SIMPLICIAL VOLUMES OF ALEXANDROV SPACES

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(Received 22 August 1996)

0. Introduction

In a paper by Gromov [G], a topological invariant of a closed manifold, the so-called simplicial volume, is defined. Roughly speaking, this invariant is the minimum *real* number of singular simplices needed to cover the manifold. In that paper the geometry and topology of this invariant for Riemannian manifolds are developed. It is known, for instance, citing the work of Thurston [T], Gromov [G] and Inoue and Yano [IY] that the simplicial volume of a closed negatively curved manifold is bounded below by a positive constant depending on volume and, in particular, it does not vanish. Furthermore, Gromov made use of averaging techniques of various types to provide an upper bound in terms of volume for the invariants of manifolds whose Ricci curvatures are bounded below. Thus the simplicial volume appears as an obstruction for collapsing. Namely, if the simplicial volume of a closed manifold does not vanish, such a manifold cannot collapse under a lower Ricci curvature bound.

Gromov's paper [G] consists of differential geometric arguments and topological arguments involving bounded cohomology and related topics. The topological part of that paper is written in a general setting. The purpose of this paper is to extend the geometric part of Gromov's work to singular spaces.

By an n -dimensional singular space, we mean a locally compact space X with singular set S such that $X - S$ is a topological n -manifold and S has a lower (topological) dimension $< n$. We say that a singular space X is orientable if the regular set $X - S$ is orientable. To define the notion of simplicial volume for an n -dimensional singular space, we assume that X has a fundamental class $[X] \in H_n(X; \mathbf{R})$ in the sense that under the inclusion homomorphism, $[X]$ projects to the local orientation class at each point of $X - S$. For instance, these conditions are satisfied by pseudomanifolds. The notion of pseudomanifolds defined in this paper is more general than the usual one, which requires a polyhedral structure of the space in question. It will become clear later that an orientable pseudomanifold

X has a fundamental class. We define the simplicial volume of a singular space X , denoted by $\|X\|$, having a fundamental class as

$$\|X\| = \inf_{[X]} \|[X]\|_1,$$

where $\|[X]\|_1$ denotes the ℓ^1 -norm of $[X]$, and $[X]$ runs over all the fundamental classes of X .

As in [G], we make use of the notion of mass and comass. To make the concept of mass of a homology class reasonable, it is natural to assume that the homology class can be represented by a Lipschitz cycle. This is not likely to be possible for an arbitrary metric space. However, for metric spaces satisfying the LLC (locally Lipschitz contractible)-condition defined in this paper, this is possible.

In our argument, the following result showing the equivalence of the volume and the mass of fundamental class is crucial.

In what follows, $\mathcal{H}^n(\cdot)$ denotes the Hausdorff n -measure.

THEOREM 0.1. *Let X be a compact orientable n -dimensional singular space having a fundamental class $[X]$. Suppose that X has a length metric satisfying:*

- (1) $X - S$ is an n -dimensional Lipschitz manifold;
- (2) the Hausdorff n -measure of the singular set S of X is zero;
- (3) X satisfies the LLC-condition.

Then

$$\mathcal{H}^n(X) = \text{mass}([X]).$$

The main objects we consider are singular spaces equipped with the notion of curvature, the so-called Alexandrov spaces. Notice that any n -dimensional Alexandrov space with curvature bounded below satisfies conditions (1) and (2) in Theorem 0.1 (Burago, Gromov and Perelman [BGP], Otsu and Shioya [OS]). If the curvature is bounded above, it satisfies condition (3), and furthermore if the space is geodesically complete, it also satisfies (1) and (2) (see Otsu and Tanoue [OT]).

In the case when X is a Riemannian manifold, Theorem 0.1 is a direct consequence of Stokes' theorem and the existence of a triangulation of X , neither of which can be applied to our situation. In the proof of Theorem 0.1 we develop a cancellation technique which might be considered as a replacement of Stokes' theorem.

In [G], the idea of Thurston's straightening procedure plays an important role. To extend this procedure to singular spaces, we give an estimate of the Hausdorff

measure of straight simplices in a negatively curved space and obtain a generalization of [T], [G] and [IY].

THEOREM 0.2. *Let X be an n -dimensional compact, geodesically complete Alexandrov space with curvature $K \leq -1$. Suppose further that X is orientable and has a fundamental class (e.g. an orientable pseudomanifold). Then*

$$\mathcal{H}^n(X) \leq \frac{\pi}{(n-1)!} \|X\|.$$

The inequality above is sharp only when $n = 2$. This point will be discussed later. Let v_n be the largest volume of straight ideal simplices in the simply connected hyperbolic space of constant curvature $K = -1$. In the case when X is a Riemannian manifold with constant curvature $K = -1$, Gromov [G] proved that $\text{vol}(X) = v_n \|X\|$. In view of this fact, the constant $\pi/(n-1)!$ should be replaced by the sharp constant v_n for $n \geq 3$.

Example 0.3. Let M_1 and M_2 be n -dimensional closed orientable Riemannian manifolds with curvature $K \leq -1$. Suppose that there are (not necessarily connected) closed totally geodesic submanifolds $S_i \subset M_i$ ($i = 1, 2$) and an isometry $f : S_1 \rightarrow S_2$. Then the quotient space

$$X = M_1 \coprod_f M_2$$

is an orientable Alexandrov space with curvature $K \leq -1$ and satisfies the assumptions of Theorem 0.2.

For singular spaces, we can obtain a systematic approach of calculating or estimating the simplicial volume of a complicated space. For instance, let Σ_g be a closed orientable surface of genus $g > 1$. It is shown in [G] that $\|\Sigma_g\| = 4(g-1)$. Now, for a non-contractible simple closed curve C in Σ_g , let X_n be the quotient space obtained from the disjoint union of n copies of Σ_g by identifying C in each copy of Σ_g . Then we have

$$\|X_n\| = 4n(g-1).$$

Obviously, $\|X_n\| \leq 4n(g-1)$. By performing a deformation, we may assume that C is a closed geodesic for a hyperbolic metric of Σ_g . Then X_n has a length metric of curvature $K \leq -1$ whose restriction to each piece Σ_g is the original metric. It follows from Theorem 0.2 that

$$\|X_n\| \geq \frac{1}{\pi} n \mathcal{H}^2(\Sigma_g) = 4n(g-1).$$

As mentioned earlier, Theorem 0.2 is sharp for $n = 2$. More precisely, we obtain the following.

THEOREM 0.4. *Let X be a two-dimensional compact geodesically complete Alexandrov space with curvature $K \leq -1$ which is an orientable pseudomanifold, and assume that $\|X\| = (1/\pi)\mathcal{H}^2(X)$. Then the completion of the length metric of $X - S$ is a smooth surface of constant curvature $K = -1$.*

From [G], we see that $\|X\|$ depends only on the fundamental group Γ of X and the classifying map $X \rightarrow K(\Gamma, 1)$. Since one can easily construct examples of singular spaces with complicated fundamental groups, there is no reason to consider only manifolds.

For Alexandrov spaces with curvature bounded below, we do not need to assume the existence of a fundamental class because this existence is automatically satisfied for such a space without boundary. Namely, they are pseudomanifolds. By using the averaging methods appearing in [G], we obtain the following result on the simplicial volume of Alexandrov spaces with curvature bounded below. This is a natural extension of a main result in [G] (cf. [BCG]) to singular spaces.

THEOREM 0.5. *Let X be an n -dimensional compact orientable Alexandrov space with curvature $K \geq -1$ and without boundary which satisfies the LLC-condition. Then*

$$\|X\| \leq n!(n-1)^n \mathcal{H}^n(X).$$

It is easy to see that Alexandrov spaces with curvature bounded above satisfy the LLC-condition. However, this is not the case for Alexandrov spaces with curvature bounded below. This is the reason we need the LLC-assumption in Theorem 0.5. However, it may be possible to remove this condition in the near future.

The organization of this paper is as follows. In Section 1, we first recall the pseudonorms ℓ^1 and ℓ^∞ on homology and cohomology groups, and their duality. We consider the Lipschitz homology groups of a metric space to introduce the notion of mass of a chain and show that for a metric space satisfying the LLC-condition, its Lipschitz homology group coincides with the singular homology group. The pseudonorms defined by mass and comass on homology and cohomology groups are also dual. After an argument concerning pseudomanifolds, we prove Theorem 0.1 in Section 2. In Section 3, to prove Theorem 0.2, we give an estimate for the measure of straight simplices in singular spaces with negative curvature. In Section 4, we discuss surfaces with curvature bounded above and prove Theorem 0.4. In the Appendix,

we describe how to apply Gromov's averaging methods to singular spaces and prove Theorem 0.5.

1. Gromov's pseudonorms, mass and comass

First let us recall the pseudonorms on the singular homology and cohomology groups $H_*(X)$ and $H^*(X)$ of a topological space X , where real coefficients are understood for homology and cohomology groups unless otherwise stated. For a singular chain $c = \sum r_i \sigma_i \in C_*(X)$, the ℓ^1 -norm is defined by

$$\|c\|_1 = \sum |r_i|.$$

This gives a pseudonorm on $H_* = H_*(X)$ defined by

$$\|\alpha\|_1 = \inf_c \|c\|_1,$$

where c runs over all cycles representing the homology class $\alpha \in H_*$.

For a singular cochain $u \in C^*(X)$, the ℓ^∞ -norm is defined by

$$\|u\|_\infty = \sup_\sigma |u(\sigma)|,$$

where σ runs over all singular simplices. This gives a pseudonorm on $H^* = H^*(X)$ defined by

$$\|\beta\|_\infty = \inf_u \|u\|_\infty,$$

where u runs over all cocycles representing the cohomology class $\beta \in H^*$.

The ℓ_∞ -norm on C^* is the dual of the ℓ_1 -norm on C_* . Then, the Hahn–Banach theorem implies that $\|\cdot\|_\infty$ on H^* and $\|\cdot\|_1$ on H_* are also dual. Namely,

$$\|\beta\|_\infty = \sup_{\alpha \in H_*} \frac{|\beta(\alpha)|}{\|\alpha\|_1}, \quad \|\alpha\|_1 = \sup_{\beta \in H^*} \frac{|\beta(\alpha)|}{\|\beta\|_\infty}. \quad (1.1)$$

Now let X be a metric space. We define the mass of a singular simplex $\sigma : \Delta^m \rightarrow X$ as follows. Let T be a triangulation of Δ^m , and let $\{\Delta_i\}$ be the m -simplices of T . Then

$$\text{mass}(\sigma) = \sup_T \sum \mathcal{H}^m(\sigma(\Delta_i)),$$

where $\mathcal{H}^m$ denotes the Hausdorff m -measure. Since the mass may be infinite, we restrict our attention to singular *Lipschitz* simplices. For a Lipschitz chain $c = \sum r_i \sigma_i$

(that is, each σ_i is Lipschitz), the mass of c is defined as $\text{mass}(c) = \sum |r_i| \text{mass}(\sigma_i)$. We denote by $C_*^L(X) \subset C_*(X)$ the subcomplex of Lipschitz chains and by $H_*^L(X)$ the homology groups of $C_*^L(X)$. Then the mass of $\alpha \in H_*^L(X)$ is defined as

$$\text{mass}(\alpha) = \inf_c \text{mass}(c),$$

where c runs over all cycles representing α . For most spaces, $H_*^L(X)$ is isomorphic to the singular homology group $H_*(X)$. A natural question arising here is under what geometric condition is $H_*^L(X)$ isomorphic to the singular homology groups $H_*(X)$? We give an answer to this question below.

Definition 1.2. We say that X is locally Lipschitz contractible (LLC) if for every $p \in X$ there are positive numbers $r \leq R$ such that there exists a Lipschitz map $F : B_p(r) \times [0, 1] \rightarrow B_p(R)$ with $F(x, 0) = x$ and $F(x, 1) = p$ for all $x \in X$.

For instance, Alexandrov spaces with curvature bounded above are LLC (see the next section).

PROPOSITION 1.3. *If X is LLC, then $H_*^L(X) \simeq H_*(X)$.*

Proof. By using LLC, one can deform a small simplex σ to a Lipschitz simplex $\hat{\sigma}$. This is done as follows. First connect vertices of σ by Lipschitz curves. Then extend each Lipschitz triangle to a Lipschitz surface, and use induction to complete the process. Now for any cycle $c = \sum r_i \sigma_i$, we may assume that each σ_i is so small that we can carry out the Lipschitz deformation procedure for each σ_i . Here we take such deformations $\sigma_i \rightarrow \hat{\sigma}_i$ in such a way that if τ_i and τ_j are faces of σ_i and σ_j respectively with $\tau_i = \pm \tau_j$, then $\hat{\tau}_i = \pm \hat{\tau}_j$. Then $\hat{c} = \sum r_i \hat{\sigma}_i$ is a cycle homologous to c . Thus the natural homomorphism $h : H_*^L(X) \rightarrow H_*(X)$ is surjective. Next, let c be a Lipschitz boundary, $c = \partial b$, for some singular chain b . As above, deform b to a Lipschitz chain while keeping the boundary c fixed. This shows that h is injective. $\square$

Let X be an LLC-space. For a singular cochain $u \in C^*(X)$, we define the comass of u as

$$\text{comass}(u) = \sup_{\sigma} \frac{|u(\sigma)|}{\text{mass}(\sigma)},$$

where σ runs over all the Lipschitz simplices. Then the comass of $\beta \in H^*(X)$ is defined by

$$\text{comass}(\beta) = \inf_u \text{comass}(u),$$

where u runs over all the cocycles representing β . As in the case of the ℓ_1 -norm and ℓ^∞ -norm, we see that $(H_*(X), \text{mass})$ and $(H^*(X), \text{comass})$ are dual. More precisely,

$$\text{comass}(\beta) = \sup_{\alpha \in H_*} \frac{|\beta(\alpha)|}{\text{mass}(\alpha)}, \quad \text{mass}(\alpha) = \sup_{\beta \in H^*} \frac{|\beta(\alpha)|}{\text{comass}(\beta)}. \quad (1.4)$$

2. Mass and measure

We defined the simplicial volume for a singular space having a fundamental class in the Introduction. We now give a sufficient condition for a singular space to possess a fundamental class.

Definition 2.1. We say that a locally compact Hausdorff space X is a pseudomanifold of n -dimension if it satisfies the following conditions:

- (1) There exists a closed subset $S \subset X$ such that:
 - (a) $X - S$ is a topological n -manifold;
 - (b) $\dim S \leq n - 1$;
 - (c) $\dim(S \cap (\overline{X - S})) \leq n - 2$,
 where $\dim$ denotes the topological dimension.
- (2) X is homologically locally n -connected. That is, for every $p \in X$ and neighborhood U , there exists a neighborhood V of p in U such that $\tilde{H}_i(V) \rightarrow \tilde{H}_i(U)$ is trivial for $i \leq n$, where $\tilde{H}_*$ is the reduced homology group.

Note that $X - S$ is not necessarily connected. The Alexandrov spaces with curvature bounded below and without boundary are pseudomanifolds (see [BGP], [P]). Notice also that our definition of pseudomanifolds is weaker than the usual one, which requires $\dim S \leq n - 2$ and a polyhedral structure of X .

PROPOSITION 2.2. *Let X be an n -dimensional compact orientable pseudomanifold. Then X has a fundamental class $[X] \in H_n(X)$.*

This was proved in [GP] when X is an Alexandrov space with curvature bounded below.

Proof. We denote by $\bar{H}^*$ the Alexander–Spanier cohomology group (see [Sp] for its basic properties), and by X' the closure of $X - S$. Put $S' = X' \cap S$. Since X is homologically locally n -connected, $H^n(X, \mathbb{Z}) \simeq \bar{H}^n(X, \mathbb{Z})$, which is isomorphic to $\bar{H}^n(X', \mathbb{Z}) \simeq \bar{H}^n(X', S', \mathbb{Z})$. This follows from the Myer–Vietoris exact sequence and the assumption on S . The important property of the

Alexander–Spanier cohomology is that $\bar{H}^*(X', S', \mathbf{Z}) \simeq \bar{H}_c^*(X' - S', \mathbf{Z})$, where $\bar{H}_c^*$ is the Alexander–Spanier cohomology with compact support. Thus we have $H^n(X, \mathbf{Z}) \simeq H_c^n(X - S, \mathbf{Z}) \simeq \mathbf{Z}^k$, where $k \leq \infty$ is the number of connected components of $X - S$. Let $X_1, \dots, X_k$ be the connected components of $X - S$. The above argument shows that $H^n(\bar{X}_i, \mathbf{Z}) \simeq \mathbf{Z}$. Now suppose that $k = \infty$. Take a point x_i in each X_i and let p be an accumulation point of the sequence x_i . Since X is compact, any neighborhood of p contains infinitely many $\bar{X}_i$. This contradicts the homologically locally n -connectedness of X . Therefore k is finite. Let $\beta = (\beta_1, \dots, \beta_k) \in H^n(X, \mathbf{Z})$ be the element corresponding to $(1, \dots, 1) \in \mathbf{Z}^k$. Then for each $x \in X - S$, the orientation class $\beta_x \in H^n(X, X - x)$ projects to β_i , where i corresponds to the component containing x . If $\mu_i \in H_n(X)$, $1 \leq i \leq k$, are the elements dual to α_i , then $[X] = (\mu_1, \dots, \mu_k)$ is a fundamental class of $[X]$. $\square$

We now give a proof of Theorem 0.1. First, we need a lemma from geometric measure theory. Let X be as in Theorem 0.1. The symbol $O(\epsilon)$ denotes a function satisfying $\lim_{\epsilon \rightarrow 0} O(\epsilon) = 0$.

LEMMA 2.3. *Given $\epsilon > 0$ and a Lipschitz map $f : \Delta^n \rightarrow X$, there exists a Lipschitz map $g : \Delta^n \rightarrow X$ such that:*

- (1) $g = f$ on a small neighborhood of the boundary $\partial\Delta^n$;
- (2) $\mathcal{H}^n(\{x \mid g(x) \neq f(x)\}) < \epsilon$;
- (3) g is locally bi-Lipschitz homeomorphic, except on the preimage of a subset of $g(\Delta^n)$ with Hausdorff n -measure $< \epsilon$;
- (4) $|\text{mass}(f) - \text{mass}(g)| < O(\epsilon)$.

Proof. Take a neighborhood U of S such that $\mathcal{H}^n(U) \ll \epsilon$. We require the map g to be equal to f on $f^{-1}(U)$ and on a small neighborhood of the boundary $\partial\Delta^n$. Next we must extend this map on Δ^n . Using the a.e. Lipschitz structure of X , we may assume that f is a map into $\mathbf{R}^n$. One can approximate f by a C^1 map g satisfying (1) and (2), which is possible by [F, 3.1.15]. Then use Sard's theorem for the map g to obtain (3). (4) is clear by construction. $\square$

Proof of Theorem 0.1. Let a Lipschitz cycle $f = \sum r_i f_i$ represent the fundamental class $[X]$. Take a deformation $g = \sum r_i g_i$ of f , where g_i is chosen for f_i as in Lemma 2.3. Note that g also represents $[X]$. Now using Lemma 2.3, we can subdivide Δ^m into small simplices so that the decomposition $\sum_j g_{ij}$ of g_i associated with the subdivision satisfies the following. There exists an open set U of measure $< \epsilon$ containing S such that $g_{ij} : \Delta^n \rightarrow X$ is bi-Lipschitz homeomorphic outside

$g_{ij}^{-1}(U)$. We put $V = U \cup (\cup_{ij} g_{ij}(\partial \Delta^n))$. For $x \in X - V$, we set

$$\epsilon_{ij}(x) = \begin{cases} +1 & \text{if } g_{ij} \text{ preserves the orientation at } x \\ -1 & \text{if } g_{ij} \text{ reverses the orientation at } x \\ 0 & \text{if } g_{ij}^{-1}(x) = \emptyset. \end{cases}$$

If we consider g_{ij} as the cycle in $C_*(X, X - x)$, its homology class coincides with $\epsilon_{ij}(x) \in H_*(X, X - x; \mathbf{Z}) \simeq \mathbf{Z}$. Since $g = \sum r_i g_{ij}$ represents the fundamental class $[X]$, it follows that

$$\sum_{ij} r_i \epsilon_{ij}(x) = 1 \quad \text{on } X - V.$$

Hence

$$\begin{aligned} \mathcal{H}^n(X) &= \mathcal{H}^n(X - V) + O(\epsilon) \\ &= \int_{X-V} \sum r_i \epsilon_{ij}(x) d\mathcal{H}^n(x) + O(\epsilon) \\ &\leq \sum |r_i| \mathcal{H}^n(g_{ij}(\Delta^n)) + O(\epsilon) \\ &\leq \text{mass}(f) + O(\epsilon). \end{aligned}$$

Thus we obtain $\mathcal{H}^n(X) \leq \text{mass}([X])$.

Conversely, for every $\epsilon > 0$, take a cycle $f = \sum r_i \sigma_i$ representing $[X]$ such that $\text{mass}(f) < \text{mass}([X]) + \epsilon$. Since the singular homology group is naturally isomorphic to the *ordered* singular homology group, by changing the order of the vertices of the simplices f_i if necessary, we may assume that $r_i > 0$ for every i . By taking the ' C^1 -approximation' of each σ_i as above, we may assume that for some set $U \supset S$ with $\mathcal{H}^n(U) \ll \epsilon$, if σ_i does not intersect U , then it is a bi-Lipschitz homeomorphism. We put $W = U \cup (\cup_i \partial \sigma_i)$. Then as above,

$$\sum_i r_i \epsilon_i(x) = 1 \quad \text{on } X - W. \quad (2.4)$$

For a set $\tau \subset X - W$ such that for every i , $\tau \subset \sigma_i$ or τ does not meet σ_i , we set

$$\begin{aligned} I_+(\tau) &= \{i \mid \epsilon_i(\tau) = 1\}, \quad I_-(\tau) = \{i \mid \epsilon_i(\tau) = -1\}, \quad I(\tau) = I_+(\tau) \cup I_-(\tau), \\ E &= \{x \in X - W \mid I_-(x) \neq \emptyset\}. \end{aligned}$$

ASSERTION 2.5. $\mathcal{H}^n(E) < O(\epsilon)$.

Proof. Take a sequence $x_1, x_2, \dots$ in E and a sequence $\tau_1, \tau_2, \dots$ of bi-Lipschitz n -simplices in $X - W$ with positive orientation satisfying the following:

- (1) x_i is contained in the interior of τ_i ;
- (2) $\{\tau_i\}_i$ is mutually disjoint;
- (3) $\mathcal{H}^n(E - \cup \tau_i) = 0$;
- (4) τ_i is contained in the intersection of all σ_j with $j \in I(x_i)$.

For any large integer N , put $\tau = \tau_1 + \cdots + \tau_N$. We consider τ as both a chain and a set. For each i , $\sigma_i^{-1}(\tau)$ is either empty or a disjoint union of curved n -simplices. Take a triangulation K_i of $\Delta^n - \sigma_i^{-1}(\tau)$ such that $K_i \cup \sigma_i^{-1}(\tau)$ is a triangulation of Δ^n . This gives rise to chains $\sum_j \sigma_{ij}$ homologous to $\sigma_i|_{K_i|}$. We denote by J_+ (respectively J_-) the set of j for which σ_j is orientation preserving (respectively orientation reversing). We remark that if $i \in J_+$, σ_i is homologous to $\sum \sigma_{ij} + \tau^{(i)}$ (respectively homologous to $\sum \sigma_{ij} - \tau^{(i)}$), where $\tau^{(i)}$ is the sum of τ_j contained in σ_i . Consider the chain $g = \sum r_i \sigma_{ij} + \tau$. Using (2.4), we have

$$\begin{aligned} g &= \sum r_i \sigma_{ij} + \sum_{\alpha=1}^N \left(\sum_{i \in J_+(\tau_\alpha)} r_i \tau_\alpha - \sum_{i \in J_-(\tau_\alpha)} r_i \tau_\alpha \right) \\ &= \sum_{i \in J_+} r_i (\sigma_{ij} + \tau^{(i)}) + \sum_{i \in J_-} r_i (\sigma_{ij} - \tau^{(i)}). \end{aligned}$$

It follows that g is homologous to f . Note that

$$\begin{aligned} \text{mass}(g) &= \sum r_i \text{mass}(\sigma_{ij}) + \text{mass}(\tau) \\ &= \sum r_i (\text{mass}(\sigma_i) - \text{mass}(\tau^{(i)})) \\ &\quad + \left(\sum_{i \in J_+} r_i \text{mass}(\tau^{(i)}) - \sum_{i \in J_-} r_i \text{mass}(\tau^{(i)}) \right) \\ &= \text{mass}(f) - 2 \sum_{i \in J_-} r_i \text{mass}(\tau^{(i)}). \end{aligned}$$

Therefore,

$$\sum_{i \in J_-} r_i \text{mass}(\tau^{(i)}) < \epsilon/2, \quad (2.6)$$

and hence $\text{mass}(\tau) < \epsilon/2\delta$, where $\delta = \min_i r_i$. This implies that $\mathcal{H}^n(E) \leq \epsilon/\delta$ and completes the proof of the assertion. $\square$

We obtain from (2.6) that

$$\sum_{i \in J_-} r_i \text{mass}(\sigma_i) < O(\epsilon).$$

It then follows from Assertion 2.5 that

$$\begin{aligned}
 \mathcal{H}^n(X) &\sim \mathcal{H}^n(X - V - E) \\
 &= \int_{X-V-E} \sum r_i \epsilon_i(x) d\mathcal{H}^n(x) \\
 &\sim \sum_{i \in J_+} r_i \text{mass}(\sigma_i) \\
 &\sim \text{mass}(f),
 \end{aligned}$$

where we have used the notation $a \sim b$ to indicate $|a - b| < O(\epsilon)$. Thus we have completed the proof of Theorem 0.1. $\square$

3. Volumes of straight simplices

We begin with the definition of singular spaces equipped with the notion of curvature. Let X be a complete locally compact length space. For a real number κ and for a geodesic triangle Δxyz in X with vertices x, y and z , we denote by $\tilde{\Delta}_\kappa xyz$ a *comparison triangle* $\tilde{\Delta} \tilde{x} \tilde{y} \tilde{z}$ in the simply connected complete surface M_κ with constant curvature κ . This means that each side length of $\tilde{\Delta}_\kappa xyz$ is equal to the corresponding one for Δxyz . Here we suppose that the perimeter of Δxyz is less than $2\pi/\sqrt{\kappa}$ if $\kappa > 0$. We say that an open set U in X satisfies the $\text{CAT}_-(\kappa)$ -inequality (respectively $\text{CAT}_+(\kappa)$ -inequality) if for any geodesic triangle in U with vertices x, y and z and for any point w on the geodesic segment yz joining y to z , we have $|x, w| \leq |\tilde{x}, \tilde{w}|$ (respectively $|x, w| \geq |\tilde{x}, \tilde{w}|$), where $\tilde{w}$ is the point on $\tilde{y}\tilde{z}$ corresponding to w . We say that X is a space with curvature $K \leq \kappa$ (respectively $K \geq \kappa$) if each point in X has a neighborhood satisfying the $\text{CAT}_-(\kappa)$ -inequality (respectively $\text{CAT}_+(\kappa)$ -inequality).

In this section we assume X to have curvature $K \leq \kappa$. Note that every pair of points in a $\text{CAT}_-(\kappa)$ -domain can be joined by a *unique* minimal geodesic. Note also that X satisfies the LLC-condition given in the previous section. For finitely many points $p_0, \dots, p_m$ in U satisfying the $\text{CAT}_-(\kappa)$ -inequality, the straight simplex $\Delta = \Delta(p_0, \dots, p_m)$ is defined as follows. First let $\Delta(p_0, p_1)$ be the minimal geodesic segment joining p_1 to p_0 . Assuming that $\Delta(p_0, \dots, p_{m-1})$ has been constructed, we define $\Delta(p_0, \dots, p_m)$ to be the set of all geodesic segments joining p_m to the points in $\Delta(p_0, \dots, p_{m-1})$. It follows from the $\text{CAT}_-(\kappa)$ -inequality that a straight simplex can be considered as a Lipschitz simplex.

First we consider straight 2-simplices. An essential idea of the following result is

due to Alexandrov [A], but our proof is slightly different technically. By a geodesic triangle, we also mean a geodesic surface spanned by the triangle.

THEOREM 3.1. *Let $\Delta = \Delta(p, q, r)$ be a straight simplex in a space X with $K \leq \kappa$, and let $\tilde{\Delta} = \tilde{\Delta}(p, q, r)$ be a straight simplex in M_κ with the same edge lengths. Then $\mathcal{H}^2(\Delta) \leq \mathcal{H}^2(\tilde{\Delta})$, where the equality holds only if Δ is isometric to $\tilde{\Delta}$.*

Proof. Take an ϵ -decomposition $p = p_0, \dots, p_N = q$ of the segment pq . On the comparison triangle $\Delta \tilde{p}\tilde{q}\tilde{r}$ we draw the comparison triangles $\Delta \tilde{p}\tilde{p}_{1-}\tilde{r}$ and $\Delta \tilde{p}_{1+}\tilde{q}\tilde{r}$ for Δpp_1r and Δp_1qr , respectively (see Figure 3.2). Next, on the comparison triangle $\Delta \tilde{p}_{1+}\tilde{q}\tilde{r}$, we draw the comparison triangles $\Delta \tilde{p}_{1+}\tilde{p}_{2-}\tilde{r}$ and $\Delta \tilde{p}_{2+}\tilde{q}\tilde{r}$ for Δp_1p_2r and Δp_2qr respectively. Repeating this procedure, we have comparison triangles $\Delta \tilde{p}_{i+}\tilde{p}_{i+1-}\tilde{r}$ for $\Delta p_i p_{i+1}r$ on Δpqr , $i = 0, \dots, N-1$, where $\tilde{p}_{0+} = \tilde{p}$, $\tilde{p}_{N-} = \tilde{q}$.

FIGURE 3.2.

Let $\tilde{\Delta}_\epsilon$ be the union of the segments $\tilde{p}_{i-}\tilde{p}_{i+}$ and the edges of all triangles $\Delta\tilde{p}_{i+}\tilde{p}_{i+1-}\tilde{r}$. Then the map $f_\epsilon: \tilde{\Delta}_\epsilon \rightarrow \Delta$ is naturally defined by $f_\epsilon(\tilde{r}\tilde{p}_{i\pm}) = rp_i$, and $f_\epsilon(\tilde{p}_{i-}\tilde{p}_{i+}) = p_i$. By using the $\text{CAT}_-(\kappa)$ -inequality, we shall show that f_ϵ is contracting.

For every $x, y \in \tilde{\Delta}_\epsilon$, we assume that x is on $\tilde{r}\tilde{p}_{i+}$ and y is on $\tilde{r}\tilde{p}_{j-}$ ($i < j$). The other cases are similar, as we now see. Let L be the union of the segments $\tilde{p}_{k+}\tilde{p}_{k+1-}$ and $\tilde{p}_{k+1-}\tilde{p}_{k+1+}$ ($i \leq k \leq j-1$). Then we have the following.

Case 1. The segment xy does not meet L .

Let xy meet $\tilde{\Delta}_\epsilon$ with $x = x_i, x_{i+1-}, x_{i+1+}, \dots, x_{j-} = y$, where $x_{k\pm} \in \tilde{r}\tilde{p}_{k\pm}$. From the $\text{CAT}_-(\kappa)$ -inequality, $|x_{k+}x_{k+1-}| \geq |f_\epsilon(x_{k+})f_\epsilon(x_{k+1-})|$. Hence

$$\begin{aligned} |x, y| &\geq \sum_{k=i}^{j-1} |x_{k+}x_{k+1-}| \\ &\geq \sum_{k=i}^{j-1} |f_\epsilon(x_{k+})f_\epsilon(x_{k+1-})| \\ &\geq |f_\epsilon(x)f_\epsilon(y)|. \end{aligned}$$

Case 2. xy meets L .

Here we assume that xy meets L at only two points, say z_1, z_2 , where $z_1 \in \tilde{p}_{k+}\tilde{p}_{k+1-}$, $z_2 \in \tilde{p}_{\ell+}\tilde{p}_{\ell+1-}$, $k < \ell$. The other cases are similar.

First consider the special case $k = i, \ell = i+1$. Note that z_1z_2 is contained in the convex set $\Delta\tilde{p}_{i+}\tilde{q}\tilde{r}$. Let z'_1 and z'_2 be points on $\tilde{p}_{i+}\tilde{q}$ such that $|\tilde{p}_{i+}z'_1| = |\tilde{p}_{i+}, z_1|$ and $|\tilde{q}, z'_2| = |\tilde{q}, z_2|$, and let p'_{i+1} be the point on $\tilde{p}_{i+}\tilde{q}$ such that $|\tilde{p}_{i+}, p'_{i+1}| = |p_i, p_{i+1}|$ and $|p'_{i+1}, \tilde{q}| = |p_{i+1}, q|$. Let A_- (respectively A_+) be the arc joining $\tilde{p}_{i+1-}$ to p'_{i+1} (respectively $\tilde{p}_{i+1+}$ to p'_{i+1}), each point of which is equidistant from $\tilde{p}_{i+}$ (respectively to $\tilde{q}$). Let z_1z_2 intersect A_- and A_+ at w_- and w_+ , respectively. Then, $|z_1, w_-| \geq |z_1, \tilde{p}_{i+1-}|$ and $|z_2, w_+| \geq |z_2, \tilde{p}_{i+1+}|$. It follows that $|z_1, z_2| \geq |z_1, \tilde{p}_{i+1-}| + |\tilde{p}_{i+1+}, z_2| = |f_\epsilon(z_1), f_\epsilon(z_2)|$. From Case 1, we see that $|x, z_1| \geq |f_\epsilon(x), f_\epsilon(z_1)|$ and $|y, z_2| \geq |f_\epsilon(y), f_\epsilon(z_2)|$. Thus,

$$\begin{aligned} |x, y| &= |x, z_1| + |z_1, z_2| + |z_2, y| \\ &\geq |f_\epsilon(x), f_\epsilon(z_1)| + |f_\epsilon(z_1), f_\epsilon(z_2)| + |f_\epsilon(z_2), f_\epsilon(y)| \\ &\geq |f_\epsilon(x), f_\epsilon(y)|. \end{aligned}$$

Next we consider the general case. From Case 1, it suffices to show that $|z_1, z_2| \geq |f_\epsilon(z_1), f_\epsilon(z_2)|$. We prove this by induction on $\ell - k$. The previous

argument shows that the inequality holds for $\ell - k = 1$. We may assume that the triangles $\Delta \tilde{p}_{k+} \tilde{p}_{k+1-} \tilde{r}$ and $\Delta \tilde{p}_{k+1+} \tilde{q} \tilde{r}$ and the segment $z_1 z_2$ are all contained in the comparison triangle $\Delta \tilde{p}_k \tilde{q} \tilde{r}$, and that $\Delta \tilde{p}_{\ell+} \tilde{p}_{\ell+1-} \tilde{r}$ is contained in $\Delta \tilde{p}_{k+1+} \tilde{q} \tilde{r}$, as before. Let $z_1 z_2$ meet $\tilde{p}_{k+1+} \tilde{q}$ at w . Let z'_1 and w' be the points on $\tilde{p}_k \tilde{q}$ for which the equalities $|\tilde{p}_k z_1| = |\tilde{p}_k z'_1|$ and $|\tilde{q} w| = |\tilde{q} w'|$ hold. As above, we have $|z_1, w| \geq |z'_1, w'| = |f_\epsilon(z_1), p_{k+1}| + |\tilde{p}_{k+1+}, w|$. By the inductive assumption, $|\tilde{p}_{k+1+}, z_2| \geq |p_{k+1}, f_\epsilon(z_2)|$. Consequently,

$$\begin{aligned} |z_1, z_2| &= |z_1, w| + |w, z_2| \\ &\geq |f_\epsilon(z_1), p_{k+1}| + |\tilde{p}_{k+1+}, z_2| \\ &\geq |f_\epsilon(z_1), f_\epsilon(z_2)|. \end{aligned}$$

Let K be the Hausdorff limit of $\tilde{\Delta}_\epsilon$ in $\tilde{\Delta}$ as $\epsilon \rightarrow 0$. We may assume that f_ϵ converges to a surjective contracting map $f : K \rightarrow \Delta$. Thus

$$\mathcal{H}^2(\Delta) \leq \mathcal{H}^2(K) \leq \mathcal{H}^2(\tilde{\Delta}).$$

If the equality holds in both cases, it follows from the proof of the inequality above that for every $s \in pq$, $|r, s| = |\tilde{r}, \tilde{s}|$, where $\tilde{s}$ is the point of $\tilde{p}\tilde{q}$ corresponding to s . Then the rigidity theorem by Alexandrov [A] (cf. [Sh]) implies that Δ is isometric to $\tilde{\Delta}$. $\square$

THEOREM 3.3. *Let X have curvature $K \leq -1$. For every straight m -simplex Δ ($m \geq 2$), we have*

$$\mathcal{H}^m(\Delta) \leq \frac{\pi}{(m-1)!}.$$

For the Riemannian case, see [G].

Let $\Delta = \Delta(p_0, \dots, p_m)$, $\Delta' = \Delta(p_0, \dots, p_{m-1})$, and consider Δ as the geodesic cone over Δ' with vertex p_m . We say that a subset K of X is a spherical geodesic cone over a set B with vertex p if it is a geodesic cone over B and if the distance between p and each point of B is always the same number, say ℓ , called the radius of K . For the proof of Theorem 3.3, we show the following assertion by induction on m .

ASSERTION 3.4. *For every $\epsilon > 0$, we can take disjoint spherical geodesic cones $K_1, \dots, K_N \subset \Delta$ and spherical geodesic cones $\tilde{K}_1, \dots, \tilde{K}_N \subset \mathbf{H}^n$ over $\tilde{B}_i$ such that the following are true:*

- (1) $\mathcal{H}^m(\Delta - \cup K_i) < \epsilon$;

- (2) $\sum \mathcal{H}^{m-1}(\tilde{B}_i) < \mathcal{H}^{m-1}(\Delta') + \epsilon;$
 (3) *there exists an $O(\epsilon)$ -contraction $f_i : \tilde{K}_i \rightarrow K_i$, in the sense that*

$$|f_i(x), f_i(y)| \leq (1 + O(\epsilon))|x, y|$$

for every $x, y \in \tilde{K}_i$.

We prove Theorem 3.3 by assuming Assertion 3.4. This is also done by induction on m . Theorem 3.1 implies Theorem 3.3 for $m = 2$. Let K_i and $\tilde{K}_i$ be as in the assertion. From a direct calculation,

$$\mathcal{H}^m(\tilde{K}_i) < \frac{1}{(m-1)} \mathcal{H}^{m-1}(\tilde{B}_i).$$

It follows that

$$\begin{aligned} \mathcal{H}^m(\Delta) &< \sum \mathcal{H}^m(K_i) + \epsilon \\ &< \sum (1 + O(\epsilon))^m \mathcal{H}^m(\tilde{K}_i) + \epsilon \\ &< \sum \frac{(1 + O(\epsilon))^m}{(m-1)} \mathcal{H}^{m-1}(\tilde{B}_i) + \epsilon \\ &< \frac{(1 + O(\epsilon))^m}{(m-1)} \mathcal{H}^{m-1}(\Delta') + 2\epsilon \\ &< \frac{1 + O(\epsilon)}{(m-1)!} \pi + 2\epsilon \quad (\text{by inductive assumption}). \end{aligned}$$

This completes the proof of Theorem 3.3. $\square$

Proof of Assertion 3.4. For $m = 2$, the assertion follows from the proof of Theorem 3.1. Let Δ_ϵ be the set of points x of Δ for which $|x, \Delta'| \geq \epsilon$. Let K'_i , $\tilde{K}'_i$ and $f_i : \tilde{K}'_i \rightarrow K'_i$ be as in the assertion for Δ' . Decompose each K'_i into small disjoint measurable sets $\{A_{ij}\}$ such that both $\tilde{A}_{ij} = f_i^{-1}(A_{ij})$ and A_{ij} have diameter $\ll \epsilon$. Let ℓ_{ij} be the distance from p_m to A_{ij} . Then let K_{ij} be the spherical geodesic cone over a set B_{ij} with vertex p_m and of radius ℓ_{ij} constructed by cutting off the geodesic cone over A_{ij} . Take some point $\tilde{p}_{ij} \in \mathbf{H}^m$ such that a geodesic segment from p_{ij} to a point in $\tilde{A}_{ij}$ meets $\mathbf{H}^{n-1}$ orthogonally and such that $|\tilde{p}_{ij}, \mathbf{H}^{n-1}| = \ell_{ij}$. Let $\tilde{K}_{ij}$ be the spherical geodesic cone over a set $\tilde{B}_{ij}$ with vertex $\tilde{p}_{ij}$ and of radius ℓ_{ij} constructed by cutting off the geodesic cone over $\tilde{A}_{ij}$. By modifying f_i , one can easily construct an $O(\epsilon)$ -contraction $\hat{f}_{ij} : \tilde{K}_{ij} \rightarrow K_{ij}$. This completes the proof of Assertion 3.3. $\square$

THEOREM 3.5. *Let X have curvature $K \leq -1$. Then for every class $\alpha \in H_m(X)$, $m \geq 2$, we have*

$$\text{mass}(\alpha) \leq \frac{\pi}{(m-1)!} \|\alpha\|_1.$$

Proof. This follows from Thurston's straightening procedure as follows. Let $c = \sum r_i \sigma_i$ represent α . Take a lift $\tilde{\sigma}_i$ of σ_i to the universal covering of X , and consider the straight simplex spanned by the vertices of $\tilde{\sigma}_i$. Let $\hat{\sigma}_i$ be the projection of $\tilde{\sigma}_i$. Then $\hat{c} = \sum r_i \hat{\sigma}_i$ is homologous to c , and hence

$$\begin{aligned} \text{mass}(\alpha) &\leq \text{mass}(\tilde{c}) = \sum |r_i| \text{mass}(\hat{\sigma}_i) \\ &\leq \frac{\pi}{(m-1)!} \sum |r_i|. \end{aligned}$$

This implies the required inequality. $\square$

Theorem 0.2 is a direct consequence of Theorems 0.1 and 3.5.

4. Alexandrov pseudosurface

In this section, we give a proof of Theorem 0.4. Let X be a two-dimensional, compact, geodesically complete Alexandrov pseudomanifold with curvature $K \leq \kappa$, and let S be its singular set. First we reduce the problem to the case $\dim S = 0$.

LEMMA 4.1. *There exists a two-dimensional, geodesically complete Alexandrov pseudomanifold X' with curvature $K \leq \kappa$, and with singular set S' such that:*

- (1) $\mathcal{H}^2(X') = \mathcal{H}^2(X)$;
- (2) $\|X'\| = \|X\|$;
- (3) $\dim S' = 0$.

Proof. We may assume that $\dim S = 1$. By the definition of a pseudomanifold, S is a graph whose endpoints, the vertices of order one, are contained in the closure of $X - S$. We define an equivalence relation on $\overline{X - S}$ by designating two points in $\overline{X - S} \cap S$ equivalent if they are joined by the graph S . Let X' be the quotient space with the quotient length metric. The new space satisfies (1) and (3). Since the identification map $\pi : X \rightarrow X'$ has degree one, we have $\|X'\| \leq \|X\|$. It is also easy to construct a map $X' \rightarrow X$ of degree one. Therefore, X' satisfies (2). $\square$

In view of the previous lemma, to prove Theorem 0.4 we may assume that the dimension of S is zero.

LEMMA 4.2. *For any $p \in S$, a small neighborhood of p is homeomorphic to the cone over the disjoint union of finitely many circles. In particular, S is finite.*

Proof. Take a small metric ball B around p such that B does not contain a δ -branch point of p for small δ .

SUBLEMMA 4.3. *For any $x \in B$ distinct from p , Σ_x is homeomorphic to a circle.*

Proof. First we note that for any $x \in B$ distinct from p , Σ_x is connected. Otherwise, $B - \{x\}$ must be disconnected. This contradicts the assumption $\dim S = 0$.

It is known that Σ_x is a one-dimensional, geodesically complete, compact Alexandrov space satisfying the following:

- (1) for every $\xi \in \Sigma_x$ there is a $\xi^* \in \Sigma_x$ such that the angle between ξ and ξ^* is π ;
- (2) if the angle between ξ_1 and ξ_2 in Σ_x is strictly less than π , then there is a unique minimal geodesic joining them.

Suppose now that Σ_x is not homeomorphic to a circle. Then Σ_x must contain a set Σ with two points ξ_1 and ξ_2 such that Σ consists of three geodesic segments joining ξ_1 and ξ_2 , where the angle between ξ_1 and ξ_2 is equal to π . It turns out that $\dim S \geq 1$, a contradiction. $\square$

Sublemma 4.3 now implies that the length of Σ_x is less than $2\pi + \delta$. It follows that there is a point q such that (p, q) is a $(1, O(\delta))$ -strainer at x and that the map $y \rightarrow (|p, y|, |q, y|)$ gives a homeomorphism of a neighborhood of x onto an open subset of $\mathbf{R}^2$. Thus the small metric spheres around p are one-dimensional compact manifolds without boundary, the disjoint union, say S , of finitely many circles. Therefore, the small neighborhood of p is homeomorphic to the cone over S . This completes the proof of Lemma 4.2. $\square$

Let Y denote the completion of $X - S$ with respect to the length metric of $X - S$, which is a closed topological surface. It is easy to verify that Y is an Alexandrov space with curvature $K \leq \kappa$. Let $\pi : Y \rightarrow X$ be the identification map, and let $S' = \pi^{-1}(S)$.

LEMMA 4.4. $\|X\| = \|Y\|$.

Proof. We may assume that the finite set S' is contained in a coordinate neighborhood U of Y . Define a map $f : X \rightarrow Y$ of degree one by requiring $f(\pi(B))$ to be a point. Since the degree of π is also one, we have $\|X\| = \|Y\|$. $\square$

Proof of Theorem 0.4. Suppose that $\|X\| = (1/\pi)\mathcal{H}^2(X)$. Then we have $2\pi\chi(Y) = -\mathcal{H}^2(Y)$. We shall apply results of Machigashira and Ohtuska [MO] concerning

the excess of Alexandrov surfaces. Let $e(Y)$ be the excess of Y defined there. It follows from [MO] that $e(Y) = 2\pi\chi(Y)$ and that Y has the Gaussian curvature G almost everywhere satisfying $G \leq -1$ and $e(Y) \leq \int_Y G$. This implies that $G = -1$ almost everywhere and that Y is metrically regular; the space of direction at any point of Y is isometric to the circle of length 2π . For every small triangle $\Delta \subset Y$, $e(\Delta) = \alpha + \beta + \gamma - \pi$ and $e(Y - \Delta) + e(\Delta) = e(Y)$, where α, β and γ are the angles of Δ . By [MO], we have

$$e(\Delta) \leq \int_{\Delta} G = -\mathcal{H}^2(\Delta), \quad e(Y - \Delta) \leq \int_{Y-\Delta} G = -\mathcal{H}^2(Y - \Delta).$$

Thus we have $e(\Delta) = -\mathcal{H}^2(\Delta)$. It follows easily that $e(\Delta) = e(\tilde{\Delta})$ and $\mathcal{H}^2(\Delta) = \mathcal{H}^2(\tilde{\Delta})$, where $\tilde{\Delta}$ is a comparison triangle for Δ in $M^2(-1)$. Therefore Δ is isometric to $\tilde{\Delta}$, and Y has constant curvature $K = -1$. $\square$

Appendix

Let X be a locally simply connected length space and let $\pi : Y \rightarrow X$ be the universal covering space with the induced length structure and with the covering transformation group $\Gamma = \pi_1(X)$. A singular cochain $c \in C^m(Y)$ is called *straight* and Γ -*invariant* if the following hold.

- (1) For any simplex $\sigma : \Delta^m \rightarrow Y$, $c(\sigma)$ depends only on the vertices of σ , say $y_0, \dots, y_m$. We write $\sigma(y_0, \dots, y_m)$ in place of $c(\sigma)$.
- (2) $c(\gamma y_0, \dots, \gamma y_m) = c(y_0, \dots, y_m)$ for all $y_i \in Y$ and $\gamma \in \Gamma$.
- (3) $c : Y \times \dots \times Y \rightarrow \mathbf{R}$ is a Borel function.
- (4) $c(y_0, \dots, y_m)$ is antisymmetric.

We denote by $C^*(Y; \Gamma)$ the subcomplex of $C^*(Y)$ consisting of all Γ -invariant straight cochains. We also denote by $H^*(Y; \Gamma)$ the cohomology of $C^*(Y; \Gamma)$. This has a natural ℓ^∞ -pseudonorm $\|\cdot\|$. Since the elements of $C^*(Y; \Gamma)$ are Γ -invariant, there exists a natural inclusion $I : C^*(Y; \Gamma) \rightarrow C^*(X)$ which induces a homomorphism $I_* : H^*(Y; \Gamma) \rightarrow H^*(X)$. Now, a new norm on $H^*(X)$ is defined as follows. For $\beta \in H^*(X)$, we put

$$\|\beta\|'_\infty = \begin{cases} \inf_{\alpha \in I_*^{-1}(\beta)} \|\alpha\|_\infty & \text{if } I_*^{-1}(\beta) \neq \emptyset \\ \infty & \text{if } I_*^{-1}(\beta) = \emptyset, \end{cases}$$

and by duality, we put for $\alpha \in H_m(X)$,

$$\|\alpha\|'_1 = \sup_{\beta \in H^m(X)} \frac{|\beta(\alpha)|}{\|\beta\|'_\infty}.$$

Gromov proved the following in the general situation.

THEOREM A.1. ([G]) $\|\cdot\|'_\infty = \|\cdot\|_\infty$ and hence $\|\cdot\|'_1 = \|\cdot\|_1$.

Therefore we use the new norm $\|\cdot\|'$ in place of the old norm $\|\cdot\|$, implicitly.

Let $\hat{C}^*(Y; \Gamma)$ be the subcomplex of *bounded* cochains (those with finite ℓ^∞ -norm) of $C^*(Y; \Gamma)$, and let $\mathcal{M}$ be the Banach space consisting of all finite measures μ on X with norm

$$\|\mu\| = \int_Y |\mu|.$$

Note that Γ acts naturally on $\mathcal{M}$ by $\int f(g\mu) = \int (f \circ g)\mu$. Let $\Lambda^m \mathcal{M}$ be the linear space of all $(m+1)$ -multilinear antisymmetric functions on the product space $\mathcal{M} \times \cdots \times \mathcal{M}$. Define $\Phi : \hat{C}^m(Y; \Gamma) \rightarrow \Lambda^m \mathcal{M}$ by

$$\Phi(c)(\mu_0, \dots, \mu_m) = \int \cdots \int c(y_0, \dots, y_m) \mu_0(y_0) \cdots \mu_m(y_m).$$

Now we define an averaging operator. To do this we assume that X is an n -dimensional Alexandrov space with curvature bounded below/above. Note that in this case X as well as Y has a C^1 -local coordinate system in the sense of [OS] and [OT] defined almost everywhere.

Let $\mathcal{M}_+ \subset \mathcal{M}$ be the cone of positive measures. We say that a Lipschitz map $S : Y \rightarrow \mathcal{M}_+$ is a differentiable averaging operator if:

- (1) S is differentiable almost everywhere with respect to the C^1 -local coordinate system defined almost everywhere; and
- (2) S is Γ -equivariant.

Now define $S_* : C^*(Y; \Gamma) \rightarrow C^*(Y; \Gamma)$ by $S_*(c) = S_1^*(\Phi(c))$, where $S_1(y) = S(y)/\|S(y)\|$ is the normalized map. Explicitly,

$$(S_*(c))(y_0, \dots, y_m) = \int \cdots \int c(z_0, \dots, z_m) \frac{S(y_0)(z_0) \cdots S(y_m)(z_m)}{\|S(y_0)\| \cdots \|S(y_m)\|}.$$

As remarked in [G], S_* is a cochain map and induces the identity on $H^*(Y; \Gamma)$. Obviously, $\|S_*(c)\|_\infty \leq \|c\|_\infty$. Given a differentiable averaging operator $S : Y \rightarrow \mathcal{M}_+$ and $y \in Y$, we put

$$\begin{aligned} \mathcal{L}S_y &= \limsup_{z \rightarrow y} \frac{\|S(y) - S(z)\|}{|y, z|}, \\ \mathcal{L}[S]_y &= \frac{\mathcal{L}S_y}{\|S(y)\|}, \quad \mathcal{L}[S] = \sup_{y \in Y} [S]_y. \end{aligned}$$

Next, we exhibit an example of differentiable averaging operators. Let X be an n -dimensional Alexandrov space with curvature $K \geq -\kappa^2$ and $\pi : Y \rightarrow X$, as above. We have the following volume comparison theorem of the Bishop and Gromov type:

$$\frac{\text{vol } B_y(R)}{\text{vol } B_y(r)} \leq \frac{v_\kappa^n(R)}{v_\kappa^n(r)} \quad (\text{A.2})$$

for every $y \in Y$ and $R \geq r > 0$, where $v_\kappa^n(r)$ denotes the volume of an r -ball in $M_{-\kappa^2}^n$, the simply connected complete space with constant curvature $-\kappa^2$ (see [BGP], [Y]).

For $R > 0$, consider the map $S_R : Y \rightarrow \mathcal{M}_+$ defined by

$$S_R(y) = \begin{cases} \text{the Hausdorff } n\text{-measure} & \text{on } B_y(R) \\ 0 & \text{on } Y - B_y(R). \end{cases}$$

Namely, if $s_R : Y \times Y \rightarrow \mathbf{R}$ is given by

$$s_R(y, z) = \begin{cases} 1 & \text{if } d(y, z) \leq R \\ 0 & \text{if } d(y, z) > R, \end{cases}$$

then $S_R(y) = s(y, \cdot) d\mathcal{H}^n$. For every $\epsilon > 0$, define $S_{R,\epsilon} : Y \rightarrow \mathcal{M}_+$ by

$$S_{R,\epsilon}(y) = \frac{1}{\epsilon} \int_{R-\epsilon}^R S_{R'}(y) dR'.$$

PROPOSITION A.3. $S_{R,\epsilon}$ is a differentiable averaging operator satisfying

$$\lim_{\substack{R \rightarrow \infty, \\ \epsilon \rightarrow 0}} \sup \mathcal{L}[S_{R,\epsilon}] \leq (n-1)\kappa.$$

Proof. Obviously, $S_{R,\epsilon}$ is Γ -equivariant, and

$$\|S_{R,\epsilon}(y)\| = \frac{1}{\epsilon} \int_{R-\epsilon}^R \text{vol } B_y(R') dR'.$$

Since

$$\begin{aligned} \frac{\|S_{R,\epsilon}(y) - S_{R,\epsilon}(z)\|}{|y, z|} &\leq \frac{1}{\epsilon} \int_{R-\epsilon}^R \frac{\|S_{R'}(y) - S_{R'}(z)\|}{|y, z|} dR' \\ &\leq \frac{1}{\epsilon} \int_{R-\epsilon}^R \frac{v_\kappa^n(R' + |y, z|) - v_\kappa^n(R')}{|y, z|} dR', \end{aligned}$$

$S_{R,\epsilon}$ is Lipschitz. Next we show that $S_{R,\epsilon}$ is differentiable almost everywhere. For every $y \in Y$ and direction $\xi \in \Sigma_y$, the directional derivative of $S_{R,\epsilon}$ is given by

$$dS_{R,\epsilon}(\xi) = \int_{R-\epsilon \leq |y,z| \leq R} \max_{\eta} \cos \langle \xi, \eta \rangle d\mathcal{H}^n(z),$$

where ξ runs over all directions of minimal geodesics joining y to z (see [BGP]). By [OS], the set of points z for which minimal geodesics joining y to z are not unique has measure zero with respect to the Hausdorff n -measure. Thus

$$dS_{R,\epsilon}(\xi) = \int_{R-\epsilon \leq |y,z| \leq R} -\langle \nabla d_z, \xi \rangle d\mathcal{H}^n(z),$$

where ∇d_z denotes the gradient of the distance function d_z from z . Since the linear operator $dS_{R,\epsilon} : T_y(Y) \rightarrow \mathcal{M}$ is bounded, $S_{R,\epsilon}$ is differentiable at almost all points y . Furthermore,

$$\begin{aligned} \frac{\mathcal{L}(S_{R,\epsilon})_y}{\|S_{R,\epsilon}(y)\|} &= \limsup_{t \rightarrow 0} \frac{\left((1/\epsilon) \int_{R-\epsilon}^R (\text{vol } B_y(R' + t) - \text{vol } B_y(R'))/t \, dR' \right)}{\left((1/\epsilon) \int_{R-\epsilon}^R \text{vol } B_y(R') \, dR' \right)} \\ &\leq \limsup_{t \rightarrow 0} \frac{1}{\epsilon} \int_{R-\epsilon}^R \frac{\text{vol } B_y(R' + t) - \text{vol } B_y(R')}{t(\text{vol } B_y(R') + O(\epsilon))} \, dR' \\ &\leq \lim_{t \rightarrow 0} \frac{1}{\epsilon} \int_{R-\epsilon}^R \frac{v_\kappa^n(R' + t) - v_\kappa^n(R')}{t(v_\kappa^n(R') + O(\epsilon))} \, dR' \\ &= \frac{1}{\epsilon} \int_{R-\epsilon}^R \frac{s_\kappa^{n-1}(R')}{(v_\kappa^n(R') + O(\epsilon))} \, dR', \end{aligned}$$

where $s_\kappa^{n-1}(R)$ denotes the volume of a metric sphere of radius R in $M_{-\kappa^2}^n$. This completes the proof. $\square$

We denote by $C_\Gamma(Y)$ the complex of Γ -invariant singular cochains in Y . Note that $C^*(Y; \Gamma) \subset C_\Gamma^*(Y) \subset C^*(Y)$.

PROPOSITION A.4. *For a differentiable averaging operator $S : Y \rightarrow \mathcal{M}_+$, and for any bounded n -cocycle $c \in C^n(Y; \Gamma)$, there exists a Γ -invariant cocycle $\omega \in C_\Gamma^n(Y)$ such that:*

- (1) ω is cohomologous to c in $C_\Gamma^*(Y)$; and
- (2) $\text{comass}(\omega) \leq n! \|c\|_\infty (\mathcal{L}[S])^n$.

Proof. Let $\tilde{c}$ be the differential m -form on $\mathcal{M}$ such that

$$\tilde{c}(\mu_1, \dots, \mu_n) = n! \Phi(c)(\mu, \mu_1, \dots, \mu_n), \quad \mu_i \in T_\mu(\mathcal{M}).$$

Then for a linear simplex $\Delta(\mu, \mu_1, \dots, \mu_n) \subset \mathcal{M}$,

$$\int_{\Delta(\mu, \mu_1, \dots, \mu_n)} \tilde{c} = \Phi(c)(\mu, \mu_1, \dots, \mu_n).$$

Thus $\Phi(c)$ can be considered as the straightening of $\tilde{c}$. Let $\mathcal{M}_1$ be the unit sphere in $\mathcal{M}$ around the zero. It is easily verified that $\tilde{c}|_{\mathcal{M}_1}$ is closed. It follows that $\Phi(c)$ is cohomologous to $\tilde{c}$ in $C_\Gamma^*(\mathcal{M})$, the complex of Γ -invariant singular cochains in $\mathcal{M}$. Note that

$$\begin{aligned} \text{comass}(\tilde{c}) &= \sup_{\|\mu_i\|=1} |\tilde{c}(\mu_1, \dots, \mu_n)| \\ &= \sup \left| n! \int \cdots \int \Phi(c)(y, y_1, \dots, y_n) \mu(y) \mu_1(y_1) \cdots \mu_n(y_n) \right| \\ &\leq n! \|c\|_\infty. \end{aligned}$$

Consider the restriction $\tilde{c}|_{\mathcal{M}_1}$ as the Γ -invariant n -cycle in $\mathcal{M}_1$ by

$$\tilde{c}(\sigma) = \int_\sigma \tilde{c}$$

for Lipschitz simplices $\sigma : \Delta^n \rightarrow \mathcal{M}_1$. Then the induced cochain $\omega = S_1^*(\tilde{c}|_{\mathcal{M}_1})$ on Y is Γ -invariant. Pointing out that

$$\begin{aligned} c &\sim S_*(c) = S_1^*(\Phi(c)) \quad \text{in } C^*(Y; \Gamma) \\ &\sim S_1^*(\tilde{c}) = \omega \quad \text{in } C_\Gamma^*(Y), \end{aligned}$$

the proposition then follows from the following assertion.

ASSERTION A.5. $\text{comass}(\omega) \leq n! \|c\|_\infty (\mathcal{L}[S])^n$.

Proof. Let $\sigma : \Delta^n \rightarrow X$ be a singular n -simplex that is bi-Lipschitz homeomorphic. Observe that $S_1 \circ \sigma$ is differentiable almost everywhere. Let $\{e_i\}$ be an orthonormal frame on (Δ^m, g) , and let $\sigma_*(e_i) = \Sigma_j a_{ij} \xi_j$, where $\{\xi_j\}$ is an orthonormal basis of the tangent space $T_\sigma(Y)$ defined almost everywhere. Since $\tilde{c}$ vanishes in the radial direction, it follows that

$$\begin{aligned}
|\tilde{c}(d(S_1 \circ \sigma)(e_1), \dots, d(S_1 \circ \sigma)(e_n))| &= |\det(a_{ij})| |\tilde{c}(dS(\xi_1), \dots, dS(\xi_n))| / \|S(y)\|^n \\
&\leq |\det(a_{ij})| \left(\frac{\|\mathcal{L}S\|}{\|S\|} \right)^n \text{comass}(\tilde{c}) \\
&\leq n! |\det(a_{ij})| \|c\|_\infty (\mathcal{L}[S])^n,
\end{aligned}$$

and hence,

$$\begin{aligned}
|\omega(\sigma)| &\leq \int_{\Delta^n} |\tilde{c}(d(S_1 \circ \sigma)(e_1), \dots, d(S_1 \circ \sigma)(e_n))| dx^1 \cdots dx^n \\
&\leq \text{mass}(\sigma) n! \|c\|_\infty (\mathcal{L}[S])^n.
\end{aligned}$$

For any Lipschitz n -simplex σ , approximate it as in Lemma 2.3 and apply the above argument. This completes the proof. $\square$

From Propositions A.3 and A.4, we immediately have the following.

THEOREM A.6. *Let X be an n -dimensional Alexandrov space with curvature $K \geq -\kappa^2$ satisfying the LLC-condition. Then for any $\beta \in H^n(X)$ and for any $\alpha \in H_n(X)$, we have:*

- (1) $\text{comass}(\beta) \leq n!((n-1)\kappa)^n \|\beta\|_\infty$;
- (2) $\|\alpha\| \leq n!((n-1)\kappa)^n \text{mass}(\alpha)$.

Now suppose further that X is compact and without boundary. Then it is an n -dimensional pseudomanifold (see [BGP] and [P] for details). Thus, applying Theorem A.6 to the fundamental class of X , we complete the proof of Theorem 0.5.

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