THE RELATIVE LUBIN–TATE FORMAL GROUPS AND p -ADIC INTERPOLATING FUNCTIONS†

Kazuhito KOZUKA

(Received 9 February 1996 and revised 16 December 1996)

1. Introduction

Let p be a prime number, $\mathbf{Q}_p$ the rational p -adic number field and $\mathbf{Z}_p$ the ring of integers of $\mathbf{Q}_p$.

Let $F(X, Y) \in \mathbf{Z}_p[[X, Y]]$ be a Lubin–Tate formal group and denote by $\lambda_F(X)$ and $e_F(X)$ respectively the logarithmic series and the exponential series of $F(X, Y)$ with $\lambda'_F(0) = e'_F(0) = 1$. Further, we denote the completion of the algebraic closure of $\mathbf{Q}_p$ by $\mathbf{C}_p$ and the ring of integers of $\mathbf{C}_p$ by $\mathcal{O}$ and take any meromorphic series $h(X) \in \mathcal{O}((X))^\times$ with some unit as the leading coefficient.

In order to explain many well-known p -adic interpolating functions in a unified manner, Shiratani and Imada [6] defined the numbers $B_n(F, h)$ by

$$\frac{Xh'(e_F(X))}{\lambda'_F(e_F(X))h(e_F(X))} = \sum_{n=0}^{\infty} \frac{B_n(F, h)}{n!} X^n \quad (1.1)$$

and constructed a p -adic interpolating function $\zeta_p(s, F, h)$. For example, if $F(X, Y)$ is the formal multiplicative group $\mathbf{G}_m(X, Y) = (X+1)(Y+1) - 1$ and if $h(X) = X$, then $B_n(F, h) = B_n$, the usual n th Bernoulli number and $\zeta_p(s, F, h)$ coincides with the Kubota–Leopoldt p -adic zeta function. In the case where $F = \mathbf{G}_m$ and $h(X) = X - u + 1$ with $u - 1 \in \mathcal{O}^\times$, $B_n(F, h)$ can be expressed by using Euler numbers $H^n(u)$ belonging to u and $-\zeta_p(s, F, h)$ is equal to the p -adic interpolating function $G_p(s, u)$ for the Euler numbers constructed by Shiratani and Yamamoto [7]. By choosing $F(X, Y)$ and $h(X)$ suitably, the function $\zeta_p(s, F, h)$ also becomes the p -adic zeta function associated with elliptic curves defined over $\mathbf{Z}$ with complex multiplication with ordinary reduction (Lichtenbaum [4]).

In this paper, we extend the result above in the following manner. We suppose that $F(X, Y)$ is a relative Lubin–Tate formal group defined over the ring of integers of any finite unramified extension of $\mathbf{Q}_p$. Note that the definition (1.1) of $B_n(F, h)$

† Dedicated to Professor Katsumi Shiratani on his sixtieth birthday.

is also valid in this case. For any primitive Dirichlet character χ with conductor a power of p , we define the numbers $B_{n,\chi}(F, h)$ as analogues to the generalized Bernoulli numbers and construct a p -adic interpolating function $L_p(s, \chi, F, h)$, which is an extension of $\zeta_p(s, F, h)$. Finally, we calculate the value $L_p(1, \chi, F, h)$ and deduce a natural generalization of Leopoldt's formula on the value at $s = 1$ of the Kubota–Leopoldt p -adic L -functions.

2. Notations and definitions

For each positive integer d , let k_d denote the unique unramified extension of $\mathbf{Q}_p$ of degree d and $\mathcal{J}_d$ the ring of integers of k_d . We denote by K the completion of the maximal unramified extension $\bigcup_{d=1}^{\infty} k_d$ of $\mathbf{Q}_p$ and by $\mathcal{J}$ the ring of integers of K . Further, we write ψ for the Frobenius automorphism of K over $\mathbf{Q}_p$.

Let ξ be an element of $p^d \mathbf{Z}_p - p^{d+1} \mathbf{Z}_p$ and consider the set

$$\begin{aligned} \mathcal{F}_\xi = \{f(X) \in \mathcal{J}_d[[X]] \mid f(X) \equiv \pi X \pmod{\deg 2}, N_{k_d/\mathbf{Q}_p}(\pi) = \xi \\ \text{and } f(X) \equiv X^p \pmod{p\mathcal{J}}\}. \end{aligned}$$

For each $f(X) \in \mathcal{F}_\xi$, there exists a unique formal group $F_f(X, Y) \in \mathcal{J}_d[[X, Y]]$ such that f is a homomorphism from F_f to F_f^ψ and it is called the relative Lubin–Tate formal group associated with f (de Shalit [1], see also [2] and [5]). Note that f^ψ is also in $\mathcal{F}_\xi$ and that $F_f^\psi = F_{f^\psi}$. In what follows, we fix a power series $f(X) \in \mathcal{F}_\xi$, put $f'(0) = \pi$ and write $F(X, Y)$ or $X +_F Y$ for $F_f(X, Y)$. As in the Introduction, we denote by $\lambda_F(X)$ and $e_F(X)$, respectively, the logarithmic series and the exponential series of $F(X, Y)$ with $\lambda'_F(0) = e'_F(0) = 1$. Further, choose and fix an element Ω of $\mathcal{J}^\times$ such that

$$\Omega^\psi / \Omega = p / \pi. \quad (2.1)$$

Then, there is a unique isomorphism $\phi_F : \mathbf{G}_m \rightarrow F$ over $\mathcal{J}$ such that $\phi'_F(0) = \Omega^{-1}$ and we have

$$(f \circ \phi_F)(T) = \phi_F^\psi((T + 1)^p - 1) \quad (2.2)$$

(de Shalit [1, 1.6 and 3.2]). We also have

$$\phi_F(e^Z - 1) = e_F(\Omega^{-1} Z) \quad (2.3)$$

and

$$(\lambda_F \circ \phi_F)(T) = \Omega^{-1} \log(1 + T). \quad (2.4)$$

In addition, ϕ_F^ψ is an isomorphism from $\mathbf{G}_m$ to F^ψ over $\mathcal{J}$ with $(\phi_F^\psi)'(0) = (\Omega^\psi)^{-1}$ and it is reasonable to put

$$\phi_{F^\psi}(T) = \phi_F^\psi(T). \quad (2.5)$$

Next, let χ be a Dirichlet character with conductor p^{m_χ} and fix a primitive p^{m_χ} th root of unity ζ_χ arbitrarily. We put $\tau(\chi^{-1}, \zeta_\chi) = \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) \zeta_\chi^a$, the Gauss sum attached to χ^{-1} and ζ_χ . For any meromorphic series $h(X) \in \mathcal{O}((X))^\times$, we put

$$G_{F,h,\chi}(Z) = \frac{1}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \frac{\chi^{-1}(a) Z h'(\phi_F(\zeta_\chi^a - 1) + {}_F e_F(Z))}{\lambda'_F(\phi_F(\zeta_\chi^a - 1) + {}_F e_F(Z)) h(\phi_F(\zeta_\chi^a - 1) + {}_F e_F(Z))}. \quad (2.6)$$

Note that $G_{F,h,\chi}(Z)$ depends on Ω in the case of $\chi \neq 1$ but is independent of the choice of ζ_χ in any case. Since $h'(X)/h(X)$ is in $(1/X)\mathcal{O}[[X]]$, $G_{F,h,\chi}(Z)$ is in $\mathbf{C}_p[[Z]]$. Note that we can also express (2.6) as

$$G_{F,h,\chi}(Z) = \frac{1}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \frac{\chi^{-1}(a) Z \{h(\phi_F(\zeta_\chi^a - 1) + {}_F e_F(Z))\}'}{h(\phi_F(\zeta_\chi^a - 1) + {}_F e_F(Z))}. \quad (2.6')$$

Now we define the numbers $B_{n,\chi}(F, h)$ by

$$G_{F,h,\chi}(Z) = \sum_{n=0}^{\infty} \frac{B_{n,\chi}(F, h)}{n!} Z^n. \quad (2.7)$$

If $\chi = 1$, then $B_{n,\chi}(F, h) = B_n(F, h)$. In the case $\xi = p$, $f(X) = (X+1)^p - 1$, $\Omega = 1$ and $h(X) = X$, we have $F = \mathbf{G}_m$ and we can deduce

$$G_{F,h,\chi}(Z) = \sum_{a=1}^{p^{m_\chi}} \frac{\chi(a) Z e^{aZ}}{e^{p^{m_\chi} Z} - 1},$$

which means $B_{n,\chi}(\mathbf{G}_m, X) = B_{n,\chi}$, the usual n th generalized Bernoulli number for χ . For any $u \in \mathbf{C}_p$ with $1/(1-u) \in \mathcal{O}$, let $H_\chi^n(u)$ be the n th generalized Euler number for χ belonging to u defined by formula (5) of Tsumura [8]. Then, for any $aX + b \in \mathcal{O}[X]$ with $b \in \mathcal{O}^\times$, if we put $u = 1 - (b/a)$, then $1/(1-u) \in \mathcal{O}$ and we have $B_{0,\chi}(\mathbf{G}_m, aX + b) = 0$ and

$$\frac{B_{n,\chi}(\mathbf{G}_m, aX + b)}{n} = \frac{u}{1 - u^{p^{m_\chi}}} H_\chi^{n-1}(u)$$

for any $n \geq 1$ except for the case where $n = 1$ and $\chi = 1$ and in this case, we have

$$B_{1,\chi}(\mathbf{G}_m, aX + b) = \frac{1}{1 - u}.$$

3. p -adic interpolation

In order to construct a p -adic interpolating function of the numbers $B_{n,\chi}(F, h)$, we first recall some of the basic properties of the p -adic Γ -transform.

Let $\mathcal{M}$ denote the ring of $\mathcal{O}$ -valued measures on $\mathbf{Z}_p$. For each $g(T) \in \mathcal{O}[[T]]$, there exists a unique measure $\mu_g \in \mathcal{M}$ satisfying

$$g(T) = \sum_{n=0}^{\infty} \left(\int_{\mathbf{Z}_p} \binom{x}{n} d\mu_g(x) \right) T^n, \quad \text{where } \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!},$$

the binomial coefficient function. The map $g(T) \rightarrow \mu_g$ gives an isomorphism of $\mathcal{O}[[T]]$ with $\mathcal{M}$. Put $e_0 = 2$ or $p-1$ according as $p = 2$ or $p \geq 3$. For each $g(T) \in \mathcal{O}[[T]]$ and each integer i modulo e_0 , a Γ -transform

$$\Gamma_g^{(i)} : \mathbf{Z}_p \rightarrow \mathcal{O}$$

is defined by

$$\Gamma_g^{(i)}(s) = \int_{\mathbf{Z}_p^\times} \langle x \rangle^s \omega^i(x) d\mu_g(x), \quad (3.1)$$

where ω is the Teichmüller character for p and $\langle x \rangle = x/\omega(x)$ for any $x \in \mathbf{Z}_p^\times$. Fix a topological generator γ of the multiplicative group $1 + 2p\mathbf{Z}_p$. We can define a homomorphism $\ell : \mathbf{Z}_p^\times \rightarrow \mathbf{Z}_p$ by $\langle x \rangle = \gamma^{\ell(x)}$ for any $x \in \mathbf{Z}_p^\times$ and if we put

$$g^{(i)}(T) = \sum_{n=0}^{\infty} \left(\int_{\mathbf{Z}_p^\times} \binom{\ell(x)}{n} \omega^i(x) d\mu_g(x) \right) T^n \in \mathcal{O}[[T]],$$

we have

$$\Gamma_g^{(i)}(s) = g^{(i)}(\gamma^s - 1). \quad (3.2)$$

We put $\tilde{g}(T) = g(T) - (1/p) \sum_{\zeta \neq 1} g(\zeta(1+T) - 1)$, which is also in $\mathcal{O}[[T]]$. Then, $\mu_{\tilde{g}}$ is the measure on $\mathbf{Z}_p$ obtained by restricting μ_g to $\mathbf{Z}_p^\times$ and extending by 0. Note that we have $g^{(i)}(T) = \tilde{g}^{(i)}(T)$.

As in the previous section, let χ be a primitive Dirichlet character with conductor p^{m_χ} . For each $g(T) \in \mathcal{O}[[T]]$, we put

$$g_\chi(T) = \frac{1}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) g(\zeta_\chi^a(1+T) - 1), \quad (3.3)$$

which is independent of the choice of ζ_χ and lies in $\mathcal{O}[[T]]$. Let D be the operator $(1+T)d/dT$. Then, for each integer $n \geq 0$, we have

$$\Gamma_g^{(i)}(n) = (D^n g_{\omega^{i-n}})(0) = \left(\frac{d}{dZ}\right)^n (g_{\omega^{i-n}}(e^Z - 1))|_{Z=0}. \quad (3.4)$$

If we write $(D^{-1}g)(T)$, for any power series in $\mathbf{C}_p[[T]]$ such that $D(D^{-1}g)(T) = g(T)$, then

$$\int_{\mathbf{Z}_p^\times} x^{-1} d\mu_g(x) = (D^{-1}g)(0) - \frac{1}{p} \sum_{\zeta^p=1} (D^{-1}g)(\zeta - 1). \quad (3.5)$$

If φ is a Dirichlet character of the second kind for p , then

$$(g_\varphi)_{\omega^j}(T) = g_{\varphi\omega^j}(T) \quad (3.6)$$

and

$$(g_{\varphi\omega^j})^{(i)}(T) = g^{(i+j)}(\varphi(\gamma)(1+T) - 1) \quad (3.7)$$

for any integer j modulo e_0 .

Now, let $f(X) \in \mathcal{F}_\xi$, $F(X, Y) \in \mathcal{J}_d[[X, Y]]$, $\phi_F : \mathbf{G}_m \rightarrow F$ and $h(X) \in \mathcal{O}((X)^\times)$ be as in the previous section and we put

$$g_{F,h}(T) = \frac{h'(\phi_F(T))}{\lambda'_F(\phi_F(T))h(\phi_F(T))}, \quad (3.8)$$

which is in $(1/T)\mathcal{O}[[T]]$. By virtue of (2.4), equation (3.8) is also expressed as

$$g_{F,h}(T) = \Omega \cdot D \log\{h(\phi_F(T))\}. \quad (3.8')$$

For each $c \in \mathbf{Z}_p^\times$, we put

$$g_{F,h,c}(T) = c g_{F,h}((1+T)^c - 1) - g_{F,h}(T). \quad (3.9)$$

Then, $g_{F,h,c}(T) \in \mathcal{O}[[T]]$. Let χ be as before. We can express uniquely

$$\chi = \varphi_\chi \omega^{i_\chi}$$

with a character φ_χ of the second kind for p and an integer i_χ modulo e_0 . Let $\mathcal{N}_F$ denote Coleman's norm operator, that is, $\mathcal{N}_F h(X) \in \mathcal{O}((X)^\times)$ is a uniquely determined power series satisfying

$$\mathcal{N}_F h(f(X)) = \prod_{\alpha \in W_f} h(X + {}_F\alpha), \quad (3.10)$$

where $W_f = \{\alpha \in \mathcal{O} \mid f(\alpha) = 0\}$ (de Shalit [1, 2.1]).

Further, we put $\chi_n = \chi \omega^{-n}$ for each integer n .

PROPOSITION 3.1. *Let $c \in \mathbf{Z}_p^\times$. Then, for each integer $n \geq 1$, we have*

$$(g_{F,h,c})^{(i_\chi-1)}(\varphi_\chi(\gamma)\gamma^{n-1} - 1) \\ = (\chi_n(c)c^n - 1)\Omega^{1-n} \left\{ \frac{B_{n,\chi_n}(F,h)}{n} - \frac{\chi_n(p)\pi^n}{p} \cdot \frac{B_{n,\chi_n}(F^\psi, \mathcal{N}_F h)}{n} \right\}.$$

In order to prove this, we use the following lemma, which is a generalization of Lemma 7 of Shiratani and Imada [6].

LEMMA 3.2. *We have*

$$\frac{\pi(\mathcal{N}_F h)'(f(X))}{\lambda'_{F^\psi}(f(X))\mathcal{N}_F h(f(X))} = \sum_{\alpha \in W_f} \frac{h'(X + F\alpha)}{\lambda'_F(X + F\alpha)h(X + F\alpha)}.$$

Proof. By logarithmic differentiation of formula (3.10), we have

$$\frac{(\mathcal{N}_F h)'(f(X))f'(X)}{(\mathcal{N}_F h)(f(X))} = \sum_{\alpha \in W_f} \frac{h'(X + F\alpha)}{h(X + F\alpha)} \cdot \frac{\partial F}{\partial X}(X, \alpha).$$

Note that we have $\lambda_F(X + FY) = \lambda_F(X) + \lambda_F(Y)$ and $(\lambda_{F^\psi} \circ f)(X) = \pi \lambda_F(X)$. Hence, $\lambda'_F(X + F\alpha)(\partial F/\partial X)(X, \alpha) = \lambda'_F(X)$ for each $\alpha \in W_f$ and $\lambda'_{F^\psi}(f(X))f'(X) = \pi \lambda'_F(X)$. Consequently, we obtain the required equation. $\square$

Proof of Proposition 3.1. From (3.2), (3.4), (3.6) and (3.7), we deduce

$$(g_{F,h,c})^{(i_\chi-1)}(\varphi_\chi(\gamma)\gamma^{n-1} - 1) = \Gamma_{(g_{F,h,c})\varphi_\chi}^{(i_\chi-1)}(n-1) \\ = \left(\frac{d}{dZ}\right)^{n-1} ((g_{F,h,c})_{\chi_n}(e^Z - 1))|_{Z=0}. \quad (3.11)$$

In the case $\chi_n = 1$, we have

$$(g_{F,h,c})_{\chi_n}(T) = c g_{F,h}((1+T)^c - 1) - g_{F,h}(T) \\ - 1/p \sum_{\zeta^p=1} \{c g_{F,h}(\zeta^c(1+T)^c - 1) - g_{F,h}(\zeta(1+T) - 1)\}.$$

From (3.8) and Lemma 3.2 and from the fact that $W_f = \{\phi_F(\zeta - 1) \mid \zeta^p = 1\}$, we see

$$\sum_{\zeta^p=1} g_{F,h}(\zeta(1+T) - 1) = \sum_{\zeta^p=1} \frac{h'(\phi_F(\zeta(1+T) - 1))}{\lambda'_F(\phi_F(\zeta(1+T) - 1)h(\phi_F(\zeta(1+T) - 1))}$$

$$\begin{aligned}
&= \sum_{\alpha \in W_f} \frac{h'(\phi_F(T) + {}_F\alpha)}{\lambda'_F(\phi_F(T) + {}_F\alpha)h(\phi_F(T) + {}_F\alpha)} \\
&= \frac{\pi(\mathcal{N}_F h)'((f \circ \phi_F)(T))}{\lambda'_{F^\psi}((f \circ \phi_F)(T))\mathcal{N}_F h((f \circ \phi_F)(T))}.
\end{aligned}$$

From (2.2), (2.5) and (3.8), it follows that

$$\begin{aligned}
\sum_{\zeta^p=1} g_{F,h}(\zeta(1+T)-1) &= \frac{\pi(\mathcal{N}_F h)'(\phi_{F^\psi}((1+T)^p-1))}{\lambda'_{F^\psi}(\phi_{F^\psi}((1+T)^p-1))(\mathcal{N}_F h)(\phi_{F^\psi}((1+T)^p-1))} \\
&= \pi g_{F^\psi, \mathcal{N}_F h}((1+T)^p-1).
\end{aligned}$$

Hence, we also have

$$\sum_{\zeta^p=1} g_{F,h}(\zeta^c(1+T)^c-1) = \pi g_{F^\psi, \mathcal{N}_F h}((1+T)^{cp}-1).$$

Consequently,

$$\begin{aligned}
(g_{F,h,c})_{\chi_n}(T) &= c g_{F,h}((1+T)^c-1) - g_{F,h}(T) \\
&\quad - \frac{\pi}{p} \{c g_{F^\psi, \mathcal{N}_F h}((1+T)^{cp}-1) - g_{F^\psi, \mathcal{N}_F h}((1+T)^p-1)\}.
\end{aligned}$$

In addition, from (2.3) and (3.8), we have

$$g_{F,h}(e^Z-1) = \frac{h'(e_F(\Omega^{-1}Z))}{\lambda'_F(e_F(\Omega^{-1}Z))h(e_F(\Omega^{-1}Z))}.$$

Hence, from (1.1) and (2.1), we obtain

$$\begin{aligned}
&(g_{F,h,c})_{\chi_n}(e^Z-1) \\
&= \sum_{n=1}^{\infty} (c^n-1) \left\{ \Omega^{1-n} B_n(F, h) - \frac{\pi}{p} (\Omega^\psi)^{1-n} p^{n-1} B_n(F^\psi, \mathcal{N}_F h) \right\} \frac{Z^{n-1}}{n!} \\
&= \sum_{n=1}^{\infty} (c^n-1) \Omega^{1-n} \left\{ B_n(F, h) - \frac{\pi^n}{p} B_n(F^\psi, \mathcal{N}_F h) \right\} \frac{Z^{n-1}}{n!}
\end{aligned}$$

and our assertion follows from (3.11).

Next, suppose that $\chi_n \neq 1$. Then a simple calculation shows

$$\sum_{\zeta^p=1} (g_{F,h,c})_{\chi_n}(\zeta(1+T)-1) = 0.$$

Hence, we have

$$\begin{aligned}
 & (\widetilde{g_{F,h,c}})_{\chi_n}(T) \\
 &= (g_{F,h,c})_{\chi_n}(T) \\
 &= \frac{1}{\tau(\chi_n^{-1}, \zeta_{\chi_n})} \sum_{a=1}^{p^m \chi_n} \chi_n^{-1}(a) g_{F,h,c}(\zeta_{\chi_n}^a (1+T) - 1) \\
 &= \frac{1}{\tau(\chi_n^{-1}, \zeta_{\chi_n})} \sum_{a=1}^{p^m \chi_n} \chi_n^{-1}(a) \{c g_{F,h}(\zeta_{\chi_n}^{ca} (1+T)^c - 1) - g_{F,h}(\zeta_{\chi_n}^a (1+T) - 1)\}.
 \end{aligned}$$

Since $\phi_F(\zeta_{\chi_n}^{ca} (1+T)^c - 1) = \phi_F(\zeta_{\chi_n}^{ca} - 1) + {}_F\phi_F((1+T)^c - 1)$, we deduce from (2.3), (2.6), (2.7) and (3.8) that

$$\begin{aligned}
 (g_{F,h,c})_{\chi_n}(e^Z - 1) &= \frac{\Omega}{Z} \{ \chi_n(c) G_{F,h,\chi_n}(\Omega^{-1} c Z) - G_{F,h,\chi_n}(\Omega^{-1} Z) \} \\
 &= \sum_{n=1}^{\infty} (\chi_n(c) c^n - 1) \Omega^{1-n} B_{n,\chi_n}(F, h) \frac{Z^{n-1}}{n!}.
 \end{aligned}$$

Therefore, from (3.11), our assertion also follows in the case $\chi_n \neq 1$. $\square$

For each $c \in \mathbf{Z}_p^\times$ and each integer i modulo e_0 , put

$$\mathcal{G}_{F,h,c}^{(i)} = (g_{F,h,c})^{(i-1)}(\gamma^{-1}(1+T) - 1)$$

and if $\omega^i(c)(1+T)^{\ell(c)} - 1 \neq 0$, we put

$$\mathcal{G}_{F,h}^{(i)}(T) = \mathcal{G}_{F,h,c}^{(i)} / \Omega(\omega^i(c)(1+T)^{\ell(c)} - 1).$$

Then, Proposition 3.1 shows that $\mathcal{G}_{F,h}^{(i)}(T)$ is independent of c . Moreover, $\mathcal{G}_{F,h}^{(i)}(T)$ is in $\mathcal{O}[[T]]$ or $(1/T)\mathcal{O}[[T]]$ according to whether $i \not\equiv 0 \pmod{e_0}$ or $i \equiv 0 \pmod{e_0}$. Now, we define a function

$$L_p(s, \chi, F, h) = -\mathcal{G}_{F,h}^{(i_\chi)}(\varphi_\chi(\gamma)\gamma^{1-s} - 1),$$

which is analytic on $\mathbf{Z}_p$ except possibly for $s = 1$, at which it has at most a pole.

From Proposition 3.1 and from the construction of $L_p(s, \chi, F, h)$, we immediately obtain the following.

THEOREM 3.3. *For each integer $n \geq 1$, we have*

$$L_p(1-n, \chi, F, h) = -\Omega^{-n} \left\{ \frac{B_{n,\chi_n}(F, h)}{n} - \frac{\chi_n(p)\pi^n}{p} \cdot \frac{B_{n,\chi_n}(F^\psi, \mathcal{N}_F h)}{n} \right\}.$$

Example. (1) If $F = \mathbf{G}_m$ and if we take $\phi_F(X) = X$, then we have $L_p(s, \chi, \mathbf{G}_m, X) = L_p(s, \chi)$, the Kubota–Leopoldt p -adic L -function.

(2) Let $h(X) = aX + b$ with $a \in \mathcal{O}$ and $b \in \mathcal{O}^\times$ and put $u = 1 - (b/a)$. Then, under the same assumption as in (1), we have $-L_p(s, \chi\omega, \mathbf{G}_m, h) = l_p(u, s, \chi)$, the p -adic interpolating function of the generalized Euler numbers constructed by Tsumura ([8], see also [3]).

4. Calculation of $L_p(1, \chi, F, h)$

In this section, we calculate the value at $s = 1$ of $L_p(s, \chi, F, h)$ in the case $\chi \neq 1$.

THEOREM 4.1. *If $\chi \neq 1$, then*

$$L_p(1, \chi, F, h) = -\frac{1}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) \log_p \{h(\phi_F(\zeta_\chi^a - 1))\}.$$

Proof. From the construction of $L_p(s, \chi, F, h)$, we have

$$L_p(1, \chi, F, h) = -(g_{F,h,c})^{(i_\chi-1)}(\varphi_\chi(\gamma)\gamma^{-1} - 1)/\Omega(\chi(c) - 1) \quad (4.1)$$

for any $c \in \mathbf{Z}_p^\times$ with $\chi(c) \neq 1$. From (3.1), (3.2) and (3.7), we see

$$\begin{aligned} (g_{F,h,c})^{(i_\chi-1)}(\varphi_\chi(\gamma)\gamma^{-1} - 1) &= ((g_{F,h,c})_\chi)^{(-1)}(\gamma^{-1} - 1) \\ &= \Gamma_{(g_{F,h,c})_\chi}^{(-1)}(-1) \\ &= \int_{\mathbf{Z}_p^\times} x^{-1} d\mu_{(g_{F,h,c})_\chi}(x). \end{aligned} \quad (4.2)$$

Further, from (3.3), (3.8) and (3.9), we deduce that

$$\begin{aligned} &(g_{F,h,c})_\chi(T) \\ &= \frac{1}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) g_{F,h,c}(\zeta_\chi^a(1+T) - 1) \\ &= \frac{1}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) \{c g_{F,h}(\zeta_\chi^{ac}(1+T)^c - 1) - g_{F,h}(\zeta_\chi^a(1+T) - 1)\} \\ &= \frac{\Omega}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) D \left\{ \log \frac{h(\phi_F(\zeta_\chi^{ac}(1+T)^c - 1))}{h(\phi_F(\zeta_\chi^a(1+T) - 1))} \right\}. \end{aligned}$$

It follows from (3.5) that

$$\begin{aligned}
 & \int_{\mathbf{Z}_p^\times} x^{-1} d\mu_{(g_{F,h,c})_\chi}(x) \\
 &= \frac{\Omega}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) \\
 & \quad \times \left\{ \log_p \frac{h(\phi_F(\zeta_\chi^{ac} - 1))}{h(\phi_F(\zeta_\chi^a - 1))} - \frac{1}{p} \sum_{\zeta^p=1} \log_p \frac{h(\phi_F(\zeta_\chi^{ac} \zeta^c - 1))}{h(\phi_F(\zeta_\chi^a \zeta - 1))} \right\} \\
 &= \frac{\Omega(\chi(c) - 1)}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a) \\
 & \quad \times \left\{ \log_p h(\phi_F(\zeta_\chi^a - 1)) - \frac{1}{p} \sum_{b=1}^p \log_p h(\phi_F(\zeta_\chi^{a+bp^{m_\chi-1}} - 1)) \right\}. \quad (4.3)
 \end{aligned}$$

A simple calculation shows that

$$\sum_{a=1}^{p^{m_\chi}} \sum_{b=1}^p \chi^{-1}(a) \log_p h(\phi_F(\zeta_\chi^{a+bp^{m_\chi-1}} - 1)) = 0.$$

Hence, combining (4.1), (4.2) and (4.3), we obtain the required equation. $\square$

REFERENCES

- [1] E. de Shalit. Iwasawa Theory of Elliptic Curves with Complex Multiplication. Academic Press, 1987.
- [2] K. Iwasawa. Local Class Field Theory. Oxford University Press, 1986.
- [3] K. Kozuka. On a p -adic interpolating power series of the generalized Euler numbers. J. Math. Soc. Japan. **42** (1990), 113–125.
- [4] S. Lichtenbaum. On p -adic L-functions associated to elliptic curves. Invent. Math. **56** (1980), 19–55.
- [5] J. Lubin and J. Tate. Formal complex multiplication in local fields. Ann. of Math. **81** (1965), 380–387.
- [6] K. Shiratani and T. Imada. The exponential series of the Lubin–Tate groups and p -adic interpolation. Mem. Fac. Sci., Kyushu Univ. **46** (1992), 351–365.
- [7] K. Shiratani and S. Yamamoto. On a p -adic interpolation of the Euler numbers and its derivatives. Mem. Fac. Sci., Kyushu Univ. **39** (1985), 113–125.
- [8] H. Tsumura. On a p -adic interpolation of the generalized Euler numbers and its applications. Tokyo J. Math. **10** (1987), 281–293.

*Kazuhito Kozuka
Department of Mathematics
Miyakonojo National College of Technology
Miyakonojo, Miyazaki 885
Japan*