - p\|_{L^2(\mathcal{D})} \leq C(\|F_j - F\|_{L^2(\mathcal{D})} + \|\phi_j - \phi\|_{H^{1/2,2}(\partial\mathcal{D})}), \quad j = 1, \dots,$$

where $C = C(\mathcal{D})$. Hence, by (3.6) we have

$$u_j \rightarrow u \text{ in } H^{1,2}(\mathcal{D}), \quad p_j \rightarrow p \text{ in } L^2(\mathcal{D}) \quad (3.8)$$

as $j \rightarrow \infty$. Since $\{u_j, p_j\}$ ($j = 1, \dots$) are smooth solutions of (3.7), we have the representation

$$\begin{aligned} u_j(x) &= \int_{\mathcal{D}} \nabla E(x-y) F_j(y) \, dy - \int_{\partial\mathcal{D}} \phi_j(\xi) T(E, e)(x-\xi) \cdot \nu_\xi \, dS_\xi \\ &\quad + \int_{\partial\mathcal{D}} E(x-\xi) (T(u_j, p_j)(\xi) + F_j(\xi)) \cdot \nu_\xi \, dS_\xi \end{aligned} \quad (3.9)$$

for all $x \in \mathcal{D}$. Since $|\nabla E(x-y)| \leq C|x-y|^{-2}$ for all $x, y \in \mathbb{R}^3$, the Hardy–Littlewood–Sobolev inequality [26, p. 119, Theorem 1] yields

$$\int_{\mathcal{D}} \nabla E(\cdot - y) F_j(y) \, dy \rightarrow \int_{\mathcal{D}} \nabla E(\cdot - y) F(y) \, dy \quad \text{in } L^6(\mathcal{D})$$

as $j \rightarrow \infty$. Hence there is a subsequence of $\{F_j\}_{j=1}^\infty$, which we also denote by $\{F_j\}_{j=1}^\infty$ for simplicity, such that

$$\int_{\mathcal{D}} \nabla E(x-y) F_j(y) dy \rightarrow \int_{\mathcal{D}} \nabla E(x-y) F(y) dy \quad \text{for almost all } x \in \mathcal{D} \quad (3.10)$$

as $j \rightarrow \infty$. Clearly, we have by (3.6)

$$\int_{\partial \mathcal{D}} \phi_j(\xi) T(E, e)(x - \xi) \cdot \nu_\xi dS_\xi \rightarrow \int_{\partial \mathcal{D}} \phi(\xi) T(E, e)(x - \xi) \cdot \nu_\xi dS_\xi \quad (3.11)$$

for almost all $x \in \mathcal{D}$ as $j \rightarrow \infty$. Moreover, (3.3) and (3.7) state that

$$\operatorname{div}(T(u_j, p_j) + F_j - (T(u, p) + F)) = 0 \quad \text{in the sense of distributions in } \mathcal{D}.$$

Hence by (3.6), (3.8) and (3.1), it follows that

$$(T(u_j, p_j) + F_j) \cdot \nu \rightarrow (T(u, p) + F) \cdot \nu \quad \text{in } H^{-1/2,2}(\partial \mathcal{D}),$$

which yields

$$\begin{aligned} & \int_{\partial \mathcal{D}} E(x - \xi) (T(u_j, p_j)(\xi) + F_j(\xi)) \cdot \nu_\xi dS_\xi \\ & \rightarrow \langle E(x - \cdot), (T(u, p) + F) \cdot \nu \rangle_{\partial \mathcal{D}} \quad \text{for all } x \in \mathcal{D} \text{ as } j \rightarrow \infty. \end{aligned} \quad (3.12)$$

On the other hand, by (3.8), there is a subsequence $\{u_j\}_{j=1}^\infty$, which we also denote by $\{u_j\}_{j=1}^\infty$ for simplicity, such that

$$u_j(x) \rightarrow u(x) \quad \text{for almost all } x \in \mathcal{D} \text{ as } j \rightarrow \infty. \quad (3.13)$$

Now letting $j \rightarrow \infty$ in (3.9), from (3.10)–(3.13), we obtain the desired formula (3.5). $\square$

Proof of Theorem 1. There is a constant C such that

$$|T(E, e)(x - \xi)| \leq \frac{C}{|x - \xi|^2}, \quad |E(x - \xi)| \leq \frac{C}{|x - \xi|} \quad \text{for all } x, \xi \in \mathbb{R}^3.$$

Since $\hat{H}_{0,\sigma}^{1,2} \subset L^6$, we have

$$u \in L^6, \quad T(u, p) + F - u \otimes u \in L^2 + L^3.$$

Then it follows from Lemma 2.2(2) that for every fixed $R > 0$ with $B_R \supset \partial\Omega$, there is a sequence $\{R_m\}_{m=1}^\infty$ with $2R < R_1 < \dots < R_m < \dots \rightarrow \infty$ such that

$$\left\{ \begin{array}{l} \int_{|\xi|=R_m} u(\xi) T(E, e)(x - \xi) \cdot v_\xi dS_\xi \rightarrow 0, \\ \int_{|\xi|=R_m} E(x - \xi)(T(u, p)(\xi) + F(\xi) - u \otimes u(\xi)) \cdot v_\xi dS_\xi \rightarrow 0 \end{array} \right. \quad (3.14)$$

uniformly in $x \in \Omega_R$ as $m \rightarrow \infty$. For each domain $\Omega_{R_m} \equiv \Omega \cap B_{R_m}$, we have by Proposition 1 that

$$u \in H^{1,2}(\Omega_{R_m}), p \in L^2(\Omega_{R_m}) \quad \text{and} \quad F - u \otimes u \in L^2(\Omega_{R_m}).$$

Hence it follows from Lemma 3.1 that

$$\begin{aligned} u(x) &= \int_{\Omega_{R_m}} \nabla E(x - y)(-u \otimes u(y) + F(y)) dy \\ &\quad + \langle E(x - \cdot), (T(u, p) + F) \cdot v \rangle_{\partial\Omega} \\ &\quad - \int_{|\xi|=R_m} u(\xi) T(E, e)(x - \xi) \cdot v_\xi dS_\xi \\ &\quad + \int_{|\xi|=R_m} E(x - \xi)(T(u, p)(\xi) + F(\xi) - u \otimes u(\xi)) \cdot v_\xi dS_\xi \end{aligned}$$

for almost all $x \in \Omega_R$. Letting $m \rightarrow \infty$ in this identity, we see by (3.14) that the volume integral $\int_{\Omega} \nabla E(x - y)(u \otimes u(y) - F(y)) dy$ converges absolutely and that $u(x)$ admits the representation

$$u(x) = \int_{\Omega} \nabla E(x - y)(u \otimes u(y) - F(y)) dy + \langle E(x - \cdot), (T(u, p) + F) \cdot v \rangle_{\partial\Omega}$$

for almost all $x \in \Omega_R$. Since R can be taken arbitrarily large, we get the desired result. This completes the proof of Theorem 1. $\square$

3.2. Force exerted on $\partial\Omega$: proof of Theorem 2

(1) Let $\{r_1, r_2, r_3\}$ be a canonical orthonormal basis in $\mathbb{R}^3$. By Theorem 1, the weak solution $u = (u^1, u^2, u^3)$ is expressed as

$$\begin{aligned}
u^i(x) &= \sum_{k,l=1}^3 \int_{\Omega} \frac{\partial E_{k,i}}{\partial x_l}(x-y)(-u^l u^k(y) + F_{lk}(y)) dy \\
&\quad + \langle E(x - \cdot) r_i - E(x) r_i, (T(u, p) + F) \cdot \nu \rangle_{\partial\Omega} \\
&\quad + E(x) \langle r_i, (T(u, p) + F) \cdot \nu \rangle_{\partial\Omega} \\
&\equiv U(x) + V(x) + W(x) \quad (i = 1, 2, 3)
\end{aligned} \tag{3.15}$$

for almost all $x \in \Omega$. Since

$$\frac{\partial E_{k,i}}{\partial x_l}(x-y) \leq C|x-y|^{-2}$$

for all $x, y \in \mathbb{R}^3$ and since $-u \otimes u + F \in L^{3/2}$ by the assumption, the Hardy–Littlewood–Sobolev inequality [26, p. 119, Theorem 1] yields

$$U \in L^3. \tag{3.16}$$

Let $R > 0$ be as $B_R \supset \partial\Omega$. For $\xi \in \partial\Omega$ and $x \in \Omega$ with $|x| > 2R$, we have

$$\begin{aligned}
|E(x - \xi) - E(x)| &= \left| \int_0^1 \frac{d}{d\theta} E(x - \theta\xi) d\theta \right| = \left| \int_0^1 \nabla E(x - \theta\xi) \cdot (-\xi) d\theta \right| \\
&\leq C \int_0^1 \frac{|\xi|}{|x - \theta\xi|^2} d\theta \\
&\leq C \int_0^1 \frac{|\xi|}{|x - \xi|^2} d\theta \\
&\leq C \frac{R}{|x|^2}, \\
\left| \frac{\partial E}{\partial \xi}(x - \xi) \right| &\leq \frac{C}{|x - \xi|^2} \leq \frac{C}{|x|^2}
\end{aligned}$$

with C independent of x, ξ . Hence it follows that

$$\|E(x - \cdot) r_i - E(x) r_i\|_{H^{1/2,2}(\partial\Omega)} \leq \frac{C}{|x|^2} \quad (i = 1, 2, 3) \quad \text{for all } |x| \geq 2R$$

with $C = C(R)$. Since $(T(u, p) + F) \cdot \nu \in H^{-1/2,2}(\partial\Omega)$, we obtain from the above estimate that

$$V \in L^q(B_{2R}^c) \quad \text{for all } 3/2 < q \leq \infty. \tag{3.17}$$

By (3.15)–(3.17) together with $u^i \in L^3$, it follows that $W \in L^3(B_{2R}^c)$. But

$$\int_{|x| \geq 2R} |E(x)|^3 dx = \infty;$$

hence

$$\langle r_i, (T(u, p) + F) \cdot \nu \rangle_{\partial\Omega} = 0, \quad i = 1, 2, 3$$

and we get (1.3).

(2) We shall next prove that (1.3) and (1.4) imply (1.5) and (1.6). Our argument is essentially deduced from Miyakawa [24, Theorem C] in which the smooth solutions u of (N–S) in the class

$$\sup_{x \in \Omega} |x| |u(x)| < \infty, \quad \sup_{x \in \Omega} |x|^2 |\nabla u(x)| < \infty \quad (3.18)$$

were considered. He imposed the condition that there is a sequence $\{\phi_j\}_{j=1}^\infty$ in $C_{0,\sigma}^\infty$ such that $\|\nabla \phi_j - \nabla u\|_{3/2,\infty} \rightarrow 0$. Here we shall prove for arbitrary weak solutions without any decay properties like (3.18) and make it clear that the existence of such a sequence $\{\phi_j\}_{j=1}^\infty$ can be replaced by (1.3).

By (1.1), it follows that $T(u, p) - u \otimes u + F \in L_{\text{loc}}^2(\bar{\Omega})$ with

$$\operatorname{div}(T(u, p) - u \otimes u + F) = 0 \quad \text{in the sense of distributions on } \Omega.$$

Applying the generalized Stokes formula as in (3.2), we have by (1.3) that

$$\int_{|\xi|=R} (T(u, p) - u \otimes u + F) \cdot \nu dS = 0,$$

for all $R > 0$ with $B_R \supset \partial\Omega$, where ν denotes the unit outer normal to the sphere $\{\xi \mid |\xi| = R\}$. In the same way as Theorem 1 and (3.15), it is easy to see that u admits the representation

$$\begin{aligned} u(x) &= \int_{B_R^c} \nabla E(x-y)(-u \otimes u(y) + F(y)) dy \\ &\quad - \int_{|\xi|=R} u(\xi) T(E, e)(x-\xi) \cdot \nu_\xi dS_\xi \\ &\quad + \int_{|\xi|=R} (E(x-\xi) - E(x))(T(u, p)(\xi) - u \otimes u(\xi) + F(\xi)) \cdot \nu_\xi dS_\xi \end{aligned} \quad (3.19)$$

for almost all $x \in B_R^c (= \{x \in \mathbb{R}^3; |x| \geq R\})$.

Let us consider the following integral equation for v :

$$v(x) = - \int_{B_R^c} \nabla E(x-y) u \otimes v(y) dy + u_0(x), \quad x \in B_R^c, \quad (3.20)$$

where

$$\begin{aligned} u_0(x) &\equiv \int_{B_R^c} \nabla E(x-y) F(y) dy - \int_{|\xi|=R} u(\xi) T(E, e)(x-\xi) \cdot v_\xi dS_\xi \\ &\quad + \int_{|\xi|=R} (E(x-\xi) - E(x))(T(u, p)(\xi) - u \otimes u(\xi) + F(\xi)) \cdot v_\xi dS_\xi \end{aligned}$$

Then we have the following lemma.

LEMMA 3.2.

- (1) *There is a constant λ_0 such that if u is in $L^{3,\infty}(B_R^c)$ with $\|u\|_{L^{3,\infty}(B_R^c)} \leq \lambda_0$, then for every $F \in L^{3/2,\infty}(B_R^c)$, we have a unique solution v of (3.20) in $L^{3,\infty}(B_R^c)$.*
- (2) *For $1 \leq q \leq 3/2$, there is a constant $\lambda_* = \lambda_*(q) \leq \lambda_0$ such that if $F \in L^{3/2,\infty}(B_R^c) \cap L^q(B_R^c)$ and if $\|u\|_{L^{3,\infty}(B_R^c)} \leq \lambda_*$, then the solution v given by (1) above has the additional properties*

$$v \in L^{3q/(3-q)}(B_R^c), \quad \nabla v \in L^q(B_R^c) \quad \text{provided } 1 < q \leq \frac{3}{2}, \quad (3.21)$$

$$v \in L^{3/2,\infty}(B_R^c), \quad \nabla v \in L^{1,\infty}(B_R^c) \quad \text{provided } q = 1. \quad (3.22)$$

By (3.19) and Lemma 3.2(1) we have $u(x) \equiv v(x)$ for $|x| \geq R$. Since $u \in L^6$ with $\nabla u \in L^2$, the desired properties (1.5) and (1.6) follow from (3.21) and (3.22), respectively. This proves Theorem 2. $\square$

Now it remains to prove Lemma 3.2.

Proof of Lemma 3.2. (1) Let us define the linear operator T_u defined by

$$(T_u v)(x) \equiv - \int_{B_R^c} \nabla E(x-y) u \otimes v(y) dy. \quad (3.23)$$

By the Hardy–Littlewood–Sobolev and the Hunt–Marcinkiewicz inequalities we have

$$\begin{aligned} T_u &\in \mathcal{B}(L^r(B_R^c)) \quad \text{for } 1 < r \leq 3, \\ T_u &\in \mathcal{B}(L^{r,\infty}(B_R^c)) \quad \text{for } 1 \leq r \leq 3 \end{aligned}$$

$(B(X))$; the set of bounded operators on X) with

$$\|T_u v\|_{L^r(B_R^c)} \leq C \|u\|_{L^{3,\infty}(B_R^c)} \|v\|_{L^r(B_R^c)}, \quad v \in L^r(B_R^c) \quad \text{for } 1 < r \leq 3, \quad (3.24)$$

$$\|T_u v\|_{L^{r,\infty}(B_R^c)} \leq C \|u\|_{L^{3,\infty}(B_R^c)} \|v\|_{L^{r,\infty}(B_R^c)}, \quad v \in L^{r,\infty}(B_R^c) \quad \text{for } 1 \leq r \leq 3, \quad (3.25)$$

where $C = C(r)$ is independent of R , u and v . Since $F \in L^{3/2}$ and since $|T(E, e)(x - \xi)| \leq C|x - \xi|^{-2}$, we have similar to (3.16), (3.17) and above that

$$u_0 \in L^{3,\infty}(B_R^c). \quad (3.26)$$

Consider the linear iteration scheme for $\{v_j\}_{j=0}^\infty$ defined by

$$\begin{cases} v_0 = u_0, \\ v_{j+1} = T_u v_j + u_0, \quad j = 0, 1, \dots \end{cases} \quad (3.27)$$

By (3.25) with $r = 3$, we have

$$\begin{aligned} \|v_{j+1} - v_j\|_{L^{3,\infty}(B_R^c)} &= \|T_u(v_j - v_{j-1})\|_{L^{3,\infty}(B_R^c)} \\ &\leq C \|u\|_{L^{3,\infty}(B_R^c)} \|v_j - v_{j-1}\|_{L^{3,\infty}(B_R^c)}. \end{aligned}$$

Taking $\lambda_0 = 1/2C$, we obtain from the above estimate that

$$\|v_{j+1} - v_j\|_{L^{3,\infty}(B_R^c)} \leq \frac{1}{2} \|v_j - v_{j-1}\|_{L^{3,\infty}(B_R^c)} \leq \dots \leq \left(\frac{1}{2}\right)^{j+1} \|u_0\|_{L^{3,\infty}(B_R^c)}$$

for all $j = 1, \dots$. Hence there exists a limit v of $\{v_j\}_{j=0}^\infty$ in $L^{3,\infty}(B_R^c)$. Letting $j \rightarrow \infty$ in (3.27), we easily see that this limit v gives a unique solution of (3.20) in $L^{3,\infty}(B_R^c)$.

(2) Let $F \in L^{3/2,\infty}(B_R^c) \cap L^q(B_R^c)$ for $1 \leq q \leq 3/2$. First notice that the function $\nabla^2 E(x - y)$ is a Calderón–Zygmund kernel (see Stein [26]). By the well-known properties of singular integrals and of the double layer potential, we see that u_0 belongs to the same class as (3.21) and (3.22). Taking $r = 3q/(3 - q)$ in (3.24) and (3.25), we have as (1) above that

$$\begin{aligned} &\|v_{j+1} - v_j\|_{L^{3q/(3-q)}(B_R^c)} \\ &\leq C \|u\|_{L^{3,\infty}(B_R^c)} \|v_j - v_{j-1}\|_{L^{3q/(3-q)}(B_R^c)} \quad \text{provided } 1 < q \leq \frac{3}{2}, \end{aligned}$$

$$\|v_{j+1} - v_j\|_{L^{3/2,\infty}(B_R^c)} \leq C \|u\|_{L^{3,\infty}(B_R^c)} \|v_j - v_{j-1}\|_{L^{3/2,\infty}(B_R^c)} \quad \text{provided } q = 1,$$

where $C = C(q)$ is independent of $j = 1, \dots$. Defining $\lambda_* = \lambda_*(q) = 1/2C$, under the condition that $\|u\|_{L^{3,\infty}(B_R^c)} \leq \lambda_*$, we see from the above scheme that the solution v of (3.20) given by (1) belongs to $L^{3q/(3-q)}(B_R^c)$ for $1 < q \leq 3/2$ and to $L^{3/2,\infty}(B_R^c)$ for $q = 1$. Since $\nabla^2 E(x - y)$ is a Calderón–Zygmund kernel, the assertion on ∇v can be derived from expression (3.20). $\square$

Proof of Corollary 1. By Theorem 2, it suffices to prove that u is in $L^{3,\infty}$ with (1.4). By Kozono and Yamazaki [19, Theorem], for any $\varepsilon > 0$, there is a constant $\delta = \delta(\varepsilon)$ such that if $F \in L^{3/2,\infty} \cap L^2$ satisfies $\|F\|_{3/2,\infty} \leq \delta$, then there exists a weak solution v of (N–S) with $\nabla v \in L^{3/2,\infty}$, $v \in L^{3,\infty}$ satisfying $\|v\|_{3,\infty} \leq \varepsilon$. Let us first choose $\varepsilon = \varepsilon_1 = \lambda(q)$ so that the condition $\|F\|_{3/2,\infty} \leq \delta(\varepsilon_1)$ yields such a solution v as

$$\|v\|_{3,\infty} \leq \lambda(q),$$

where $\lambda(q)$ is the same constant as in Theorem 2.

On the other hand, a uniqueness criterion on weak solutions of (N–S) given by [20] states that there is a constant $\varepsilon_2 > 0$ such that if $\|v\|_{3,\infty} \leq \varepsilon_2$, then we have

$$u \equiv v \quad \text{on } \Omega.$$

Hence, defining μ in the assertion of Corollary 1 as

$$\mu \equiv \min\{\delta(\varepsilon_1), \delta(\varepsilon_2)\},$$

we see that if $\|F\|_{3/2,\infty} \leq \mu$, then it follows that $u = v$. This implies that u is in $L^{3,\infty}$ with (1.4). This proves Corollary 1. $\square$

3.3. Energy identity: proof of Theorem 3

Let us first show that the pressure p in Proposition 1 can be taken as

$$p = p_1 + p_2, \quad \text{where } p_1 \in L^2, \quad p_2 \in L^{9/4}. \quad (3.28)$$

In fact, since $u \in L^{9/2}$, we have $u \otimes u \in L^{9/4}$ and it follows that

$$\begin{aligned} -\Delta u + u \cdot \nabla u - \operatorname{div} F \\ = \operatorname{div}(-\nabla u + u \otimes u - F) &\in (\hat{H}_0^{1,2} \cap \hat{H}_0^{1,9/5})^* \\ &= (\hat{H}_0^{1,2})^* + (\hat{H}_0^{1,9/5})^*, \end{aligned} \quad (3.29)$$

where $\hat{H}_0^{1,r}$ is the closure of $C_0^\infty(\Omega)$ with respect to the homogeneous norm $\|\nabla u\|_r$, $1 < r < \infty$. By Giga and Sohr [14, Corollary 2.2] it follows that

$$\begin{aligned} (\hat{H}_{0,\sigma}^{1,r})^\perp &\equiv \{f \in (\hat{H}_0^{1,r})^*; \langle f, \phi \rangle = 0 \text{ for all } \phi \in \hat{H}_{0,\sigma}^{1,r}\} \\ &= \{f \in (\hat{H}_0^{1,r})^*; f = \nabla p \text{ with some } p \in L^{r/(r-1)}, \\ &\quad \text{i.e. } \langle f, \psi \rangle = -(p, \operatorname{div} \psi) \text{ for all } \psi \in \hat{H}_0^{1,r}\}, \end{aligned} \quad (3.30)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(\hat{H}_0^{1,r})^*$ and $\hat{H}_0^{1,r}$. Since $C_{0,\sigma}^\infty$ is dense in $\hat{H}_{0,\sigma}^{1,2} \cap \hat{H}_{0,\sigma}^{1,9/5}$ [17, Theorem 1], we see by (1.1), (3.29) and (3.30) that the pressure p associated with u can be written as (3.28).

By the Hölder inequality, it follows that

$$|u|(|\nabla u| + |p_1| + |F|) \in L^{18/13}, \quad |u|(|p_2| + |u|^2) \in L^{3/2}.$$

Then it follows from Lemma 2.1 that there exists a sequence $\{R_m\}_{m=1}^\infty$ with $2^m < R_m < 2^{m+1}$ ($m = 1, \dots$) such that

$$\int_{|\xi|=R_m} |u|(|\nabla u| + |p_1| + |F|) dS + \int_{|\xi|=R_m} |u|(|p_2| + |u|^2) dS \rightarrow 0 \quad (3.31)$$

as $m \rightarrow \infty$. Now applying the generalized Stokes formula (3.2) with $\mathcal{D} = \Omega_{R_m}$, we have

$$\begin{aligned} 0 &= \int_{\Omega_{R_m}} \operatorname{div}(\nabla u - u \otimes u + pI + F) \cdot u \, dx \\ &= - \int_{\Omega_{R_m}} |\nabla u|^2 \, dx - \int_{\Omega_{R_m}} F \cdot \nabla u \, dx \\ &\quad + \int_{|\xi|=R_m} u(\nabla u + p_1 I + F) \cdot \nu \, dS + \int_{|\xi|=R_m} (p_2 + \tfrac{1}{2}|u|^2) u \cdot \nu \, dS. \end{aligned} \quad (3.32)$$

Here we used the identity

$$\int_{\Omega_{R_m}} u \cdot \nabla u \cdot u \, dx = \frac{1}{2} \int_{|\xi|=R_m} |u|^2 u \cdot \nu \, dS.$$

Letting $m \rightarrow \infty$ in (3.32), we obtain from (3.31) that

$$\int_{\Omega} |\nabla u|^2 \, dx = - \int_{\Omega} F \cdot \nabla u \, dx.$$

This proves Theorem 3. □

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