Φ -DENSITY THEOREM ON A METRIC SPACE†

Ryoji FUKUDA and Hiroshi SATO‡

(Received 20 May 1996)

1. Introduction

Let (E, d) be a separable metric space, let $B(x, r)$ be the closed ball with center $x \in E$ and radius $r > 0$ and let λ be a locally finite Borel measure on E ; that is $\lambda(B(x, r)) < \infty$ for any $x \in E$ and $r > 0$. We say that the measure space (E, λ) satisfies the *local density theorem* (LDT) if

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} = \frac{d\mu}{d\lambda}(x), \quad \lambda\text{-a.e.} \quad (1)$$

for any non-negative finite Borel measure μ , where $d\mu/d\lambda$ denotes the Radon–Nikodym density of μ with respect to λ in the sense of the Lebesgue decomposition. We will denote the family of all non-negative finite Borel measures on E by $\mathcal{M}(E)$.

Let $\Phi = \{\Phi_T(x, y)\}_{T \geq 1}$ be a family of Borel measurable functions on $E \times E$. We say that (E, λ) satisfies the Φ -density theorem (Φ -DT) if

$$\lim_{T \rightarrow \infty} \int_E \Phi_T(x, y) \mu(dy) = \frac{d\mu}{d\lambda}(x), \quad \lambda\text{-a.e.} \quad (2)$$

for any $\mu \in \mathcal{M}(E)$. Consider a function ϕ on $(0, \infty)$ which satisfies the following properties:

(F.1) ϕ is non-increasing and $\lim_{r \rightarrow \infty} \phi(r) = 0$ ($\phi(0+) \leq \infty$);

(F.2) $\phi(r_0) > 0$ for some $r_0 > 0$;

(F.3) $\int_E \phi(d(x, y)) \lambda(dy) < \infty$, λ -a.e.

Then for any $T \geq 1$, and $x, y \in E$, set

$$C_x(T) = \int_E \phi(Td(x, y)) \lambda(dy),$$

$$\Phi_T^\phi(x, y) = \frac{1}{C_x(T)} \phi(Td(x, y)).$$

† Dedicated to Professor Satoru Igari for his sixtieth birthday.

‡ This research was partially supported by SFB 170, ‘Geometry and Analysis’, Göttingen.

If $\{\Phi_T^\phi\}_{T \geq 1}$ satisfies (2) for any $\mu \in \mathcal{M}(E)$, we say that (E, λ) satisfies the ϕ -density theorem (ϕ -DT). It is well known that the Lebesgue measure space satisfies the LDT (see, for example, Dunford and Schwartz [2]). Using this LDT, Calderón and Zygmund [1] proved (2) for $\{\Phi_T^\phi(x, y)\}_{T \geq 1}$ when ϕ satisfies (F.1)–(F.3) and μ is λ -absolutely continuous.

In this paper we shall give a necessary and sufficient condition for the LDT to imply ϕ -DT, and as an application we give an extension of a theorem of Calderón and Zygmund [1].

We shall prove the following necessary and sufficient condition.

THEOREM 1.1. *Let E be a separable metric space, let λ be a locally finite Borel measure, and assume the LDT for (E, λ) . Then, if ϕ , a function on $(0, \infty)$, satisfies (F.1)–(F.3), the following assertions (I)–(III) are equivalent.*

- (I) (E, λ) satisfies ϕ -DT.
- (II) (E, λ) and ϕ satisfy the following conditions (A.1) and (A.2):

$$(A.1) \text{ For any } r > 0, \lim_{T \rightarrow \infty} \frac{\phi(rT)}{C_x(T)} = 0, \lambda\text{-a.e.}$$

$$(A.2) \lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int_{B(x, \delta)^c} \phi(Td(x, y)) \lambda(dy) = 0 \text{ for some } \delta > 0, \lambda\text{-a.e.}$$

$$(III) \text{ For any } \delta > 0, \lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int_{B(x, \delta)^c} \phi(Td(x, y)) \lambda(dy) = 0, \lambda\text{-a.e.}$$

In the case when (E, λ) satisfies the doubling condition,

$$\beta(x) \equiv \sup \left\{ \frac{\lambda(B(x, 2r))}{\lambda(B(x, r))}; r > 0, \lambda(B(x, r)) > 0 \right\} < \infty, \quad \lambda\text{-a.e.}, \quad (3)$$

the LDT for (E, λ) is implied by the theorems of Federer [3, Theorems 2.9.7, 2.9.15 and 2.9.17], and the ϕ -DT with $\phi(r) = 1/(1 + r^\alpha)$ ($\alpha > \log \beta / \log 2$) was proved by Sato [4] without the use of the LDT. As an application of Theorem 1.1, we have the following.

THEOREM 1.2. *Let E be a separable metric space and let λ be a locally finite Borel measure on E satisfying the doubling condition (3), and let*

$$-\int_{[1, \infty)} r^{H(x)} d\phi(r) < \infty, \quad \lambda\text{-a.e.}, \quad (4)$$

where $H(x) = \log \beta(x) / \log 2$. Then (E, λ) satisfies ϕ -DT.

Moreover, using the argument appearing in Sato [4, p. 591], we obtain the following theorem which is a generalization of Theorem 4 in [4].

THEOREM 1.3. *Let (E, λ) satisfy ϕ -DT, and let $\Phi = \{\Phi_T(x, y)\}_{T \geq 1}$ be a family of Borel measurable functions on $E \times E$ satisfying:*

$$(H.1) \quad \int_E \Phi_T(x, y) \lambda(dy) = 1, \quad x \in E, \quad T \geq 1;$$

$$(H.2) \quad \sup_{T \geq 1, y \in E} \frac{|\Phi_T(x, y)| C_x(T)}{\phi(Td(x, y))} < \infty, \quad \lambda\text{-a.e.}$$

Then (E, λ) satisfies Φ -DT.

When (E, λ) satisfies the doubling condition (3), the factor $C_x(T)$ on the left-hand side of (H.2) may be replaced by $\lambda(B(x, 1/T))$ (see Sato [4, Lemma 2]).

2. Proof of the theorems

To begin, we shall prove the following lemma.

LEMMA 2.1. *Let ϕ satisfy (F.1)–(F.3). Then (A.1) implies*

$$\lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int_{B(x, \delta)^c} \phi(Td(x, y)) \mu(dy) = 0, \quad \lambda\text{-a.e.}$$

for any $\mu \in \mathcal{M}(E)$ and $\delta > 0$.

Proof. Using (F.1), we have

$$\begin{aligned} \frac{1}{C_x(T)} \int_{B(x, \delta)^c} \phi(Td(x, y)) \mu(dy) &\leq \frac{1}{C_x(T)} \int_{B(x, \delta)^c} \phi(T\delta) \mu(dy) \\ &\leq \frac{1}{C_x(T)} \mu(E) \phi(T\delta). \end{aligned}$$

(A.1) implies that the right-hand side tends to 0 as $T \rightarrow \infty$, λ -a.e. □

Proof of Theorem 1.1.

We shall prove (II) $\Rightarrow$ (III), (III) $\Rightarrow$ (I) and (I) $\Rightarrow$ (II).

Let D be a countable dense subset of E and set

$$S = \bigcup_{y \in D} \bigcup_{n \in \mathbb{N}, \lambda(B(y, 1/n))=0} B\left(y, \frac{1}{n}\right).$$

Then $\lambda(S) = 0$, and for any $x \in S^C$ and $r > 0$, $\lambda(B(x, r)) > 0$. Therefore, without loss of generality, we may assume that $E = S^C$, that is, $\lambda(B(x, r)) > 0$ for any $x \in E$ and $r > 0$.

If there exists $x_0 \in E$ such that $\lambda(\{x_0\}) > 0$, (F.3) implies $\phi(0+) < \infty$, and by (F.1) we have

$$\lim_{T \rightarrow \infty} C_{x_0}(T) = \lim_{T \rightarrow \infty} \int_E \phi(Td(x_0, y)) \lambda(dy) = \lambda(\{x_0\}) \phi(0+).$$

This implies (II), (III) (with $x = x_0$), and the following:

$$\lim_{T \rightarrow \infty} \frac{1}{C_{x_0}(T)} \int_E \phi(Td(x_0, y)) \mu(dy) = \frac{\mu(\{x_0\})}{\lambda(\{x_0\})} = \frac{d\mu}{d\lambda}(x_0).$$

Therefore we may assume that $\lambda(\{x\}) = 0$ for any $x \in E$. Then, for each $\mu \in \mathcal{M}(E)$, we may also assume that $\mu(\{x\}) = 0$ for any $x \in E$, since $\{x; \mu(\{x\}) > 0\}$ is a λ -null set.

(II) $\Rightarrow$ (III). It is enough to prove that (A.1) implies the equality

$$\lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int_{B(x, \delta_2) \setminus B(x, \delta_1)} \phi(Td(x, y)) \lambda(dy) = 0, \quad \lambda\text{-a.e.},$$

for any $0 < \delta_1 < \delta_2$, but this is a special case of Lemma 2.1.

(III) $\Rightarrow$ (I). First, we shall prove (III) $\Rightarrow$ (A.1). Set

$$L_1 \equiv \left\{ x; \lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int_{B(x, 1/n)^c} \phi(Td(x, y)) \lambda(dy) = 0 \text{ for any } n \in \mathbb{N} \right\}.$$

Then (III) implies $\lambda(L_1^C) = 0$. Fix any $x_0 \in L_1$ and $r_0 > 0$. We shall prove

$$\lim_{T \rightarrow \infty} \frac{\phi(r_0 T)}{C_{x_0}(T)} = 0. \quad (5)$$

In fact, since we have $\alpha_0 = \lambda(B(x_0, r_0)) > 0$ and $\lambda(\{x_0\}) = \inf_{n \in \mathbb{N}} \lambda(B(x_0, 1/n)) = 0$, there exists $n \in \mathbb{N}$ such that $\lambda(B(x_0, 1/n)) < \alpha_0/2$. Thus we have $\lambda(B(x_0, r_0) \setminus B(x_0, 1/n)) > \alpha_0/2$. This implies

$$\begin{aligned} \frac{\phi(r_0 T)}{C_{x_0}(T)} &\leq \frac{2}{\alpha_0} \frac{1}{C_{x_0}(T)} \int_{B(x_0, r_0) \setminus B(x_0, 1/n)} \phi(Td(x_0, y)) \lambda(dy) \\ &\leq \frac{2}{\alpha_0} \frac{1}{C_{x_0}(T)} \int_{B(x_0, 1/n)^c} \phi(Td(x_0, y)) \lambda(dy) \rightarrow 0, \quad \text{as } T \rightarrow \infty. \end{aligned}$$

Thus we have obtained (A.1).

Consider a particular version of $d\mu/d\lambda(x)$, and set

$$L_2 \equiv \left\{ x; \lim_{\varepsilon \rightarrow 0} \frac{\mu(B(x, \varepsilon))}{\lambda(B(x, \varepsilon))} = \frac{d\mu}{d\lambda}(x) \right\}$$

Then the LDT implies $\lambda(L_2^c) = 0$.

Fix $x_1 \in L_2$, and define $\gamma_{x_1}(r) \equiv \lambda(B(x_1, r))$ and $m_{x_1}(r) \equiv \mu(B(x_1, r))$ for any $r > 0$. By the definition of L_2 , for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{r < \delta} \left| \frac{m_{x_1}(r)}{\gamma_{x_1}(r)} - \frac{d\mu}{d\lambda}(x_1) \right| < \varepsilon. \quad (6)$$

On the other hand, the relation

$$\lim_{r \rightarrow 0} \phi(Tr) \gamma_{x_1}(r) \leq \lim_{r \rightarrow 0} \int_{B(x_1, r)} \phi(Td(x_1, y)) \lambda(dy) = 0$$

implies

$$\lim_{r \rightarrow 0} \phi(Tr) m_{x_1}(r) = \lim_{r \rightarrow 0} \phi(Tr) \gamma_{x_1}(r) \frac{d\mu}{d\lambda}(x_1) = 0.$$

Therefore we have

$$\begin{aligned} & \left| \frac{1}{C_{x_1}(T)} \int_E \phi(Td(x_1, y)) \mu(dy) - \frac{d\mu}{d\lambda}(x_1) \right| \\ &= \frac{1}{C_{x_1}(T)} \left| \int_E \phi(Td(x_1, y)) \mu(dy) - \frac{d\mu}{d\lambda}(x_1) \int_E \phi(Td(x_1, y)) \lambda(dy) \right| \\ &\leq \frac{1}{C_{x_1}(T)} \left| \int_{[0, \delta)} \phi(Tr) dm_{x_1}(r) - \frac{d\mu}{d\lambda}(x_1) \int_{[0, \delta)} \phi(Tr) d\gamma_{x_1}(r) \right| \\ &\quad + \frac{1}{C_{x_1}(T)} \int_{B(x_1, \delta)^c} \phi(Td(x_1, y)) \mu(dy) \\ &\quad + \frac{1}{C_{x_1}(T)} \frac{d\mu}{d\lambda}(x_1) \int_{B(x_1, \delta)^c} \phi(Td(x_1, y)) \lambda(dy) \\ &\leq \frac{-1}{C_{x_1}(T)} \int_{(0, T\delta]} \left| m_{x_1}\left(\frac{r}{T}\right) - \frac{d\mu}{d\lambda}(x_1) \gamma_{x_1}\left(\frac{r}{T}\right) \right| d\phi(r) \\ &\quad + \frac{\phi(T\delta)}{C_{x_1}(T)} \left(m_{x_1}(\delta) + \frac{d\mu}{d\lambda}(x_1) \gamma_{x_1}(\delta) \right) + \frac{1}{C_{x_1}(T)} \int_{B(x_1, \delta)^c} \phi(Td(x_1, y)) \mu(dy) \\ &\quad + \frac{1}{C_{x_1}(T)} \frac{d\mu}{d\lambda}(x_1) \int_{B(x_1, \delta)^c} \phi(Td(x_1, y)) \lambda(dy) \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Condition (A.1), Lemma 2.1 and (III) imply the λ -almost everywhere convergence of I_2 , I_3 and I_4 as $T \rightarrow \infty$, and by (6) we have

$$\begin{aligned} I_1 &\leq \frac{-1}{C_{x_1}(T)} \int_{[0, T\delta)} \varepsilon \gamma_{x_1} \left(\frac{r}{T} \right) d\phi(r) \\ &= \left(\int_{(0, \infty]} \gamma_{x_1} \left(\frac{r}{T} \right) d\phi(r) \right)^{-1} \varepsilon \int_{(0, T\delta]} \gamma_{x_1} \left(\frac{r}{T} \right) d\phi(r) \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{C_{x_1}(T)} \int_E \phi(Td(x_1, y)) \mu(dy) = \frac{d\mu}{d\lambda}(x_1).$$

(I) $\Rightarrow$ (II). Let D be a countable dense subset of E . Then, for each $y \in D$, the Dirac measure $\delta_y \in \mathcal{M}(E)$ is singular with respect to λ , so that $d\delta_y/d\lambda(x) = 0$, λ -a.e. Set

$$E_y \equiv \left\{ x; \lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int \phi(Td(x, z)) \delta_y(dz) = 0 \right\} \quad \text{and} \quad E_D \equiv \bigcap_{y \in D} E_y.$$

Then (I) (ϕ -DT) implies $\lambda(E_D^C) = 0$. Fix an arbitrary $r > 0$. Then for any $x \in E_D$, there exists $y \in D$ such that

$$r' \equiv d(x, y) < r.$$

Since $x \in E_D \subset E_y$, we have

$$\begin{aligned} 0 &= \lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int_E \phi(Td(x, z)) \delta_y(dz) \\ &= \lim_{T \rightarrow \infty} \frac{\phi(Tr')}{C_x(T)} \geq \lim_{T \rightarrow \infty} \frac{\phi(Tr)}{C_x(T)}, \end{aligned}$$

that is, (A.1) holds.

Let $\mu \in \mathcal{M}(E)$ be equivalent to λ , (that is, $d\mu/d\lambda > 0$ λ -a.e.) and let $\varepsilon > 0$ be an arbitrary positive number. Then, the reasoning used in the proof of (III) $\Rightarrow$ (I), together with the LDT, implies the existence of $\delta > 0$ such that $I_1 \leq \varepsilon$ and that

$$I_4 \leq I_1 + I_2 + I_3 + \left| \frac{1}{C_x(T)} \int_E \phi(Td(x, y)) \mu(dy) - \frac{d\mu}{d\lambda}(x) \right|, \quad \lambda\text{-a.e.} \quad (7)$$

Assertion (I) implies that the last term above tends to 0 as $T \rightarrow \infty$. By Lemma 2.1 and (A.1), we have $I_2 \rightarrow 0$ and $I_3 \rightarrow 0$ as $T \rightarrow \infty$, λ -a.e. Therefore we obtain the

relation $\limsup_{T \rightarrow \infty} I_4 \leq \varepsilon$. Since I_4 is non-decreasing with respect to δ , for some fixed $\delta_0 > 0$, we have

$$\left(\frac{d\mu}{d\lambda}(x) \right)^{-1} \lim_{T \rightarrow \infty} I_4 = \lim_{T \rightarrow \infty} \int_{B(x, \delta_0)^c} \phi(Td(x, y)) \lambda(dy) = 0, \quad \lambda\text{-a.e.}$$

Therefore (A.2) holds. $\square$

COROLLARY 2.2. Assume (F.1)–(F.3) and (A.1) for (E, λ, ϕ) . Then we have:

- (a) $\{\Phi_T^\phi\}_{T \geq 1}$ satisfies (2) when $\mu \in \mathcal{M}(E)$ is singular with respect to λ ;
- (b) ϕ -DT holds when λ is a finite measure.

Proof. By the arguments used in the proof of (III) $\Rightarrow$ (I), we have only to prove $\lim_{T \rightarrow \infty} I_4 = 0$ for each case.

When μ is singular with respect to λ , we have $d\mu/d\lambda(x) = 0$ λ -a.e., that is, $I_4 = 0$ λ -a.e., and if λ is a finite measure; the equality $\lim_{T \rightarrow \infty} I_4 = 0$ is obtained from Lemma 2.1. $\square$

Proof of Theorem 1.2.

We shall prove (III).

As remarked in the proof of Theorem 1.1, we may assume that neither λ nor μ possess a point mass. By (F.2), there exists $r_0 > 0$ such that $\phi(r_0) > 0$. Considering $\tilde{\phi}(r) = \phi(r_0 r)$ if necessary, we may also assume that $r_0 = 1$, that is, $\phi(1) > 0$.

Since λ satisfies the doubling condition (3), by Lemma 1 in Sato [4], we have

$$\gamma_x(rs) \leq \beta(x) r^{H(x)} \gamma_x(s), \quad \lambda\text{-a.e.} \quad (8)$$

for any $s > 0$ and $r \geq 1$. Fix an arbitrary $\delta > 0$. Then (8) implies

$$\begin{aligned} & \frac{1}{C_x(T)} \int_{B(x, \delta)^c} \phi(Td(x, y)) \lambda(dy) \\ &= \frac{1}{C_x(T)} \int_{(\delta, \infty)} \phi(Tr) d\gamma_x(r) \\ &\leq \frac{-1}{C_x(T)} \int_{(T\delta, \infty)} \gamma_x\left(\frac{r}{T}\right) d\phi(r) \\ &\leq \frac{-1}{C_x(T)} \int_{(T\delta, \infty)} \beta(x) r^{H(x)} \gamma_x\left(\frac{1}{T}\right) d\phi(r) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{-\beta(x)\gamma_x(1/T)}{\int_{B(x,1/T)} \phi(Td(x,y))\lambda(dy)} \int_{(T\delta,\infty)} r^{H(x)} d\phi(r) \\
&\leq \frac{-\beta(x)}{\phi(1)} \int_{(T\delta,\infty)} r^{H(x)} d\phi(r).
\end{aligned}$$

By (4), the right-hand side converges to 0 as $T \rightarrow \infty$. $\square$

3. Applications and examples

Let ϕ satisfy (F.1)–(F.3), and let λ be a locally finite measure on E satisfying the doubling condition (3) or

$$\beta_1(x) \equiv \sup_{0 < r < 1} \frac{\lambda(B(x, 2r))}{\lambda(B(x, r))} < \infty, \quad \lambda\text{-a.e.} \quad (9)$$

Then the theorems of Federer [3] imply the LDT.

For a separable metric space and Borel measures on such a space, several types of dimensions are defined and the doubling conditions (3) and (9) are closely related to these. For example, the Hausdorff dimension and the packing dimension of λ are majorized by $\text{ess. sup}_{x \in E} H(x)$ when $\beta(x) < \infty$, λ -a.e. (see Tamashiro [5]).

First we give a sufficient condition for (A.1).

LEMMA 3.1. *Let ϕ satisfy (F.1)–(F.3), and let (E, λ) satisfy the LDT and the local doubling condition (9). Then*

$$\lim_{T \rightarrow \infty} \frac{\phi(T)}{\lambda(B(x, 1/T))} = 0, \quad \lambda\text{-a.e.} \quad (10)$$

is a sufficient condition for (A.1).

Proof. By (F.2), there exists $r_0 > 0$ such that $\phi(r_0) > 0$, and we have

$$C_x(T) \geq \int_{B(x, r_0/T)^c} \phi(T(d(x, y)))\lambda(dy) \geq \phi(r_0)\lambda\left(B\left(x, \frac{r_0}{T}\right)\right) > 0, \quad \lambda\text{-a.e.}$$

Then by using (9),

$$M_1(x) \equiv \sup_{T \geq 1} \frac{\lambda(B(x, 1/T))}{\lambda(B(x, r_0/T))} < \infty, \quad \lambda\text{-a.e.}$$

Therefore, for any $T \geq 1$, we have

$$C_x(T) \geq \frac{\phi(r_0)\lambda(B(x, 1/T))}{M_1(x)} > 0, \quad \lambda\text{-a.e.}$$

Fix an r satisfying $0 < r \leq 1$. Then, using the argument appearing in Lemma 1 of Sato [4], we have

$$\lambda\left(B\left(x, \frac{1}{Tr}\right)\right) \leq \beta_1(x) \left(\frac{1}{r}\right)^{H_1(x)} \lambda\left(B\left(x, \frac{1}{T}\right)\right), \quad \lambda\text{-a.e.}$$

for any $T \geq 1/r$, where $H_1(x) = \log \beta_1(x) / \log 2$. Hence

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{\phi(rT)}{C_x(T)} &\leq \limsup_{T \rightarrow \infty} \frac{M_1(x)\phi(rT)}{\phi(r_0)\lambda(B(x, 1/T))} \\ &\leq \frac{\beta_1(x)M_1(x)}{\phi(r_0)r^{H_1(x)}} \limsup_{T \rightarrow \infty} \frac{\phi(rT)}{\lambda(B(x, 1/Tr))} = 0. \end{aligned}$$

□

LEMMA 3.2. *Let ϕ satisfy (F.1)–(F.3) with $r_0 = 1$, and let (E, λ) satisfy the LDT. Assume condition (A.1) and that there exists a non-negative function ψ on $(0, \infty)$ such that the following inequalities hold:*

$$(S.1) \quad \int_E \psi(d(x, y))\lambda(dy) < \infty, \quad \lambda\text{-a.e.}$$

$$(S.2) \quad K(x) = \sup_{r \geq 1, T \geq 1} \frac{\phi(Tr)}{\lambda(B(x, 1/T))\psi(r)} < \infty, \quad \lambda\text{-a.e.}$$

Then (E, λ) satisfies ϕ -DT.

Proof. We have only to prove

$$\lim_{T \rightarrow \infty} \frac{1}{C_x(T)} \int_{B(x, 1)^c} \phi(Td(x, y))\lambda(dy) = 0. \quad (11)$$

In fact, for λ -a.e. $x \in E$, (A.1) implies

$$\lim_{T \rightarrow \infty} \frac{\phi(Td(x, y))}{C_x(T)} = 0$$

for any $y \neq x$. Also by (S.2) we have

$$\frac{\phi(Td(x, y))}{C_x(T)} \leq \frac{K(x)}{\phi(1)} \psi(d(x, y))$$

for any $T \geq 1$ and $y \in B(x, 1)^c$, where the right-hand side is $\lambda(dy)$ -integrable by (S.1). Therefore, by the dominated convergence theorem, we obtain (11). □

Next we shall give three examples for ϕ -DT. The first example is the radial function ϕ for which ϕ -DT was proved in Sato [4].

Example 3.1. Assume the doubling condition (3) and set

$$\phi(r) = \frac{1}{1 + r^\alpha},$$

where α satisfies $\alpha > H(x)$, λ -a.e. Then ϕ -DT holds.

Indeed, it is easy to prove that

$$-\int_{[1,\infty)} r^{H(x)} d\phi(r) < \infty, \quad \lambda\text{-a.e.},$$

and thus Theorem 1.2 implies ϕ -DT. $\square$

The second example illuminates the fact that (4) is not a necessary condition for ϕ -DT.

Example 3.2. Assume the doubling condition (3) and that $\lambda(E) < \infty$. Then set

$$\phi(r) \equiv \frac{1}{1 + r^{H(x)} \log(r+1)}.$$

This function satisfies (10), and $\psi = 1$ satisfies (S.1) and (S.2). Therefore Lemmas 3.1 and 3.2 prove ϕ -DT. However, $\phi(r)$ does not satisfy (4), and thus this is not a necessary condition. $\square$

The third example is ϕ -DT for a measure space which satisfies the local doubling condition (9) but not the global doubling condition (3).

Example 3.3. Define $E \equiv \mathbb{R}$, $\lambda(dx) \equiv 2^{|x|}dx$ and the function $\phi(r) \equiv e^{-r}$.

The function ϕ satisfies (F.1)–(F.3), and (E, λ) satisfies the LDT and the local doubling condition (9), but it does not satisfy the doubling condition (3).

In this case, Lemmas 3.1 and 3.2 imply ϕ -DT. Indeed, we have

$$\lim_{T \rightarrow \infty} \frac{\phi(T)}{\lambda(B(x, 1/T))} \leq \lim_{T \rightarrow \infty} \frac{T e^{-T}}{2 \times 2^{|x|}} = 0,$$

and

$$\sup_{T \geq 1, r \geq 1} \frac{\phi(rT)}{\lambda(B(x, 1/T))\phi(r)} \leq \sup_{T \geq 1, r \geq 1} \frac{T e^{-rT+r}}{2 \times 2^{|x|}} \leq 2^{-|x|}.$$

Therefore ϕ satisfies the assumptions of Lemmas 3.1 and 3.2 with $\psi = \phi$. $\square$

In general, (F.1)–(F.3) do not imply (A.1), and (A.1) does not imply (10). To finish the paper, we give two illuminating examples.

Example 3.4. Define $E \equiv \{x \in \mathbb{R}^3; |x| \leq 1\}$, $\phi(r) \equiv 1/(1+r^2)$, let $d(\cdot, \cdot)$ be the Euclidean metric and let λ be the Lebesgue measure. Then ϕ satisfies (F.1)–(F.3). On the other hand,

$$\begin{aligned} C_x(T) &= \int_E \phi(Td(x, y))\lambda(dy) \\ &\leq 4\pi \int_0^2 \frac{r^2}{1+T^2r^2}dr \leq \frac{8\pi}{T^2}. \end{aligned}$$

Therefore

$$\frac{\phi(T)}{C_x(T)} \geq \frac{T^2}{8\pi(1+T^2)} \geq \frac{1}{8\pi}.$$

Hence (A.1) does not hold. $\square$

Example 3.5. Define $E \equiv [0, 1] \subset \mathbb{R}$, $\phi(r) \equiv \log(1+r)/(1+r)$, let $d(\cdot, \cdot)$ be the Euclidean metric and let λ be the Lebesgue measure on E . Then we have

$$\lim_{T \rightarrow \infty} \frac{\phi(T)}{\lambda(B(x, 1/T))} = \infty.$$

On the other hand, for any $r > 0$, we have

$$\limsup_{T \rightarrow \infty} \frac{\phi(rT)}{C_x(T)} \leq \limsup_{T \rightarrow \infty} \frac{\log(1+Tr)}{(1+Tr) \int_0^{1/2} \log(1+Ts)/(1+Ts) ds} = 0.$$

Therefore ϕ satisfies (A.1) but does not satisfy (10). $\square$

REFERENCES

- [1] A. P. Calderón and A. Zygmund. On the existence of certain singular integrals. *Acta Math.* **88** (1952), 85–139.
- [2] N. Dunford and J. T. Schwartz. *Linear Operators, Part 1*. Interscience, New York, 1957.
- [3] H. Federer. *Geometric Measure Theory*. Springer, Berlin, Heidelberg, New York, 1969.
- [4] H. Sato. Global density theorem for a Federer measure. *Tôhoku Math. J.* **44** (1992), 581–595.
- [5] M. Tamashiro. Dimensions in a separable metric space. *Kyushu J. Math.* **49** (1994), 143–162.

*Ryoji Fukuda
Department of Basic engineering
Faculty of Engineering
Oita University
Oita 870-11, Japan*

*Hiroshi Sato
Graduate School of Mathematics
Kyushu University-33
Fukuoka 812-81, Japan*