SPECTRUM OF LAPLACIAN ON NON-COMPACT RIEMANNIAN MANIFOLDS

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1. Introduction

Let H be a Hilbert space and $A : H \rightarrow H$ a self-adjoint operator. According to the spectral theory $\lambda \in \mathbb{R}$ is called a regular point of A if $(\lambda I - A)$ exists and is a bounded linear operator and is called a spectrum point of A if λ is not a regular point. We denote the set of spectrum points of A by $\sigma(A)$. $\lambda \in \sigma(A)$ is called an essential spectrum point of A (which we denote by $\sigma_{\text{ess}}(A)$) if $\dim E(\lambda - \epsilon, \lambda + \epsilon) = +\infty$, for all $\epsilon > 0$, where $E(\lambda)$ is a projection valued measure on the real line such that

$$A = \int \lambda dE(\lambda).$$

So $\sigma(A) \setminus \sigma_{\text{ess}}(A) = \{\text{isolated eigenvalues of finite multiplicity}\}$.

Let M be an n -dimensional complete Riemannian manifold and let Δ be the Laplacian on M (with convention that $\Delta = d^2/dx^2$ on $\mathbb{R}^1$). It is well known that Δ has a unique extension from $C_0^\infty(M)$ to $L^2(M)$ and is negative semidefinite. So $\sigma(-\Delta) \subset [0, +\infty)$. We shall consider the case of non-compact Riemannian manifolds.

When $M = H^n(-c)$ is the hyperbolic space of constant negative sectional curvature $-c$, $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [(n-1)^2c/4, +\infty)$. McKean [M] proved that $\sigma_{\text{ess}}(-\Delta) \subset [(n-1)^2c/4, +\infty)$ for sectional curvatures bounded above by some negative constant $-c$. Donnelly and Li [DL] showed that $\sigma(-\Delta)$ is stable under the compact perturbations and $\sigma_{\text{ess}}(-\Delta) = \emptyset$ provided that either (i) M is simply connected and $\bar{K}(r) \rightarrow -\infty$, or (ii) $\dim M = 2$, $\pi_1(M)$ is finite generated, and $\bar{K}(r) \rightarrow -\infty$, where $\bar{K}(r) = \sup\{K(x, \pi) \mid \text{dist}(x, p) \geq r, p \in M \text{ is a fixed point and } K(x, \pi) \text{ is the sectional curvature of a 2-plane } \pi \subset T_x M\}$. Donnelly [D1] proved that if the Ricci curvature of M is bounded below by $-(n-1)c$, then

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$\sigma_{\text{ess}}(-\Delta) \cap [0, (n-1)^2 c/4] \neq \emptyset$. Escobar and Freire [EF] considered the case of complete Riemannian manifolds of non-negative curvatures.

In this paper we first compute $\sigma_{\text{ess}}(-\Delta)$ for the complete non-compact Riemannian manifolds. Our results have some applications to Cartan–Hadamard manifolds, i.e. simply connected complete Riemannian manifolds of non-positive curvature. The following theorem will be proved in this paper.

THEOREM 3.2. *Let M be a complete Riemannian manifold with a pole and with Ricci curvature bounded below by $-A\rho^{-2}(x)$ for some positive constant $A < n/(n-1)$, where $\rho(x)$ is the distance function to the pole p_0 . Then $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, +\infty)$.*

Secondly, we show that L^2 eigenvalues of a Laplacian on M cannot happen when M satisfies some curvature conditions; for example we prove the following.

THEOREM 4.1. *Let M be an n -dimensional ($n \geq 3$) complete Riemannian manifold with a pole p_0 and infinite volume. Assume the sectional curvature of M satisfies*

$$-\frac{A}{\rho^2} \leq K(x) \leq \frac{A_1}{\rho^2}$$

for $\rho(x) = \text{dist}(x, p_0) > 0$, where A and $A_1 < \frac{1}{4}$ are non-negative constants satisfying $(n-2)\sqrt{1+4A} - n\sqrt{1-4A_1} \leq 2$. Then $-\Delta$ has no L^2 eigenvalues.

THEOREM 4.4. *Let M be a two-dimensional Cartan–Hadamard manifold with sectional curvatures satisfying*

$$K(x) \geq -\frac{A}{\rho^2(x)}$$

for $\rho(x) = \text{dist}(x, o) > 0$, where $A \leq 1$ is a non-negative constant. Then $-\Delta$ has no L^2 eigenvalues.

We can see from a theorem of Calabi [Ca] that $A_1 < \frac{1}{4}$ is reasonable and in Section 5 we will give a result similar to Calabi's theorem. The difference lies in that we need not impose the assumption that the Ricci curvature of manifold is non-negative.

2. The Hessian estimates of distance functions

We shall use Riccati matrix equations to obtain the estimates of the Hessian of the distance function $\rho(x)$. We first recall some useful facts from [E] for completeness.

Let E be a real n -dimensional vector space with Euclidean inner product $\langle \cdot, \cdot \rangle$. The space $S(E)$ of self-adjoint endomorphisms inherits the inner product

$$\langle A, B \rangle = \text{trace } A \cdot B \quad (2.1)$$

for $A, B \in S(E)$. We get a partial ordering $< (\leq)$ on $S(E)$ by putting $A < B$ ($A \leq B$) if $\langle Ax, x \rangle < \langle Bx, x \rangle$ ($\langle Ax, x \rangle \leq \langle Bx, x \rangle$) for every $x \in E \setminus \{0\}$. Now let $I \subset \mathbb{R}^1$ be an open interval and $R : I \rightarrow S(E)$ a smooth curve. We consider

$$Y'(t) + Y^2(t) + R(t) = 0. \quad (2.2)$$

A solution $Y(t)$ may have poles on I , i.e. some eigenvalues of $Y(t)$ may tend to $-\infty$ as $t \rightarrow t_1 - 0$, for some $t_1 \in I$ and $t_1 \neq +\infty$. Let $I = (t_-, t_+)$ with $-\infty \leq t_- < t_+ \leq +\infty$. We consider two smooth curves $R_1, R_2 : I \rightarrow S(E)$ and solutions Y_j ($j = 1, 2$) of the corresponding Riccati equations. Fix an initial point $t_0 \in I$ and let $t_j > t_0$ be the first pole of Y_j if there is some pole, otherwise put $t_j = t_+$. The following two propositions were proved in [E] and will be used in our proof.

PROPOSITION 2.1. *Let Y_1, Y_2 be the solutions of (2.2) corresponding to R_1, R_2 respectively. Assume that (a) $R_1(t) \geq R_2(t)$ for all $t \in I$, and (b) $Y_1(t_0) \leq Y_2(t_0)$. Then $t_1 \leq t_2$ and $Y_1(t) \leq Y_2(t)$ for $t_0 < t < t_1$.*

LEMMA 2.1. *$U : (0, T) \rightarrow S(E)$ is a solution of the Riccati equation (2.2) defined on a maximal interval with the asymptotes*

$$U(t) = t^{-1}I_E + O(t), \quad \text{as } t \rightarrow 0. \quad (2.3)$$

Then:

- (i) *if $R(t) \geq -(A/t^2)I_E$ for some constant $A > 0$, then $U(t) \leq (B_1/t)I_E$ on $(0, T)$, where $B_1 = (1 + \sqrt{1 + 4A})/2$;*
- (ii) *if $R(t) \geq -(A/t^2)I_E$ for some constant $A > 0$ and $T = +\infty$, then $U(t) + (A/t)I_E$ is positive semidefinite for all $t > 0$;*
- (iii) *if $R(t) \leq (A_1/t^2)I_E$ for $t \in (0, +\infty)$, where $0 < A_1 < \frac{1}{4}$, then U is defined on $(0, +\infty)$ and $U(t) \geq (B_2/t)I_E$ on $(0, +\infty)$, where $B_2 = (1 + \sqrt{1 - 4A_1})/2$.*

Proof. (i) Since $U_1(t) = (B_1/t)I_E$ is an exact solution of (2.2) with $R(t) = -(A/t^2)I_E$ and $B_1 > 1$, then the conclusion follows from Proposition 2.1 and (2.3).

To prove (ii), we only need to prove that $\langle U(t)v + (A/t)v, v \rangle \geq 0$ for any $v \in E$. First we shall prove that any global solution $f(t)$ of

$$\begin{cases} f' + f^2 + q(t) \leq 0 \\ f(t) = t^{-1} + O(t) \quad \text{as } t \rightarrow 0+ \end{cases} \quad (2.4)$$

satisfies $f(t) \geq -A/t$ if $q(t) \geq -A/t^2$. We claim that the global solution of (2.4) satisfies

$$\liminf_{t \rightarrow +\infty} f(t) \geq 0.$$

Otherwise $c := \liminf_{t \rightarrow +\infty} f(t) < 0$, if $c > -\infty$, and then we can find $t_0 > -4\sqrt{A}/c$ such that $f(t_0) < c/2$. Thus for $c_0 = 3c^2/16$ we can assume for the sake of contradiction that there exists $t_0 > 0$ such that $f(t_0) < 0$ and $f^2(t_0) - A/t_0^2 > c_0 > 0$. Whenever $t_1 > 0$ satisfies $f(t_1) < 0$ and $f^2(t_1) - A/t_1^2 > c_0 > 0$,

$$f'(t_1) \leq -f^2(t_1) - q(t_1) \leq -f^2(t_1) + \frac{A}{t_1^2} \leq -c_0 < 0.$$

Thus we can conclude that $f'(t) \leq -c_0 < 0$ if $t > t_0$, and then $f(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. Then there exists some constant $T_0 > \sqrt{A}$ such that $f^2(t) > 2$, $t > T_0$, and thus $\frac{1}{2}f^2(t) > 1 > A/t^2$ when $t > T_0$.

$$\begin{aligned} f'(t) &\leq -f^2(t) - q(t) \\ &\leq -f^2(t) + \frac{A}{t^2} \\ &= -\frac{1}{2}f^2(t) - \left(\frac{1}{2}f^2(t) - \frac{A}{t^2}\right) \leq -\frac{1}{2}f^2(t), \end{aligned}$$

then

$$\left(\frac{1}{f}\right)' \geq \frac{1}{2},$$

and for some large $T > 0$ and $t > T$

$$\frac{1}{f(t)} - \frac{1}{f(T)} \geq \frac{1}{2}(t - T);$$

this implies that

$$\frac{1}{f(t)} \geq \frac{1}{2}(t - T) + \frac{1}{f(T)} \quad \text{as } t \geq T_0.$$

So we have arrived at a contradiction. Since

$$f'(t) = -q(t) - f^2(t) \leq -q(t) \leq \frac{A}{t^2},$$

then

$$f(T) - f(t) \leq \frac{A}{t} - \frac{A}{T}$$

for any $T > t > 0$. Let $T \rightarrow +\infty$. So $f(t) \geq -A/t$.

For any fixed $v \in E$ with $\|v\| = 1$ we denote $f(t) = \langle v, U(t)v \rangle$. Then from (2.2)

$$f'(t) + f^2(t) + q(t) \leq 0,$$

where $q(t) = \langle v, K(t)v \rangle \geq -A/t^2$. From the discussion above we know that $f(t) \geq -A/t$. Then for any $v \in E$ we have

$$\langle v, U(t)v \rangle \geq -\frac{A}{t}\|v\|^2,$$

namely $U(t) \geq -(A/t)I_E$.

(iii) is essentially the same as the corresponding result in [EF]. $\square$

Let S be a soul of M such that the normal exponential map $\exp_S : NS \rightarrow M$ is a diffeomorphism (if $\dim(S) = 0$, S is a pole of M). Denote by $\rho(x) : M \rightarrow R$ the distance function to S . Then ρ is smooth and $|\nabla\rho| = 1$ on $M \setminus S$. A basic observation in comparison theory in [E] is the fact that the self-adjoint operator $U(x) : E_x \rightarrow E_x$ defined by $H\rho|_{E_x}$ is a solution of the matrix Riccati equation

$$U' + U^2 + K = 0, \tag{2.5}$$

where E_x is the tangent space to the level sets of ρ , U' denotes the covariant derivative of U in the direction $\nabla\rho$ and $K : E_x \rightarrow E_x$ is the radial curvature operator $K(X_T) = R(X_T, \nabla\rho)\nabla\rho$.

Any $x \in M$ may be connected to S by a unique minimizing geodesic $\gamma(t)$ whose initial vector $v \in N_p S$, $p \in S$. Thus, x defines a splitting $T_p M = T_0 \oplus N_0 \oplus Rv$, where $N_p S = N_0 \oplus Rv$ and $T_0 = T_p S$, the tangent space to S at p . We may extend this splitting by parallel transport along γ to obtain an orthogonal splitting

$$E_x = T_x \oplus N_x, \quad x \in M \setminus S,$$

where one of the factors may be trivial. Given $x \in M \setminus S$, we will often denote $U(x)$, E_x , T_x , N_x by $U(t)$, E_t , T_t , N_t respectively, where $t = r(x)$. As $t \rightarrow 0+$,

$U(t) = t^{-1}P_{N_0} + A \circ P_{T_0} + O(t)$, where P_{T_0} and P_{N_0} are orthogonal projections onto T_p and N_p respectively and $A \in S(E_p)$, $E_p = T_p \oplus N_p$. If S is a point, T_0 is trivial, so $E_0 = N_0$, and $U^{-1}(t) \rightarrow 0$ as $t \rightarrow 0+$. In general, $U(t)$ is singular at $t = 0$ along ‘ N_0 directions’ and regular along ‘ T_0 directions’. Let $\gamma : [0, +\infty) \rightarrow M$ be a ray of M , $\gamma(0) = p \in S$, $\dot{\gamma}(0) = v \in N_p S$. Consider an arbitrary embedded geodesic arc $\tau : (-\epsilon, \epsilon) \rightarrow S$, $\tau(0) = p$, and $\dot{\tau}(0) = w$. By [CE, 8.22(4)] the parallel transport $v(s)$ of v along τ gives rise to a ‘flat strip’ in M , i.e. a totally geodesic isometric embedding

$$\begin{aligned}\Phi &= \Phi(s, t) : (-\epsilon, \epsilon) \times (0, \infty) \rightarrow M \\ \Phi(s, t) &= \exp_{\tau(s)} t v(s)\end{aligned}$$

of a flat Euclidean strip into M . (Here, we use the assumption that γ is a ray of M .) The geodesic arcs $\tau_t(s) = \Phi(s, t)$ ($s \in (-\epsilon, \epsilon)$) are at a constant distance from $S : r(\tau_t(s)) \equiv t$; therefore,

$$Hr(\dot{\tau}_t(0), \dot{\tau}_t(0)) = \frac{d^2}{ds^2}(r(\tau_t(s))) \Big|_{s=0} = 0,$$

for all $t > 0$. Since $\dot{\tau}_t(0) \equiv w(t)$ is the parallel transport of w along $\gamma(t)$, we have $U(\gamma(t))w(t) \equiv 0$. This shows that $U|_{T_t} \equiv 0$. In addition, since S is totally geodesic, we have

$$U(t) = t^{-1}P_{N_0} + O(t).$$

Applying Lemma 2.1 we have the following.

LEMMA 2.2. *Let M^n ($n \geq 2$) be a complete non-compact Riemannian manifold with sectional curvatures $K \geq -A/\rho^2$. Assume M has a soul S such that $\exp_S : NS \rightarrow M$ is a diffeomorphism. Let $E_x = T_x \oplus N_x$ be the splitting of E_x described as above and denote by $H\rho$ the Hessian form of the distance function $\rho(x) = \text{dist}(x, S)$. Then:*

(a)

$$\begin{aligned}-\frac{A}{\rho}|X|^2 &\leq H\rho(X, X) \leq \frac{B_1}{\rho}|X|^2 \quad \forall X \in N_x, \\ H\rho(X, X) &= 0 \quad \forall X \in T_x;\end{aligned}$$

(b) if

$$-\frac{A}{\rho^2} \leq K \leq \frac{A_1}{\rho^2}$$

for some constants $A \geq 0$, $A_1 \in [0, \frac{1}{4})$, then

$$\begin{aligned} \frac{B_2}{\rho}|X|^2 \leq H\rho(X, X) &\leq \frac{B_1}{\rho}|X|^2 \quad \forall X \in N_x, \\ H\rho(X, X) &= 0 \quad \forall X \in T_x, \end{aligned}$$

where B_1 and B_2 are defined as in Lemma 2.1.

If $\dim\{S\} = 0$, i.e. S is a pole, we have the following.

LEMMA 2.3. Let M^n ($n \geq 2$) be a complete non-compact Riemannian manifold with a pole p_0 and with Ricci curvature bounded below by $-A/\rho^2$, where $\rho(x) = \text{dist}(x, p_0)$. Then

$$|\Delta\rho(x)| \leq \frac{C}{\rho(x)} \quad \forall x \in M \setminus \{p_0\},$$

where C is a constant depending on n and A .

Proof. Since $\Delta\rho = \text{trace}(H\rho) = \text{trace } U$, then

$$\text{trace } U^2 = \sum_{i,j} u_{ij}^2 \geq \sum_i u_{ii}^2 \geq \frac{1}{n-1} (\text{trace } U)^2,$$

and for any unit speed geodesic $\gamma(t)$ starting from p_0 , denote $f(t) = \Delta\rho(\gamma(t))$. From (2.5), we have

$$f'(t) + \frac{1}{n-1} f^2(t) + \text{Ric}(\dot{\gamma}(t)) \leq 0.$$

Thus,

$$f'(t) + \frac{1}{n-1} f^2(t) - \frac{A}{t^2} \leq 0.$$

Denote $g(t) = f/(n-1)$. Hence,

$$g'(t) + g^2(t) - \frac{A}{(n-1)t^2} \leq 0.$$

We know from the fact that p_0 is a pole that $g(t)$ is defined on $(0, +\infty)$. It follows from Lemma 2.1(i) and (ii) that there exist positive constants C_1, C_2 , such that

$$-\frac{C_1}{t} \leq g(t) \leq \frac{C_2}{t}.$$

This proves Lemma 2.3. □

3. Proof of the results on the essential spectrum

Throughout this section we will always assume that M is a complete Riemannian manifold with a soul S such that the exponential map of S is a diffeomorphism unless otherwise stated. We can define the distance function as $\rho(x) = \text{dist}(x, S)$ for any point $x \in M \setminus S$. It is well known that every point in a Cartan–Hadamard manifold is a pole and the distance function $\rho(x)$ is smooth in $M \setminus \{0\}$. The following lemma is straightforward from Lemma 2.2.

LEMMA 3.1. *If the sectional curvature of M is bounded below by $-A\rho^{-2}(x)$, then*

$$\Delta\rho(x) \leq \frac{B}{\rho(x)} \quad (3.1)$$

holds for $x \in M \setminus S$, where B is a constant depending only on A and n .

Denote by B_r the ball with radius r , i.e. $B_r = \{x \in M \mid \rho(x) \leq r\}$. The level sets of ρ , $S_r = \{x \in M \mid \rho(x) = r\}$, are smooth embedded hypersurfaces. We denote by $v(r)$ the volume of S_r and by $V(r)$ the volume of B_r . We want to prove the following lemma.

LEMMA 3.2. *Let M be a complete Riemannian manifold with a soul S such that the normal exponential map of S is a diffeomorphism. If the sectional curvature of M is bounded below by $-A\rho^{-2}(x)$, then*

$$\frac{V(a) - V(b)}{a^{B+1} - b^{B+1}} \geq \frac{V(c) - V(d)}{c^{B+1} - d^{B+1}} \quad (3.2)$$

holds for any constants $a > b \geq c > d > 0$, where B is the constant in Lemma 3.1.

Proof. Since

$$\begin{aligned} v'(r) &= \int_{S_r} \Delta\rho \, dS \\ &\leq \frac{B}{r} v(r), \end{aligned}$$

then

$$\left[\frac{v(r)}{r^B} \right]' \leq 0. \quad (3.3)$$

So we can find $\xi_1 \in [b, a]$ and $\xi_2 \in [d, c]$ such that

$$\frac{V(a) - V(b)}{a^{B+1} - b^{B+1}} = \frac{v(\xi_1)}{(B+1)\xi_1^B} \quad (3.4)$$

and

$$\frac{V(c) - V(d)}{c^{B+1} - d^{B+1}} = \frac{v(\xi_2)}{(B+1)\xi_2^B}. \quad (3.5)$$

Combining (3.3)–(3.5) we can get (3.2). $\square$

To show that a real number λ is in the essential spectrum of operator A we will use the following proposition.

PROPOSITION 3.1. **([D1, D2])** $\lambda \in \sigma_{\text{ess}}(A)$ if and only if there exists a sequence $\{f_j\} \subset H$, $j = 1, 2, \dots$, such that

$$\dim(\text{span}\{f_j, j = 1, 2, \dots\}) = +\infty$$

and

$$\frac{\|(A - \lambda I)f_j\|}{\|f_j\|} \rightarrow 0. \quad (3.6)$$

Now we are in a position to state and prove one of our theorems.

THEOREM 3.1. *Let M be a complete Riemannian manifold with a soul S such that the normal exponential map of S is a diffeomorphism. If the sectional curvature of M is bounded below by $-A\rho^{-2}(x)$ for some non-negative constant $A \in (0, n/(n-1)]$, where $\rho(x) = \text{dist}(x, S)$, then $\sigma_{\text{ess}}(-\Delta) = \sigma(-\Delta) = [0, +\infty)$.*

Before proving this theorem we give a corollary and a remark.

COROLLARY 3.1. *Let M be a complete Riemannian manifold with non-negative sectional curvature and with a soul S such that the normal exponential map of S is a diffeomorphism. Then $\sigma_{\text{ess}}(-\Delta) = \sigma(-\Delta) = [0, +\infty)$.*

Remark 3.1. In [EF] the conclusion of the above corollary was proved under the additional assumptions that M has non-negative sectional curvature and some integrable condition of Ricci curvature. The corollary was obtained in [Z] and says that the integrable condition is unnecessary. Li [L] proved the same result in the case when S is a pole. Recently, we were informed that Kumura [K] also obtained some related results.

Proof of Theorem 3.1. We shall prove that $(0, +\infty) \subset \sigma_{\text{ess}}(-\Delta)$ and then the conclusion of our theorem follows from the closeness of $\sigma(-\Delta)$ and the fact that $-\Delta$ is a semidefinite positive operator. For any $\lambda = -\eta^2 \in (0, +\infty)$ we need only show that $\lambda \in \sigma_{\text{ess}}(-\Delta)$, where η is a positive constant.

Let $\chi_j(t)$ be the smooth function on $[0, +\infty)$ such that

$$\chi_j(t) = \begin{cases} 0, & t \leq \frac{\pi}{\eta}, \\ 1, & \frac{2\pi}{\eta} \leq t \leq \frac{j\pi}{\eta}, \\ 0, & t \geq \frac{(j+1)\pi}{\eta}, \end{cases}$$

and $|\chi_j'(t)| \leq C_1$, $|\chi_j''(t)| \leq C_1$ for some fixed constant C_1 , where $j = 3, 4, \dots$. Define

$$u_j(x) = \chi_j(\rho(x)) \sin(\eta\rho(x)),$$

where $\rho(x) = \text{dist}(x, S)$. It is clear that $\dim(\text{span}\{u_j, j = 3, 4, \dots\}) = +\infty$. We are going to prove that

$$\frac{\|\Delta u_j + \eta^2 u_j\|^2}{\|u_j\|^2} \rightarrow 0, \quad (3.7)$$

where $\|\cdot\|$ is the L^2 -norm. We divide the remainder of the proof into three steps.

Step 1. We want to estimate $\|u_j\|^2$:

$$\begin{aligned} \|u_j\|^2 &= \int_M |u_j|^2 dV \\ &= \int_0^{+\infty} \left(\int_{S_r} \chi_j^2(\rho(x)) \sin^2(\eta\rho(x)) dS \right) dr \\ &\geq \int_{2\pi/\eta}^{j\pi/\eta} v(r) \sin^2 \eta r dr \\ &= \sum_{k=2}^{j-1} \int_{k\pi/\eta}^{(k+1)\pi/\eta} v(r) \sin^2 \eta r dr \\ &\geq \frac{1}{2} \sum_{k=2}^{j-1} \left[V \left[\left(k + \frac{3}{4} \right) \frac{\pi}{\eta} \right] - V \left[\left(k + \frac{1}{4} \right) \frac{\pi}{\eta} \right] \right]. \end{aligned}$$

From (3.2) we know that

$$\begin{aligned} \frac{V[(k + \frac{3}{4})\pi/\eta] - V[(k + \frac{1}{4})\pi/\eta]}{V[(k+2)\pi/\eta] - V[(k+1)\pi/\eta]} &\geq \frac{[(k + \frac{3}{4})\pi/\eta]^{B+1} - [(k + \frac{1}{4})\pi/\eta]^{B+1}}{[(k+2)\pi/\eta]^{B+1} - [(k+1)\pi/\eta]^{B+1}} \\ &\geq \frac{B(k + \frac{1}{4})^B (\frac{3}{4} - \frac{1}{4})}{B(k+2)^B} \end{aligned}$$

$$\geq \frac{1}{(1+2/k)^B} \geq \frac{1}{2^{2B+1}}, \quad \text{when } k \geq 2,$$

where we have used the mean value theorem; that is for any C^1 function on $[a, b]$ there exists $\xi \in (a, b)$ such that $f(b) - f(a) = f'(\xi)(b - a)$. Then,

$$\begin{aligned} \|u_j\|^2 &\geq \frac{1}{2^{2(B+1)}} \sum_{k=2}^{j-1} \left[V \left[(k+2) \frac{\pi}{\eta} \right] - V \left[(k+1) \frac{\pi}{\eta} \right] \right] \\ &= \frac{1}{2^{2(B+1)}} \left\{ V \left[(j+1) \frac{\pi}{\eta} \right] - V \left(\frac{3\pi}{\eta} \right) \right\}. \end{aligned} \quad (3.8)$$

Step 2. We want to estimate this upper bound of $\|\Delta u_j + \eta^2 u_j\|^2$.

$$\begin{aligned} \nabla u_j &= (\chi_j'(\rho(x)) \sin \eta \rho(x) + \eta \chi_j(\rho(x)) \cos \eta \rho(x)) \nabla \rho \\ \Delta u_j &= (\chi_j''(\rho(x)) \sin \eta \rho(x) + 2\eta \chi_j'(\rho(x)) \cos \eta \rho(x) - \eta^2 \chi_j(\rho(x)) \sin \eta \rho(x)) \\ &\quad + (\chi_j'(\rho(x)) \sin \eta \rho(x) + \eta \chi_j(\rho(x)) \cos \eta \rho(x)) \Delta \rho. \end{aligned}$$

Then

$$\begin{aligned} |\Delta u_j + \eta^2 u_j|^2 &\leq 3\{|\chi_j''(\rho(x)) \sin \eta \rho(x) + 2\eta \chi_j'(\rho(x)) \cos \eta \rho(x)|^2 \\ &\quad + |\chi_j'(\rho(x)) \sin \eta \rho(x) \Delta \rho|^2 + |\eta \chi_j(\rho(x)) \cos \eta \rho(x)|^2\} \\ &\leq C_2 + 2\eta^2 |\Delta \rho|^2. \end{aligned}$$

So

$$\begin{aligned} \|\Delta u_j + \eta^2 u_j\|^2 &= \int_0^{+\infty} \left(\int_{S_r} |\Delta u_j + \eta^2 u_j|^2 dS \right) dr \\ &\leq \int_{[\pi/\eta, 2\pi/\eta] \cup [j\pi/\eta, (j+1)\pi/\eta]} \left(\int_{S_r} 2|\chi_j'' \sin \eta \rho(x) + 2\eta \chi_j' \cos \eta \rho(x)|^2 dS \right) dr \\ &\quad + \int_{[\pi/\eta, 2\pi/\eta] \cup [j\pi/\eta, (j+1)\pi/\eta]} \left(\int_{S_r} 2|\chi_j' \sin \eta \rho(x) \Delta \rho|^2 dS \right) dr \\ &\quad + \int_{\pi/\eta}^{(j+1)\pi/\eta} \left(\int_{S_r} |\eta \chi_j(\rho(x)) \cos \eta \rho(x) \Delta \rho|^2 dS \right) dr \\ &\leq C_3 + C_4 \left[V \left[(j+1) \frac{\pi}{\eta} \right] - V \left(\frac{j\pi}{\eta} \right) \right] + 2\eta^2 \int_{\pi/\eta}^{(j+1)\pi/\eta} v(r) \frac{B^2}{r^2} dr \end{aligned}$$

$$\begin{aligned}
&\leq C_3 + C_4 \left[V \left[(j+1) \frac{\pi}{\eta} \right] - V \left(\frac{j\pi}{\eta} \right) \right] \\
&\quad + 2\eta^2 \sum_{k=1}^j \frac{B^2}{(k\pi/\eta)^2} \left[V \left[\frac{(k+1)\pi}{\eta} \right] - V \left(\frac{k\pi}{\eta} \right) \right], \tag{3.9}
\end{aligned}$$

where C_3, C_4 are positive constants.

Step 3. We will prove (3.7) in this step. We can see from [CGT, Theorem 4.9] that $V(r) \rightarrow +\infty$, as $t \rightarrow +\infty$, and $C_n r^n \geq V(r) \geq C \log r$ for large r and some constants C and C_n . Then

$$0 \leq \frac{V((j+1)\pi/\eta) - V(j\pi/\eta)}{V((j+1)\pi/\eta)} \leq \frac{(j+1)^{B+1} - j^{B+1}}{j^{B+1} - 1} \rightarrow 0$$

and

$$\begin{aligned}
0 &\leq \frac{\sum_{k=1}^j (B^2/(k\pi/\eta)^2)(V((k+1)\pi/\eta) - V(k\pi/\eta))}{V((j+1)\pi/\eta)} \\
&= \frac{\left(\sum_{k=1}^{\lfloor \log \log j \rfloor} + \sum_{k=\lfloor \log \log j \rfloor+1}^j \right) (B^2/(k\pi/\eta)^2)(V((k+1)\pi/\eta) - V(k\pi/\eta))}{V((j+1)\pi/\eta)} \\
&\leq \frac{V(\lfloor \log \log j + 1 \rfloor \pi/\eta)}{V((j+1)\pi/\eta)} + \frac{1}{(\lfloor \log \log j + 1 \rfloor)^2} \\
&\quad \times \frac{V((j+1)\pi/\eta) - V(\lfloor \log \log j + 1 \rfloor \pi/\eta)}{V((j+1)\pi/\eta)},
\end{aligned}$$

where $[x]$ denotes the largest integer which is not bigger than x . Both terms of the last formula tend to zero as $j \rightarrow +\infty$. This proves our theorem. $\square$

Remark 3.2. There are many models satisfying the condition of our theorem in [GW].

When M is a manifold with a pole, by Lemma 2.3 and the proof of Theorem 3.1 we have the following.

THEOREM 3.2. *Let M be a complete Riemannian manifold with a pole and with Ricci curvature bounded below by $-A\rho^{-2}(x)$ for some positive constant $A < n/(n-1)$, where $\rho(x)$ is the distance function to the pole p_0 . Then $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, +\infty)$.*

THEOREM 3.3. *Let M be a Cartan–Hadamard manifold with Ricci curvature bounded below by $-A\rho^{-2}(x)$ for some positive constant, where $\rho(x)$ is the distance function to the pole p_0 . Then $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, +\infty)$.*

4. The non-existence of L^2 eigenvalues

In this section we use the results proved in [EF] and the estimates in Section 2 to obtain a criterion for the non-existence of L^2 eigenfunctions on complete Riemannian manifolds with some curvature restrictions.

The following proposition was obtained in [EF, Theorem 3.3] and is quoted here for the sake of completeness.

PROPOSITION 4.1. *Let M be a complete Riemannian manifold with infinite volume. Assume there exists a function $f \in C^2(M)$ satisfying the following conditions:*

- (i) $|\nabla f| \leq \Phi \circ \rho$ and $|Hf|$ is bounded on M , where $\int_1^{+\infty} 1/\Phi(t) dt = +\infty$;
- (ii) for some constant $k \in \mathbb{R}$

$$|X|^2 \Delta f - 2Hf(X, X) \leq k|X|^2 \leq |X|^2 \Delta f \quad (4.1)$$

for all $x \in M$, $X \in T_x M$.

- (iii) f is strictly convex at some point.

Then there are no eigenvalues for $-\Delta$ in $[0, +\infty)$.

LEMMA 4.1. *Let M be an n -dimensional ($n \geq 3$) complete Riemannian manifold with a pole p_0 and with infinite volume. Assume there exist constants $B_1 \in [1, +\infty)$ and $B_2 \in (0, 1]$ such that $B_2 \geq ((n-2)/n)B_1$ and*

$$\frac{B_2}{\rho} |X|^2 \leq H\rho(X, X) \leq \frac{B_1}{\rho} |X|^2 \quad (4.2)$$

for all $X \in T_x M$ and $\rho(x) = \text{dist}(x, p_0)$. Then $-\Delta$ has no L^2 eigenvalues.

Proof. Let $f(x) = \frac{1}{2}\rho^2(x)$. (i) and (iii) in Proposition 4.1 are satisfied; we only need to prove that (ii) is also satisfied under the assumptions of Lemma 4.1. In fact, we can write $X \in T_x M$ as $X = X_T + X_\rho \nabla \rho$, $\langle X_T, \nabla \rho \rangle = 0$. Define

$$h_0 = \inf\{Hf(X, X) \mid X \in T_x M, |X| = 1\},$$

$$h_1 = \sup\{Hf(X, X) \mid X \in T_x M, |X| = 1\}.$$

Since $Hf(X, X) = X_\rho^2 + \rho H\rho(X_T, X_T)$, then $h_0(x) \geq B_2$, $h_1(x) \leq B_1$.

Set $k = (nB_1 + (n-2)B_2)/2$. Then

$$\begin{aligned} \Delta f - k &\geq (n-1)B_2 + B_1 - k \\ &= \frac{(2-n)B_1 + nB_2}{2} \geq 0 \end{aligned}$$

$$\begin{aligned}
\Delta f - k &\leq (n-1)B_1 + B_2 - k \\
&= \frac{(n-2)B_1 - (n-4)B_2}{2} \\
&\leq \frac{(n-2)B_1(1 - (n-4)/n)}{2} \\
&= \frac{2(n-2)}{n}B_1 \leq 2B_2.
\end{aligned}$$

Thus (ii) is satisfied. This completes the proof. $\square$

The proof of the following theorem is straightforward by Lemma 2.2 and Lemma 4.1.

THEOREM 4.1. *Let M be an n -dimensional ($n \geq 3$) complete Riemannian manifold with a pole p_0 and infinite volume. Assume the sectional curvature of M satisfies*

$$-\frac{A}{\rho^2} \leq K(x) \leq \frac{A_1}{\rho^2} \quad (4.3)$$

for $\rho(x) = \text{dist}(x, p_0) > 0$, where A and $A_1 < \frac{1}{4}$ are non-negative constants satisfying $(n-2)\sqrt{1+4A} - n\sqrt{1-4A_1} \leq 2$. Then $-\Delta$ has no L^2 eigenvalues.

THEOREM 4.2. *Let M be an n -dimensional ($n \geq 3$) Cartan–Hadamard manifold with sectional curvatures satisfying*

$$K(x) \geq -\frac{A}{\rho^2(x)} \quad (4.4)$$

for $\rho(x) = \text{dist}(x, 0) > 0$, where A is a non-negative constant satisfying $A \leq 2n/(n-2)^2$. Then $-\Delta$ has no L^2 eigenvalues.

Proof. In this case, the assumptions of Lemma 4.1 except (4.2) are clearly satisfied. (4.2) is satisfied with $B_2 = 1$, $B_1 = (1 + \sqrt{1+4A})/2$ since $A \leq 2n/(n-2)^2$ implies

$$B_1 \leq \frac{1 + (n+2)/(n-2)}{2} = \frac{n}{n-2}.$$

Thus the conclusion of Theorem 4.2 follows from Lemma 4.1. $\square$

By similar methods presented above, we have the results in the two-dimensional case. We shall omit some details.

LEMMA 4.2. *Let M be a two-dimensional complete Riemannian manifold with a pole p_0 and with infinite volume. Assume there exist constants $B_1 \in (0, +\infty)$ and $B_2 \in (0, +\infty)$ such that $B_1 \leq 3B_2$,*

$$\frac{B_2}{\rho} |X|^2 \leq H\rho(X, X) \leq \frac{B_1}{\rho} |X|^2$$

for all $X \in T_x M$ and $\rho(x) = \text{dist}(x, p_0)$. Then $-\Delta$ has no L^2 eigenvalues.

Proof. Let $f(x) = \frac{1}{2}\rho^2(x)$. Then $f \in C^2(M)$ and $|\nabla f| \leq \rho(x)$, $X = X_\rho \nabla \rho + X_T$, with $\langle X_T, \nabla \rho \rangle = 0$. We know that

$$Hf(X, X) = X_\rho + \rho \Delta \rho |X_T|^2.$$

Since $\Delta f = 1 + \rho \Delta \rho$, $1 + B_2 \leq \Delta f \leq 1 + B_1$ and

$$|X|^2 \Delta f - 2Hf(X, X) \leq |X|^2 (\Delta f - 2B_2) \leq |X|^2 (1 + B_2).$$

This shows that f satisfies condition (ii) of Proposition 4.1 for $k = 1 + B_2$. $\square$

THEOREM 4.3. *Let M be a two-dimensional complete Riemannian manifold with a pole p_0 and infinite volume. Assume the sectional curvature of M satisfies*

$$-\frac{A}{\rho^2} \leq K(x) \leq \frac{A_1}{\rho^2}$$

for $\rho(x) = \text{dist}(x, p_0) > 0$, where $A \leq \frac{3}{4}$ and $A_1 < \frac{1}{4}$ are non-negative constants. Then $-\Delta$ has no L^2 eigenvalues.

Proof. For $B_1 = (1 + \sqrt{1 + 4A})/2$ and $B_2 = (1 + \sqrt{1 - 4A_1})/2$, it is straightforward to show that $A \leq \frac{3}{4}$ and $A_1 \leq \frac{1}{4}$ implies $B_1 \leq B_2$. $\square$

THEOREM 4.4. *Let M be a two-dimensional Cartan–Hadamard manifold with sectional curvatures satisfying*

$$K(x) \geq \frac{-A}{\rho^2(x)}$$

for $\rho(x) = \text{dist}(x, 0) > 0$, where $A \leq 1$ is a non-negative constant. Then $-\Delta$ has no L^2 eigenvalues.

5. Compactness theorem

One of the most well-known results relating the curvature and topology of a complete Riemannian manifold M is the classical theorem of Myers [My] which states that if the Ricci curvature with respect to unit vectors on M has a positive lower bound then M is compact. In 1957 Ambrose [A] published an interesting generalization of Myers' theorem. He proved that if there is a point $q \in M$ such that along each geodesic $\gamma : [0, +\infty) \rightarrow M$ emanating from q (and parametrized by arc length t) the Ricci curvature satisfies

$$\int_0^{+\infty} \text{Ric} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right) dt = +\infty, \quad (5.1)$$

then M is compact. One of the important features of this result is that the Ricci curvature is not required to be everywhere non-negative.

Another generalization of Myers' theorem is due to Calabi [Ca]. In [Ca] the non-negativity of the Ricci curvature is proposed and the condition of [Ca] is best among a certain class of conditions.

One can easily see from [A] and [Ca] that (5.1) is not optimal, though it is not contained in [Ca]. In this paper we will assume (5.1) fails and present a general technique for establishing compactness criteria for complete Riemannian manifolds. As an application of this technique we obtain a generalization of Calabi's theorem. Precisely, we will prove the following.

THEOREM 5.1. *Let M be a complete Riemannian manifold and let $r(x) = d(x, x_0)$ be the distance function from some fixed point $x_0 \in M$. Assume that $\text{Ric}_M(\partial/\partial r, \partial/\partial r) \geq (n-1)K(r(x))$ for $r(x) \geq r_0$, where r_0 is a constant and $K(r)$ is some continuous function. If $K(r)$ satisfies*

(H) $\int_t^{+\infty} K(s) ds < +\infty$ and set $P(t) = \int_t^{+\infty} K(s) ds$ we assume $P(t) > 0$ when $t \geq t_0$ for some t_0 and $\int_t^{+\infty} P^2(s) ds \geq bP(t)$, where $b > \frac{1}{4}$ is a constant, then M is compact.

The proof of Theorem 5.1 is straightforward from Lemmas 5.1 and 5.2. An application of Theorem 5.1 is the following.

COROLLARY 5.1. *Let M be a complete Riemannian manifold and let $r(x) = \text{dist}(x_0, x)$ be the distance function from some fixed point $x_0 \in M$. Assume that*

$$\text{Ric}_M \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \geq (n-1)\beta \frac{2 + \sin r - r \cos r}{r^2}$$

for $r \geq r_0$ and $\beta > \frac{3}{4}$, where r_0 is a constant. Then M is compact.

Proof. For $K(r)$ defined in (5.2) we shall use Theorem 5.1. Without loss of generality we assume $K(r_0) \geq 0$ and we know that when $t \geq r_0$

$$\begin{aligned} P(t) &= \int_t^{+\infty} K(s) ds \\ &= \beta \int_t^{+\infty} \frac{2 + \sin s - s \cos s}{s^2} ds \\ &= \frac{\beta(2 + \sin t)}{t}. \end{aligned}$$

Then for $t \geq r_0$

$$\begin{aligned} \int_t^{+\infty} P^2(s) ds &\geq \int_t^{+\infty} \frac{\beta^2}{s^2} ds \\ &= \frac{\beta^2}{t} \\ &\geq \frac{\beta}{2 + \sin t} P(t) \\ &\geq \frac{\beta}{3} P(t). \end{aligned}$$

Since $\beta > \frac{3}{4}$, the conclusion follows from Theorem 5.1 □

Remark 5.1. The condition $\int_t^{+\infty} K(s) ds < +\infty$ is not a restriction by the result of Ambrose [A]. If otherwise, the conclusion of Theorem 5.1 holds. It is easy to see that if $K(r)$ is defined as

$$K(r) = \frac{\beta(2 + \sin r - r \cos r)}{r^2}, \quad \text{for } r \geq r_0 \quad (5.2)$$

and $= K(r_0)$ for $r \in [0, r_0)$, then $\int_0^{+\infty} K(s) ds < +\infty$ and $K(s)$ changes sign. So one cannot use the results of [A] and [C] to conclude the result of Corollary 5.1.

Remark 5.2. If $K(r) = bt^{-2}$ with $b > \frac{1}{4}$, then (H) is satisfied. So Theorem 5.1 generalizes the Calabi's theorem [C].

To prove Theorem 5.1, we first prove the following.

LEMMA 5.1. *Let M be a non-compact complete Riemannian manifold and let $r(x) = d(x, x_0)$ be the distance function from some fixed point $x_0 \in M$. Assume*

that $\text{Ric}_M(\partial/\partial r, \partial/\partial r) \geq (n-1)K(r)$ for $r \geq r_0$, where r_0 is a constant and $K(r)$ is some continuous function. Then the following boundary value problem

$$\begin{cases} f'' + K(r)f = 0 \\ f(a) = f(b) = 0 \end{cases} \quad (5.3)$$

does not possess any positive solutions on (a, b) for any $b > a > 0$.

Proof. Assume for the sake of contradiction that (5.3) has a positive solution f . By Hopf–Rinow’s theorem we can find a normal geodesic $\gamma : [0, +\infty) \rightarrow M$ without conjugate points and then choose an orthonormal basis $\{e_i\}$, $i = 1, 2, \dots, n$, at x such that $e_1 = \dot{\gamma}(x)$. By defining $X_i(t) = f(t)e_i(t)$, $i = 2, \dots, n$, for some function $f(t)$ we have

$$\begin{aligned} \sum_{i=1}^n I(X_i, X_i) &= \int_a^{b_1} \{(n-1)f'^2 - (n-1)f^2 \text{Ric}(\dot{\gamma}, \dot{\gamma})\} dt \\ &\leq \int_a^{b_1} (n-1)(f'^2 - K(s)f^2) ds. \end{aligned}$$

We can find a point b_0 such that $f'_0(b_0) < 0$ and then choose $f = f_0$ and $b_1 = b_0$; we have

$$\begin{aligned} \sum_{i=1}^n I(X_i, X_i) &\leq \int_a^{b_0} (n-1)(f_0'^2 - K(s)f_0^2) ds \\ &= (n-1)f'_0(b_0)f_0(b_0) < 0. \end{aligned}$$

This is a contradiction which shows our lemma. □

We shall analyze the equation with the initial values

$$\begin{cases} f'' + K(r)f = 0 \\ f(a) = 0, \quad f'(a) = 1. \end{cases} \quad (5.4)$$

For any fixed $a > 0$ there exists a solution $f(t)$ for $t \geq a$. From Lemma 5.1 we know that $f(t) > 0$ for any $t > a$. Then we can define $w(t) = f'(t)/f(t)$, with $w(t)$ satisfying the following Ricatti equation:

$$w'(t) + w^2(t) + K(t) = 0 \quad (5.5)$$

$$\lim_{t \rightarrow a} w(t) = +\infty. \quad (5.6)$$

We shall prove the following.

LEMMA 5.2. *If $K(r)$ satisfies (H) for $t > t_0$, then (5.5) does not possess any solution on $[t_0, +\infty)$.*

Proof. Assume for the sake of contradiction that there exists a function $w(t)$ satisfying (5.5) for $t \geq t_0$. Then

$$\int_{t_0}^t w^2(s) ds + w(t) - w(t_0) + P(t) - P(t_0) = 0 \quad (5.7)$$

$$\int_{t_0}^t w(s) ds + \int_{t_0}^t ds \int_{t_0}^s w^2(\tau) d\tau = w(t_0)(t - t_0) + \int_{t_0}^t [P(s) - P(t_0)] ds.$$

Divide both sides of the above equation by t . Then

$$\frac{1}{t} \int_{t_0}^t w(s) ds + \frac{1}{t} \int_{t_0}^t ds \int_{t_0}^s w^2(s) ds = (w(t_0) - P(t_0)) \frac{(t - t_0)}{t} + \frac{1}{t} \int_{t_0}^t P(s) ds \leq C.$$

By the Schwartz inequality we know that

$$\begin{aligned} \left| \frac{1}{t} \int_{t_0}^t w(s) ds \right| &\leq \left| \frac{1}{t} \left[\int_{t_0}^t w^2(s) ds \right]^{1/2} (t - t_0)^{1/2} \right| \\ &= \left(\frac{1}{t} \int_{t_0}^t w^2(s) ds \right)^{1/2} \left(\frac{t - t_0}{t} \right)^{1/2}. \end{aligned} \quad (5.8)$$

Denote $Q(t) = \int_{t_0}^t \int_{t_0}^s w^2(\tau) d\tau ds$. Then

$$\begin{aligned} \frac{1}{t} Q(t) &\leq C + \left(\frac{1}{t} Q'(t) \right)^{1/2} \left(\frac{t - t_0}{t} \right)^{1/2} \\ &\leq C + \left(\frac{1}{t} Q'(t) \right)^{1/2}. \end{aligned}$$

Since $w^2(s) \not\equiv 0$, then $Q(t) \rightarrow +\infty$.

(i) We will show $W(s) = \lim_{T \rightarrow +\infty} \int_s^T w^2(\tau) d\tau < +\infty$. If not, we can find T such that $\frac{1}{2} Q(t) - Ct \geq 0$ for $t \geq T$. Thus,

$$\frac{1}{2} Q(t) \leq (t Q'(t))^{1/2} \quad (5.9)$$

$$\begin{aligned} \frac{Q'(t)}{Q^2(t)} &\geq \frac{1}{4t} \\ \frac{1}{Q(T_0)} - \frac{1}{Q(t)} &\geq \frac{1}{4} \log \frac{t}{T} \end{aligned} \quad (5.10)$$

for any $t > T$. Let $t \rightarrow +\infty$ then $Q(t) \rightarrow +\infty$. (5.8) yields a contradiction which shows that $W(s) < +\infty$.

(ii) From (5.6) we have

$$[w(t) - P(t)]' = -w^2(t).$$

Writing $v(t) = w(t) - P(t)$, then $v'(t) \leq 0$. So $\lim_{t \rightarrow +\infty} v(t) = c > -\infty$ or $-\infty$. Then $\lim_{t \rightarrow +\infty} w(t) = 0$, thus

$$w(t) = \int_t^{+\infty} w^2(s) ds + P(t). \quad (5.11)$$

We have $w(t) \geq P(t)$, $t \geq T$. Then

$$\begin{aligned} w(t) &\geq \int_t^{+\infty} P^2(s) ds + P(t) \\ &\geq (1+b)P(t). \end{aligned}$$

Then use (5.11) again to obtain

$$w(t) \geq [1 + (1+b)^2 b]P(t).$$

If we set $b_0 = b$, $b_1 = 1 + b$, $b_2 = 1 + b_1^2 b$, $\dots$, $b_n = 1 + b_{n-1}^2 b$ we know that

$$w(t) \geq b_n P(t).$$

So $\{b_n\}$ is bounded and $\lim_{n \rightarrow +\infty} b_n = l < +\infty$. Then $l = 1 + bl^2$; this contradicts the fact that $b > \frac{1}{4}$. $\square$

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